Journal of Mathematical Sciences, Vol. 194, No. 1, October, 2013

THE LEONTOVICH–FOCK PARABOLIC EQUATION METHOD IN PROBLEMS OF SHORT-WAVE DIFFRACTION BY PROLATE BODIES

N. Ya. Kirpichnikova* **and M. M. Popov***

UDC 517.9

Application of the parabolic equation to problems of short-wave diffraction by prolate convex bodies in the rotational symmetry case is considered. The wave field is constructed in the Fock domain and in the shaded part of the body, where creeping waves appear. In the problems under consideration, the following two large parameters arise: $\mathbf{M} = (k\rho/2)^{1/3}$ *and* $\mathbf{\Lambda} = \rho/f$*, where* k *is the wave number,* ρ *is the radius of curvature along geodesics (meridians) and* f *is the radius of curvature in the transversal direction. The first one is the so-called Fock parameter, and the second one* $\mathbf{\Lambda}$ *characterizes the prolateness of the body. Under the condition* $\mathbf{\Lambda} = \mathbf{M}^{2-\varepsilon}$*,* $0 < \varepsilon < 2$*, the parabolic equation method in classical form is valid and describes the wave field in terms of the Airy function and integrals of it. In the case of* $\varepsilon = 0$*, some coefficients in the corresponding recurrent equations become singular and the question on the solvability of the equations in terms of regular and smooth functions remains open. Bibliography: 9 titles.*

Introduction

Recently a number of articles concerning the short-wave diffraction by elongated bodies have been published (see [1, 2] and the bibliography therein).

Actually the problem therein is set in a simplified version: the scatterer represents a strictly convex body of revolution and the incident plane wave propagates along the axis of revolution, i.e., an axisymmetric problem is considered. The wave field has been studied in the vicinity of the light-shadow boundary (the Fock domain) and in the shaded part of this zone, where creeping waves appear. The body prolateness is characterized by the ratio of the longitudinal curvature radius of geodesics (meridians), orthogonal to the light-shadow boundary on the surface of the scatterer, to the curvature cross-radius of the light-shadow boundary. Moreover, both curvature radii are assumed to be large in comparison to the length of the incident wave. This suggests the usage of the parabolic Leontovich–Fock equation method. In the articles under discussion (see [1, 2]), the usage of the parabolic equation method is indeed declared and the Fock's scales for local coordinates in the Fock domain are preserved, but at the same time the proposed equation differs essentially from the Fock's one.

We would like to recall that the method by V. A. Fock (see [3]) is based upon the following heuristic propositions:

(a) it is necessary to single out correctly a multiplier that describes the most rapid oscillations of the wave field. In this case, this is $\exp\{iks\}$, where k is the wave number and s represents the meridian arc length (geodesic), because the geodesic coincides with the direction of incident wave ray at the point where this ray touches the light-shadow boundary;

(b) after expansion of the equation coefficients in the Fock domain, one needs to choose the asymptotically most significant summands in the equation;

(c) to choose coordinate scales in such a way that these main summands become of the same order as $k \to \infty$ in the parabolic equation obtained.

We emphasize that the choice of scales is of fundamental importance, because it determines the scale of functions where the asymptotics of the solution is constructed, and in a fixed scale of functions no different asymptotics of the same function and of the solution of the problem considered may exist.

In the present paper, a different approach to the problem of short-wave diffraction by an elongated body is proposed. At that we also, as in [1], consider an axisymmetric problem of plane wave diffraction by a strictly convex body of revolution. To simplify it, we assume that the wave field satisfies the Helmholtz equation and the Dirichlet boundary conditions on the scatterer's surface.

The proposed method is based upon the classical parabolic equation approach, but represents a two-scale asymptotic expansion in two independent parameters: Fock's parameter $\mathbf{M}_0 = (k\rho(0)/2)^{1/3}$ and the new one $\mathbf{\Lambda}_0 = \rho(0)/f(0)$, which characterizes the oblongness of the scatterer. Here, $\rho(0) = \rho_0$ and $f(0)$ are the main radii of the scatterer surface curvature along the meridian and the equator on the light-shadow boundary,

*St.Petersburg Department of the Steklov Mathematical Institute, Russian Academy of Sciences, St.Petersburg, Russia, e-mail: nkirp@pdmi.ras.ru, mpopov@pdmi.ras.ru.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 409, 2012, pp. 55–79. Original article submitted November 27, 2012.

 1072-3374/13/1941-0030

respectively. In this paper, creeping waves, which appear in the shaded part of the surface behind the Fock domain, are constructed as well.

This two-scale expansion enables one to obtain formulas for calculating the wave field, in particular, the current on the surface, for different relations between $\mathbf{M}_0$ and $\mathbf{\Lambda}_0$ satisfying the condition $\mathbf{\Lambda}_0 = \mathbf{M}_0^{2-\varepsilon}$. The parameter ε is contained in the interval $0 < \varepsilon < 2$. Moreover, the classical method of the parabolic Fock equation is used and the wave field is described in terms of Airy functions and integrals of them. However, for $\varepsilon = 0$ a considerable rearrangement of operators in the recurrence equations occurs in the Fock domain, as well as in the domain of creeping waves. It turns out that in the Fock domain, the coefficients in the arising recurrence equations become singular.

1. Statement of the problem

We assume that the surface $\partial\Sigma$ of the scatterer Σ is formed by rotation of a plane convex curve

$$x = f(z), \quad r = \sqrt{x^2 + y^2}, \quad x = r\cos\varphi, \quad y = r\sin\varphi$$

around the axis z of the orthogonal coordinate system x, y, z, and the cross-section of $\partial\Sigma$ by the plane $z = 0$ represents an equator and coincides with the light-shadow boundary of the incident plane wave $U^{\mathrm{inc}} = \exp(ikz)$.

The wave field U satisfies the Helmholtz equation and the Dirichlet condition on the surface $\partial\Sigma$ of the body of revolution

$$(\Delta + k^2)U = 0, \quad U\big|_{\partial\Sigma} = 0. \tag{1.1}$$

We consider the wave field close to $\partial\Sigma$ in the Fock domain and in the shaded part, where creeping waves appear. On the surface of the body, we take the orthogonal net of main curvature lines – meridians and parallels; in the vicinity of the body, the length n along the outer normal $\mathbf{n}$ is added to them. The length along the meridians is denoted by s and is counted starting from the equator. The connection between s and z is determined by the relation

$$s = \int\limits_0^z \sqrt{1 + (f'(z))^2}\, dz. \tag{1.2}$$

For the unit vector of the outer normal $\mathbf{n}$, the following expression is valid:

$$\mathbf{n} = (1 + (f'(z))^2)^{-1/2}(\cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y - f'(z)\, \mathbf{e}_z), \tag{1.3}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ represent the unit vectors of the corresponding Cartesian coordinate system and φ is the azimuth angle. The coordinate system s, n, φ is orthogonal and regular in the considered domain; the square of the length element dS^2 has the form

$$dS^2 = h_s^2 ds^2 + dn^2 + h_\varphi^2 d\varphi^2,$$

where the Lamé coefficients h_s and h_φ are described by the formulas

$$\begin{aligned} h_s &= 1 - n\frac{f''(z(s))}{[1 + (f'(z(s)))^2]^{3/2}} = 1 + \frac{n}{\rho(s)}, \\ h_\varphi &= f(z(s)) + \frac{n}{\sqrt{1 + (f'(z(s)))^2}}. \end{aligned} \tag{1.4}$$

We note that in formulas (1.4), the coordinate z is regarded as a function of s obtained by inverting relation (1.2), and the function $\rho(s)$ serves as the meridian curvature radius at the point s. We are interested in an axisymmetric solution of problem (1.1), i.e., in the solution for which the condition $\dfrac{\partial U}{\partial\varphi} \equiv 0$ is fulfilled. In this case, the answer is sought in the form

$$U = \exp(iks)W(s, n),$$

where the multiplier $\exp(iks)$ describes the main oscillations of the wave field and W is called the attenuation function. The equation for W can be represented in the form

$$\left[(\Delta + k^2)e^{iks}W\right] = e^{iks}g\left\{\frac{\rho_0^2}{4\mathbf{M}_0^4}(AW + BW)\right\} = 0, \tag{1.5}$$

where $g = (4\mathbf{M}_0^4)/\rho_0^2$ and the operators A and B are equal to

$$AW = k^2(1 - h_s^{-2})W + \left(2ik\frac{\partial W}{\partial s} + \frac{\partial^2 W}{\partial s^2}\right)h_s^{-2} + \frac{\partial^2 W}{\partial n^2} - \left(ikW + \frac{\partial W}{\partial s}\right)h_s^{-2}\frac{\partial \ln h_s}{\partial s} + \frac{\partial W}{\partial n}\frac{\partial \ln h_s}{\partial n}, \quad (1.6)$$

$$BW = \frac{1}{h_\varphi}\left[\frac{df}{ds}\frac{1}{h_s}\left(ikW + \frac{\partial W}{\partial s}\right) + \frac{\partial h_\varphi}{\partial n}\frac{\partial W}{\partial n}\right]. \quad (1.7)$$

We emphasize that the operator A corresponds to the two-dimensional problem of diffraction by the curve in the cross-section $\varphi = \text{const}$ and the operator B includes all specificity of the three-dimensional initial problem and is of principal interest. Concerning the two-dimensional problem, book [3] contains a detailed construction of the principal asymptotic term, and in books [4, 5] the reader will find a mathematically clear and justified procedure of calculating subsequent terms.

