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# ANALYTICAL METHODS OF THE THEORY OF INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

**Yu. E. Anikonov** *

Sobolev Institute of Mathematics SB RAS
4, pr. Akad. Koptyuga and
Novosibirsk State University
2, ul. Pirogova, Novosibirsk 630090, Russia
anikon@math.nsc.ru

**M. V. Neshchadim**

Sobolev Institute of Mathematics SB RAS
4, pr. Akad. Koptyuga and
Novosibirsk State University
2, ul. Pirogova, Novosibirsk 630090, Russia
neshch@math.nsc.ru

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*We obtain new presentations for solutions and coefficients of some equations of mathematical physics which are used for studying multidimensional inverse problems. Bibliography*: 13 *titles.*

Representations of solutions and coefficients of differential equations are well presented in the literature. In particular, the following topics are discussed: construction of functional-invariant solutions to the hyperbolic equation [1], analytic representations of solutions and coefficients of the parabolic equation [2], representation of solutions and coefficients of the Sturm–Liouville equation with application to inverse problems of scattering theory, construction of harmonic and other potentials for computing solutions (velocity) and coefficients (pressure) of the gas dynamic equations etc.

In this paper, we propose new representations of solutions and coefficients of parabolic equations. Such representations are partially used for studying multidimensional inverse problems.

## 1 Representation of Solutions and Coefficients of Parabolic Equations

A solution $w(x, y, t)$ to the heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - q(y)w$$

admits the representation [3]

$$w = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \int\limits_0^\infty e^{-\lambda t} \varphi(y, \lambda) d\sigma(\lambda),$$

* To whom the correspondence should be addressed.

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where $\varphi(y,\lambda)$ is such that

$$-\frac{\partial^2\varphi}{\partial y^2}+q(y)\varphi=\lambda\varphi.$$

This representation was used to solve the inverse problem for finding the coefficient $q(y)$.

According to [4], the diffusion equation is represented by the parabolic equation

$$\rho(x,t)\frac{\partial w}{\partial t}=\operatorname{div}\left(p(x,t)\nabla w\right)-q(x,t)w+F(x,t).$$

Moreover, the heat equation has the form

$$c(x,t)\rho(x,t)\frac{\partial w}{\partial t}=\operatorname{div}\left(k(x,t)\nabla w\right)+F(x,t),$$

where $w$ is the temperature, $\rho$ is the density, $c$ is the heat capacity, $k$ is the heat conductivity, and $F$ is the source function. The parabolic diffusion equation with transfer taken into account and applications can be found in [5]:

$$\frac{\partial w}{\partial t}=V_1\frac{\partial w}{\partial x}+V_2\frac{\partial w}{\partial y}+V_3\frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left(\gamma\frac{\partial w}{\partial z}\right)+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+F(x,y,z,t).$$

An interesting interpretation of solutions and coefficients of the parabolic equation is given in [6]. Thus, for the equation

$$\frac{\partial w}{\partial t}=\operatorname{div}\left(k(x,y)\nabla w\right)-q(x,y),\quad (x,y)\in D\subset\mathbb{R}^2,\quad t>0,$$

the solution $w(x,y,t)$ and coefficients $k(x,y)$, $q(x,y)$ are interpreted as follows: $w(x,y,t)$ is the goods cost, $k(x,y)$ is the the goods conductivity coefficient, and $q(x,y)$ is the difference between demand and supply.

We denote by $F_j(z,p)$, $j=1,2$, the fundamental system of solutions to the following linear second order ordinary differential equation with parameter $0\leqslant p<\infty$:

$$F^{''}(z)+b(z)F^{'}(z)+(pa(z)+c(z))F(z)=0,$$

where $c(z)$, $b(z)$, and $a(z)$ are meromorphic functions. Let $a_{ij}(x)$, $a_j(x)$, $i,j=1,2,\dots,n$, $x=(x_1,\dots,x_n)$ be fixed continuously differentiable functions in $D\subset\mathbb{R}^n$, $n\geqslant 1$, and let $\psi(x)$ be a solution to the linear equation

$$L\psi\equiv\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2\psi}{\partial x_i\partial x_j}+\sum_{j=1}^{n}a_j(x)\frac{\partial\psi}{\partial x_j}=0.$$

We introduce a function $\varphi(x)$ by the equality

$$\psi(x)=\int_0^{\varphi(x)}\exp\left(-\int_0^{s}b(z)dz\right)ds$$

assuming that such a function exists [7]. The function $\varphi(x)$ satisfies the nonlinear equation [7]

$$\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}+\sum_{j=1}^{n}a_j(x)\frac{\partial\varphi}{\partial x_j}=b(\varphi(x))\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}.$$

We denote by $Q(p)$, $R(p)$, $p \in \mathbb{R}$, integrable functions that are rapidly decreasing at infinity (for example, smooth and compactly supported functions).

**Lemma 1.** *The functions $w(x,t)$, $\lambda(x)$, and $\mu(x)$ defined by*

$$w(x,t) = \int_0^\infty [Q(p)F_1(\varphi(x),p) + R(p)F_2(\varphi(x),p)]\, e^{-pt} dp,$$

$$\lambda(x) = a(\varphi(x)) \sum_{i,j=1}^n a_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j},$$

$$\mu(x) = c(\varphi(x)) \sum_{i,j=1}^n a_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}$$

*satisfy the first order evolution equation*

$$\lambda(x) \frac{\partial w}{\partial t} = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{j=1}^n a_j(x) \frac{\partial w}{\partial x_j} + \mu(x) w.$$

**Example 1.** Let $F_1(y,p)$ and $F_2(y,p)$ be the fundamental system of solutions to the Sturm–Liouville equation $F''(y) = (c(y) - p)F(y)$ with parameter $p \geqslant 0$, and let $u(x) \neq 0$, $x \in D \subset \mathbb{R}^n$ be a harmonic function such that $\nabla u(x) \neq 0$. Then the function

$$w(x,t) = \int_0^\infty [Q(p)F_1(u(x),p) + R(p)F_2(u(x),p)]\, e^{-pt} dp$$

satisfies the parabolic equation

$$\frac{\partial w}{\partial t} = \frac{1}{|\nabla u|^2} \Delta w - c(u) w.$$

For nonlinear equations the following assertion holds.

