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CONJUGATE POINTS IN NILPOTENT SUB-RIEMANNIAN PROBLEM ON THE ENGEL GROUP

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ABSTRACT. The left-invariant sub-Riemannian problem on the Engel group is considered. This problem is very important as nilpotent approximation of nonholonomic systems in four-dimensional space with two-dimensional control, for instance of a system which describes motion of mobile robot with a trailer. We study local optimality of extremal trajectories and estimate conjugate time in this article.

CONTENTS

1. Introduction

This work deals with the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 4)$. This problem evolves on the Engel group, which is the 4-dimensional nilpotent Lie group represented by matrices as follows:

$$\left\{ \begin{pmatrix} 1 & b & c & d \\ 0 & 1 & a & a^2/2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a,\ b,\ c,\ d \in \mathbb{R} \right\}.$$

The Lie algebra of the Engel group is the 4-dimensional nilpotent Lie algebra $L = \operatorname{span}(X_1, X_2, X_3, X_4)$ with the multiplication table

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.$$

We consider the sub-Riemannian problem on the Engel group G for the left-invariant sub-Riemannian structure generated by the orthonormal frame X_1, X_2:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad q \in G, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min;$$

see, e.g., the book by R. Montgomery [6] as a reference on sub-Riemannian geometry.

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In appropriate coordinates $q = (x, y, z, v)$ on the Engel group $G \cong \mathbb{R}^4$, the problem is stated as follows:

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix}, \quad q = (x, y, z, v) \in \mathbb{R}^4, \quad u \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0 = (x_0, y_0, z_0, v_0), \quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1), \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{3}$$

Since the problem is invariant under left shifts on the Engel group, we can assume that the initial point is an identity of the group $q_0 = (x_0, y_0, z_0, v_0) = (0, 0, 0, 0)$.

The paper continues the study of this problem started in the work [5]. The main result of [5] is the upper bound of the cut time (i.e., the time of loss of *global* optimality) along extremal trajectories of the problem. The aim of this paper is to investigate the first conjugate time (i.e., the time of loss of *local* optimality) along the trajectories. We show that the function that gives the upper bound of the cut time provides the lower bound of the first conjugate time. In order to state this main result exactly, we recall necessary facts from the previous work [5].

The existence of optimal solutions of problem (1)–(3) is implied by Filippov's theorem [3]. By the Cauchy–Schwarz inequality, it follows that sub-Riemannian length minimization problem (3) is equivalent to the action minimization problem

$$\int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min. \tag{4}$$

Pontryagin's maximum principle [3, 7] was applied to the resulting optimal control problem (1), (2), (4). Explicit formulas for abnormal extremals were obtained.

Denote vector fields near controls on the right-hand side of system (1):

$$X_1 = \left(1, 0, -\frac{y}{2}, 0\right)^T, \qquad X_2 = \left(0, 1, \frac{x}{2}, \frac{x^2 + y^2}{2}\right)^T,$$

their Lie brackets

$$X_3 = [X_1, X_2] = (0, 0, 1, x)^T, \qquad X_4 = [X_1, X_3] = (0, 0, 0, 1)^T,$$

and the corresponding Hamiltonians $h_i(\lambda) = \langle \lambda, X_i(q) \rangle$, $\lambda \in T^*M$, $i = 1, \dots, 4$, which are linear on fibers of the cotangent bundle T^*M. The normal extremals satisfy the Hamiltonian system

$$\dot{\lambda} = \vec{H}(\lambda), \qquad \lambda \in T^*M, \tag{5}$$

where $H = \frac{1}{2}\left(h_1^2 + h_2^2\right)$.

Introduce coordinates (θ, c, α) on the level surface $\left\{\lambda \in T^*M \mid H = \frac{1}{2}\right\}$ by the following formulas:

$$h_1 = \cos(\theta + \pi/2), \qquad h_2 = \sin(\theta + \pi/2), \qquad h_3 = c, \qquad h_4 = \alpha.$$

On this surface the normal Hamiltonian system (5) takes the following form:

$$\dot{\theta} = c, \qquad \dot{c} = -\alpha \sin\theta, \qquad \dot{\alpha} = 0, \tag{6}$$
$$\dot{q} = \cos\theta\, X_1(q) + \sin\theta\, X_2(q), \qquad q(0) = q_0.$$

The family of all normal extremals is parameterized by points of the phase cylinder of pendulum

$$C = \left\{ \lambda \in T^*_{q_0} M \mid H(\lambda) = \frac{1}{2} \right\} = \left\{ (\theta, c, \alpha) \mid \theta \in S^1, \ c, \alpha \in \mathbb{R} \right\},$$

and is given by the exponential mapping

$$\begin{aligned} &\mathrm{Exp} \,:\, N = C \times \mathbb{R}_+ \to M, \\ &\mathrm{Exp}(\lambda, t) = q_t = (x_t, y_t, z_t, v_t). \end{aligned}$$

The energy integral of pendulum (6) is expressed by $E = \frac{c^2}{2} - \alpha \cos\theta$. The cylinder C has the following stratification corresponding to the particular type of trajectories of the pendulum:

$$\begin{aligned} &C = \cup_{i=1}^{7} C_i, \quad C_i \cap C_j = \emptyset, \ i \neq j, \quad \lambda = (\theta, c, \alpha), \\ &C_1 = \{\lambda \in C \mid \alpha \neq 0, E \in (-|\alpha|, |\alpha|)\}, \\ &C_2 = \{\lambda \in C \mid \alpha \neq 0, E \in (|\alpha|, +\infty)\}, \\ &C_3 = \{\lambda \in C \mid \alpha \neq 0, E = |\alpha|, c \neq 0\}, \\ &C_4 = \{\lambda \in C \mid \alpha \neq 0, E = -|\alpha|\}, \\ &C_5 = \{\lambda \in C \mid \alpha \neq 0, E = |\alpha|, c = 0\}, \\ &C_6 = \{\lambda \in C \mid \alpha = 0, \ c \neq 0\}, \\ &C_7 = \{\lambda \in C \mid \alpha = c = 0\}. \end{aligned}$$

Further, the sets C_i, $i = 1, \dots, 5$, are divided into subsets determined by the sign of α:

$$C_i^+ = C_i \cap \{\alpha > 0\}, \qquad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\}.$$

In order to parameterize extremal trajectories, coordinates (φ, k, α) in the domains C_1 and C_2 were introduced in [5] in the following way. In the domain C_1^+:

$$\begin{aligned} &k = \sqrt{\frac{E + \alpha}{2\alpha}} = \sqrt{\frac{c^2}{4\alpha} + \sin^2 \frac{\theta}{2}} \in (0, 1), \\ &\sin\frac{\theta}{2} = k \,\mathrm{sn}(\sqrt{\alpha}\varphi), \quad \cos\frac{\theta}{2} = \mathrm{dn}(\sqrt{\alpha}\varphi), \quad \frac{c}{2} = k\sqrt{\alpha}\,\mathrm{cn}(\sqrt{\alpha}\varphi), \quad \varphi \in [0, 4K]. \end{aligned}$$

In the domain C_2^+:

$$\begin{aligned} &k = \sqrt{\frac{2\alpha}{E + \alpha}} = \frac{1}{\sqrt{\frac{c^2}{4\alpha} + \sin^2 \frac{\theta}{2}}} \in (0, 1), \\ &\sin\frac{\theta}{2} = \mathrm{sgn}\, c \,\mathrm{sn}\, \frac{\sqrt{\alpha}\varphi}{k}, \quad \cos\frac{\theta}{2} = \mathrm{cn}\, \frac{\sqrt{\alpha}\varphi}{k}, \quad \frac{c}{2} = \mathrm{sgn}\, c \,\frac{\sqrt{\alpha}}{k}\,\mathrm{dn}\,\frac{\sqrt{\alpha}\varphi}{k}, \quad \varphi \in [0, 2kK]. \end{aligned}$$

Here and below $\mathrm{dn}\, p$, $\mathrm{sn}\, p$, $\mathrm{cn}\, p$ are Jacobi's functions [15].

In the domains C_1^- and C_2^-, the coordinates φ and k are defined as follows:

$$\varphi(\theta, c, \alpha) = \varphi(\theta - \pi, c, -\alpha), \quad k(\theta, c, \alpha) = k(\theta - \pi, c, -\alpha).$$

Immediate differentiation shows that system (6) rectifies in the coordinates (φ, k, α), i.e., $\dot\varphi = 1$, $\dot k = 0$, $\dot\alpha = 0$. In [5] these coordinates were used for parameterization of extremal trajectories.

Further, in the work [5] discrete symmetries of the exponential mapping were described. The corresponding Maxwell sets were constructed. Recall that a point q_t of an extremal trajectory $q_s = \mathrm{Exp}(\lambda, s)$ is called a Maxwell point if there exists another extremal trajectory $\tilde q_s = \mathrm{Exp}(\tilde\lambda, s)$, $\tilde q_s \not\equiv q_s$, such that $\tilde q_t = q_t$. The instant t is called a Maxwell time. It is known [12] that an extremal trajectory cannot be optimal after a Maxwell time.

The main result of the paper [5], given by Theorem 1 below, provides an upper bound of the cut time along extremal curves

$$t_{\mathrm{cut}}(\lambda) = \sup\{t > 0 \mid \mathrm{Exp}(\lambda, s) \text{ is optimal for } s \in [0, t]\}.$$

Define the following function $t^1_{\mathrm{MAX}} : C \to (0, +\infty]$:

$$\lambda \in C_1 \quad \Rightarrow \quad t^1_{\mathrm{MAX}} = \min(2p^1_z, 4K)/\sigma, \tag{7}$$

$$\lambda \in C_2 \quad \Rightarrow \quad t^1_{\mathrm{MAX}} = 2Kk/\sigma, \tag{8}$$

$$\lambda \in C_6 \quad \Rightarrow \quad t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|}, \tag{9}$$

$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \quad \Rightarrow \quad t^1_{\mathrm{MAX}} = +\infty. \tag{10}$$

where $\sigma = \sqrt{|\alpha|}$,

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}};$$

$p^1_z(k) \in (K(k), 3K(k))$ is the first positive root of the function

$$f_z(p, k) = \mathrm{dn}\, p \ \mathrm{sn}\, p + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}\, p,$$

and

$$\mathrm{E}(p) = \int_0^p \mathrm{dn}^2\, t\, dt.$$

Theorem 1 (see [5, Theorem 3]). *For any* $\lambda \in C$

$$t_{\mathrm{cut}}(\lambda) \le t^1_{\mathrm{MAX}}(\lambda). \tag{11}$$

In this article we study local optimality of extremal trajectories and estimate the first conjugate time. Denote the Jacobian of the exponential map by

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \cdots & \frac{\partial x}{\partial t} \\ \vdots & \ddots & \vdots \\ \frac{\partial v}{\partial \theta} & \cdots & \frac{\partial v}{\partial t} \end{vmatrix}.$$

A point $q_t = \mathrm{Exp}(\lambda, t)$ is called *a conjugate point* for q_0 if $\nu = (\lambda, t)$ is a critical point of the exponential mapping and that is why q_t is the corresponding critical value:

$$d_\nu \,\mathrm{Exp} : T_\nu N \to T_{q_t} M \text{ is degenerate},$$

i.e.,

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)}(\nu) = 0.$$

The instant t is called *a conjugate time* along the extremal trajectory $q_s = \mathrm{Exp}(\lambda, s)$, $s \ge 0$.