In constructing a solution, we need to obtain explicit analytic formulas for three successive asymptotic terms, which leads, unfortunately, to quite cumbersome calculations.

We would like to note that the mathematical validity of this scheme follows from the fact that the short-wave asymptotics of the diffraction problem by a strictly convex body is mathematically justified on the whole (see [6–8]).

2. Incident wave field in the Fock domain

The Fock domain, according to V. M. Babich's terminology, is an "embryo." Knowing the wave field in this domain, one can construct the wave field asymptotics in the vicinity of the limit ray and in the shaded part of the scatterer. In this case, this zone represents a circular domain close to the light-shadow boundary (near the equator), where the coordinates s and n are considered small.

Further, one should expand the coefficients in Eq. (1.5) in powers of s and n and then pass to scaled (stretched) variables σ and ν:

$$\sigma = \frac{k^{1/3}s}{2^{1/3}\rho_0^{2/3}} = \frac{\mathbf{M}_0 s}{\rho_0}, \quad \nu = \frac{2^{1/3}k^{2/3}n}{\rho_0^{1/3}} = \frac{2n\mathbf{M}_0^2}{\rho_0}, \quad (2.1)$$

where ρ_0 is the curvature radius of geodesics (meridians) on the light-shadow boundary (equator), i.e., for $s = 0$, and by $\mathbf{M}_0$ the large dimensionless Fock parameter $\mathbf{M}_0 = \left(\frac{k\rho_0}{2}\right)^{1/3}$ is denoted, with respect to which the asymptotic expansion is carried out. The inverse formulas for s and n are given in the form

$$s = \rho_0\frac{\sigma}{\mathbf{M}_0}, \quad n = \frac{\rho_0}{2}\frac{\nu}{\mathbf{M}_0^2}. \quad (2.2)$$

Let us dwell on several key points, which allow us to simplify calculations to some extend.

(1) The Lamé coefficients (1.4) are viewed as functions of the geodesic length s, which demands the inversion of function (1.2). In the Fock domain, we need to expand them in powers of z and s, respectively, i.e.,

$$s = \sum_{n=1}^{N}\alpha_n z^n + O(z^{N+1}), \quad z = \sum_{n=1}^{N}\beta_n s^n + O(s^{N+1}). \quad (2.3)$$

The coefficients α_n are easily obtained from formula (1.2), while the β_n are expressed in terms of α_n by inverting the series for s. To calculate three terms of the wave field asymptotics in relations (2.3), we should take $N = 5$. It is obvious that $\alpha_1 = \beta_1 = 1$ and, what is of importance in the sequel, $\alpha_2 = \beta_2 = 0$. Relation (1.2) also implies that

$$\frac{dz}{ds} = \frac{1}{\sqrt{1 + (f'(z(s)))^2}}$$

and therefore h_φ can be introduced in the form

$$h_\varphi(s, n) = f(z(s)) + n\frac{dz}{ds}, \quad (2.4)$$

which is convenient for calculations.

(2) The Lamé coefficient h_s may be regarded as given in the form $h_s = 1 + \dfrac{n}{\rho(s)}$, where $\rho(s)$ is the geodesic curvature radius at the point s. Then, the coefficients of the expansion of $\rho^{-1}(s)$ in powers of s can be expressed in terms of derivatives of $f(z(s))$ with the help of the relation

$$\frac{1}{\rho(s)} = \frac{-f^{''}(z(s))}{[1 + (f'(z(s)))^2]^{3/2}}. \tag{2.5}$$

So, for example, from formula (2.5) and relation (2.3) we get

$$f_z'(0) = 0, \quad f_z^{(2)} = f_{zz}^{''}(0) = -\frac{1}{\rho_0}, \quad f_z^{(3)} = f_{zzz}^{'''}(0) = -\frac{d}{ds}\frac{1}{\rho}\bigg|_{s=0}, \ldots .$$

(3) Now we turn to the incident plane wave $U = \exp(ikz)$. We shall need its expansion in the local coordinates s and n in the Fock domain in order to construct the reflected (diffracted) wave U^r, in view of boundary condition (1.1) on the surface $\partial\Sigma$ of the scatterer. The derivation of it is based on a formula that describes the connection between the coordinate z and the coordinates s and n:

$$z = z(s) + n(\mathbf{n}, \mathbf{e}_z) = z(s) + n\frac{-f'(z(s))}{\sqrt{1 + [f'(z(s))]^2}}, \tag{2.6}$$

where $\mathbf{e}_z$ is the unit vector of the axis z of the initial Cartesian coordinates. Expanding the right-hand side in powers of s, we get

$$z = s + \left(-\frac{s^3}{3!\rho_0^2} + \frac{ns}{\rho_0}\right) + \left(\frac{3\rho_0' s^4}{4!\rho_0^3} - \frac{ns^2\rho_0'}{2!\rho_0^2}\right) + \left(\frac{\alpha s^5}{5!\rho_0^4} - \frac{\beta n s^3}{3!\rho_0^3}\right) + O(s^6, ns^4). \tag{2.7}$$

Henceforth in all formulas related to the Fock domain, functions of s are calculated at the point $s = 0$: $\rho(s)\big|_{s=0} = \rho_0$, $\rho'(s)\big|_{s=0} = \rho_0'$, and $\rho''(s)\big|_{s=0} = \rho_0''$.

The coefficients α and β equal

$$\alpha = 4\rho_0\rho_0^{''} - 11{\rho_0'}^2 + 1, \quad \beta = \rho_0\rho_0^{''} - 2{\rho_0'}^2 + 1. \tag{2.8}$$

Passing in the expansion (2.7) from s, n to scaled (stretched) coordinates σ, ν, we discover that all summands enclosed in parentheses are of the same order with respect the wave number k, and the order of each subsequent parenthesis is obtained from the previous one by multiplying by $k^{-1/3}$. This gives rise to the following expansion for the attenuation function W^{inc} of the incident plane wave:

$$U^{\mathrm{inc}} = e^{iks}W^{\mathrm{inc}}, \quad W^{\mathrm{inc}} = \sum_{m=0} W_m^{\mathrm{inc}} k^{-m/3}, \tag{2.9}$$

where $W_m^{\mathrm{inc}} = W_0^{\mathrm{inc}} P_m^{\mathrm{inc}}$ and P_m^{inc} represent polynomials in σ and ν. Note that expansion (2.9) can be represented as an asymptotic series in negative powers of the large dimensionless Fock parameter $\mathbf{M}_0 = (k\rho_0/2)^{1/3}$. To this end it is required to employ explicit formulas for the coefficients β_q, $q = 3, 4, 5, \ldots$, and for the derivatives of the function $f(z)$ with respect to z at $z = 0$, following from Eq. (2.5).

In the present article, three terms of the asymptotic expansion of the form (2.9) for the incident and reflected waves are calculated. Not to overload the paper with too cumbersome formulas, we omit numerous intermediate computations, replacing them by verbal comments on their derivation. We hope that such an approach will not make this article less readable.