**Lemma 2.** *Suppose that*

1) $y \in \mathbb{R}$, $t_0 \leqslant t \leqslant t_1$, $x = (x_1, \ldots, x_n) \in D$, $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$,

2) $\varphi = \varphi(x,t)$ *and* $\psi = \psi(x,t) \neq 0$ *are twice continuously differentiable functions such that*

$$\beta\, |\nabla \rho|^2 \frac{\partial \varphi}{\partial t} + \alpha \Delta \varphi = 0,$$

$$\beta\, |\nabla \rho|^2 \frac{\partial \psi}{\partial t} + \alpha \Delta \psi = 0, \quad \rho = \frac{\varphi}{\psi},$$

3) $F(y,t) = (F_1, \ldots, F_m)$ *and* $a(y,z) = (a_1, \ldots, a_m)$ *are twice continuously differentiable vector-valued functions such that*

$$\beta \frac{\partial F}{\partial t} + \alpha \frac{\partial^2 F}{\partial y^2} = a(y, F(y,t)).$$

*Then the vector-valued function $w(x,t) = \psi(x,t) F(\rho(x,t),t)$ is a solution to the system*

$$\beta\, |\nabla \rho|^2 \frac{\partial w}{\partial t} + \alpha \Delta w = \psi\, |\nabla \rho|^2 a(\rho, w/\psi).$$

If $F$ is independent of $t$, then the following assertion holds.

**Lemma 3.** *Suppose that* $y \in \mathbb{R}$, $t_0 \leqslant t \leqslant t_1$, $x = (x_1, \dots, x_n) \in D \subset \mathbb{R}^n$, $u = u(x,t)$, $F = (F_1, \dots, F_m)(y)$, $a = (a_1, \dots, a_m)(y, z)$, $z = (z_1, \dots, z_m) \in \mathbb{R}^m$. *If*

$$\alpha \frac{\partial^2 F}{\partial y^2} = a(y, F(y)),$$

$$\beta \frac{\partial u}{\partial t} + \alpha \Delta u = 0$$

*for some, generally speaking, complex constants* $\alpha$ *and* $\beta$, *then the vector-valued function* $w(x,t) = F(u(x,t))$ *is a solution to the system*

$$\beta \frac{\partial w}{\partial t} + \alpha \Delta w = |\nabla u|^2 a(u(x,t), w).$$

**Example 2.** If $\beta = i\hbar$, $\alpha = \hbar^2/(2m_0)$, $\dfrac{\partial F}{\partial t} = 0$, and $\psi = 1$, then we obtain, generally speaking, the nonlinear Schródinger equations

$$i\hbar \frac{\partial w}{\partial t} + \frac{\hbar^2}{2m_0} \Delta w = |\nabla \varphi|^2 a(\varphi(x,t), w).$$

**Lemma 4.** *Suppose that*

1) $V(x,t)$, $x = (x_1, \dots, x_n) \in D$, $t \in \mathbb{R}$, *is an arbitrary twice differentiable function such that* $\nabla V \neq 0$,

2) $B$ *and* $C$ *are constants such that* $B - CV(x,t) \neq 0$,

3) $F(y,t)$ *is an arbitrary solution to the equation*

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial y^2}, \quad y \in \mathbb{R}.$$

*Then the functions* $w(x,t)$, $k(x,t)$, $A^i(x,t)$, $i = 1, \dots, n$, *defined by*

$$w(x,t) = \frac{1}{B - CV(x,t)} F(V(x,t), t), \tag{1}$$

$$k(x,t) = \frac{1}{|\nabla V(x,t)|^2}, \tag{2}$$

$$A^i(x,t) = A_0^i(x,t) - \frac{1}{|\nabla V(x,t)|^2} \left( \frac{2C}{B - CV(x,t)} + \frac{\Delta V(x,t)}{|\nabla V(x,t)|^2} - \frac{\partial V}{\partial t} \right) \frac{\partial V}{\partial x_i},$$

*satisfy the equation*

$$\frac{\partial w}{\partial t} = k(x,t) \Delta w + \sum_{i=1}^{n} A^i \frac{\partial w}{\partial x_i},$$

*where* $A_0^i(x,t)$, $i = 1, \dots, n$, *are functions such that*

$$\sum_{i=1}^{n} A_0^i \frac{\partial V}{\partial x_i} = 0.$$

**Remark 1.** To the solution $w(x,t)$ we can add any solution $\widetilde{w}(x,t)$ to the equation with the same coefficients, i.e.,

$$w(x,t) = \frac{1}{B - CV(x,t)} F(V(x,t),t) + \widetilde{w}(x,t).$$

Such a procedure is often required in order to satisfy initial-boundary conditions.

**Corollary 1.** *Let the assumptions of Lemma* 4 *be satisfied, and let* $A_0^i(x,t) = 0$, $i = 1, \dots, n$. *Then*

$$\frac{\partial w}{\partial t} = \frac{1}{|\nabla V(x,t)|^2} \Delta w - \frac{1}{|\nabla V(x,t)|^2} \left( \frac{2C}{B - CV(x,t)} + \frac{\Delta V}{|\nabla V(x,t)|^2} - \frac{\partial V}{\partial t} \right) \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial V}{\partial x_i},$$

*where* $w(x,t)$ *is defined by* (1).

**Example 3.** In the identity of Corollary 1, we put $n = 1$,

$$F(y,t) = \frac{1}{\sqrt{\pi t}} \exp(-(y - y_0)^2/(4t)),$$

$B = 1$, $C = 0$, and $V(x,t) = a(t)x + b(t)$, where $a(t)$ and $b(t)$ are continuously differentiable functions, $a(t) \neq 0$. Then we obtain relations similar to the Kolmogorov formulas [2]:

$$w(x,t) = F(V(x,t),t) = \frac{1}{\sqrt{\pi t}} \exp\left( -\frac{(x - P(t))^2}{Q(t)} \right),$$

$$\frac{\partial w}{\partial t} = C(t) \frac{\partial^2 w}{\partial x^2} + (A(t)x + B(t)) \frac{\partial w}{\partial x},$$

where

$$Q(t) = 4ta^2(t), \quad P(t) = \frac{b(t) - y_0}{a(t)}, \quad C(t) = \frac{1}{a^2(t)}, \quad A(t) = \frac{a'(t)}{a(t)}, \quad B(t) = \frac{b'(t)}{a(t)}.$$

**Corollary 2.** *Let the assumptions of Corollary* 1 *be satisfied, and let* $V(x,t)$ *be a solution to the equation*

$$\frac{2C}{B - CV} + \frac{\Delta V}{|\nabla V|^2} - \frac{\partial V}{\partial t} = 0.$$