As explained in Sec. 3, conjugate times are separated from each other; thus, any normal extremal contains not more than a countable family of conjugate points (see [14]). The first conjugate time along a trajectory $\mathrm{Exp}(\lambda, s)$ is denoted by

$$t^1_{\mathrm{conj}} = \min\{t > 0 \mid t \text{ is a conjugate time along } \mathrm{Exp}(\lambda, s),\ s \ge 0\}.$$

The trajectory $\mathrm{Exp}(\lambda, s)$ loses local optimality at the moment $t = t^1_{\mathrm{conj}}(\lambda)$ (see [3]). Our main aim is to prove the following lower bound of the first conjugate time.

Theorem 2. *For any* $\lambda \in C$

$$t^1_{\mathrm{conj}}(\lambda) \geq t^1_{\mathrm{MAX}}(\lambda). \tag{12}$$

In Sections 4–7 we prove inequality (12), $\lambda \in C_i$ for all $i = 1, \ldots, 7$ (see Theorems 4, 5, 6, 7).

2. Conjugate Time and Symmetries of Exponential Mapping

The normal Hamiltonian system for the considered problem has the following symmetries (see [5]): reflection

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t) \tag{13}$$

and dilations

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto \left(\theta, \frac{c}{\sqrt{\gamma}}, \frac{\alpha}{\gamma}, \sqrt{\gamma}x, \sqrt{\gamma}y, \gamma z, \gamma^{\frac{3}{2}}v, \sqrt{\gamma}t\right), \quad \gamma > 0. \tag{14}$$

We consider the corresponding symmetries of the exponential mapping and their action on conjugate points.

2.1. Reflection. Define the action of reflection in the preimage and image of the exponential mapping according to (13):

$$i : N \to N, \qquad i(\nu) = i(\theta, c, \alpha, t) = \widetilde{\nu} = (\theta - \pi, c, -\alpha, t),$$
$$i : M \to M, \qquad i(q) = i(x, y, z, v) = \widetilde{q} = (-x, -y, z, -v).$$

The existence of symmetry (13) of a Hamiltonian system implies that the reflection i is the symmetry of the exponential mapping: $\mathrm{Exp} \circ i = i \circ \mathrm{Exp}$. Hence we obtain $d\,\mathrm{Exp} \circ di = di \circ d\,\mathrm{Exp}$. The reflection i is nondegenerate ($\mathrm{Ker}\ di = 0$) and therefore $\nu = (\lambda, t) = (\theta, c, \alpha, t)$ is a critical point of Exp if and only if $\widetilde{\nu} = i(\nu) = (\widetilde{\lambda}, t) = (\theta - \pi, c, -\alpha, t)$ is a critical point of Exp. Hence $t^1_{\mathrm{conj}}(\widetilde{\lambda}) = t^1_{\mathrm{conj}}(\lambda)$. Using the definition of Maxwell time t^1_{MAX} (see (7)–(10)), we easily get a similar equality $t^1_{\mathrm{MAX}}(\widetilde{\lambda}) = t^1_{\mathrm{MAX}}(\lambda)$. Therefore it is enough to prove the required inequality (12) only for $\alpha \geq 0$.

2.2. Dilations. According to formula (14) define the action of dilations in the preimage and image of the exponential mapping:

$$\Phi_\gamma : N \to N, \quad \Phi_\gamma(\nu) = \Phi_\gamma(\lambda, t) = \Phi_\gamma(\theta, c, \alpha, t) = (\tilde{\lambda}, \tilde{t}) = \left(\theta, \frac{c}{\sqrt{\gamma}}, \frac{\alpha}{\gamma}, \sqrt{\gamma}, \sqrt{\gamma}t\right),$$
$$\Phi_\gamma : M \to M, \quad \Phi_\gamma(q) = \Phi_\gamma(x, y, z, v) = \tilde{q} = \left(\sqrt{\gamma}x, \sqrt{\gamma}y, \gamma z, \gamma^{\frac{3}{2}}v\right), \quad \gamma > 0.$$

These formulas define the action of the multiplicative Lie group $\mathbb{R}_+$ in N and M, such that

$$\mathrm{Exp} \circ \Phi_\gamma = \Phi_\gamma \circ \mathrm{Exp} \qquad \forall \gamma > 0.$$

Thus, there is a one-dimensional symmetry group $G = \{\Phi_\gamma | \gamma > 0\}$ of the exponential map.

It is easy to see that the symmetries preserve the sets of critical points and critical values of the exponential mapping.

Lemma 1. (1) *If* $q \in M$ *is the critical value of* Exp *corresponding to a critical point* $\nu \in N$*, then* $\Phi_\gamma(q)$ *is also the critical value of* Exp *corresponding to the critical point* $\Phi_\gamma(\nu)$ *for any* $\gamma > 0$.

(2) *Let* $\gamma > 0$, $\lambda = (\theta, c, \alpha)$, $\widetilde{\lambda} = \left(\theta, \frac{c}{\sqrt{\gamma}}, \frac{\alpha}{\gamma}\right) \in C$. *Then* $t^1_{\mathrm{conj}}(\lambda) = \frac{1}{\sqrt{\gamma}} t^1_{\mathrm{conj}}(\widetilde{\lambda})$.

Proof. (1) follows from the equality $d\,\mathrm{Exp} \circ d\Phi_\gamma = d\Phi_\gamma \circ d\,\mathrm{Exp}$.

(2) follows from (1). □

Let $\alpha > 0$. Suppose $\gamma = \alpha$; then from Lemma 1, we get the following:

$$t^1_{\text{conj}}(\theta, c, \alpha) = \frac{1}{\sqrt{\alpha}} t^1_{\text{conj}}\left(\theta, \frac{c}{\sqrt{\alpha}}, 1\right).$$

From the definition of Maxwell time t^1_{MAX}, a similar equation follows:

$$t^1_{\text{MAX}}(\theta, c, \alpha) = \frac{1}{\sqrt{\alpha}} t^1_{\text{MAX}}\left(\theta, \frac{c}{\sqrt{\alpha}}, 1\right).$$

Therefore, it is sufficient to prove the required inequality (12) in two cases: for $\alpha = 1$ and $\alpha = 0$.

2.3. Transformation of Jacobian of the exponential mapping. For a fixed $\lambda = (\theta, c, \alpha)$, conjugate times are roots $t > 0$ of the Jacobian $\dfrac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)}$. First, we transform this Jacobian by using the symmetry group $G = \{\Phi_\gamma | \gamma > 0\}$. Since $\operatorname{Exp} \circ \Phi_\gamma(\lambda, t) = \Phi_\gamma \circ \operatorname{Exp}(\lambda, t)$, we have

$$\operatorname{Exp} \;:\; \left(\theta, \frac{c}{\sqrt{\gamma}}, \frac{\alpha}{\gamma}, \sqrt{\gamma} t\right) \mapsto \left(\sqrt{\gamma} x(t), \sqrt{\gamma} y(t), \gamma z(t), \gamma^{\frac{3}{2}} v(t)\right), \tag{15}$$

where $(x(t), y(t), z(t), v(t)) = \operatorname{Exp}(\lambda, t)$, $\lambda = (\theta, c, \alpha)$. Differentiating equality (15) w.r.t. γ for $\gamma = \alpha$, we get

$$-\frac{c}{2}\frac{\partial q}{\partial c} - \frac{\partial q}{\partial \alpha} + \frac{t}{2}\frac{\partial q}{\partial t} = \left(\frac{x}{2}, \frac{y}{2}, z, \frac{3}{2} v\right) =: L.$$

Therefore, when $\alpha = 1$

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)} = \det\left(\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial c}, \frac{\partial q}{\partial \alpha}, \frac{\partial q}{\partial t}\right) = \det\left(\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial c}, -\frac{c}{2}\frac{\partial q}{\partial c} + \frac{t}{2}\frac{\partial q}{\partial t} - L, \frac{\partial q}{\partial t}\right)$$

$$= \det\left(\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial c}, \frac{\partial q}{\partial t}, L\right) = \frac{1}{2}\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial c} & \frac{\partial x}{\partial t} & x \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial c} & \frac{\partial y}{\partial t} & y \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial c} & \frac{\partial z}{\partial t} & 2z \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial c} & \frac{\partial v}{\partial t} & 3v \end{vmatrix}.$$

3. Conjugate Points and Homotopy

In this section we recall some necessary facts from the theory of conjugate points in optimal control problems. For details see, e.g., [1, 3, 14].

Consider an optimal control problem of the form

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{16}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed}, \tag{17}$$

$$J = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, \tag{18}$$

where M is a finite-dimensional analytic manifold, $f(q, u)$ and $\varphi(q, u)$ are respectively analytic in (q, u) families of vector fields and functions on M depending on the control parameter $u \in U$, and U is an open subset of $\mathbb{R}^m$. Admissible controls are $u(\cdot) \in L_\infty([0, t_1], U)$, and admissible trajectories $q(\cdot)$ are Lipschitzian. Let

$$h_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u), \qquad \lambda \in T^*M, \quad q = \pi(\lambda) \in M, \quad u \in U,$$

be the normal Hamiltonian of PMP for problem (16)–(18). Fix a triple $(\widetilde{u}(t), \lambda_t, q(t))$ consisting of a normal extremal control $\widetilde{u}(t)$, the corresponding extremal λ_t, and the extremal trajectory $q(t)$ for the problem (16)–(18).