The principal summand in (2.9) has the simple form

$$W_0^{\mathrm{inc}}(\sigma, \nu) = \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{i\sigma\zeta} v(\zeta - \nu)\,d\zeta, \tag{2.10}$$

where $v(\zeta - \nu)$ is the real-valued Airy function in the Fock definition, because the polynomial P_0^{inc} turns out to be equal to one. The second summand in the expansion of the incident wave is a bit more complicated, because

$$P_1^{\mathrm{inc}} = i\left(\frac{2}{\rho_0}\right)^{1/3}\frac{\rho_0'}{4}(\sigma^4 - 2\sigma^2\nu).$$

The polynomial P_2^{inc} already contains σ^8 and ν^4 and is cumbersome; however it is easily reconstructed in expanding $\exp(ikz)$ with regard to formula (2.7).

We mention a technical point important in the sequel. In the scaled coordinates σ and ν, the problem of constructing the reflected wave in the Fock domain is to be solved in the half-plane $\{-\infty < \sigma < \infty,\ \nu > 0\}$. For this reason, it is convenient to separate σ in the form of the Fourier transform, as is represented in relation (2.10). Such a transform is carried out on the basis of the following formal relation:

$$(-i\sigma)^m \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\sigma\zeta} \frac{d^m}{d\zeta^m} v(\zeta - \nu) d\zeta. \tag{2.11}$$

For the mathematical justification of this relation and normalization of the integrals, we refer to book [5].

Using the Airy equation $v''(t) = tv(t)$, relation (2.11) can be rewritten in the form

$$(-i\sigma)^m \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\sigma\zeta} [P_m(\zeta,\nu) v(\zeta - \nu) + Q_m(\zeta,\nu) v'(\zeta - \nu)] d\zeta, \tag{2.12}$$

where the polynomials P_m and Q_m in ζ and ν quickly become more complicated with increase of m.

Thus, for $W_1^{\mathrm{inc}}(\sigma,\nu)$ we obtain the formula

$$W_1^{\mathrm{inc}}(\sigma,\nu) = i\left(\frac{2}{\rho_0}\right)^{1/3} \frac{\rho_0'}{4\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\sigma\zeta} \left[(\zeta^2 - \nu^2) v(\zeta - \nu) + 2v'(\zeta - \nu)\right] d\zeta, \tag{2.13}$$

where $v'(\zeta - \nu)$ is the derivative of $v(t)$ with respect to the argument t.

Now the third term W_2^{inc} and subsequent ones of the incident wave expansion in the Fock domain gain the following structure (see (2.12)). For example, the summand at $m = 2$ in expansion (2.9) is represented in the form

$$W_2^{\mathrm{inc}}(\sigma,\nu) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\rho_0}\right)^{2/3} \int_{-\infty}^{\infty} e^{i\sigma\zeta} [P_2(\zeta,\nu) v(\zeta - \nu) + Q_2(\zeta,\nu) v'(\zeta - \nu)] d\zeta. \tag{2.14}$$

Here, the polynomials $P_2(\zeta,\nu)$ and $Q_2(\zeta,\nu)$ are as follows:

$$P_2 = \frac{8\alpha}{5!}(\zeta - \nu) + i\frac{\beta}{3!}\nu - \frac{\rho_0'^2}{8}[4\zeta + 3(\zeta - \nu) + 1/4(\zeta - \nu)^4 + \nu(\zeta - \nu)^3 + \nu^2(\zeta - \nu)^2];$$
$$Q_2 = \frac{2\alpha}{5!}(\zeta - \nu)^2 + i\frac{\beta}{3!}\nu(\zeta - \nu) - \frac{\rho_0'^2}{4}[3/2(\zeta - \nu)^2 + 3\nu(\zeta - \nu) + \nu^2];$$

and the coefficients α, β are determined by relations (2.8).

The formulas for $W_m^{\mathrm{inc}}(\sigma,\nu)$, $m = 0, 1, 2$, give an idea of the analytic structure of the reflected field.

3. The Fock domain. Construction of the reflected wave

We build the reflected wave $U^{\mathrm{ref}} = e^{iks} W^{\mathrm{ref}}$ in the form of an asymptotic series in powers $k^{-m/3}$, $m = 0, 1, 2, \ldots$, similarly to series (2.9):

$$W^{\mathrm{ref}} = \sum_{m=0} W_m^{\mathrm{ref}} k^{-m/3}. \tag{3.1}$$

The first step consists of the derivation of a system of recurrence equations for successive calculation of the coefficients W_m^{ref} of the attenuation function W^{ref}. Upon expanding the coefficients of the initial Helmholtz equation in powers of s, n and passing to scaled variables σ, ν (see (2.1)), the result can be written in the form

$$(\Delta + k^2) e^{iks} W = e^{iks} g \sum_{m=0} k^{(-m)/3} L_m W = 0, \tag{3.2}$$

where a common multiplier g equals $g = (4\mathbf{M}_0^4)/(\rho_0^2)$, and the L_m, $m = 0, 1, 2, \ldots$, represent differential operators in σ and ν (of the first and second order), which contain polynomials in σ and ν as multipliers. As m increases, the explicit formulas for operators become noticeably more complicated and their derivation demands though simple but unwieldy calculations. In the sequel we shall give their explicit form for $m = 0, 1, 2$. Substituting into

(3.2) the expansion (3.1) for the attenuation function $W^{\rm ref}$ and equating the coefficients at identical powers of k to zero, we obtain the sought recurrence system

$$\begin{aligned} &L_0 W_0^{\rm ref} = 0, \\ &L_0 W_1^{\rm ref} + L_1 W_0^{\rm ref} = 0, \\ &L_0 W_2^{\rm ref} + L_1 W_1^{\rm ref} + L_2 W_0^{\rm ref} = 0 \end{aligned} \tag{3.3}$$

for successive construction of the functions $W_m^{\rm ref}$. Equations (3.3) should be solved simultaneously with the boundary conditions on the scatterer's surface at $\nu = 0$:

$$(W_m^{\rm ref} + W_m^{\rm inc})\big|_{\nu=0} = 0, \quad m = 0, 1, 2, \ldots,$$

which follow from the Dirichlet condition in (1.1), and with conditions as $\nu \to +\infty$ such that $W^{\rm ref}$ describes a wave moving off from the body of revolution. These conditions are discussed in full in [3–5].

Let us note that since the incident wave satisfies the Helmholtz equation, the functions $W_m^{\rm inc}$ automatically satisfy Eqs. (3.3), and this can be used to check the correctness of the calculations.

The main feature of Eqs. (3.3) in the case of an elongated body, in comparison to the traditional method of the parabolic Fock equation, consists of the circumstance that the operators L_m contain a new large parameter $\mathbf{\Lambda}_0 = \rho(0)/f(0)$ that characterizes the extent of scatterer's oblongness near the equator. We emphasize that the parameter $\mathbf{\Lambda}_0$ in our calculations is assumed to be independent of the large Fock parameter $\mathbf{M}_0 = (k\rho(0)/2)^{1/3}$ and that it is bounded from above, because the body of revolution must be strictly convex and $kf(0) \gg 1$ by the initial assumption. Our closest goal is to clarify how the parameter $\mathbf{\Lambda}_0$ occurs in Eqs. (3.3). To this end we turn to formulas (1.5)–(1.7). As was mentioned, the selected part A in the Helmholtz operator coincides with the 2D problem of the incident plane wave diffraction by the curve $x = f(z)$, i.e., by the body of revolution crossed by the plane $\varphi =$ const. Therefore it does not contain the cross-radius of the surface curvature of $\partial\Sigma$ (does not contain h_φ). Consequently, it remains only to determine in what manner the part B of the Helmholtz equation, defined by formula (1.7), is transformed in the Fock domain.