*Then the function* $w(x,t)$ *defined by* (1) *satisfies the equation*

$$\frac{\partial w}{\partial t} = \frac{1}{|\nabla V(x,t)|^2} \Delta w$$

*which takes the following form in the case* $n = 1$*:*

$$\frac{\partial V}{\partial t} = \frac{2C}{B - CV(x)} + \frac{\partial^2 V}{\partial x^2} \bigg/ \left( \frac{\partial V}{\partial x} \right)^2.$$

*If* $T(t)$ *and* $X(x)$ *are defined by the equalities*

$$T(t) = \pm \frac{1}{(C_1 - 2\alpha C t)^{1/2}}, \qquad \int \exp\left( \frac{\alpha}{2X^2} \right) dX = C_2 x + C_3,$$

*where* $\alpha$, $C_1$, $C_2$, *and* $C_3$ *are constants, then the function*

$$V = \frac{B}{C} - \frac{1}{CT(t)X(x)}$$

*is a solution to this equation.*

**Lemma 5.** *Let the assumptions of Lemma 4 be satisfied, and let $V(x,t)$ be a solution to the equation*

$$\frac{\partial V}{\partial t} = \frac{\Delta V}{|\nabla V|^2} + \frac{2C}{B - CV} - \frac{2}{|\nabla V|^4} \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j}. \tag{3}$$

*Then the function $w(x,t)$ defined by (1) satisfies the divergence form equation*

$$\frac{\partial w}{\partial t} = \operatorname{div}\,(k(x,t)\nabla w),$$

*where the coefficient $k(x,t)$ is defined by (2).*

In the case $n = 1$, Equation (3) takes the form

$$\frac{\partial V}{\partial t} = \frac{2C}{B - CV(x)} - \frac{\partial^2 V}{\partial x^2} \bigg/ \left(\frac{\partial V}{\partial x}\right)^2.$$

If $T(t)$ and $X(x)$ are defined by the equalities

$$T(t) = \pm \left(6\alpha C^2 t + C_1\right)^{3/2}, \qquad \int \exp\left(\frac{3\alpha}{2} X^{2/3}\right) dX = C_2 x + C_3,$$

where $\alpha$, $C_1$, $C_2$, and $C_3$ are constants, then the function $V = \frac{1}{C}(B - \sqrt[3]{T(t)X(x)})$ is a solution.

Consider the differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i(x,t) \frac{\partial}{\partial x_i} + c(x,t),$$

where $x = (x_1, \dots, x_n)$, $a_{ij}(x,t) = a_{ji}(x,t)$, $b_i(x,t)$, and $c(x,t)$ are sufficiently smooth functions of $(x,t)$. Then for the parabolic equation

$$\frac{1}{\lambda^2} \frac{\partial w}{\partial t} = Lw \tag{4}$$

the following assertion holds [8, 9].

**Lemma 6.** *Let $\varphi(x,t)$ and $\psi(x,t)$ satisfy the nonlinear system*

$$\begin{aligned} \frac{1}{\lambda^2} \frac{\partial \varphi}{\partial t} &= L\varphi, \\ \frac{1}{\lambda^2} \frac{\partial \psi}{\partial t} &= L\psi, \end{aligned} \tag{5}$$

*where*

$$\frac{1}{\lambda^2} = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial}{\partial x_i}\left(\frac{\varphi}{\psi}\right) \frac{\partial}{\partial x_j}\left(\frac{\varphi}{\psi}\right),$$

*and let $F(y,t)$ be an arbitrary solution to the parabolic equation*

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial y^2}.$$

*Then the function $w(x,t) = \psi F(\varphi/\psi, t)$ is a solution to Equation (4).*

**Corollary 3.** *If the coefficients $a_{ij}$, $b_i$, and $c$ of the operator $L$ are independent of $t$, then the solution to the nonlinear system* (5) *are functions $\varphi(x)$ and $\psi(x)$ such that $L\varphi = 0$ and $L\psi = 0$. Moreover, if $\varphi(x) \neq 0$ and $\psi(x) \neq 0$, then*

$$\frac{1}{\lambda^2} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i}\left(\frac{\varphi}{\psi}\right) \frac{\partial}{\partial x_j}\left(\frac{\varphi}{\psi}\right), \quad w(x,t) = \psi F\left(\frac{\varphi}{\psi}, t\right)$$

*and, consequently, the search of $\lambda(x)$ can be conditioned by the boundary data $\varphi(x)$ and $\psi(x)$.*

The above formulas can be further generalized, in particular, by simultaneous transformation of coordinates and solutions [10]–[12]. We give a general scheme.

**Lemma 7.** *Suppose that*

*1) $D \subset \mathbb{R}^n$ is a domain with smooth boundary $\gamma = \partial D$ and $\nu$ is the unit normal vector to $\gamma$,*

*2) $v(x)$ is a vector-valued function realizing a diffeomorphism from the domain $D$ onto its image $v(D) \subset \mathbb{R}^n$,*

*3) $u(x) = (u_{ij}(x))$, $i, j = 1, \dots, m$, $m \geqslant 1$, $x \in D$, is a matrix-valued function possessing the inverse $u^{-1}(x)$ (in the sense of matrix multiplicataion),*

*4) $G(z)$, $z = (z_1, \dots, z_m)$, is a vector-valued function realizing a diffeomorphism from $\mathbb{R}^m$ onto itself,*

*5) $w_0(x) = (w_{01}(x), \dots, w_{0m}(x))$ is a vector-valued function.*

*We assume that $F(y,t) = (F_1, \dots, F_m)$ solve the evolution equation*

$$\frac{\partial F}{\partial t} = L(F), \quad y \in D \subset \mathbb{R}^n, \quad t \geqslant 0,$$

*with the initial condition*

$$F|_{t=0} = u^{-1}(v^{-1}(y)) G(w_0(v^{-1}(y))), \quad y \in D,$$

*where $L$ is a nonlinear operator. Then the implicit solution $w(x,t)$ to the equation*

$$G(w(x,t)) = u(x) F(v(x), t) \tag{6}$$

*satisfies the evolution equation*

$$\frac{\partial w}{\partial t} = B_G(w)$$

*and the initial condition*

$$w|_{t=0} = w_0(x), \quad x \in D,$$

*where*

$$B_G(w) = \left(\frac{\partial G}{\partial w}\right)^{-1} u(x) L(u^{-1}(x) G(w));$$