Let the following hypotheses hold:

(**H1**) *For all* $\lambda \in T^*M$ *and* $u \in U$*, the quadratic form* $\dfrac{\partial^2 h_u}{\partial u^2}(\lambda)$ *is negative definite.*

(**H2**) *For any* $\lambda \in T^*M$*, the function* $u \mapsto h_u(\lambda)$*,* $u \in U$*, has a maximum point* $\bar{u}(\lambda) \in U$*:*

$$h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \qquad \lambda \in T^*M.$$

(**H3**) *The extremal control* $\widetilde{u}(\cdot)$ *is a corank one critical point of the endpoint mapping.*

(**H4**) *The Hamiltonian vector field* $\vec{H}(\lambda)$*,* $\lambda \in T^*M$*, is forward complete, i.e., all its trajectories are defined for* $t \in [0, +\infty)$*.*

An instant $t_* > 0$ is called a conjugate time (for the initial instant $t = 0$) along the extremal λ_t if the restriction of the second variation of the endpoint mapping to the kernel of its first variation is degenerate; see [3] for details. In this case the point $q(t_*) = \pi(\lambda_{t_*})$ is called conjugate for the initial point q_0 along the extremal trajectory $q(\cdot)$.

Under hypotheses (**H1**)–(**H4**), we have the following:

(1) Normal extremal trajectories lose their local optimality (both strong and weak) at the first conjugate point, see [3].
(2) An instant $t > 0$ is a conjugate time iff the exponential mapping $\mathrm{Exp}_t = \pi \circ e^{t\vec{H}}$ is degenerate, see [1].
(3) Along each normal extremal trajectory, conjugate times are isolated one from another, see [14].

We will apply the following statement for the proof of absence of conjugate points via homotopy.

Theorem 3 (see [8, Corollary 2.2])**.** *Let* $(u^s(t), \lambda_t^s)$*,* $t \in [0, +\infty)$*,* $s \in [0, 1]$*, be a continuous in parameter s family of normal extremal pairs in the optimal control problem* (16)–(18) *satisfying hypotheses* (**H1**)–(**H4**).

Let $s \mapsto t_1^s$ *be a continuous function,* $s \in [0, 1]$*,* $t_1^s \in (0, +\infty)$*. Assume that for any* $s \in [0, 1]$ *the instant* $t = t_1^s$ *is not a conjugate time along the extremal* λ_t^s*.*

If the extremal trajectory $q^0(t) = \pi(\lambda_t^0)$*,* $t \in (0, t_1^0]$*, does not contain conjugate points, then the extremal trajectory* $q^1(t) = \pi(\lambda_t^1)$*,* $t \in (0, t_1^1]$*, also does not contain conjugate points.*

One can easily check that the sub-Riemannian problem (1), (2), (4) satisfies all hypotheses (**H1**)–(**H4**), so the results cited in this section are applicable to this problem.

4. Estimate of Conjugate Time for $\lambda \in C_1$

4.1. Evaluation of Jacobian. We use the elliptic coordinates (φ, k, α) in C_1; see Sec. 1. For a fixed $\lambda = (\theta, c, \alpha) \in C_1$, conjugate times are roots $t > 0$ of the Jacobian

$$J = \frac{\partial(x, y, z, v)}{\partial(t, \varphi, k, \alpha)}.$$

We transform this Jacobian in the same way as the determinant $\dfrac{\partial(x,y,z,v)}{\partial(\theta,c,\alpha,t)}$ in Subsection 2.3, then get

$$J = -\frac{1}{2}\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial x}{\partial k} & x \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial \varphi} & \dfrac{\partial y}{\partial k} & y \\ \dfrac{\partial z}{\partial t} & \dfrac{\partial z}{\partial \varphi} & \dfrac{\partial z}{\partial k} & 2z \\ \dfrac{\partial v}{\partial t} & \dfrac{\partial v}{\partial \varphi} & \dfrac{\partial v}{\partial k} & 3v \end{vmatrix}.$$

(Here and below we assume $\alpha = 1$ according to Subsec. 2.2.) Explicit calculation of the function by parameterization of the exponential mapping obtained in [5] gives the following expression of the determinant:

$$J = R \cdot J_1,$$
$$R = -\frac{32}{k(1-k^2)(1-k^2\sin^2 u_1 \sin^2 u_2)^2} \neq 0,$$
$$J_1 = d_0 + d_2 \sin^2 u_2 + d_4 \sin^4 u_2,$$
$$d_i = d_i(u_1, k), \quad i = 0, 2, 4,$$
$$u_1 = \operatorname{am}(p,k), \quad u_2 = \operatorname{am}(\tau,k), \tag{19}$$
$$p = \frac{t}{2}, \quad \tau = \varphi + \frac{t}{2}, \tag{20}$$
$$d_0 = a_1 \cdot \sin u_1, \tag{21}$$
$$d_4 = k^2 a_2 \cdot f_{zu}, \tag{22}$$
$$d_0 + d_2 = -a_2 \cdot f_{zu}, \tag{23}$$
(24)

$$a_1 = \frac{1}{2}\Big[4(1-k^2)\cos u_1\left(1-2k^2\sin^2 u_1\right)\sqrt{1-k^2\sin^2 u_1}F^2(u_1) + 4k^2\cos u_1 \sin^2 u_1\left(1-k^2\sin^2 u_1\right)^{\frac{3}{2}}$$
$$+ 4\sin u_1\left(1-k^2\sin^2 u_1\right)E^3(u_1) - 2\left(1-k^2\right)\sin u_1\left(1-k^2\sin^2 u_1\right)F^3(u_1)$$
$$+ 2F(u_1)\left(\sin u_1 - 2k^2\left(3-2k^2\right)\sin^3 u_1 + k^4\left(5-4k^2\right)\sin^5 u_1\right)$$
$$+ E^2(u_1)\left(2\left(4k^2-5\right)\left(1-k^2\sin^2 u_1\right)\sin u_1 F(u_1) + 6\cos u_1\left(1-2k^2\sin^2 u_1\right)\sqrt{1-k^2\sin^2 u_1}\right)$$
$$+ E(u_1)\Big(2\left(4k^2-5\right)\cos u_1\left(1-2k^2\sin^2 u_1\right)\sqrt{1-k^2\sin^2 u_1}F(u_1)$$
$$+ 8\left(1-k^2\right)\left(1-k^2\sin^2 u_1\right)\sin u_1 F^2(u_1) - 2\left(1+k^2+3k^2\cos(2u_1)\right)\sin u_1\left(1-k^2\sin^2 u_1\right)\Big)\Big],$$

$$a_2 = -\cos u_1\left((E(u_1)-F(u_1))^2 + k^2 F(u_1)\left(2E(u_1)-F(u_1)\right)\right)$$
$$- \sin u_1\sqrt{1-k^2\sin^2 u_1}\left(E(u_1) - \left(1-k^2\right)F(u_1)\right),$$

$$f_{zu} = \sin u_1\sqrt{1-k^2\sin^2 u_1} + \left(F(u_1) - 2E(u_1)\right)\cos u_1,$$

where $F(u) = \int\limits_0^u \frac{dt}{\sqrt{1-k^2\sin^2 t}}$ and $E(u) = \int\limits_0^u \sqrt{1-k^2\sin^2 t}\,dt$ are elliptic integrals of the first and second kinds respectively.

Denote

$$x = \sin^2 u_2. \tag{25}$$

4.2. Estimates of functions. In this subsection we present two simple lemmas used in this section in order to obtain required bounds of functions.

Lemma 2. *Let real analytic functions $f(u), g(u)$ satisfy on $(0, u_0)$ the conditions*

$$f(u) \not\equiv 0, \qquad g(u) > 0, \qquad \left(\frac{f(u)}{g(u)}\right)' \geq 0, \tag{26}$$

$$\lim_{u\to 0} \frac{f(u)}{g(u)} = 0. \tag{27}$$

Then $f(u) > 0$ for $u \in (0, u_0)$.

If functions f and g satisfy conditions (26), (27), then we say that g is a comparison function for f on the interval $(0, u_0)$.

Proof. The function $\left(\frac{f}{g}\right)'$ is real analytic; thus it either has isolated zeros or is identically zero. It is not hard to prove that the second case is impossible: if $\left(\frac{f}{g}\right)' \equiv 0$, then $\frac{f}{g} \equiv \text{const}$, hence $\frac{f}{g} \equiv 0$ (because $\lim\limits_{u\to 0} \frac{f(u)}{g(u)} = 0$), whence $f \equiv 0$; this contradiction proves the case.

So the function $\left(\frac{f}{g}\right)' \geq 0$ has isolated zeros, and, therefore, $\frac{f}{g}$ strictly increases for $u \in (0, u_0)$. The inequality $\frac{f(u)}{g(u)} > 0$ follows from the equality (27), so the inequality $f(u) > 0$ for $u \in (0, u_0)$ follows from $g(u) > 0$. □

Lemma 3. *If $f(0, y) > 0, f(1, y) > 0$ and $a(y) \leq 0$ for the function $f(x, y) = a(y)x^2 + b(y)x + c(y)$ with $y \in (0, y_0)$, then $f(x, y) > 0$ for $y \in (0, y_0), x \in [0, 1]$.*

Proof. We obviously have the inequality $f(x, y) > 0$ for $x = 0$ and $x = 1$, $y \in (0, y_0)$. Since $a(y) \leq 0$, it follows that $f(x, y)$ is convex with respect to the variable x (possibly not strictly). Consequently we get $f(x, y) > 0$ for $y \in (0, y_0), x \in [0, 1]$. □

4.3. Conjugate points as $k \to 0$. We show that extremals corresponding to sufficiently small values of the parameter k have no conjugate points for $t < t^1_{\text{MAX}}(\lambda)$.

The function J_1 has the following asymptotics as $k \to 0$:

$$J_1(u_1, x, k) = k^2 J_1^0(u_1, x) + o(k^2), \qquad x = \sin^2 u_2,$$
$$J_1^0(u_1, x) = d_0^0(u_1) + d_2^0(u_1)x,$$
$$d_0^0(u_1) = \frac{1}{2}\sin u_1\bigl(2u_1^3 \sin u_1 + 3u_1^2 \cos u_1 + u_1 \sin^3 u_1 - 6u_1 \sin u_1 + 3\cos u_1 \sin^2 u_1\bigr),$$
$$d_2^0(u_1) = -u_1 \sin^4 u_1 - 2\cos u_1 \sin^3 u_1 + 3u_1 \sin^2 u_1 - u_1^3.$$

4.3.1. Auxiliary lemmas. In the following three lemmas we analyze the sign of the function $J_1^0(u_1, x)$, which is the dominant term of the asymptotics for the function $J_1(u_1, x, k)$ as $k \to 0$.