From the explicit formulas (1.4) and (2.4) for the Lamé coefficients h_s and h_φ, we get

$$\frac{\partial h_\varphi}{\partial n} = \frac{dz(s)}{ds}, \quad \frac{\partial h_\varphi}{\partial s} = f'(z(s))\frac{dz(s)}{ds} + n\frac{d^2 z(s)}{ds^2}$$

and their expansions in powers of s and n do not contain the curvature cross-radius $f(0)$. Obviously, this holds true for the expansion of h_s^{-2} in powers of s and n as well. Thus, all coefficients contained in the brackets of the operator B expand in the Fock domain only in negative powers of the parameter $\mathbf{M}_0$. In order to implement that, we pass from s and n to the variables σ and ν in the brackets and separate out of them a common multiplier g occurring in formula (3.2). After not complicated calculations we obtain

$$g^{-1}\left[\frac{1}{h_s^2}\frac{\partial h_\varphi}{\partial s}\left(ikW + \frac{\partial W}{\partial s}\right) + \frac{\partial h_\varphi}{\partial n}\frac{\partial W}{\partial n}\right] = \frac{\rho_0}{2\mathbf{M}_0^2}\left[-i\sigma W + \frac{\partial W}{\partial \nu} + O\left(\frac{1}{\mathbf{M}_0}\right)\right]. \tag{3.4}$$

Now we turn to the coefficient $h_\varphi(s, n)$. By expanding the right-hand side in relation (2.4) in powers of s and passing to the scaled variables σ and ν, we successively get

$$\begin{aligned} h_\varphi(\sigma,\nu) &= f(0) - \frac{s^2}{2\rho_0} + n + \frac{s^3\rho_0'}{3!\rho_0^2} + \frac{s^4(\rho_0\rho_0'' - 2\rho_0'^2 + 1)}{3!\rho_0^3} - \frac{s^2 n}{2\rho_0^2} + O(s^5) \\ &= f(0)\left\{1 + \frac{\mathbf{\Lambda}_0}{2\mathbf{M}_0^2}\left[(\nu - \sigma^2) + \frac{\sigma^3\rho_0'}{12\mathbf{M}_0} + \frac{\sigma^2}{2\mathbf{M}_0^2}\left(\frac{\sigma^2}{3!}(\rho_0\rho_0'' - 2\rho_0'^2 + 1) - \nu\right) + O(\mathbf{M}_0^{-3})\right]\right\}, \end{aligned} \tag{3.5}$$

where the expression in the brackets represents an expansion in negative integer powers of the Fock parameter $\mathbf{M}_0$. Hence we obtain the following expansion for h_φ^{-1}:

$$h_\varphi^{-1}(\sigma,\nu) = \frac{1}{f(0)}\sum_{m=0}(-1)^m\left(\frac{\mathbf{\Lambda}_0}{2\mathbf{M}_0^2}\right)^m\left[(\nu - \sigma^2) + O\left(\frac{1}{\mathbf{M}_0}\right)\right]^m. \tag{3.6}$$

We would like to recall that the parameter $\mathbf{\Lambda}_0$ characterizes the oblongness of the body of revolution close to the equator. Finally, substituting (3.4) and (3.6) into formula (1.7) for the part B of the initial operator, we arrive at the following structure of the operators L_m relative to the independent additional parameter $\mathbf{\Lambda}_0$: it occurs in the recurrence system (3.3) only in the form of the ratio $\mathbf{\Lambda}_0/\mathbf{M}_0^2$ and its integer powers, and appears for the first time only in the third equation of (3.3). Further, as the curvature radius $f(0)$ of the equator decreases, the parameter $\mathbf{\Lambda}_0$ increases and thus in this ratio $\mathbf{\Lambda}_0$ begins to compensate the influence of the large Fock parameter. Hence it follows that the solution of the recurrence system (3.3) preserves the asymptotic character with respect to $\mathbf{M}_0$, provided that

$$\mathbf{\Lambda}_0 = \mathbf{M}_0^{2-\varepsilon}, \tag{3.7}$$

where ε is any positive number from the interval $0 < \varepsilon < 2$. This condition represents a limitation on the possibility of applying the classical method of the parabolic Fock equation to the problem of diffraction of the incident field by a strictly convex body of revolution in the vicinity of the light-shadow boundary (the Fock domain).

Note that the possibility of assuming that $\varepsilon = 0$ in (3.7) requires an additional investigation.

Now we turn to the recurrence system (3.3) the solution of which is the reflected wave. In the 3D case, the algorithm or method of finding it remains the same as in the 2D case. We shall briefly give an account of it, following papers [3–5] and omitting many nontrivial mathematical details, concerning, for example, the normalization of emerging integrals by deformation of the contour of integration and so on.

The first equation $L_0 W_0^{\rm ref} = 0$ in (3.3), upon cancellation of the common multiplier g, becomes

$$\left(i\frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\nu^2} + \nu\right) W_0^{\rm ref} = 0. \tag{3.8}$$

Here the variables σ and ν are separated, and the Airy equation arises for ν. The solution is taken in the form

$$W_0^{\rm ref} = \int_{-\infty}^{+\infty} e^{i\sigma\zeta} B_0(\zeta) w_1(\zeta - \nu) d\zeta, \tag{3.9}$$

where $B_0(\zeta)$ is a function, which is arbitrary for the time being, and $w_1(\zeta - \nu)$ is the Airy function in the Fock definition [3]. The choice of exactly this function is based on the principle of limit absorption, which consists of the fact that for a small positive imaginary part of the wave number k, $\operatorname{Im} k > 0$, the function $w_1(\zeta - \nu)$ tends to zero as $\nu \to +\infty$ (see [3]). Moreover, one can also say that formula (3.9) describes the wave moving away from the scatterer as $\nu \to +\infty$.

The function $B_0(\zeta)$ is determined from the boundary condition $W_0^{\rm inc} + W_0^{\rm ref} = 0$ for $\nu = 0$:

$$B_0(\zeta) = -\frac{1}{\sqrt{\pi}} \frac{v(\zeta)}{w_1(\zeta)}.$$

Therefore $W_0^{\rm ref}$ is found uniquely. The total field in the principal asymptotic approximation in the Fock domain has the form

$$U_0(s, n) = e^{iks} W_0(\sigma, \nu) = e^{iks}(W_0^{\rm inc} + W_0^{\rm ref}) = \frac{1}{\sqrt{\pi}} \int_{\mathcal{L}} e^{i\sigma\zeta} \left[v(\zeta - \nu) - \frac{v(\zeta)}{w_1(\zeta)} w_1(\zeta - \nu) \right] d\zeta. \tag{3.10}$$

Here the contour of integration $\mathcal{L}$ goes along the ray $\arg\zeta = 2\pi/3$ from infinity to 0 and along the ray $\arg\zeta = -\pi/3$ from zero to infinity.

The other equations (3.3) are nonhomogeneous equations, and we need to construct the general solution for each of them; moreover, the general solution of the homogeneous equation has the form (3.9) with its own independent function $B_m(\zeta)$, $m = 1, 2, \ldots$. A particular solution of the nonhomogeneous equation is found in a form similar to formula (2.14), but instead of the function $v(\zeta - \nu)$ and its derivative $v'(\zeta - \nu)$ it is necessary to take $w_1(\zeta - \nu)$ and $w_1'(\zeta - \nu)$ from the same considerations as in the principal term of the reflected wave.

In what follows, we shall give explicit formulas for the operators $\widetilde{L}_m$, which arise from L_m in Eqs. (3.3) after separation of the coordinate σ with the help of the Fourier transform by formula (2.11), taking into account the fact that $w_1(\zeta - \nu)$ is substituted for the function $v(\zeta - \nu)$ and the corresponding normalization contour is substituted for the contour of integration:

$$\widetilde{L}_0 = \frac{\partial^2}{\partial \nu^2} + (\nu - \zeta); \tag{3.11}$$

$$\widetilde{L}_1 = -i\left(\frac{2}{\rho}\right)^{1/3} \rho' \nu \frac{\partial}{\partial \zeta}; \tag{3.12}$$

$$\widetilde{L}_2 = \left(\frac{2}{\rho_0}\right)^{2/3} \left[\frac{\rho_0\rho_0'' - 2\rho_0'^2}{2}\nu\frac{\partial^2}{\partial\zeta^2} - \frac{3\nu^2}{4} + \nu\zeta - \frac{1}{4}\zeta^2 + \frac{1}{2}\frac{\partial}{\partial\nu} + \frac{\mathbf{\Lambda}_0}{2}\left(\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\nu}\right)\right]. \tag{3.13}$$

Omitting cumbersome calculations, we finally write down formulas for the reflected field in the Fock domain:

$$W_0^{\text{ref}}(\sigma,\nu) = -\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} e^{i\sigma\zeta} \frac{v(\zeta)}{w_1(\zeta)} w_1(\zeta - \nu)\,d\zeta; \tag{3.14}$$

$$W_1^{\text{ref}}(\sigma,\nu) = \left(\frac{2}{\rho_0}\right)^{1/3} \frac{i\rho_0'}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} e^{i\sigma\zeta} \{B_1(\zeta) w_1(\zeta-\nu) + P_1^{(2)} \nu^2 w_1(\zeta - \nu) + (Q_1^{(1)}\nu + Q_1^{(0)}) w_1'(\zeta-\nu)\}\,d\zeta. \tag{3.15}$$