*here, $\dfrac{\partial G}{\partial w}$ is the Jacobi matrix of the mapping $G(w)$. Moreover, if*

$$w|_\gamma = \alpha(s,t), \quad \frac{\partial w}{\partial \nu}\Big|_\gamma = \beta(s,t), \quad u|_\gamma = 1, \quad \frac{\partial u}{\partial \nu}\Big|_\gamma = 0, \quad v|_\gamma = x, \quad \frac{\partial v}{\partial \nu}\Big|_\gamma = \nu, \tag{7}$$

*then*

$$F|_\gamma = G(\alpha(s,t)), \quad \frac{\partial F}{\partial \nu}\Big|_\gamma = \frac{\partial G}{\partial \alpha} \beta(s,t).$$

We note that if $G(z) = z$, then the boundary data $\alpha(s,t)$, $\beta(s,t)$ for $w(x,t)$ and $F(y,t)$ coincide, which is important in applications; moreover, if we additionally suppose that the leading part of the operator $B_G$ is selfadjoint or the coefficients at lower order derivatives in the operator vanishes, then $u(x)$ and $y = v(x)$ satisfy a differential equation (cf. Example 4).

For the first example, we consider the transformed Poisson formula with an essential arbitrariness, which can be used in the inverse and some other problems.

**Example 4.** Let $m = 1$ in Lemma 7, and let $F(y,t)$ be a solution to the parabolic equation

$$\frac{\partial F}{\partial t} = \Delta F.$$

Then

$$\widetilde{u}(y) = \frac{1}{u(V^{-1}(y))}, \quad y \in \mathbb{R}^n, \, a_{ij}(x) = \sum_{k=1}^{n} \frac{\partial V_i^{-1}}{\partial y_k} \frac{\partial V_j^{-1}}{\partial y_k}\bigg|_{y=V(x)},$$

$$a_j(x) = \frac{1}{\widetilde{u}(y)} \left[ \sum_{k=1}^{n} 2 \frac{\partial \widetilde{u}}{\partial y_k} \frac{\partial V_j^{-1}}{\partial y_k} + \widetilde{u}(y) \Delta V_j^{-1} \right]\bigg|_{y=V(x)}, \quad a(x) = \frac{1}{\widetilde{u}(y)} \Delta \widetilde{u}\bigg|_{y=V(x)},$$

$$w(x,t) = G^{-1}\left( \frac{u(x)}{\pi^{n/2}} \int_{\mathbb{R}^n} \frac{G(w_0(V^{-1}(V(x) + 2\xi\alpha\sqrt{t})))}{u(V^{-1}(V(x) + 2\xi\alpha\sqrt{t}))} e^{-\xi^2} d\xi \right),$$

$$\frac{\partial w}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \frac{G''(w)}{G'(w)} + \sum_{j=1}^{n} a_j(x) \frac{\partial w}{\partial x_j} + a(x) \frac{G(w)}{G'(w)},$$

$$w\Big|_{t=0} = w_0(x), \quad x \in \mathbb{R}^n.$$

In the case $u(x) = 1$, $G(z) = z$, and $V(x) = x$, we obtain the Poisson formula. We emphasize some interesting combinations of direct and inverse mappings in the formula for $w(x,t)$.

**Remark 2.** If it is known that $a_j(x) = 0$ and $a(x) = 0$, then we obtain the following differential relations for $V(x)$ and $u(x)$:

$$\Delta \widetilde{u}(y) = 0, \widetilde{u}(y) = \frac{1}{u(V^{-1}(y))}, \quad y \in \mathbb{R}^n,$$

$$2 \sum_{k=1}^{n} \frac{\partial \widetilde{u}}{\partial y_k} \frac{\partial V_j^{-1}}{\partial y_k} + \widetilde{u}(y) \Delta V_j^{-1} = 0, \quad j = 1, \ldots, n,$$

which can be used for computing $a_{ij}(x)$ in some particular inverse problems, possibly, with (7).

An interesting example of the general scheme of Lemma 7 is obtained by using formulas for the solution to the Cauchy problem for the parabolic equation with variable coefficients.

**Example 5.** Following [13], we consider the equation

$$\frac{\partial F}{\partial t} = \Delta_\mu F, \tag{8}$$

where $F = F(y,t)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\Delta_\mu$ is the second order differential operator defined by

$$\Delta_\mu = \frac{1-|y|^2}{4}\left((1-|y|^2)\sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} - 2\mu \sum_{j=1}^n y_j \frac{\partial}{\partial y_j} + \mu(2-n-\mu)\right),$$

where $|y|^2 = y_1^2 + \ldots + y_n^2$. If $\mu = 2-n$, then $\Delta_{2-n}$ is the Laplace–Beltrami operator associated with the metric $\dfrac{4|dy|^2}{(1-|y|^2)^2}$, where $|dy|^2 = dy_1^2 + \ldots + dy_n^2$.

We recall that the Gauss hypergeometric function is defined by the formula

$${}_2F_1(a,b,c,s) = \sum_{k=0}^\infty \frac{(a)_k (b)_k s^k}{(c)_k k!},$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\dots(a+k-1)$, $k \geqslant 1$. We define the functions

$$\Phi_\lambda(y) = (1-|y|^2)^{\frac{1-\mu+i\lambda}{2}} {}_2F_1\left(\frac{n-1+i\lambda}{2}, \frac{1-i\lambda}{2}, \frac{n}{2}, |y|^2\right),$$

$$C(\lambda) = \frac{2^{1-\mu-i\lambda}\Gamma\left(\frac{n}{2}\right)\Gamma(i\lambda)}{\Gamma\left(\frac{n-1+i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}, C_{n,\mu} = \begin{cases} 1, & n \in (2-n, 0), \\ \dfrac{\Gamma\left(\dfrac{n}{2}\right)\Gamma(1-\mu)}{\Gamma\left(\dfrac{2-\mu}{2}\right)\Gamma\left(\dfrac{n-\mu}{2}\right)}, & \end{cases}$$

$$p(t,q) = C_{n,\mu}\int_0^\infty \exp\left(-t\frac{(1-\mu)^2+\lambda^2}{4}\right)\Phi_\lambda(q)\,|C(\lambda)|^{-2}\,d\lambda.$$

Then the function

$$F(y,t) = \int_{\mathbb{R}^n} p(t, y-z)F_0(z)dz$$

is a solution to Equation (8) and satisfies the initial condition $F|_{t=0} = F_0(y)$.