Lemma 4. *The function* $d_0^0(u_1) = \frac{1}{2}\sin u_1 \big(2u_1 \sin u_1 (u_1^2 - 3) + 3\cos u_1 (u_1^2 + \sin^2 u_1) + u_1 \sin^3 u_1\big) < 0$ *for* $u_1 \in (0, \pi)$.

Proof. We show that the function $g(u_1) = \sin u_1(\sin u_1 - u_1 \cos u_1)$ is a comparison function for $-d_0^0(u_1)$ for $u_1 \in (0, \pi)$.

The inequality $d_0^0(u_1) \not\equiv 0$ follows from the expansion $d_0^0(u_1) = -\frac{4}{4725}u^{11} + o(u^{11})$.

If $u_1 \in (0, \pi)$, then $\sin u_1 > 0$. Further, $\phi(u_1) = \sin u_1 - u_1 \cos u_1 > 0$ for $u_1 \in (0, \pi)$, since $\phi(0) = 0, \phi'(u_1) = u_1 \sin u_1 > 0$. Therefore $g(u_1) > 0$ for $u_1 \in (0, \pi)$.

Finally, we get the equalities

$$\left(\frac{-d_0^0(u_1)}{g(u_1)}\right)' = \frac{(2(-1 + u_1^2 + \cos(2u_1)) + u_1 \sin(2u_1))^2}{4(\sin u_1 - u_1 \cos u_1)^2}$$

and $\dfrac{-d_0^0(u_1)}{g(u_1)} = \dfrac{4}{1575}u_1^7 + o(u_1^7)$.

So $g(u_1)$ is a comparison function for $-d_0^0(u_1)$; thus, it follows from Lemma 2 that $d_0^0(u_1) < 0$ for $u_1 \in (0, \pi)$. □

Lemma 5. *If* $u_1 \in (0, \pi)$, *then*

$$d_0^0(u_1) + d_2^0(u_1) = \frac{1}{2}\big(- 2u_1^3 - \sin u_1(-2u_1^3 \sin u_1 + u_1 \sin^3 u_1 + \cos u_1(-3u_1^2 + \sin^2 u_1))\big) < 0.$$

Proof. We check that the function $g(u_1) = \sin u_1(\sin u_1 - u_1 \cos u_1)$ is a comparison function for $-(d_0^0(u_1) + d_2^0(u_1))$ for $u_1 \in (0, \pi)$.

The inequality $d_0^0(u_1) + d_2^0(u_1) \not\equiv 0$ follows from the expansion $d_0^0(u_1) + d_2^0(u_1) = -\frac{4}{135}u^9 + o(u^9)$.

Note that $g(u_1) > 0$ for $u_1 \in (0, \pi)$ (see the proof of Lemma 4).

Also, there hold the equalities

$$\left(\frac{-(d_0^0(u_1) + d_2^0(u_1))}{g(u_1)}\right)' = \frac{(-2u_1 + \sin(2u_1))^2}{4\sin^2 u_1}$$

and

$$\frac{-(d_0^0(u_1) + d_2^0(u_1))}{g(u_1)} = \frac{4}{45}u_1^5 + o(u_1^5).$$

Finally, $g(u_1)$ is a comparison function for $-(d_0^0(u_1) + d_2^0(u_1))$ therefore it follows from Lemma 2 that $d_0^0(u_1) + d_2^0(u_1) < 0$ for $u_1 \in (0, \pi)$. □

Lemma 6. *For any* $u_1 \in (0, \pi)$, $x \in [0, 1]$, *we have* $J_1^0(u_1, x) < 0$

Proof. If $u_1 \in (0, \pi)$, then $J_1^0(u_1, 0) < 0$ (see Lemma 4) and $J_1^0(u_1, 1) < 0$ (see Lemma 5), and so it follows from Lemma 3 that $J_1^0(u_1, x) < 0$ for $u_1 \in (0, \pi), x \in [0, 1]$. □

4.3.2. Estimate of conjugate time as $k \to 0$.

Proposition 1. *There exists* $\bar{k} \in (0, 1)$ *such that for any* $k \in \big(0, \bar{k}\big)$, $u_1 \in (0, \pi)$, $x \in [0, 1]$ *we have* $J_1(u_1, x, k) < 0$.

Proof. Assume the converse. Then there exist sequences $\{k_n\}$, $\{u_1^n\}$, $\{x_n\}$, $n \in \mathbb{N}$, such that $k_n \in (0, 1)$, $k_n \to 0$, $u_1^n \in (0, \pi)$, $x_n \in [0, 1]$, and $J_1(u_1^n, x_n, k_n) \geq 0$ for all $n \in \mathbb{N}$. By passing to subsequences, we can assume that $u_1^n \to \hat{u}_1 \in [0, \pi]$, $x_n \to \hat{x} \in [0, 1]$.

(1) Let $\hat{u}_1 \in (0, \pi)$. From Lemma 6, we get $J_1^0(u_1, x) < 0$ for all $u_1 \in (0, \pi)$, $x \in [0, 1]$. Thus $J_1(u_1^n, x_n, k_n) = k_n^2 \big(J_1^0(u_1^n, x_n) + o(1)\big) < 0$ for large values of n, a contradiction.

(2) Let $\hat{u}_1 = 0$. As $k^2 + u_1^2 \to 0$ we have

$$d_0 = -\frac{4}{4725} k^2 u_1^{11} + o(k^2 u_1^{11}),$$
$$d_2 = -\frac{4}{135} k^2 u_1^9 + o(k^2 u_1^9),$$
$$d_4 = \frac{4}{135} k^4 u_1^9 + o(k^4 u_1^9).$$

(2.1) If $\hat{x} \neq 0$, then $J_1 = -\frac{4}{135} k^2 u_1^9 x + o(k^2 u_1^9)$. Therefore $J_1(u_1^n, x_n, k_n) < 0$ for large values of n, a contradiction.

(2.2) If $\hat{x} = 0$, then

$$J_1 = -\frac{4}{4725} k^2 u_1^{11} + o(k^2 u_1^{11}) - \frac{4}{135} k^2 u_1^9 x + o(k^2 u_1^9 x)$$

and $J_1(u_1^n, x_n, k_n) < 0$ as $n \to \infty$, a contradiction.

(3) Let $\hat{u}_1 = \pi$. As $k^2 + (\pi - u_1)^2 \to 0$ we get

$$d_0 = -\frac{3}{2} \pi^2 k^2 (\pi - u_1) + o\left(k^2 (\pi - u_1)\right),$$
$$d_2 = -\pi^3 k^2 + o(k^2),$$
$$d_4 = \pi^3 k^4 + o(k^4).$$

(3.1) If $\hat{x} \neq 0$, then $J_1 = k^2 \left(-\pi^3 x + o(1)\right)$. Whence $J_1\left(u_1^n, x_n, k_n\right) < 0$ as $n \to \infty$, a contradiction.

(3.2) If $\hat{x} = 0$, then

$$J_1 = -\frac{3}{2} \pi^2 k^2 (\pi - u_1) + o\left(k^2 (\pi - u_1)\right) - \pi^3 k^2 x + o(k^2 x).$$

Hence $J_1(u_1^n, x_n, k_n) < 0$ as $n \to \infty$. The contradiction completes the proof. □

Going back from the variables (u_1, x, k) to (t, φ, k) by formulas (25), (19), (20), we get the following statement from Proposition 1.

Corollary 1. *There exists $\bar{k} \in (0, 1)$, such that for any $k \in (0, \bar{k})$, $\varphi \in \mathbb{R}$, an arc of the extremal curve* $\operatorname{Exp}(\lambda, t)$, $\lambda = (\varphi, k, \alpha)$, $t \in (0, t^1_{\mathrm{MAX}}(\lambda))$, *does not contain conjugate points.*

4.4. Conjugate points at $t = t^1_{\mathrm{MAX}}$. In this subsection we find sufficient conditions that ensure that the instant $t = t^1_{\mathrm{MAX}}$ is a conjugate time. Let us recall that $t^1_{\mathrm{MAX}}(\lambda) = \min(2p_z^1(k), 4K(k))$ for $\lambda \in C_1$, $\alpha = 1$, where $p = p_z^1(k) \in (K, 3K)$ is the first positive root of the function $f_z(p, k) = \operatorname{dn} p \operatorname{sn} p + (p - 2\operatorname{E}(p)) \operatorname{cn} p$ (see [5]).

It is shown in [12] that

$$k \in (0, k_0) \quad \Rightarrow \quad p_z^1(k) \in (3K, 2K),$$
$$k = k_0 \quad \Rightarrow \quad p_z^1(k) = 2K,$$
$$k \in (k_0, 1) \quad \Rightarrow \quad p_z^1(k) \in (K, 2K),$$

where $k_0 \approx 0.9$ is the unique root of the equation $2E(k) - K(k) = 0$. Therefore

$$t^1_{\mathrm{MAX}}(\lambda) = \begin{cases} 4K(k) \text{ for } k \in (0, k_0], \\ 2p_z^1(k) \text{ for } k \in [k_0, 1). \end{cases}$$

Changing the variable t by $u_1 = \operatorname{am}\left(\frac{t}{2}, k\right)$, we get

$$u^1_{\mathrm{MAX}}(k) = \begin{cases} \pi \text{ for } k \in (0, k_0], \\ u_z^1(k) \text{ for } k \in [k_0, 1), \end{cases}$$

where $u_1 = u_z^1(k) = \operatorname{am}(p_z^1(k), k) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is the first positive root of the function $f_{zu}(u_1, k) = f_z(\operatorname{am} u_1, k)$.

Lemma 7. *The function* $f_3(k) = E(k) + 2(k^2 - 1)E(k)K(k) - (k^2 - 1)K(k) > 0$ *on the interval* $k \in (0, 1)$.

Proof. Let us prove that the function $g(k) = 1 - k^2$ is a comparison function for $f_3(k)$ on the interval $k \in (0, 1)$.

The inequality $f_3(k) \not\equiv 0$ follows from the expansion $f_3(k) = \frac{\pi^2}{4}k^2 + o(k^2)$.

Notice that $g(k) > 0$ for $k \in (0, 1)$.

Also, we have the equalities $\left(\dfrac{f_3(k)}{g(k)}\right)' = \dfrac{2kE^2(k)}{(k^2-1)^2}$ and $\dfrac{f_3(k)}{g(k)} = \dfrac{\pi^2}{4}k^2 + o(k^2)$.