Here the coefficients B_1, $P_1^{(2)}$, $Q_1^{(1)}$, and $Q_1^{(0)}$ have the form

$$\begin{gathered} B_1(\zeta) = \left[-\frac{\zeta^2 v(\zeta)}{4w_1(\zeta)} + \frac{1}{2}\frac{1}{w_1^2(\zeta)} - \frac{2}{3}\frac{\zeta w_1'(\zeta)}{w_1^3(\zeta)}\right], \quad P_1^{(2)}(\zeta) = \frac{v(\zeta)}{4w_1(\zeta)}, \\ Q_1^{(1)}(\zeta) = \frac{1}{3w_1^2(\zeta)}, \quad Q_1^{(0)}(\zeta) = \left(\frac{2\zeta}{3w_1^2(\zeta)} - \frac{v(\zeta)}{2w_1(\zeta)}\right). \end{gathered} \tag{3.16}$$

Further,

$$\begin{aligned} W_2^{\text{ref}}(\sigma,\nu) = \left(\frac{2}{\rho}\right)^{2/3} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} e^{i\sigma\zeta} \Bigg\{ & B_2(\zeta) w_1(\zeta - \nu) + \left[\sum_{j=1}^{4} P_2^{(j)}(\zeta)\nu^j\right] w_1(\zeta-\nu) \\ & + \left(\sum_{j=0}^{3} Q_2^{(j)}(\zeta)\nu^j\right) w_1'(\zeta-\nu)\Bigg\}\,d\zeta. \end{aligned} \tag{3.17}$$

Henceforth in all formulas related to the Fock domain, the functions ρ, ρ', ρ'' of s are calculated at the point $s = 0$. In view of the unhandiness of those formulas given below, ρ, ρ', ρ'' occurring in them contain no subscript. Then the polynomial coefficients in ν are equal to

$$\begin{aligned} &P_2^{(4)}(\zeta) = \frac{\rho'^2 v(\zeta)}{32 w_1(\zeta)}, \qquad P_2^{(2)}(\zeta) = \left[\frac{(6\rho\rho'' + \rho'^2)}{24}\frac{1}{w_1^2(\zeta)} - \frac{\rho'^2\zeta^2 v(\zeta)}{16 w_1(\zeta)}\right], \\ &P_2^{(3)}(\zeta) = \left(\frac{-\rho'^2 w_1'(\zeta)}{9w_1^3(\zeta)}\right), \quad P_2^{(1)}(\zeta) = \left[\frac{(-4\rho\rho'' + \rho'^2 + 4)}{40}\frac{v(\zeta)}{w_1(\zeta)} + \frac{\rho'^2\zeta}{6w_1^2(\zeta)}\right], \\ &Q_2^{(3)}(\zeta) = \frac{\rho'^2}{12 w_1^2(\zeta)}, \qquad Q_2^{(2)}(\zeta) = \left[\frac{(\rho'^2 - 4\rho\rho'' - 6)}{40}\frac{v(\zeta)}{w_1(\zeta)} + \frac{\rho'^2}{6}\frac{\zeta}{w_1^2(\zeta)}\right], \\ &Q_2^{(1)}(\zeta) = \left[\frac{(\rho'^2 + \rho\rho'' + 4)}{30}\frac{\zeta v(\zeta)}{w_1(\zeta)} + \frac{5\rho'^2}{36}\frac{\zeta^2}{w_1^2(\zeta)} + \frac{(\rho'^2 + 3\rho\rho'')}{9}\frac{w_1'(\zeta)}{w_1^3(\zeta)} - \frac{2\rho'^2}{3}\frac{\zeta w_1'^2(\zeta)}{w_1^4(\zeta)}\right], \\ &Q_2^{(0)}(\zeta) = \Bigg[\frac{(23\rho'^2 + 8\rho\rho'' + 2)}{120}\frac{\zeta^2 v(\zeta)}{w_1(\zeta)} + \frac{5\rho'^2}{18}\frac{\zeta^3}{w_1^2(\zeta)} - \frac{4\rho'^2}{3}\frac{\zeta^2 w_1'^2(\zeta)}{w_1^4(\zeta)} + \frac{(11\rho'^2 + 6\rho\rho'')}{9}\frac{\zeta w_1'(\zeta)}{w_1^3(\zeta)} \\ &\qquad\qquad - \left(\frac{\mathbf{\Lambda}_0}{2} + \frac{(11\rho'^2 + 6\rho\rho'')}{12}\right)\frac{1}{w_1^2(\zeta)}\Bigg], \\ &B_2(\zeta) = -Q_2^{(0)}(\zeta)\frac{w_1'(\zeta)}{w_1(\zeta)} + \left[\frac{(23\rho'^2 + 8\rho\rho'' + 2)}{120}\frac{\zeta^2 v'(\zeta)}{w_1(\zeta)} + \frac{(17\rho'^2 + 32\rho\rho'' + 8)}{120}\frac{\zeta v(\zeta)}{w_1(\zeta)} + \frac{\rho'^2}{32}\frac{\zeta^4 v(\zeta)}{w_1(\zeta)}\right]. \end{aligned} \tag{3.18}$$

The two-scaled field expansion in the Fock domain, constructed in such a way, allows us to get approximate formulas for calculating the wave field, which depend upon the relative power of the two large parameters of the problem $\mathbf{M}_0$ and $\mathbf{\Lambda}_0$. This is determined by the value of ε in formula (3.7), which depends on exact parameters of the initial problem, namely, on k, ρ_0, and $f(0)$ in the vicinity of the light-shadow boundary, i.e., in the Fock domain. In what follows, we shall give such approximate formulas for the current on the scatterer's surface for two values $\varepsilon = 1/2$ and $\varepsilon = 1$.

We would like to recall that the current I on $\partial\Sigma$ in the problem under consideration is the following quantity:

$$I = \frac{\partial}{\partial n}(U^{\mathrm{inc}} + U^{\mathrm{ref}})\Big|_{n=0} = e^{iks}\frac{2\mathbf{M}_0^2}{\rho_0}\frac{\partial}{\partial \nu}(W^{\mathrm{inc}} + W^{\mathrm{ref}}). \tag{3.19}$$

For $\varepsilon = 1/2$ the principal summand from the part B of the initial operator has order $\mathbf{M}_0^{-1/2}$, as follows from formulas (3.4) and (3.6). Therefore, the summands in the aforementioned formulas (3.17) for W_2^{ref} that contain the multiplier $\mathbf{\Lambda}_0/\mathbf{M}_0^2$ must be put just after the principal summand W_0^{ref} of the expansion (3.1), because the second term of this asymptotic expansion W_1^{ref} has a higher order of smallness, namely, $\mathbf{M}_0^{-1}$. In this approximation, we arrive at the following formula for the current:

$$I \simeq e^{iks}\frac{2\mathbf{M}_0^2}{\rho_0}\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{i\sigma\zeta}}{w_1(\zeta)}\,d\zeta - \frac{1}{2\sqrt{\pi}}\frac{\mathbf{\Lambda}_0}{\mathbf{M}_0^2}\int_{-\infty}^{+\infty}e^{i\sigma\zeta}\left[\frac{w_1'^2(\zeta)}{w_1^3(\zeta)} - \frac{\zeta}{w_1(\zeta)}\right]d\zeta\right\}. \tag{3.20}$$

If we set $\varepsilon = 1$, it will be necessary to add to this formula the current obtained from the second term of the asymptotics of the total field $W_1^{\mathrm{inc}} + W_1^{\mathrm{ref}}$, because $\mathbf{\Lambda}_0/\mathbf{M}_0^2$ has the same order $\mathbf{M}_0^{-1}$ as the second term of the asymptotics. Differentiating the sum $W_1^{\mathrm{inc}} + W_1^{\mathrm{ref}}$ with respect to ν and assuming that $\nu = 0$, we get

$$\begin{aligned}&\frac{\partial}{\partial \nu}(W_1^{\mathrm{inc}} + W_1^{\mathrm{ref}})\Big|_{\nu=0}\\ &\quad= \frac{1}{4\sqrt{\pi}}\frac{i\rho'(0)}{\mathbf{M}_0}\int_{-\infty}^{+\infty}e^{i\sigma\zeta}\left\{\zeta^2\frac{v(\zeta)w_1'(\zeta)}{w_1(\zeta)} - 2(1+\zeta)v(\zeta) - \frac{5}{3}\frac{\zeta^2}{w_1(\zeta)} - \frac{2}{3}\frac{w_1'(\zeta)}{w_1^2(\zeta)} + \frac{8\zeta}{3}\frac{w_1'^2(\zeta)}{w_1^3(\zeta)}\right\}d\zeta\end{aligned} \tag{3.21}$$

and this expression should be added to the right-hand side of (3.20).