By Lemma 6, for $F_0(y)$ of the form $F_0(y) = u^{-1}(v^{-1}(y))G(w_0(v^{-1}(y)))$ the function $w(x,t)$ defined by (6) is a solution to the transformed equation

$$\frac{\partial w}{\partial t} = B_G(w)$$

and satisfies the initial condition $w|_{t=0} = w_0(x)$. The arbitrariness of $G(y)$, $u(x)$, and $v(x)$ can be used in different problems, including inverse ones.

## 2 Inverse Problems

We begin to study inverse problems with a general scheme of constructing formal solutions to linear multi-dimensional inverse problems. Similar schemes of constructing solutions are also presented for nonlinear inverse problems, but with boundary conditions.

We begin with a multi-dimensional linear inverse problem for the first order evolution equation with respect to $t$: find two functions $w(x,t)$, $\lambda(x)$, $x \in D \subset \mathbb{R}^n$, $t \geqslant 0$, such that

$$\alpha \frac{\partial w}{\partial t} = Aw + f(t)\lambda(x), \tag{9}$$

$$w|_{t=0} = w_0(x), \quad x \in D, \tag{10}$$

$$w|_{\partial D} = \varphi(s,t), \quad s \in \partial D, \quad t \geqslant 0, \tag{11}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_{\partial D} = \psi(s,t), \quad s \in \partial D, \quad t \geqslant 0, \tag{12}$$

where $A$ is a linear second order elliptic operator with smooth coefficients, $D$ is a domain in $\mathbb{R}^n$ with smooth boundary $\partial D$, $\dfrac{\partial w}{\partial \nu}$ is the normal derivative on the boundary $\partial D$ of $D$, $f(t) \neq 0$ is continuous, $\alpha$ is a constant, and the continuously differentiable functions $w_0(x)$, $\varphi(s,t)$, $\psi(s,t)$ are known.

**Theorem 1.** *Let $\widetilde{w}(x,t)$ be a solution to the initial-boundary-value problem*

$$\alpha \frac{\partial \widetilde{w}}{\partial t} = A\widetilde{w}, \quad \widetilde{w}|_{t=0} = w_0(x), \quad \widetilde{w}|_{\partial D} = \varphi(s,t),$$

*and let $\mu_k$ and $u_k(x)$ be eigenvalues and eigenfunctions of the problem*

$$Au_k + \mu_k u_k = 0, \quad u_k|_{\partial D} = 0, \quad k = 1, 2, \ldots.$$

*If $\widetilde{\psi}(s,t) = \psi(s,t) - \dfrac{\partial \widetilde{w}(s,t)}{\partial \nu}$, $s \in \partial D$, is represented as a formal series*

$$\widetilde{\psi}(s,t) = \sum_{k=1}^{\infty} a_k \frac{\partial u_k(s)}{\partial \nu} \int_0^t f(p) e^{-\frac{\mu_k}{\alpha}(t-p)} dp,$$

*where $a_k$ are constants, then the solution $w(x,t)$, $\lambda(x)$ for the inverse problem* (9)–(12) *is represented by the formal series*

$$w(x,t) = \sum_{k=1}^{\infty} a_k u_k(x) \int_0^t f(p) e^{-\frac{\mu_k}{\alpha}(t-p)} dp + \widetilde{w}(x,t), \lambda(x) = \alpha \sum_{k=1}^{\infty} a_k u_k(x).$$

The assertion of the theorem is directly verified. We describe the construction of the sought functions $w(x,t)$, $\lambda(x)$.

I. Solve the first boundary-value problem (look for $\widetilde{w}(x,t)$).

II. Find eigenvalues and eigenfunctions of the operator $A$ (look for $\mu_k$ and $u_k(x)$).

III. Construct the functions $\widetilde{\psi}(s,t) = \psi(s,t) - \left.\dfrac{\partial \widetilde{w}(x,t)}{\partial \nu}\right|_{x=s \in \partial D}$.

IV. Compute the constants $a_k$, $k = 1, 2, \ldots$, from the expansion of $\widetilde{\psi}(s,t)$:

$$\widetilde{\psi}(s,t) = \sum_{k=1}^{\infty} a_k \frac{\partial u_k(s)}{\partial \nu} \int_0^t f(p) e^{-\frac{\mu_k}{\alpha}(t-p)} dp,$$

which is related to the completeness of the system of functions

$$\frac{\partial u_k(s)}{\partial \nu} \int\limits_0^t f(p) e^{-\frac{\mu_k}{\alpha}(t-p)} dp.$$

V. Find the functions $w(x,t)$ and $\lambda(x)$ according to Theorem 1.

**Remark 3.** If $f(t) = \delta(t)$ is the Dirac function, then $\widetilde{\psi}(s,t)$ is represented as

$$\widetilde{\psi}(s,t) = \frac{1}{2} \sum_{k=1}^{\infty} a_k \frac{\partial u_k(s)}{\partial \nu} e^{-\frac{\mu_k}{\alpha} t}.$$

Thus, the question of the completeness of the system of functions $\dfrac{\partial u_k(s)}{\partial \nu} e^{-\frac{\mu_k}{\alpha} t}$ is determined by eigenfunctions and eigenvalues of the operator $A$.

We proceed with nonlinear inverse problems.

If $A = \Delta$, then for some special boundary conditions it is to determine not only the function $\lambda(x)$, but also the coefficient $k(x)$. Let us consider the inverse problem for the parabolic equation: find three functions $w(x,t)$, $k(x)$, and $\lambda(x)$, $x \in D \subset \mathbb{R}^n$, $t \geqslant 0$, such that

$$\frac{\partial w}{\partial t} = k(x) \Delta w + f(t) \lambda(x), \tag{13}$$

$$w|_{\partial D} = \varphi(s,t), \quad \left. \frac{\partial w}{\partial \nu} \right|_{\partial D} = \psi(s,t), \quad s \in \partial D, \tag{14}$$

$$w|_{t=0} = w_0(x), \quad x \in D, \tag{15}$$

where the continuously differentiable functions $f(t) \neq 0$, $\varphi(s,t)$, $\psi(s,t)$, $w_0(x)$ are known and $D$ is a domain in $\mathbb{R}^n$ with smooth boundary $\partial D$.

