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DISTRIBUTION OF ROOTS OF MITTAG-LEFFLER FUNCTIONS

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PREFACE

In 1903, in relation to the summation method for power series, Swedish mathematician Mittag-Leffler [16] introduced the new entire function

$$E_{1/\rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\rho)}, \quad \rho > 0,$$

which is now called the Mittag-Leffler function.

The appearance of this new function did not go unnoticed. It is remarkable that already in first related publications, many authors were interested in the distribution of its zeros (roots). So, Wiman (see [44]) proved that for $\rho \geq 2$, all zeros are real, negative, and simple. Later, Pólya (see [21]) by a different method reproved this fact for the case $2 \leq \rho \in \mathbb{N}$.

Later, the function $E_{1/\rho}(z)$ won new positions in the theory of functions. Along with this, its definition was modified and generalized by introduction of an additional parameter μ:

$$E_\rho(z;\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n/\rho)}, \quad \rho > 0, \quad \mu \in \mathbb{C}; \tag{1}$$

we also call this function the *Mittag-Leffler function*; obviously, $E_{1/\rho}(z) = E_\rho(z;1)$. In the literature, we encounter other names and the notation $E_\rho(z;\mu)$; for example, the notation $E_{1/\rho}(z;\mu)$ and $E_{1/\rho,\mu}(z)$ and the terms "generalized Mittag-Leffler function" and "Mittag-Leffler-type" are used. In the present work, the function $E_\rho(z;1) = E_{1/\rho}(z)$ is called the *classical Mittag-Leffler function*.

In our country, interest in the function $E_\rho(z;\mu)$ was considerably stimulated by the monograph [6] of Dzhrbashyan. We use the notation (1) introduced by Dzhrbashyan.

Definition (1) immediately implies the formulas

$$E_1(z;1) = e^z, \qquad E_1(z;-m) = z^{m+1}e^z, \quad m \in \mathbb{Z}_+, \tag{2}$$

$$E_{1/2}(z;1) = \cosh\sqrt{z}, \qquad E_{1/2}(z;-(2m-1)) = z^m \cosh\sqrt{z}, \quad m \in \mathbb{N},$$

$$E_{1/2}(z;2) = \frac{(\sinh\sqrt{z})}{\sqrt{z}}, \qquad E_{1/2}(z;-2m) = z^{m+1/2}\sinh\sqrt{z}, \quad m \in \mathbb{Z}_+,$$

$$E_1(z;c) = \frac{\Phi(1,c;z)}{\Gamma(c)}, \quad c \notin -\mathbb{Z}_+, \tag{3}$$

where $\Phi(a,c;z) = F_1(a,c;z)$ is the confluent hypergeometric function. These formulas once more indicate the breadth of the class of entire function (1).

In the sequel, we will see that the function $E_\rho(z;\mu)$, except for the cases (2), has an infinite set of zeros.

One of the important directions related to functions (1) is the theory of Fourier–Laplace-type integral transforms with Mittag-Leffler kernels. A certain hint on the possibility of construction of such transforms is contained in the first of formulas (2). The conceptual analytical part of the theory is based on asymptotic estimates of functions (1). The current state of this theory is represented in the most complete form in [6].

Recently, the scope of applications of the function $E_\rho(z;\mu)$ has been substantially extended. For example, this function is extensively used in the theory of random processes; we indicate here only the (most typical) papers [1, 5].

The problem on the distribution of zeros of Mittag-Leffler functions is central to the theory. Studies in this direction stem from the vastness of the class of entire functions (1), which have a remarkable asymptotic behavior. It is important that the need for this activity is supported by different aspects of spectral theory, theory of inverse problems, and approximation theory. Let us discuss this briefly.

The most active "consumer" (one can even say "customer") of the theory of distribution of zeros of functions (1) is the branch of spectral theory where the differential operators with fractional derivatives are considered. This is discussed in detail in the paper [17] of Nakhushev, where the problem of the number of of real roots of function $E_\rho(z;\mu)$ for $\rho > 1/2$ and real μ is discussed.