Let us give some attention to the result following immediately from the previous one. The scale of functions on which the asymptotics is constructed depends on the magnitude of the oblongness of the body of revolution in the Fock domain, i.e., on the relative magnitude of the parameters $\mathbf{\Lambda}_0$ and $\mathbf{M}_0$, because summands of order $\mathbf{M}_0^{-\varepsilon}=O(k^{-\varepsilon/3})$ arise, as well as their integer powers. For $\varepsilon = 1$, the scale of functions is $O(k^{-m/3})$, exactly coinciding with the one that arises in the classical Fock problem.

4. Wave field in the shaded part of the Fock domain

We would like to recall that the light-shadow boundary on $\partial\Sigma$ corresponds to $s = 0$, and for small n, as s increases the observation point gets into shadowy part close to the scatterer. In the local stretched coordinates σ and ν, this condition corresponds to $\sigma \to +\infty$ and bounded ν. In this case, the wave field can be obtained by means of residues of the integrals for the total field (see [3,5]).

Indeed, the integration path in the expression for the total field in all orders of the asymptotic expansion can be taken in the form of the contour running along the two rays: one ray from the point $-\infty$ to the point $\zeta = 0$ along the line $\arg\zeta = 2\pi/3$, and the other one moving along the positive semiaxis $\operatorname{Re}\zeta \geq 0$ from $\zeta = 0$ to $+\infty$. In the upper half-plane $\operatorname{Im}\zeta > 0$, the exponent $\exp\{i\sigma\zeta\}$ decreases rapidly for large positive σ, because $|\exp\{i\sigma\zeta\}| = \exp\{-\sigma \operatorname{Im}\zeta\}$ and it is more convenient to lift the contour of integration to the upper half-plane. Here, residues arise at the poles of the integrands, which appear at the roots of the Airy function $w_1(\zeta)$, positioned on the ray $\arg\zeta = \pi/3$. Unfortunately, as the number m increases in the asymptotic series for the solution W_m^{ref}, $m = 0, 1, 2, \ldots$, these calculations rapidly become more cumbrous, which is obvious from formulas (3.14)–(3.17). In these formulas, the coefficients $B_0(\zeta)$, $P_1^{(2)}(\zeta)$, and $P_2^{(j)}(\zeta)$, $j = 1, 2, 3, 4$, of polynomials in ν at the functions $w_1(\zeta - \nu)$, as well as the coefficients $Q_1^{(j)}(\zeta)$, $j = 0, 1$, $Q_2^{(j)}(\zeta)$, $j = 0, 1, 2, 3$ of polynomials in ν at the derivatives of the functions $w_1'(\zeta - \nu)$ have increasing powers of the function $w_1(\zeta)$ in the denominators, and therefore the order of poles increases as well. We emphasize that the incident field gives no contribution in this domain, because it does not contain $w_1(\zeta)$ in the denominators.

Below we give the results of calculations for three terms of the asymptotic expansion. We would like to note particularly that these formulas are necessary for unique construction of creeping waves arising in the shadowy part of the scatterer:

$$U(s,n) = e^{iks}\left[W_0(\sigma,\nu) + \frac{W_1(\sigma,\nu)}{k^{1/3}} + \frac{W_2(\sigma,\nu)}{k^{2/3}} + O(k^{-1})\right],$$

$$W_0(\sigma,\nu) = -i\,2\sqrt{\pi}\sum_{q=1} \frac{e^{i\sigma\zeta_q}}{[w_1'(\zeta_q)]^2} w_1(\zeta_q - \nu); \tag{4.1}$$

$$\frac{W_1(\sigma,\nu)}{k^{1/3}} = \frac{2\sqrt{\pi}\rho'}{\mathbf{M}_0}\sum_{q=1} \frac{e^{i\sigma\zeta_q}}{[w_1'(\zeta_q)]^2}\left\{\left(\frac{\nu^2}{12} - \frac{\sigma^2\zeta_q}{3} + \frac{i\sigma}{6} + \frac{\zeta_q^2}{4}\right) w_1(\zeta_q - \nu) - \frac{i\sigma\nu}{3} w_1'(\zeta_q - \nu)\right\}; \tag{4.2}$$

$$\frac{W_2(\sigma,\nu)}{k^{2/3}} = \frac{i2\sqrt{\pi}}{\mathbf{M}_0^2}\sum_{q=1} \frac{e^{i\sigma\zeta_q}}{[w_1'(\zeta_q)]^2}\left\{\mathcal{B}_2(\sigma,\zeta_q) w_1(\zeta-\nu) + \left[\sum_{j=1}^{4} P_2^{(j)}(\sigma,\zeta_q)\nu^j\right] w_1(\zeta-\nu)\right.$$

$$\left. + \mathcal{Q}_2^{(0)}(\sigma,\zeta_q) w_1'(\zeta-\nu) + \left(\sum_{j=1}^{3} Q_2^{(j)}(\sigma,\zeta_q)\nu^j\right) w_1'(\zeta-\nu) + \frac{\mathbf{\Lambda}_0}{4}[\nu-\sigma^2] w_1(\zeta_q-\nu)\right\}. \tag{4.3}$$

The coefficients $\mathcal{B}_2(\sigma,\zeta_q)$, $P_2^{(j)}(\sigma,\zeta_q)\nu^j$, $j = 1,2,3,4$, and $\mathcal{Q}_2^{(0)}(\sigma,\zeta_q)$, $Q_2^{(l)}(\sigma,\zeta_q)\nu^l$, $l = 1,2,3$, at the Airy function and its derivative are polynomials in σ, ν, ζ_q of respective powers. These coefficients are obtained from the coefficients $B_2(\zeta)$, $P_2^{(j)}(\zeta)$, $j = 1,2,3,4$, and $Q_2^{(l)}(\zeta)$, $l = 0,1,2,3$, of formulas (3.17) and (3.18), upon calculating the residues at the poles of the integrands, i.e., at the roots of the Airy function $w_1(\zeta)$. We note that this correspondence is violated in the coefficient $Q_2^{(0)}$, which is part of the coefficient B_2 as well, because the summand containing the multiplier $\mathbf{\Lambda}_0/\mathbf{M}_0^2$, was extracted from it (see formula (4.3)).

5. Creeping waves

In constructing creeping waves (see [4, 5, 9]), it is assumed that, unlike the Fock domain, the arc length of the geodesic satisfies the condition $s = O(1)$ and the scaled normal ν is defined by the relation $\nu(n,s) = k^{2/3}2^{1/3}\rho^{-1/3}n$, where $\rho(s)$ is the curvature radius of the geodesic at the point s. The large parameter $\mathbf{M} = (k\rho(s))^{1/3}2^{-1/3}$ depends then on the arc length s and, generally speaking, the condition $\mathbf{M} \gg 1$ leads to the limitation of the allowable variation interval for s. Consequently, we are in the vicinity of the scatterer surface only with respect to the normal, because it is assumed now that $\nu(n,s) = O(1)$ as $k \to \infty$.