**Theorem 2.** *Assume that the function $\varphi(s,t)$ in the inverse problem* (13)–(15) *admits the analytic representation*

$$\varphi(s,t) = F(v_0(s), t) + \widetilde{\varphi}(s,t),$$

*where $F(y,t) \neq 0$ is a solution to the parabolic equation*

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial y^2}, \quad t \geqslant 0, \quad y \in \mathbb{R},$$

*$v_0(s) \neq 0$ is a continuous function, $\widetilde{\varphi}(s,t)$ is a fixed differentiable supplement which can vanish, and let $v(x)$, $x \in D \subset \mathbb{R}^n$, be a harmonic function such that $v|_{\partial D} = v_0(s)$ and $|\nabla v| \neq 0$. If $\widetilde{w}(x,t)$ is the solution to the initial-boundary-value problem*

$$\frac{\partial \widetilde{w}}{\partial t} = \frac{1}{|\nabla v(x)|^2} \Delta \widetilde{w}, \quad \widetilde{w}|_{t=0} = w_0(x) - F(v(x), 0), \quad \widetilde{w}|_{\partial D} = \widetilde{\varphi}(s,t),$$

*$u_k(x)$ and $\mu_k$ are eigenfunctions and eigenvalues of the problem*

$$\frac{1}{|\nabla v(x)|^2} \Delta u_k + \mu_k u_k = 0, \quad u_k|_{\partial D} = 0, \quad k = 1, 2, \ldots,$$

*and $a_k$, $k = 1, 2, \dots,$ are defined by the equalities*

$$\sum_{k=1}^{\infty} a_k \frac{\partial u_k(s)}{\partial \nu} \int_0^t f(p) e^{-\mu_k(t-p)} dp = \psi(s,t) - \frac{\partial \widetilde{w}(s,t)}{\partial \nu} - \frac{\partial F(v_0(s),t)}{\partial y} \frac{\partial v(s)}{\partial \nu},$$

*then*

$$w(x,t) = F(v(x),t) + \widetilde{w}(x,t) + \sum_{k=1}^{\infty} a_k u_k(x) \int_0^t f(p) e^{-\mu_k(t-p)} dp,$$

$$k(x) = \frac{1}{|\nabla v(x)|^2}, \quad \lambda(x) = \sum_{k=1}^{\infty} a_k u_k(x).$$

The proof of this theorem is immediate.

As in the case of linear problems, the construction of solutions to the nonlinear inverse problem is connected with the classical problems of the theory of differential equations. Moreover, the solution $w(x,t)$ is represented as the sum of three functions

$$F(V(x),t), \quad \widetilde{w}(x,t), \quad \sum_{k=1}^{\infty} a_k u_k(x) \int_0^t f(p) e^{-\mu_k(t-p)} dp,$$

where the first function $F(V(x),t)$ determines the coefficient $k(x)$, the second function $\widetilde{w}(x,t)$ is responsible for the arbitrariness in the boundary and initial conditions, whereas the third function determines the source function. The main question is whether it is possible to obtain and justify a representation of $\varphi(s,t)$ in the form $F(v_0(s),t) + \widetilde{\varphi}(s,t)$. We only note that, for example, the functions $F(y,t)$ and $v_0(s)$ can be found by using the variational principle $\min\|\varphi(s,t) - F(v_0(s),t)\|$, where $\| \ \|$ denotes the norm.

For a more general parabolic equation and a more complicated representation of the trace of the solution we can consider the inverse problem for three coefficients and the right-hand side, i.e. it is required to find a solution $w(x,t)$, coefficients $\rho(x)$, $k(x)$, $\mu(x)$, and $\lambda(x)$, $x \in D \subset \mathbb{R}^n$, $t \geqslant 0$, of the problem

$$\rho(x) \frac{\partial w}{\partial t} = \operatorname{div}(k(x) \nabla w) + \mu(x) w + \lambda(x) f(t), \quad x \in D, \ t > 0, \tag{16}$$

$$w|_{t=0} = w_0(x), \quad x \in D, \tag{17}$$

$$w|_{\partial D} = \varphi(s,t), \ \frac{\partial w}{\partial \nu}\bigg|_{\partial D} = \psi(s,t), \quad s \in \partial D, \ t > 0, \tag{18}$$

where the smooth functions $w_0(x)$, $\varphi(s,t)$, $\psi(s,t)$ are known and $D$ is a domain in $\mathbb{R}^n$ with smooth boundary $\partial D$.

**Theorem 3.** *Assume that the function $\varphi(s,t)$ in the inverse problem* (16)–(18) *admits the representation*

$$\varphi(s,t) = \int_0^{\infty} \left(Q(p) F_1(V_0(s),p) + R(p) F_2(V_0(s),p)\right) e^{-pt} dp + \widetilde{\varphi}(s,t),$$

*where the functions $Q(p)$ and $R(p)$ rather rapidly decrease at infinity (for example, compactly supported), $V_0(s)$ is continuous $\widetilde{\varphi}(s,t)$ is a fixed differentiable addition which can vanish, the functions $F_1(z,p)$ and $F_2(z,p)$ form the fundamental system of solutions to the ordinary second order differential equation*

$$F^{''}(z) + b(z)F^{'}(z) + (pa(z) + c(z))F(z) = 0,$$

*so that*

$$a(z) = \frac{\partial}{\partial p}\left(\frac{\frac{\partial^2 F_1}{\partial z^2}\frac{\partial F_2}{\partial z} - \frac{\partial^2 F_2}{\partial z^2}\frac{\partial F_1}{\partial z}}{\frac{\partial F_1}{\partial z}F_2 - \frac{\partial F_2}{\partial z}F_1}\right), \quad b(z) = -\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}F_2 - \frac{\partial F_2}{\partial z}F_1\right),$$

$$c(z) = \frac{\frac{\partial^2 F_1}{\partial z^2}\frac{\partial F_2}{\partial z} - \frac{\partial^2 F_2}{\partial z^2}\frac{\partial F_1}{\partial z}}{\frac{\partial F_1}{\partial z}F_2 - \frac{\partial F_2}{\partial z}F_1} - p\frac{\partial}{\partial p}\left(\frac{\frac{\partial^2 F_1}{\partial z^2}\frac{\partial F_2}{\partial z} - \frac{\partial^2 F_2}{\partial z^2}\frac{\partial F_1}{\partial z}}{\frac{\partial F_1}{\partial z}F_2 - \frac{\partial F_2}{\partial z}F_1}\right);$$