Finally, $g(k)$ is a comparison function for $f_3(k)$, hence it follows from Lemma 2 that $f_3(k) > 0$ for $k \in (0, 1)$. □

Lemma 8. (1) *Let* $u_1 = \pi$, $x \in (0, 1]$; *then* $\operatorname{sgn} J_1 = -\operatorname{sgn} f_{zu}(\pi, k) = -\operatorname{sgn}(2E(k) - K(k))$, *i. e.,* $J_1 < 0$ *for* $k \in (0, k_0)$, $J_1 = 0$ *if* $k = k_0$, *and* $J_1 > 0$ *for* $k \in (k_0, 1)$.

(2) *If* $u_1 = \pi$, $x = 0$, *then* $J_1 = 0$.

Proof. From a direct calculation it follows that

$$\begin{aligned} &J_1(\pi, x, k) = -4x(1 - k^2x)f_z(\pi, k)f_3(k), \qquad (28)\\ &f_z(\pi, k) = 2E(k) - K(k),\\ &f_3(k) = E(k) + 2(k^2 - 1)E(k)K(k) - (k^2 - 1)K(k). \end{aligned}$$

Now the statement of item (1) of this lemma follows from Lemma 7 ($f_3(k) > 0$ for all $k \in (0, 1)$) and the distribution of signs of the function $2E(k) - K(k)$ [12] (this function is positive for $k < k_0$, equals zero if $k = k_0$ and is negative for $k > k_0$).

The statement of item (2) follows from formula (28). □

Lemma 9. *Let* $k \in (0, 1)$, $k \neq k_0$, $u_1 = u_z^1(k)$. *Then* $a_1(u_1, k) < 0$.

Proof. From a direct calculation it follows that if $u_2 = u_z^1(k)$, then

$$\begin{aligned} a_1 &= \left(e_0 + e_1F(u_1) + e_2F^2(u_1)\right) / \left(4\cos^3 u_1\right),\\ e_0 &= \cos^2 u_1 \sqrt{1 - k^2\sin^2 u_1}\left(1 - k^2\left(1 - \cos^4 u_1\right)\right),\\ e_1 &= -2k^2\cos u_1 \sin u_1(1 - \left(4 - 5\cos^2 u_1 + \cos^4 u_1\right) + \left(5 - \cos^2 u_1\right)k^4\sin^4 u_1 - 2k^6\sin^6 u_1),\\ e_2 &= \sin^2 u_1\left(1 - k^2\sin^2 u_1\right)^{\frac{3}{2}}\left(1 - k^2\left(1 - \cos^4 u_1\right)\right). \end{aligned}$$

To estimate the sign of the function a_1, notice first that $u_1 = u_z^1 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and therefore $\cos^3 u_1 < 0$. Further, we analyze the sign of the quadratic trinomial $h(z) = e_0 + e_1z + e_2z^2$. Its discriminant is equal to

$$D = e_1^2 - 4e_0e_2 = -16k^2\sin^4 u_1\cos^4 u_1\left(1 - k^2\sin^2 u_1\right)^4;$$

therefore, $D < 0$ for $k \neq k_0$. We have $h(z) > 0$ for $k \neq k_0$ because $e_0 > 0$, and therefore $a_1 < 0$. □

Lemma 10. (1) *Let* $k \in (0, 1)$, $u_1 = u_z^1(k)$, $x \in [0, 1)$; *then* $\operatorname{sgn} J_1 = \operatorname{sgn} f_{zu}(\pi, k) = \operatorname{sgn}(2E(k) - K(k))$, *i.e.,* $J_1 > 0$ *for* $k \in (0, k_0)$, $J_1 = 0$ *if* $k = k_0$, *and* $J_1 < 0$ *for* $k \in (k_0, 1)$.

(2) *If* $k \in (0, 1)$, $u_1 = u_z^1(k)$, $x = 1$, *then* $J_1 = 0$.

Proof. From equalities (22), (23), (21) we get for $u_1 = u_z^1(k)$, i.e., if $f_{zu}(u_1, k) = 0$:

$$\begin{aligned} &d_4 = 0, \\ &d_2 = -d_0, \\ &J_1 = d_0(1-x) = a_1 \sin u_z^1(k)(1-x). \end{aligned} \tag{29}$$

It is shown in [12] that the function $\sin u_z^1(k) = \operatorname{sn} p_z^1(k)$ is negative for $k \in (0, k_0)$, is equal to zero if $k = k_0$, and is positive for $k \in (k_0, 1)$. Thus, for $x \in [0, 1)$, the equality $\operatorname{sgn} J_1 = -\operatorname{sgn} a_1 \cdot \operatorname{sgn}(2E(k)-K(k))$ holds. To finish the proof of item (1) of the lemma we use Lemma 9: for $u = u_z^1(k)$, $k \neq k_0$ the function a_1 is negative. Also, for $k = k_0$, $u = u_z^1(k_0 = \pi)$ we have $J_1 = 2E(k_0) - K(k_0) = 0$.

Item (2) of the lemma follows from (29). □

4.5. Global bounds of conjugate time in the subdomain C_1. We prove estimate (12) and get the upper bound for the first conjugate time in this subsection.

Theorem 4. *If $\lambda \in C_1$, then $t^1_{\text{conj}}(\lambda) \geq t^1_{\text{MAX}}(\lambda)$.*

Proof. Let $\lambda = (\varphi, k, \alpha = 1) \in C_1$.

(1) Suppose $k \in (0, k_0)$. It is required to show that $t^1_{\text{conj}}(\lambda) \geq 4K(k)$.

(1.1) Let $\operatorname{sn}(\varphi, k) \neq 0$. Consider the family of extremal trajectories

$$\begin{aligned} &q^s(t) = \operatorname{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in (0, k_0), \\ &\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_1, \\ &k^s = s, \quad \varphi^s = F(\operatorname{am}(\varphi, k), s), \quad t_1^s = 4K(s). \end{aligned}$$

For any trajectory from this family

$$\begin{aligned} x^s &= \sin^2 u_2^s = \operatorname{sn}^2 \tau^s = \operatorname{sn}^2\left(\varphi^s + \frac{t_1^s}{2}\right) = \operatorname{sn}^2(\varphi^s + 2K(k^s), k^s) = \operatorname{sn}^2(\varphi^s, k^s) = \\ &= \sin^2(\operatorname{am}(\varphi^s, k^s)) = \sin^2(\operatorname{am}(\varphi, k)) = \operatorname{sn}^2(\varphi, k) \neq 0, \\ u_1^s &= \operatorname{am}\left(\frac{t_1^s}{2}, k^s\right) = \operatorname{am}(2K(s), s) = \pi; \end{aligned}$$

therefore, from Lemma 8 $J_1(u_1^s, x^s, k^s) < 0$. Namely the endpoint of a trajectory $q^s(t_1^s)$, $s \in (0, k_0)$, is not a conjugate point. According to Corollary 1, there exists $\tilde{k} \in (0, k)$, such that the trajectory $q^{\tilde{k}}(t)$, $t \in (0, t_1^{\tilde{k}}]$ does not contain conjugate points. We apply Theorem 3 to the family of the trajectories $q^s(t)$, $s \in [\tilde{k}, k]$ and see that the trajectory $q^k(t)$, $t \in (0, t_1^k]$, has no conjugate points, i. e., $t^1_{\text{conj}}(\lambda) > 4K(k)$.

(1.2) Let $\operatorname{sn}(\varphi, k) = 0$. Consider the family of trajectories

$$\begin{aligned} &q^s = \operatorname{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in (0, \varepsilon), \\ &\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_1, \\ &k^s = s, \quad \varphi^s = \varphi + s, \quad t_1^s = 4K(s), \end{aligned}$$

where $\varepsilon > 0$ is a sufficiently small number, such that $\operatorname{sn}(\varphi + s, k) \neq 0$ for $s \in (0, \varepsilon]$. For the trajectories of this family, we have

$$\begin{aligned} &x^s = \operatorname{sn}\left(\varphi^s + \frac{t_1^s}{2}\right) = \operatorname{sn}^2(\varphi + s, k) \neq 0, \quad s \in (0, \varepsilon], \\ &u_1^s = \pi, \end{aligned}$$

therefore, according to item (1.1) of this theorem, trajectories $q^s(t)$, $t \in (0, t_1^s]$, $s \in (0, \varepsilon]$, have no conjugate points. Take any $t_1 \in (0, 4K(k))$. Since conjugate times are isolated from each other, it follows that there exists $t_2 \in (t_1, 4K(k))$ that is not a conjugate time along the trajectory $q^0(t)$. Thus the instant t_2 is not a conjugate time for all trajectories of the family $q^s(t)$, $s \in [0, \varepsilon]$. Using Theorem 3, we see that the trajectory $q^0(t)$, $t \in (0, t_2]$ has no conjugate points. Therefore the instant t_1 is not a conjugate time. Since $t_1 \in (0, 4K(k))$, we obtain the required inequality $t^1_{\text{conj}}(\lambda) \geq 4K(k)$. Note that the equality is attained in this case: from Lemma 8 it follows that $J_1(u_1, x, k) = 0$, and therefore $t^1_{\text{conj}}(\lambda) = 4K(k)$.

(2) Suppose that $k = k_0$. Take any $t_1 \in (0, 4K(k_0))$ and any $t_2 \in (t_1, 4K(k_0))$, which is not a conjugate time for the trajectory $\operatorname{Exp}(\lambda, t)$. Applying Theorem 3 to the family

$$
\begin{aligned}
&q^s = \operatorname{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in (-\varepsilon, 0),\\
&\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_1,\\
&k^s = k + s, \quad \varphi^s = \varphi, \quad t_1^s = t_2,
\end{aligned}
$$

where $\varepsilon > 0$ is sufficiently small as in item (1.2) of this proof, we see that $t^1_{\text{conj}}(\lambda) \geq 4K(k_0)$. According to Lemma 8, $t^1_{\text{conj}}(\lambda) = 4K(k_0)$.