The solution of the Helmholtz equation is sought in the form

$$U = \exp\{iks + ik^{1/3}\Phi(s)\}\sum_{m=0} V_m(s,\nu)k^{-m/3}, \tag{5.1}$$

where

$$\Phi(s) = \frac{\zeta}{2^{1/3}}\int_0^s \rho^{-2/3}(s)\,ds \tag{5.2}$$

and the separation constant ζ proves then to be a root of the Airy function w_1, i.e., $w_1(\zeta) = 0$. For the sought functions V_l we obtain a recurrence system of equations

$$\mathcal{L}_0 V_0 = 0, \quad \mathcal{L}_0 V_1 + \mathcal{L}_1 V_0 = 0, \quad \mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 + \mathcal{L}_2 V_0 = 0, \ldots, \tag{5.3}$$

where $\mathcal{L}_j$, $j = 0,1,2,3,\ldots$, are differential operators containing the derivatives with respect to s and ν. Below we list explicit formulas for the first four of them:

$$\mathcal{L}_0 = \frac{\partial^2}{\partial\nu^2} + (\nu - \zeta), \quad \mathcal{L}_1 = \left(\frac{2}{\rho}\right)^{1/3}\frac{i\rho}{2}\left[2\frac{\partial}{\partial s} + \frac{\partial \ln f}{\partial s}\right], \tag{5.4}$$

$$\mathcal{L}_2 = \left(\frac{2}{\rho}\right)^{2/3}\left\{\frac{1}{2}\frac{\partial}{\partial\nu}\left(1+\mathbf{\Lambda}\right) - \frac{3\nu^2}{4} + \zeta\nu - \frac{\zeta^2}{4}\right\}, \tag{5.5}$$

$$\mathcal{L}_3 = \left(\frac{2}{\rho}\right)\frac{i\rho}{2}\left[\left(\frac{\zeta}{2} - \nu\right)\left(2\frac{\partial}{\partial s} + \frac{\partial \ln f}{\partial s}\right) - \frac{\zeta\rho'}{3\rho} + \nu\left(\frac{\partial \ln\rho}{\partial s} - \frac{\partial \ln f}{\partial s}\mathbf{\Lambda}\right)\right]. \tag{5.6}$$

In the last equations, the parameter $\mathbf{\Lambda}(s) = \rho(s)/f(s)$ arises, which characterizes the oblongness of the scatterer. In our case, this parameter also represents a function with respect to the arc length s of the geodesic on $\partial\Sigma$. For an elongated body it should be assumed that $\mathbf{\Lambda}(s) \gg 1$.

The boundary conditions $V_m = 0$ for $\nu = 0$ on the surface of the body are added to the system of equations (5.3) and below an analog of the "radiation" condition is formulated such that $V_m \to 0$ for complex ν tending to infinity. A method for solving the recurrence equations (5.3) was proposed and developed in [5, Chap. 3] and [9]. We shall employ this method in our considerations.

Similarly to the Fock domain, in solving recurrence equations (5.3) we assume that the parameters $\mathbf{M}(s)$ and $\mathbf{\Lambda}(s)$ are independent, and approximated formulas for calculating the wave field are proposed based on the estimation of the relative strength of these large parameters.

A considerable role in this is played by the structure of the part B of the Helmholtz operator, because the part A does not contain the parameter $\mathbf{\Lambda}(s)$ in our case.

The cross-radius of the scatterer's curvature occurs in the part B of the Helmholtz operator via the Lamé coefficient h_φ. Using relation (2.4) and substituting $\nu(n,s) = k^{2/3}2^{1/3}\rho^{-1/3}\, n$ for the normal n, we get the following expression for h_φ:

$$h_\varphi(s,\nu) = f(z(s))\left(1 + \frac{\mathbf{\Lambda}(s)\,\nu}{2\mathbf{M}^2(s)}\frac{dz}{ds}\right). \tag{5.7}$$

From here it follows that under the condition $\mathbf{\Lambda}(s) = \mathbf{M}^{2-\varepsilon}(s)$, $0 < \varepsilon < 2$, analogous to condition (3.7) in the Fock domain, h_φ^{-1} can be represented in the form of an expansion in powers of

$$\frac{\mathbf{\Lambda}(s)\,\nu}{2\mathbf{M}^2(s)}\frac{dz}{ds},$$

namely,

$$h_\varphi^{-1}(s,\nu) = \frac{1}{f(z(s))}\sum_{m=0}(-1)^m\left(\frac{\mathbf{\Lambda}(s)\,\nu}{2\mathbf{M}^2(s)}\frac{dz}{ds}\right)^m. \tag{5.8}$$

Moreover, the recurrence system of equations (5.3) preserves an asymptotic nature as $k \to \infty$. In the case of $\varepsilon = 0$, we conclude that $\frac{\mathbf{\Lambda}(s)}{2\mathbf{M}^2(s)}\frac{dz}{ds} = O(1)$ as $k \to \infty$. Expansion (5.8) becomes impossible and in Eqs. (5.3) a radical reorganization of the operators $\mathcal{L}_m$ is needed.

Now we turn to solving Eqs. (5.3). The solution of the main (homogeneous) equation $\mathcal{L}_0 V_0 = 0$ is taken in the form

$$V_{0,q} = A_0(s)w_1(\zeta_q - \nu),$$

where $w_1(\zeta_q - \nu)$ is the Airy function, ζ_q is the root of the Airy function with number q, and $A_0(s)$ is an arbitrary, for the time being, function of the arc length s of the geodesic line. If we begin moving along the normal away from the boundary $\partial\Sigma$ ($|\nu| \to \infty$), then for $-\pi < \arg(-\nu) < -\pi/3$ the function $w_1(\zeta_q - \nu) \to 0$ and, therefore, it satisfies the "radiation" condition. Hence, the function $V_{0,q}$ turns out to be a nonzero solution of the Sturm–Liouville homogeneous problem with the Airy operator, the Dirichlet boundary condition for $\nu = 0$, and the "radiation" condition as $|\nu| \to \infty$ for any $A_0(s)$.

All the subsequent equations in system (5.3) are nonhomogeneous, for which the homogeneous equation has a nonzero solution. This leads to the introduction of a solvability condition at every step for the Sturm–Liouville nonhomogeneous problems. This condition for the next approximation $V_{1,q}$ leads to the orthogonality of the right-hand side $-\mathcal{L}_1 V_0$ in the equation $\mathcal{L}_0 V_1 + \mathcal{L}_1 V_0 = 0$ to the solution V_0 of the homogeneous equation:

$$\int_l \mathcal{L}_1 V_0 \cdot w_1(\zeta_q - \nu)\,d\nu = 0,$$

where integration proceeds along the ray $\arg\nu = \pi/3$ and the expression $\mathcal{L}_1 V_{0,q}$ takes the form

$$\mathcal{L}_1 V_{0,q} = \left[2\frac{dA_0}{ds} + A_0\frac{d\ln f(z(s))}{ds}\right]w_1(\zeta_q - \nu) + A_0\nu w_1'(\zeta_q - \nu)\frac{2}{3}\frac{d\ln\rho(s)}{ds}.$$

Upon calculating the corresponding integrals, a transport equation arises for the function $A_0(s)$:

$$\frac{dA_0}{ds} + \left[\frac{1}{2}\frac{d\ln f(z(s))}{ds} + \frac{1}{6}\frac{d\ln\rho(s)}{ds}\right]A_0(s) = 0.$$

The initial value of $A_0(s)$ for $s = 0$ should be added to this equation:

$$A_0(0) = -\frac{2\sqrt{\pi}\,i}{w_1'^2(\zeta_q)}\sqrt{f(0)}\,\sqrt[6]{\rho(0)}.$$

It emerges from the matching requirement for creeping waves with a solution for the reflected wave in the shaded part of the Fock domain. The final formula for the principal term of the asymptotics reads

$$V_{0,q}(s,\nu) = -\frac{2\sqrt{\pi}\,i}{w_1'^2(\zeta_q)}\sqrt{\frac{f(0)}{f(z(s)}}\,\sqrt[6]{\frac{\rho(0)}{\rho(s)}}w_1(\zeta_q - \nu). \tag{5.9}$$

The subsequent terms of the asymptotic expansion for creeping waves are constructed similarly. Unfortunately, the calculations become more cumbersome with each step.