*moreover, $a(z) > 0$.*

*Let $V(x)$ be a harmonic function in $D$ such that $V|_{\partial D} = V_0(s)$ and $\nabla V \neq 0$. We set*

$$\rho(x) = a(V(x))|\nabla V|^2 \exp\left(\int_0^{V(x)} b(z)dz\right), \quad k(x) = \exp\left(\int_0^{V(x)} b(z)dz\right),$$
$$\mu(x) = c(V(x))|\nabla V|^2 \exp\left(\int_0^{V(x)} b(z)dz\right). \tag{19}$$

*Let $u_k(x)$ and $\mu_k$, $k = 1, 2, \dots$ be eigenfunctions and eigenvalues of the operator*

$$A = \frac{1}{\rho(x)}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(k(x)\frac{\partial}{\partial x_j}\right) + \frac{\mu(x)}{\rho(x)}, \quad Au_k + \mu_k u_k = 0, \quad u_k|_{\partial D} = 0.$$

*Suppose that $\widetilde{w}(x,t)$ is a solution to the first boundary value problem for the equation*

$$\frac{\partial \widetilde{w}}{\partial t} = A\widetilde{w}$$

*with the initial and boundary conditions*

$$\widetilde{w}|_{t=0} = w_0(x) - \int_0^{\infty}\left(Q(p)F_1(V(x),p) + R(p)F_2(V(x),p)\right)e^{-pt}dp, \quad \widetilde{w}|_{\partial D} = \widetilde{\varphi}(s,t),$$

*and the constants $a_k$ are determined by*

$$\widetilde{\psi}(s,t) = \psi(s,t) - \frac{\partial V(s)}{\partial \nu}\int_0^{\infty}\left(Q(p)\frac{\partial F_1(V_0(s),p)}{\partial z} + R(p)\frac{\partial F_2(V_0(s),p)}{\partial z}\right)e^{-pt}dp$$

$$-\frac{\partial \widetilde{w}(s,t)}{\partial \nu} = \sum_{k=1}^{n} a_k \frac{\partial u_k(s)}{\partial \nu}\int_0^{t} b(p)e^{-\mu_k(t-p)}dp.$$

*Then the coefficients $\rho(x)$, $k(x)$, and $\mu(x)$ are found from the equalities* (19) *and the functions*

$$w(x,t) = \int_0^\infty \left(Q(p)F_1(V(x),p) + R(p)F_2(V(x),p)\right) e^{-pt}dp + \widetilde{w}(x,t)$$

$$+ \sum_{k=1}^\infty a_k u_k(x) \int_0^t b(p) e^{-\mu_k(t-p)}dp, \quad \lambda(x) = \rho(x)\sum_{k=1}^\infty a_k u_k(x)$$

*solve the inverse problem* (16)–(18).

**Example 6.** With the trace $\varphi(s,t)$ of the solution $w(x,t)$ on the boundary $\partial D$,

$$\varphi(s,t) = \int_{-\infty}^\infty \big(Q(\omega)\exp(i\omega\sqrt[3]{u_0(s)})(i\omega\sqrt[3]{u_0(s)} - 1)$$

$$+ R(\omega)\exp(-i\omega\sqrt[3]{u_0(s)})(i\omega\sqrt[3]{u_0(s)} + 1)\big)e^{i\omega t}d\omega + \widetilde{\varphi}(s,t),$$

$u_0(s) > 0$, $s \in \partial D$, we associate the equation

$$\rho(x)\frac{\partial w}{\partial t} = \Delta w + \lambda(x) f(t);$$

where $\rho(x) = \frac{1}{9}(u(x))^{-4/3}|\nabla u|^2$ and $u(x)$ is a harmonic function such that $u|_{\partial D} = u_0(s)$, $s \in \partial D$.

Thus, in this case, the problem of finding $\rho(x)$ is reduced to the Dirichlet problem for the Laplace equation. The remaining functions are found in the same way as in Theorem 3.

If the coefficients and source function in the parabolic equation are independent of time, we can obtain an additional information for the linear inverse problem.

**Lemma 8.** *Let $w(x,t)$, $x \in D \subseteq \mathbb{R}^n$, $\alpha \leqslant t \leqslant \beta$, be a solution to the parabolic equation*

$$\rho(x)\frac{\partial w}{\partial t} = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial w}{\partial x_i}\right) + \lambda(x), \tag{20}$$

*where the $k(x)$, $\rho(x)$, and $\lambda(x)$ are independent of $t$. Then*

$$\rho(x)\left(\frac{\partial w}{\partial t}\right)^2 = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial t}\right) - \frac{1}{2}\frac{\partial}{\partial t}\left(k(x)\sum_{i=1}^n\left(\frac{\partial w}{\partial x_i}\right)^2\right) + \frac{\partial}{\partial t}\left(\lambda(x)w\right). \tag{21}$$

**Proof.** Multiplying Equation (20) by $\dfrac{\partial w}{\partial t}$ and using the identities

$$\frac{\partial}{\partial x_i}\left(k(x)\frac{\partial w}{\partial x_i}\right)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial t}\right) - \frac{1}{2}\frac{\partial}{\partial t}\left(k(x)\left(\frac{\partial w}{\partial x_i}\right)^2\right), \quad i = 1,\ldots,n,$$

we prove the required assertion. □

**Corollary 4** (uniqueness)**.** *Suppose that $k(x) > 0$, $\rho(x) > 0$, and*

$$w|_{\partial D} = 0, \quad w|_{t=\alpha} = w|_{t=\beta} = 0. \tag{22}$$

*Then $\lambda(x) = 0$ and $w(x,t) = 0$ for $x \in D$, $\alpha \leqslant t \leqslant \beta$.*

**Proof.** Integrating the identity (21) and taking into account (22), we find

$$\int_D \int_\alpha^\beta \rho(x) \left(\frac{\partial w}{\partial t}\right)^2 dxdt = 0.$$

We have $\dfrac{\partial w}{\partial t} = 0$ since $\rho(x) > 0$ and $w = 0$ since $w|_{t=\alpha} = 0$. Then $\lambda(x) = 0$ in view of (20). $\square$

**Corollary 5** (formula for $\lambda$). *If $w|_{\partial D} = \varphi(s)$, $s \in \partial D$, i.e., $\varphi(s)$ is independent of $t$ and $w|_{t=\alpha} = w|_{t=\beta} = w_0(x)$, then*

$$\lambda(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial w_0}{\partial x_i}\right).$$