(3) Suppose that $k \in (k_0, 1)$. We claim that for any $x \in [0, 1]$ the set $\{(u_1, k) | J_1(u_1, x, k) = 0\}$ is contained between the curves $u_1 = \pi$ and $u_1 = u_z^1(k)$ in a neighborhood of $(u_1, k) = (\pi, k_0)$; it can easily be checked that these curves are smooth and meet at the point $(u_1, k) = (\pi, k_0)$ at the right angle, i.e., $u_z^1(k_0) = \pi$ and $\dfrac{du_z^1}{dk}(k_0) = \infty$. At the point $(u_1, k) = (\pi, k_0)$ we have the following Taylor expansion:

$$
\begin{aligned}
J_1(u_1, x, k) = \quad & -4E^2(k_0)(\pi - u_1) + x\left[-\frac{8E^3(k_0)}{k_0(1-k_0^2)}(k - k_0) + 4E^2(k_0)(\pi - u_1)\right]\\
& + x^2 \frac{8k_0}{1-k_0^2} E^3(k_0)(k - k_0) + O((k-k_0)^2 + (\pi - u_1)^2).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
x \neq 1 \quad &\Rightarrow \quad \left.\frac{\partial J_1}{\partial u_1}\right|_{u_1 = \pi, k = k_0} = 4E^2(k_0)(x - 1) \neq 0,\\
x = 1 \quad &\Rightarrow \quad \left.\frac{\partial J_1}{\partial k}\right|_{u_1 = \pi, k = k_0} = -\frac{8E^3(k_0)}{k_0} \neq 0.
\end{aligned}
$$

Therefore for any $x \in [0, 1]$ the equation $J_1(u_1, x, k) = 0$ defines a smooth curve in a neighborhood of the point $(u_1, k) = (\pi, k_0)$. From Lemmas 8 and 10 it follows that for any $x \in [0, 1]$, $k \in (0, 1)$ there exists $u_1 \in [\pi, u_z^1(k)]$ such that $J_1(u_1, x, k) = 0$. Therefore the curve $\{J_1 = 0\}$ is contained between the curves $\{u_1 = \pi\}$ and $\{u_1 = u_z^1(k)\}$ near the point $(u_1, k) = (\pi, k_0)$. Hence for any $x \in [0, 1]$ there exists a neighborhood of the point $(u_1, k) = (\pi, k_0)$, which satisfies the inequality $J_1(u_1, x, k) \neq 0$ for $u_1 < \min(\pi, u_z^1(k)) = u^1_{\text{MAX}}(k)$.

(3.1) Let $\operatorname{sn}^2(\varphi + p_z^1(k), k) \neq 1$. For $x = \operatorname{sn}^2\left(\varphi + \frac{t^1_{\text{MAX}}}{2}\right) = \operatorname{sn}^2(\varphi + p_z^1(k))$, in a neighborhood O of the point $(u_1, k) = (\pi, k_0)$ the function $J_1(u_1, x, k)$ does not vanish for $k > k_0$, $u_1 < u_z^1(k)$. Applying Theorem 3 to the family of trajectories

$$
\begin{aligned}
&q^s(t) = \operatorname{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in [k_0, \tilde{k}],\\
&\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_1,\\
&k^s = s, \quad \varphi^s = F(\operatorname{am}(\varphi + p_z^1(k), k), s),\\
&t_1^s = 2F(u_z^1(k), s),
\end{aligned}
$$

we see that the trajectory $\tilde{q}(t) = q^{\tilde{k}}(t) = \operatorname{Exp}(\tilde{\lambda}, t)$, $\tilde{\lambda} = \lambda^{\tilde{k}} = (\tilde{\varphi}, \tilde{k}, \alpha = 1)$, $t \in [0, \tilde{t}]$, $\tilde{t_1} = t_1^{\tilde{k}}$, has no conjugate points.

Finally, applying Theorem 3 to the family

$$\begin{aligned}
&q^s(t) = \mathrm{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in [\tilde{k}, k],\\
&\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_1,\\
&k^s = s, \quad \varphi^s = F(\mathrm{am}(\varphi + p_z^1(k), k), s),\\
&t_1^s = 2p_z^1(s),
\end{aligned}$$

we see that the trajectory $q^k(t) = \mathrm{Exp}(\lambda, t)$, $t \in [0, t^1_{\mathrm{MAX}}(\lambda)]$ has no conjugate points, Q.E.D.

(3.2) In the case $\mathrm{sn}^2(\varphi + p_z^1(k), k) = 1$, the proof of $t^1_{\mathrm{conj}}(\lambda) \geq 2p_z^1(k)$ is obtained as in item (1.2).

The theorem is completely proved. □

Remark 1. *The lower bound in estimate of conjugate time* (12) *is attained. From Lemmas* 8 *and* 10 *and Theorem* 4 *we get*

$$\begin{aligned}
&k \in (0, k_0), \quad \sin\varphi = 0, \quad \alpha = 1 \quad \Rightarrow \quad t^1_{\mathrm{conj}}(\lambda) = 4K(k) = t^1_{\mathrm{MAX}}(\lambda),\\
&k = k_0, \quad \alpha = 1 \quad \Rightarrow \quad t^1_{\mathrm{conj}}(\lambda) = 4K(k_0) = t^1_{\mathrm{MAX}}(\lambda),\\
&k \in (k_0, 1),\ \sin^2(\varphi + p_z^1(k)) = 1,\ \alpha = 1 \quad \Rightarrow \quad t^1_{\mathrm{conj}}(\lambda) = 2p_z^1(k) = t^1_{\mathrm{MAX}}(\lambda).
\end{aligned}$$

In addition to the lower bound from Theorem 4, we get the upper bound of the first conjugate time in terms of the second Maxwell time

$$t^2_{\mathrm{MAX}}(\lambda) = \max(2p_z^1, 4K)/\sigma, \quad \lambda \in C_1.$$

For $\alpha = 1$ we obtain

$$\begin{aligned}
&k \in (0, k_0) \quad \Rightarrow \quad t^2_{\mathrm{MAX}}(\lambda) = 2p_z^1(k),\\
&k = k_0 \quad \Rightarrow \quad t^2_{\mathrm{MAX}}(\lambda) = 2p_z^1(k_0) = 4K(k_0),\\
&k \in (k_0, 1) \quad \Rightarrow \quad t^2_{\mathrm{MAX}}(\lambda) = 4K(k).
\end{aligned}$$

Proposition 2. *If* $\lambda \in C_1$, *then* $t^1_{\mathrm{conj}}(\lambda) \leq t^2_{\mathrm{MAX}}(\lambda)$.

Proof. Let $\lambda = (\varphi, k, \alpha) \in C_1$. Also, in the proof of the lower bound of conjugate time we can assume that $\alpha = 1$.

If $k \in (0, k_0)$, then for any $x \in [0, 1]$ we have $J_1(\pi, x, k) \leq 0$ (see Lemma 8) and $J_1(u_z^1(k), x, k) \geq 0$ (see Lemma 10), i.e., the function $J_1(u_1, x, k)$ changes sign in the segment $u_1 \in [\pi, u_z^1(k)]$. Therefore, the corresponding segment $t \in [4K(k), 2p_z^1(k)] = [t^1_{\mathrm{MAX}}(\lambda), t^2_{\mathrm{MAX}}(\lambda)]$ contains the first conjugate time.

If $k = k_0$, then for any $x \in [0, 1]$ we get $J_1(\pi, x, k_0) = 0$ (see Lemma 8), and thus $t^1_{\mathrm{conj}}(\lambda) = 4K(k_0) = t^1_{\mathrm{MAX}}(\lambda) = t^2_{\mathrm{MAX}}(\lambda)$.

Finally, if $k > k_0$, then for any $x \in [0, 1]$ $J(u_z^1(k), x, k) \leq 0$ (see Lemma 10), $J_1(\pi, x, k) \geq 0$ (see Lemma 8); therefore $t^1_{\mathrm{conj}}(\lambda) \in [2p_z^1(k), 4K(k)] = [t^1_{\mathrm{MAX}}(\lambda), t^2_{\mathrm{MAX}}(\lambda)]$. □

Remark 2. *One should not think that the segment* $[t^1_{\mathrm{MAX}}(\lambda), t^2_{\mathrm{MAX}}(\lambda)]$ *contains exactly one conjugate time. Computational experiments in the system Mathematica show that for* $\varphi = 0$ *and* $k \in (0,999; 1)$ *this segment contains two conjugate times.*

5. Estimate of Conjugate Time for $\lambda \in C_2$

The aim of this section is to prove estimate (12) in the domain C_2 for $\alpha = 1$: $t^1_{\mathrm{conj}}(\lambda) \geq 2Kk$, $\lambda \in C_2$. Using parameterization of extremal trajectories [5] for $\lambda = (\varphi, k, \alpha) \in C_2$, as well as in the domain C_1, we get the expression of the Jacobian $J = \frac{\partial(x,y,z,v)}{\partial(t,\varphi,k,\alpha)}$ for $\alpha = 1$:

$$
\begin{aligned}
&J = R \cdot J_1,\\
&R = -\frac{32}{k(1-k^2)(1-k^2\sin^2 u_1 \sin^2 u_2)^2} \neq 0,\\
&J_1 = d_0 + d_2 \sin^2 u_2 + d_4 \sin^4 u_2,\\
&u_1 = \operatorname{am}(p,k), \quad u_2 = \operatorname{am}(\tau,k),\\
&p = \frac{t}{2k}, \quad \tau = \frac{2\varphi + t}{2k},
\end{aligned}
$$

$$
\begin{aligned}
d_0 =& \frac{1}{4}\sin u_1 \cos u_1 (4E^3(u_1)\sin(2u_1) - 4(1-k^2)\cos(2u_1)\sqrt{1-k^2\sin^2 u_1}F^2(u_1)\\
&+ (8 - 8k^2 + k^4 + k^2(2-k^2)\cos(2u_1))\sin(2u_1)F(u_1) + 2(2-3k^2+k^4)\sin(2u_1)F^3(u_1)\\
&+ 2k^2\sqrt{1-k^2\sin^2 u_1}\sin^2(2u_1) + 2E^2(u_1)(6\cos(2u_1)\sqrt{1-k^2\sin^2 u_1} - (2-k^2)\sin(2u_1)\\
&\times F(u_1)) - E(u_1)(4(2-k^2)\cos(2u_1)\sqrt{1-k^2\sin^2 u_1}F(u_1) + 2(4-2k^2\\
&+ 3k^2\cos(2u_1))\sin 2u_1 + 4(1-k^2)\sin(2u_1)F^2(u_1),\\
d_2 =& -2k^2(1-k^2)\cos u_1 \sin^3 u_1 \sqrt{1-k^2\sin^2 u_1}F^2(u_1) - 2k^4\cos^3 u_1 \sin^3 u_1 \sqrt{1-k^2\sin^2 u_1}\\
&- 2(1-k^2\sin^4 u_1)E^3(u_1) - (2-3k^2+k^4)(1-k^2\sin^4 u_1)F^3(u_1)\\
&+ E^2(u_1)(6k^2\cos u_1 \sin^3 u_1\sqrt{1-k^2\sin^2 u_1} + (2-k^2)(1-k^2\sin^4 u_1)F(u_1))\\
&+ E(u_1)(k^2\cos^2 u_1 \sin^2 u_1(4 - 3k^2 + 3k^2\cos(2u_1))\\
&- 2k^2(2-k^2)\cos u_1 \sin^3 u_1\sqrt{1-k^2\sin^2 u_1}F(u_1) + 2(1-k^2)(1-k^2\sin^4 u_1)F^2(u_1))\\
&- \frac{1}{8}k^2(8 - 8k^2 + k^4 + k^2(2-k^2)\cos(2u_1))\sin^2(2u_1)F(u_1),\\
d_4 =& 2(1-k^2\sin^2 u_1)E^3(u_1) - E^2(u_1)(3k^2\cos u_1 \sin u_1\sqrt{1-k^2\sin^2 u_1}\\
&+ (2-k^2)(1-k^2\sin^2 u_1)F(u_1) + \frac{1}{4}(1-k^2)F^2(u_1)(2(2-k^2)(2-k^2+k^2\cos(2u_1))F(u_1)\\
&+ 2k^2\sqrt{1-k^2\sin^2 u_1}\sin 2u_1) + \frac{1}{4}E(u_1)(4k^4\cos^2 u_1 \sin^2 u_1\\
&- 8(1-k^2)(1-k^2\sin^2 u_1)F^2(u_1) + 2k^2(2-k^2)\sin(2u_1)\sqrt{1-k^2\sin^2 u_1}F(u_1)).
\end{aligned}
$$