The next approximation $V_{1,\,q}(s,\nu)$ to the solution of the problem is found in the form

$$V_{1,\,q}(s,\nu) = A_0(s)\left[\alpha_1(s) + \nu^2\frac{i\rho'}{12}\left(\frac{2}{\rho}\right)^{1/3}\right]w_1(\zeta_q - \nu). \tag{5.10}$$

The function $A_1(s) = A_0(s)\alpha_1(s)$ is found from the transport equation, which is a necessary and sufficient condition for solvability of the nonhomogeneous equation:

$$\int\limits_0^{\infty e^{i\pi/3}} \big[\mathcal{L}_1\,V_1\cdot V_0 + \mathcal{L}_2\,V_0\cdot V_0\big]\,d\nu = 0;$$

moreover,

$$\alpha_1(s) = \alpha_1(0) + \int\limits_0^s \frac{i\zeta_q^2}{60}\left(\frac{2}{\rho}\right)^{1/3}\left[\frac{8\rho''}{3} - \frac{16\rho'^{\,2}}{9\rho} - \frac{1}{\rho}\right]ds,$$

where the initial value of the function $\alpha_1(0)$ is obtained by matching the solutions in the penumbra (the Fock domain) and the shadow part for $\nu = O(1)$ and equals

$$\alpha_1(0) = \left(\frac{2}{\rho_0}\right)^{1/3}\frac{i\rho_0'\zeta_q^2}{4}.$$

The third term $V_{2,\,q}(s,\nu)$ of the asymptotic expansion of the solution reads

$$V_{2,\,q}(s,\nu) = \left\{A_2(s)w_1(\zeta_q - \nu) + \left[\sum_{j=1}^{4} P_2^{(j)}(s,\zeta_q)\nu^j\right]w_1(\zeta_q - \nu) + \left(\sum_{j=0}^{3} Q_2^{(j)}(s,\zeta_q)\nu^j\right)w_1'(\zeta_q - \nu)\right\}. \tag{5.11}$$

We are interested in the summands of the approximation to the solution that contain the ratio of curvature radii $\mathbf{\Lambda}(s)$. The polynomial $P_2^{(2)}$ equals

$$P^{(2)} = P_4^{(2)}\nu^4 + P_2^{(2)}\nu^2 + \nu\left[Q_2^{(2)} - \left(\frac{2}{\rho}\right)^{2/3}A_0(s)\frac{1}{4}\mathbf{\Lambda}(s)\right]. \tag{5.12}$$

The polynomial $Q_2^{(j)}$ is found, and it does not contain the ratio of curvature radii.

Then we need to derive and solve a transport equation for $A_2(s) = A_0(s)\alpha_2(s)$, i.e., for $\alpha_2(s)$. We give an equation for $\alpha_2(s)$, taking into account only summands with the ratio of curvature radii:

$$\alpha_2(s) = \alpha_2(0) + \int\limits_0^s \frac{\zeta_q}{6}\left(\frac{2}{\rho(s)}\right)^{2/3}\frac{\partial \ln f}{\partial s}\mathbf{\Lambda}\,ds + \cdots. \tag{5.13}$$

Let us note the following circumstances. In the 3D case, the construction of creeping waves is concerned with the field of geodesic lines on the scatterer's surface, coming off the light-shadow boundary on $\partial\Sigma$ in the direction of the rays belonging to the incident wave. In the axisymmetric case under consideration, the field of geodesic lines coincides, obviously, with meridians on $\partial\Sigma$. The multiplier $\sqrt{f(0)/f(z(s))}$, which turns out to be common for all terms of the asymptotic expansion, is a characteristic of the geometric spreading of this field of geodesic lines.

6. Conclusions

Based upon the analysis of the dependence of the part B of the Helmholtz equation on the parameters $\mathbf{M}(s)$ and $\mathbf{\Lambda}(s)$, it is ascertained that the wave field, in the axisymmetric diffraction problem under consideration, can be described in the traditional context of the Leontovich–Fock parabolic equation method, provided that $\mathbf{\Lambda}_0 = \mathbf{M}_0^{2-\varepsilon}$ in the Fock domain and similarly $\mathbf{\Lambda}(s) = \mathbf{M}(s)^{2-\varepsilon}$ in the domain of creeping waves if $0 < \varepsilon < 2$. The answer is also obtained in terms of Airy functions and integrals of them. Moreover, the scale of functions in which the asymptotic expansion is constructed depends on the parameter ε, the magnitude of which is determined by exact values of the parameters of the initial problem: the wave number k and the geometric characteristics of the scatterer's surface $\rho(s)$ and $f(z(s))$.

The case $\varepsilon = 0$ leads to an additional serious problem, because under this condition the expansions (3.6) and (5.8) for the Lamé coefficient h_φ^{-1} cannot be used. In this case, a significant reorganization of the operators occurs in the recurrence systems of equations both in the Fock domain and in the domain of creeping waves.

So, under the condition

$$\varkappa_0 = \frac{\mathbf{\Lambda}_0}{2\mathbf{M}_0^2} = O(1),$$

i.e., $\varepsilon = 0$, and as $k \to \infty$, we obtain the following expression for the operator L_0 in the first of the equations (3.3) (compare with (3.8)):

$$L_0 W_0 = i\frac{\partial W_0}{\partial \sigma} + \frac{\partial^2 W_0}{\partial \nu^2} + H(\sigma,\nu)\frac{\partial W_0}{\partial \nu} + (\nu - i\sigma H(\sigma,\nu))W_0 = 0,$$

where

$$H(\sigma,\nu) = \frac{\varkappa_0}{1+\varkappa_0(\nu-\sigma^2)} = \frac{1}{(\nu-\sigma^2)+1/\varkappa_0}.$$

Let us pay attention to the fact that the variable coefficient $H(\sigma,\nu)$ in this equation proves to be singular on the branches of the parabola $\nu = \sigma^2 - 1/\varkappa_0$, $\nu \geq 0$. The coefficients in all subsequent operators of the recurrence system (3.3) become singular as well (compare with [1]). However one can make certain that in the Fock domain, the principal term of the incident field satisfies this singular equation and remains a smooth function with respect to σ and ν; this fact indicates, in our opinion, that the method of the Leontovich–Fock parabolic equation fails in this situation.

In the case of creeping waves, the equation for the principal term V_0 of the asymptotic expansion reads

$$\frac{\partial^2 V_0}{\partial \nu^2} + (\nu - \zeta)V_0 + \frac{\varkappa(s)}{(1+\nu\varkappa(s))}\frac{\partial V_0}{\partial \nu} = 0,$$

where by $\varkappa(s)$ we denote the function of s

$$\varkappa(s) = \frac{\mathbf{\Lambda}(s)}{2\mathbf{M}^2(s)}\frac{dz}{ds}.$$

Here, the coefficient at $\frac{\partial V_0}{\partial \nu}$ possesses no singularity. However we do not see a way to find out the initial data for creeping waves without having a solution in the Fock domain.

We would like to draw attention to additional difficulties of the diffraction problem on elongated bodies in general, which arise in using the methods of short wavelength asymptotics. Consider, for example, an ellipsoid of revolution

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$$

elongated along the axis z, i.e., for $b > a$. For a fixed wave number k, as b increases, the conditions of applicability of the ray method for describing the diffraction on the cone's edges $z = \pm b$ are violated, because the curvature radius R of the ellipsoid surface decreases at these points. For $kR \sim 1$, the application of short-wave methods is impossible. If, in addition, the angle between the incident wave and the axis z increases, the light-shadow boundary also gets closer to the vertices $z = \pm b$, and it becomes unfeasible to apply the parabolic equation method in the Fock domain.

For this reason, in our opinion, the development of a combined method in which the direct numerical methods are used together with the short-wave asymptotics is an urgent problem.

The authors are grateful to A. P. Kachalov for useful discussion.

The paper was supported by the Russian Foundation for Basic Research (project 11-01-00407A).

Translated by P. M. Popov.

REFERENCES

1. I. V. Andronov, “Diffraction by a strongly elongated body of revolution,” *Acoust. J. RAN*, **57**, No. 2, 147–152 (2011).
2. I. V. Andronov and D. Bouche, “Diffraction by a narrow circular cone as by a strongly elongated body,” *Zap. Nauchn. Semin. POMI*, **393**, 12–22 (2011).
3. V. A. Fock, *Electromagnetic Diffraction and Propagation Problems* [in Russian], 2nd ed. Sovetskoe Radio, Moscow (1970).
4. V. M. Babich and V. S. Buldyrev, *Asymptotic Methods in Short Wavelength Diffraction Theory* [in Russian], Nauka, Moscow (1972).
5. V. M. Babich and N. Ya. Kirpichnikova, *The Boundary-Layer Method in Diffraction Problems* [in Russian], LGU, Leningrad (1974).
6. V. S. Buslaev, “Short wavelength asymptotics in the problem of diffraction by smooth convex contours,” *Trudy MIAN*, **XXIII**, No. 2, 14–117 (1964).
7. V. M. Babich, “On short wavelength asymptotics of Green’s function for the exterior of a compact convex domain,” *Dokl. AN SSSR*, **146**, No. 3, 571–573 (1962).
8. V. B. Philippov, “On mathematical justification of the short wavelength asymptotics of the diffraction problem in shadow zone,” *Zap. Nauchn. Semin. POMI*, **34**, 142–205 (1973).
9. N. Ya. Kirpichnikova, “On propagation of surface waves, which are concentrated in the vicinity of rays in an inhomogeneous elastic body of arbitrary shape,” *Trudy MIAN*, **CXV**, No. 1, 114–130 (1971).