**Proof.** Integrating the identity (21), we obtain the equality

$$\int_D \int_\alpha^\beta \rho(x) \left(\frac{\partial w}{\partial t}\right)^2 dxdt = 0.$$

Since $\rho(x) > 0$, we have $\frac{\partial w}{\partial t} = 0$, i.e., the solution $w(x,t)$ is independent of $t$. Substituting $t = \alpha$ (or $t = \beta$) into (20), we obtain the required formula for $\lambda(x)$. $\square$

Now, we consider inverse problems for parabolic equations consisting in finding a solution and two dependent coefficients. Explicit formulas for solutions are obtained up to a solution to the Cauchy problem

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial F}{\partial t}, \quad F|_{y=0} = \alpha(t), \quad \left.\frac{\partial F}{\partial y}\right|_{y=0} = \beta(t).$$

Thus, the inverse problem for the parabolic selfadjoint equation consists in finding $w(x,t)$, $A(x)$, and $C(x)$ such that

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(A(x)\frac{\partial w}{\partial x}\right) + C(x)w,$$

$$w|_{t=0} = w_0(x), \quad w|_{x=0} = \alpha(t), \quad \left.\frac{\partial w}{\partial x}\right|_{x=0} = \beta(t), \tag{23}$$

where $w_0(x) > 0$, $\alpha(t) > 0$, $\beta(t) > 0$, $w_0(0) = \alpha(0)$, $w_0'(0) = \beta(0)$ are given twice continuously differentiable functions for $0 \leqslant x < \infty$, $t \in \mathbb{R}$.

**Theorem 4.** *Let $F(y,t)$ be a solution to the Cauchy problem with respect to the variable $y$ :*

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial F}{\partial t}, \quad F|_{y=0} = \alpha(t), \quad \left.\frac{\partial F}{\partial y}\right|_{y=0} = \beta(t),$$

*and let*

$$\varphi(y) = \int_0^y F_0^2(p)dp, \quad F_0(p) = F(p,0).$$

*Then*

$$w(x,t) = \sqrt{V'(x)}F(V(x),t), \quad A(x) = \frac{1}{V'^2}, \quad C(x) = \frac{5(V'')^2 - 2V'''V'}{4(V')^4},$$

*where* $V(x) = \varphi^{-1}\left(\int\limits_0^x w_0^2(p)dp\right)$.

For the non-selfadjoint equation

$$\frac{\partial w}{\partial t} = A(x)\frac{\partial^2 w}{\partial x^2} + C(x)w$$

and the initial-boundary conditions (23) the following assertion holds.

**Theorem 5.** *Let* $F(y,t)$ *be a solution to the Cauchy problem with respect to the variable* $y$ :

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial F}{\partial t}, \quad F|_{y=0} = \alpha(t), \quad \left.\frac{\partial F}{\partial y}\right|_{y=0} = \beta(t),$$

*and let*

$$\varphi(y) = \int\limits_0^y \frac{dp}{F_0^2(p)},$$

*where* $F_0(p) = F(p,0)$. *Then*

$$w(x,t) = \frac{1}{\sqrt{V'(x)}}F(V(x),t), \quad A(x) = \frac{1}{V'^2}, \quad C(x) = \frac{2V'''V' - 3(V'')^2}{4(V')^4},$$

*where* $V(x) = \varphi^{-1}\left(\int\limits_0^x \frac{dp}{w_0^2(p)}\right)$.

Similar results for one-dimensional inverse problems for finding three connected coefficients can be found in [14].

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## References

1. S. L. Sobolev, "Functionally-invariant solutions of wave equations" [in Russian], *Tr. Fiz.-Mat. Inst. im. V. A. Steklova* **5**, 259–264 (1934).

2. A. N. Kolmogorov, "On analytic methods in probability theory" [in Russian], *Usp. Mat. Nauk* **5**, 5–41 (1938).

3. M. M. Lavrent'ev, K. G. Reznitskaya, and V. G. Yakhno, *One-Dimensional Inverse Problems of Mathematical Physics* [in Russian], Nauka, Novosibirsk, (1982).

4. V. S. Vladimirov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1981); English transl.: Marcel Dekker, New York (1971)..

5. G. I. Marchuk, "Mathematical modelling in a problem of environment" [in Russian], In: *Mathematical Modelling. Modern Problems of Mathematical Physics and Numerical Mathematics*, pp. 238–254, Nauka, Moscow (1989).

6. A. S. Malkov, G. G. Malinetskii, and D. S. Chernavskii, "Modelling of development of agrarian societies in the light of nonlinear dynamics" [in Russian], *News in Synergetics. A New Reality, New Problems, New Generation*, pp. 134–148, Nauka, Moscow (2007).

7. A. V. Bitsadze, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1982).

8. Yu. E. Anikonov, *Formulas in Inverse and Ill-Posed Problems*, VSP, Utrecht (1997).

9. Yu. E. Anikonov and S. G. Pyatkov, "On certain representations of solutions to inverse problems for second order equations" [in Russian], In: *Nonclassical Equations of Mathematical Physics*, pp. 108–111, Novosibirsk State Univ. Press, Novosibirsk (1993).

10. Yu. E. Anikonov and N. B. Ayupova, *Formulas for Solutions and Coefficients of Second Order Differential Equations and Inverse Problems* [in Russian], Preprint No. 165, IM SB RAS, Novosibirsk (2005).

11. Yu. E. Anikonov, "Constructive methods of studying the inverse problems for evolution equations" [in Russian], *Sib. Zh. Ind. Mat.* **11**, No. 2, 3-20 (2008).

12. Yu. E. Anikonov and M. V. Neshchadim, *Analytic Representations of Solutions of a Series of Inverse Problems in Mathematical Physics* [in Russian], Preprint No. 218, IM SB RAS, Novosibirsk (2009).

13. C. Liu and L. Peng, "Generalized Helgason–Fourier transforms associated to variants of the Laplace–Beltrami operators on the unit ball in $\mathbb{R}^n$," *Indiana Univ. Math. J.* **58**, No. 3, 1457–1491 (2009).

14. Yu. E. Anikonov, Yu. V. Krivtsov, and M. V. Neshchadim, *Nonlinear Problems of Control Theory* [in Russian], Preprint No. 227, IM SB RAS, Novosibirsk (2009).

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