5.1. Conjugate time as $k \to 0$. The asymptotics of the function J_1 as $k \to 0$ has the form

$$
\begin{aligned}
&J_1(u_1,x,k) = \frac{k^8}{1024}J_1^0(u_1,x) + o(k^8), \qquad x = \sin^2 u_2, \qquad (30)\\
&J_1^0(u_1,x) = d_0^0(u_1) + d_2^0(u_1)x + d_4^0(u_1)x^2,\\
&d_0^0(u_1) = \frac{1}{8}\cos u_1 \sin u_1\big((-48u_1^2 - 3)\cos(2u_1) + 3\cos(6u_1) + (42u_1 - 64u_1^3)\sin(2u_1) + 2u_1\sin(6u_1)\big),\\
&d_2^0(u_1) = -d_4^0(u_1) = -(\sin(4u_1) - 4u_1)(4u_1^2 + \sin(4u_1)u_1 + \cos(4u_1) - 1).
\end{aligned}
$$

First, we prove several auxiliary lemmas that give an estimate of the functions d_i^0.

Lemma 11. *The function $f_1(u_1) = 8u_1 + 4u_1\cos(4u_1) - 3\sin u_1 > 0$ on the interval $u_1 \in (0, \frac{\pi}{2})$.*

Proof. We show that the function $g(u_1) = 2 + \cos(4u_1)$ is a comparison function for $f_1(u_1)$ on the interval $u_1 \in (0, \frac{\pi}{2})$. The inequality $f_1(u_1) \not\equiv 0$ follows from the expansion

$$f_1(u_1) = \frac{256}{15}u^5 + o(u^5).$$

Note that $g(u_1) > 0$ for any u_1. Finally we get the equalities

$$\left(\frac{f_1(u_1)}{g(u_1)}\right)' = \frac{16\sin^4(2u_1)}{(2+\cos(4u_1))^2}, \quad \frac{f_1(u_1)}{g(u_1)} = \frac{256}{45}u_1^5 + o(u_1^5).$$

So $g(u_1)$ is a comparison function for $f_1(u_1)$; thus it follows from Lemma 2 that $f_1(u_1) > 0$ for $u_1 \in (0, \frac{\pi}{2})$. □

Lemma 12. *The function* $f_2(u_1) = -1 + 4u_1^2 + \cos(4u_1) + u_1\sin(4u_1) > 0$ *on the interval* $u_1 \in (0, \frac{\pi}{2})$.

Proof. Let us show that the function $g(u_1) = 4u_1 + \sin(4u_1)$ is a comparison function for $f_2(u_1)$ on the interval $u_1 \in (0, \frac{\pi}{2})$. The inequality $f_2(u_1) \not\equiv 0$ follows from the expansion $f_2(u_1) = \frac{128}{45}u^6 + o(u^6)$. If $u_1 > 0$, then $u_1 + \sin u_1 > 0$, and therefore $g(u_1) > 0$ for $u_1 \in (0, \frac{\pi}{2})$. Finally we have the equalities

$$\left(\frac{f_2(u_1)}{g(u_1)}\right)' = \frac{(-4u_1+\sin(4u_1))^2}{(4u_1+\sin(4u_1))^2}, \quad \frac{f_2(u_1)}{g(u_1)} = \frac{16}{45}u_1^5 + o(u_1^5).$$

So $g(u_1)$ is a comparison function for $f_2(u_1)$, and thus it follows from Lemma 2 that $f_2(u_1) > 0$ for $u_1 \in (0, \frac{\pi}{2})$. □

Lemma 13. *If* $u_1 \in (0, \frac{\pi}{2})$*, then the function*

$$d_0^0(u_1) = \frac{1}{8}\cos u_1 \sin u_1 \big((-48u_1^2 - 3)\cos(2u_1) + (42u_1 - 64u_1^3)\sin(2u_1) + 3\cos(6u_1) + 2u_1\sin(6u_1)\big) > 0.$$

Proof. We now prove that the function $g(u_1) = 4u_1 + \sin(4u_1)$ is a comparison function for $d_0^0(u_1)$ on the interval $u_1 \in (0, \frac{\pi}{2})$. The inequality $d_0^0(u_1) \not\equiv 0$ follows from the expansion $d_0^0(u_1) = \frac{4096}{4725}u^{11} + o(u^{11})$. If $u_1 \in (0, \frac{\pi}{2})$, then $g(u_1) > 0$. In the equation

$$\left(\frac{d_0^0(u_1)}{g(u_1)}\right)' = \frac{1}{8u_1^3\cos^2 u_1 \sin^2 u_1} f_1(u_1) f_2(u_1),$$

we note that $f_1(u_1) = 8u_1 + 4u_1\cos(4u_1) - 3\sin(4u_1) > 0$ on the interval $u_1 \in (0, \frac{\pi}{2})$ (see Lemma 11) and $f_2(u_1) = -1 + 4u_1^2 + \cos(4u_1) + u_1\sin(4u_1) > 0$ on the interval $u_1 \in (0, \frac{\pi}{2})$ (see Lemma 12). Meanwhile

$$\frac{d_0^0(u_1)}{g(u_1)} = \frac{4096}{4725}u_1^7 + o(u_1^7).$$

So $g(u_1)$ is a comparison function for $d_0^0(u_1)$, and therefore it follows from Lemma 2 that $d_0^0(u_1) > 0$ for $u_1 \in (0, \frac{\pi}{2})$. □

Now we estimate the function J_1^0.

Lemma 14. *For any* $u_1 \in \left(0, \frac{\pi}{2}\right)$, $x \in [0, 1]$ *the inequality* $J_1^0(u_1, x) > 0$ *holds.*

Proof. It follows from Lemma 13 that $J_1^0(u_1, 0) = J_1^0(u_1, 1) = d_0^0(u_1) > 0$ for all $u_1 \in \left(0, \frac{\pi}{2}\right)$. Further, it follows from Lemma 12 that $d_4^0(u_1) = \frac{1}{2}(\sin(4u_1) - 4u_1)f_2(u_1) < 0$ for $u_2 \in (0, \frac{\pi}{2})$. Therefore the statement of this lemma follows from Lemma 3. □

Proposition 3. *There exists* $\bar{k} \in (0, 1)$*, such that for any* $k \in (0, \bar{k})$, $u_1 \in \left(0, \frac{\pi}{2}\right)$, $x \in [0, 1]$*, the inequality* $J_1(u_1, x, k) > 0$ *holds.*

Proof. This proposition is proved in exactly the same way as Proposition 1, with the use of Lemma 14, expansions (30), and the following expansions:

$$J_1 = \frac{4}{4725}k^8 u_1^{11} + +o(k^8 u_1^{11}) + \frac{4}{135}k^8 u_1^9 x + o(k^8 u_1^9 x) - \frac{4}{135}k^8 u_1^9 x^2 + o(k^8 u_1^9 x^2), \quad k^2 + u_1^2 \to 0,$$

$$\begin{aligned} J_1 \ &= \tfrac{2\pi^2}{8192}k^8\left(\tfrac{\pi}{2} - u_1\right) + o\left(k^8\left(\tfrac{\pi}{2} - u_1\right)\right) + \tfrac{\pi^3}{512}k^8 x \quad +o(k^8 x) - \\ &\qquad -\frac{\pi^3}{512}k^8 x^2 + o(k^8 x^2), \quad k^2 + \left(\frac{\pi}{2} - u_1\right)^2 \to 0. \end{aligned}$$

□

From Proposition 3 we get the following statement in the variables (t, φ, k).

Corollary 2. *There exists* $\bar{k} \in (0,1)$ *such that for any* $k \in (0, \bar{k})$, $\varphi \in \mathbb{R}$, *the trajectory* $\mathrm{Exp}(\lambda, t)$, $\lambda = (\varphi, k, \alpha) \in C_2$, $t \in (0, t^1_{\mathrm{MAX}}(\lambda))$, *does not contain conjugate points.*

5.2. Conjugate time for $t = t^1_{\mathrm{MAX}}$.

The instant of time $t^1_{\mathrm{MAX}}(\lambda) = 2Kk$ corresponds to the value of the variable $u_1 = \frac{\pi}{2}$. We have

$$J_1\left(\frac{\pi}{2}, x, k\right) = d_2^{\frac{\pi}{2}} x + d_4^{\frac{\pi}{2}} x^2, \tag{31}$$

$$d_4^{\frac{\pi}{2}} = -d_2^{\frac{\pi}{2}} = \sqrt{1-k^2}\, g_z(K,k)\, f_4(k), \tag{32}$$

$$g_z(p,k) = ((k^2 - 2)p + 2\,\mathrm{E}(p))\,\mathrm{dn}\, p - k^2\, \mathrm{sn}\, p\, \mathrm{cn}\, p, \tag{33}$$

$$f_4(k) = E^2(k) - (1-k^2)K(k). \tag{34}$$

In the paper [12] it was proved that $g_z(p,k) < 0$ for any $p > 0$, $k \in (0,1)$; therefore $g_z(K,k) < 0$.

Lemma 15. *The function* $f_4(k) = E^2(k) + (k^2 - 1)K(k) > 0$ *on the interval* $k \in (0,1)$.

Proof. We show that the function $g(k) = 1 - k^2$ is a comparison function for $f_4(k)$ on the interval $k \in (0,1)$. The inequality $f_4(k) \not\equiv 0$ follows from the expansion $f_4(k) = \frac{\pi^2}{32}k^4 + o(k^4)$. Note that $g(k) > 0$ for $k \in (0,1)$. Finally, we have the equalities

$$\left(\frac{f_4(k)}{g(k)}\right)' = \frac{2(E(k) + (k^2-1)K(k))^2}{k(k^2-1)^2}$$

and $\dfrac{f_4(k)}{g(k)} = \dfrac{\pi^2}{32}k^4 + o(k^4)$. So $g(k)$ is a comparison function for $f_4(k)$, and thus it follows from Lemma 2 that $f_4(k) > 0$ for $k \in (0,1)$. □

Lemma 16. (1) *If* $k \in (0,1)$, $u_1 = \frac{\pi}{2}$, $x \in (0,1)$, *then* $J_1 > 0$.
(2) *If* $k \in (0,1)$, $u_1 = \frac{\pi}{2}$, $x \in \{0,1\}$, *then* $J_1 = 0$.

Proof. It follows from formula (31)–(34), inequality $g_z(K,k) < 0$, and Lemma 15. □

5.3. Global bounds of conjugate time.

Theorem 5. *If* $\lambda \in C_2$, *then* $t^1_{\mathrm{conj}}(\lambda) \geq t^1_{\mathrm{MAX}}(\lambda)$.

Proof. This theorem is proved in exactly the same way as Theorem 4 based on the homotopy invariance of the index of the second variation (the number of conjugate points); see Theorem 3. The last theorem is applied to the family of extremal trajectories

$$
\begin{aligned}
&q^s(t) = \operatorname{Exp}(\lambda^s, t), \quad t \in [0, t_1^s], \quad s \in [\tilde{k}, k],\\
&\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_2,\\
&\varphi^s = sF\left(\operatorname{am}\left(\frac{\varphi + t_1^k/2}{k}, k\right), s\right) - t_1^s/2,\\
&k^s = s, \quad t_1^s = 2K(s)s, \quad \tilde{k} \in (0, \bar{k}).
\end{aligned}
$$

□

Remark 3. *It follows from Lemma* 16 *that the lower bound from Theorem* 5 *is attained: if* $\varphi = Kkn$, $n \in \mathbb{Z}$, *then* $t^1_{\text{conj}}(\lambda) = 2Kk$, $\lambda = (\varphi, k, \alpha = 1) \in C_2$.

Remark 4. *Using the homotopy invariance of the index of the second variation, we can prove the upper bound of conjugate time:*

$$
\begin{aligned}
&t^1_{\text{conj}}(\lambda) \le t^2_{\text{MAX}}(\lambda), \quad \lambda \in C_2,\\
&t^2_{\text{MAX}}(\lambda) = 4kK, \quad \lambda \in C_2.
\end{aligned}
$$

Note that the segment $[t^1_{\text{MAX}}(\lambda), t^2_{\text{MAX}}(\lambda)]$ *contains exactly two conjugate times (with account of multiplicity).*

6. Estimate of Conjugate Time for $\lambda \in C_3$

Theorem 6. *If* $\lambda \in C_3$, *then the extremal trajectory* $\operatorname{Exp}(\lambda, t)$, $t \in (0, +\infty)$, *does not contain conjugate points.*

Proof. Let $\lambda = (\varphi, k = 1, \alpha = 1) \in C_3$ and $t_1 > 0$. We show that the trajectory $\operatorname{Exp}(\lambda, t)$, $t \in (0, t_1]$, has no conjugate points. Choose a time $t_2 > t_1$ that is not a conjugate time. There exists $k_1 \in (0, 1)$, such that $k_1 K(k_1) > 2t_2$. According to Theorem 5, all trajectories

$$
\begin{aligned}
&q^s(t) = \operatorname{Exp}(\lambda^s, t), \quad t \in (0, t_1^s], \quad s \in [k_1, 1),\\
&\lambda^s = (\varphi^s, k^s, \alpha = 1) \in C_2,\\
&\varphi^s = \varphi, \quad k^s = s, \quad t_1^s = \frac{1}{2}K(s)s,
\end{aligned}
$$

have no conjugate points. Applying Theorem 3 to the family of trajectories $q^s(t)$, $t \in (0, t_2]$, $s \in [k_1, 1]$, we conclude that the trajectory $q^1(t) = \operatorname{Exp}(\lambda, t)$, $t \in (0, t_2]$, does not contain conjugate points. □

7. Estimate of Conjugate Time for $\lambda \in \cup_{i=4}^7 C_i$

If $\lambda \in \bigcup\limits_{i=4}^{7} C_i$, then the conjugate time (and cut time) can be located by projecting the original problem (1)–(3) into simpler problems of a lower dimension using the following proposition.

Proposition 4. *Let us consider two optimal control problems:*

$$
\begin{aligned}
&\dot{q}^i = f^i(q^i, u), \quad q^i \in M^i, \quad u \in U,\\
&q^i(0) = q_0^i, \quad q^i(t_1) = q_1^i,\\
&J = \int\limits_0^{t_1} \varphi(u)\, dt \to \min,\\
&i = 1, 2.
\end{aligned}
$$

Suppose that there exists a smooth map $G : M^1 \to M^2$, such that if $q^1(t)$ is the trajectory of the first system corresponding to a control $u(t)$, then $q^2(t) = G(q^1(t))$ is the trajectory of the second system with the same control.

Further assume that $q^1(t)$ and $q^2(t)$ are such trajectories. If $q^2(t)$ is locally (globally) optimal for the second problem, then $q^1(t)$ is locally (globally) optimal for the first problem.

Proof. Assume the converse. Suppose $q^2(t)$ is optimal and $q^1(t)$ is not optimal. Then for the first problem there exists a trajectory $\tilde{q}^1(t)$, such that the value of the functional J for this trajectory is less than for $q^1(t)$. So the value of J is less on the trajectory $\tilde{q}^2(t) = G(\tilde{q}^1(t))$ than on $q^2(t)$. This contradiction proves the proposition. □

Theorem 7. (1) *If $\lambda \in C_4 \cup C_5 \cup C_7$, then $t^1_{\text{conj}}(\lambda) = t_{\text{cut}}(\lambda) = +\infty = t^1_{\text{MAX}}(\lambda)$.*

(2) *If $\lambda \in C_6$, then $t^1_{\text{conj}}(\lambda) \geq t_{\text{cut}}(\lambda) = t^1_{\text{MAX}}(\lambda)$.*

Proof. (1) Consider the mapping

$$G : \mathbb{R}^4_{x,y,z,v} \to \mathbb{R}^2_{x,y}, \qquad (x, y, z, v) \mapsto (x, y),$$

and the Riemannian problem in the Euclidean plane $\mathbb{R}^2_{x,y}$:

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad (x, y)(0) = (x_0, y_0), \quad (x, y)(t_1) = (x_1, y_1), \tag{35}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{36}$$

The mapping G sends trajectories of system (1), (2) to trajectories of (35), with the same control.

Let $\lambda \in C_4 \cup C_5 \cup C_7$, and let $\text{Exp}(\lambda, t) = (x_t, y_t, z_t, v_t)$. As was shown in [5], the curve (x_t, y_t) is a straight line. Since straight lines (x_t, y_t), $t \in [0, +\infty)$, are globally optimal for the Riemannian problem on $\mathbb{R}^2$, then, by Proposition 4, the trajectories (x_t, y_t, z_t, v_t), $t \in [0, +\infty)$, are globally optimal for the sub-Riemannian problem on the Engel group.

(2) The mapping

$$G : \mathbb{R}^4_{x,y,z,v} \to \mathbb{R}^3_{x,y,z}, \qquad (x, y, z, v) \mapsto (x, y, z),$$

sends trajectories of system (1), (2) to trajectories of the sub-Riemannian problem on the Heisenberg group $\mathbb{R}^3_{x,y,z}$:

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = -\frac{y}{2}u_1 + \frac{x}{2}u_2, \tag{37}$$

$$(x, y, z)(0) = (x_0, y_0, z_0), \quad (x, y, z)(t_1) = (x_1, y_1, z_1), \tag{38}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{39}$$

Let $\lambda \in C_6$, and let $\text{Exp}(\lambda, t) = (x_t, y_t, z_t, v_t)$. As was shown in [5],

$$x_t = \frac{\cos(ct + \theta) - \cos\theta}{c}, \quad y_t = \frac{\sin(ct + \theta) - \sin\theta}{c}, \quad z_t = \frac{ct - \sin(ct)}{2c^2}.$$

It is well known that the curve (x_t, y_t, z_t) is globally optimal for problem (37)–(39) at the segment $t \in [0, 2\pi/|c|]$, i.e., up to the first turn of the circle (x_t, y_t) (see, e.g., [6]).

It follows from Proposition 4 that $t^1_{\text{conj}}(\lambda) \geq t_{\text{cut}}(\lambda) \geq \frac{2\pi}{|c|} = t^1_{\text{MAX}}(\lambda)$ for $\lambda \in C_6$. By Theorem 1, we have $t^1_{\text{conj}}(\lambda) \geq t_{\text{cut}}(\lambda) \leq \frac{2\pi}{|c|} = t^1_{\text{MAX}}(\lambda)$ for $\lambda \in C_6$. □

Remark 5. *Passing to the limit $\alpha \to 0$, $k \to 0$, it can be shown that for $\lambda = (\theta, c, \alpha) \in C_6$, $\theta = \alpha = 0$, the equality $t^1_{\text{conj}}(\lambda) = \frac{2\pi}{|c|} = t^1_{\text{MAX}}(\lambda)$ holds. But for $\lambda \in C_6$ this equality does not hold in the general case.*

8. Conclusion

Theorem 2 follows from Theorems 4, 5, 6, 7.

Using the estimate of cut time obtained in the work [5] (Theorem 1) and the estimate of conjugate time proved in this work (Theorem 2), we can get a description of the global structure of the exponential map in the sub-Riemannian problem on the Engel group. So we can reduce this problem to solving a system of algebraic equations. This will be the subject of another paper.

The method for estimating the conjugate time used in this paper was successfully applied earlier to Euler's elastic problem [8] and the sub-Riemannian problem on the group of rototranslations [13]. There is no doubt that this method is also valid for the nilpotent sub-Riemannian problem with the growth vector (2,3,5) [9–12].

The method can be used for other invariant sub-Riemannian problems on low-dimensional Lie groups integrable in non-elementary functions. The first natural step in this direction is investigation of the invariant sub-Riemannian problem on 3D Lie groups classified by A. A. Agrachev and D. Barilari [2].

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