An interesting connection between inverse problems of special type and distribution of zeros of the function $E_\rho(z;\mu)$ for certain values of parameters ρ and μ was established by Tikhonov [41] whose reports on a special seminar at the mechanical-mathematical department of the Moscow State University influenced our investigations in this direction.

The second author constructed for the first time, for all $1 \le p < \infty$, systems of exponents $e^{i\lambda_n x}$, simultaneously complete and minimal in the spaces L^p on the line with rapidly decreasing weight (cf. [34]). Under certain conditions on weights, the indices λ_n of the systems are described as sequences of zeros of functions (1) (with parameters depending on weights).

Thus, apart from the evident independent importance, the theory of distribution of zeros of Mittag-Leffler-type functions is quite representatively motivated by important applications in analysis. Our goal consists in representing the contemporary state of this theory.

In its development, apart form the mentioned authors (see [21, 44]), many mathematicians participated: Dzhrbashyan and Nersesyan [6] Ostrovsky and Peresyolkova [19]), Pskhu [26], and others. The authors of the present work have made their own contribution to the problem.

Basically, we consider here the following sequence of problems:

- asymptotic behavior of zeros of Mittag-Leffler functions and a consistent method of enumerating *all zeros*;
- the number of real zeros of functions (1) for $\mu \in \mathbb{R}$;
- the problem, originally considered by Wiman, of describing the set of parameters (ρ, μ), $0 < \rho \le 1/2$, such that all zeros of the functions $E_\rho(z;\mu)$ are real, negative, and simple, and localization of zeros;
- the so-called nonasymptotic properties of zeros, for example, their distribution in a given angle, the multiplicity problem of zeros.

Moreover, we study distributions of zeros of some other entire functions, which are close to functions (1) in the appropriate sense, namely, the confluent hypergeometric function $\Phi(a, c; z)$ (see formula (3)) and the Laplace transforms of compactly supported monotonic functions (the indicated closeness consists of the possibility of representing the function $E_1(z;\mu)$ for certain values of μ as such a transform).

Chapters 1, 3, and 6 are written by A. Yu. Popov, and Chaps. 2, 4, and 5 by A. M. Sedletskii.

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Chapter 1

INTEGRAL REPRESENTATIONS, ASYMPTOTICS, AND ESTIMATION OF MITTAG-LEFFLER FUNCTIONS

In Chap. 1, we present theorems on complete asymptotic expansions of the functions $E_\rho(z;\mu)$ as $z \to \infty$. Proofs of these theorems are based on integral representations of Mittag-Leffler functions. We start with these integral representations. We follow [6, Chap. 3] and [7, Chap. 1, Sec. 5 and Chap. 4, Sec. 4]. Note that in [7], only the classical Mittag-Leffler function was considered. In theorems on asymptotics of $E_\rho(z;\mu)$, we repair errors and drawbacks of [6, 7] and present a more detailed analysis. A characteristic feature of our theorems on asymptotics of Mittag-Leffler functions is an explicit estimate of the remainder (without O-symbol), which is valid not only for "sufficiently large" $|z|$ but for all $z \neq 0$ (occasionally for $|z| \geq 2$).

1.1. Integral Representations of the Mittag-Leffler function

We denote by $\gamma(\sigma, \alpha_1, \alpha_2)$, where $\sigma > 0$, $\alpha_1 < 0$, and $\alpha_2 > 0$, the contour

$$\mathfrak{P} = \{\zeta = re^{i\varphi},\ r > 0,\ \varphi \in \mathbb{R}\}$$

on the Riemann surface of the argument (it is usually called the Hankel loop) oriented towards the non-decreasing argument $\operatorname{Arg}\zeta$ and consisting of the following parts:

(1) the ray $\operatorname{Arg}\zeta = \alpha_1$, $|\zeta| \geq \sigma$,
(2) the arc $|\zeta| = \sigma$, $\alpha_1 \leq \operatorname{Arg}\zeta \leq \alpha_2$,
(3) the ray $\operatorname{Arg}\zeta = \alpha_2$, $|\zeta| \geq \sigma$.

Theorem 1.1.1. *For any $\rho > 0$, $\mu \in \mathbb{C}$, $z \in \mathbb{C}$, $\sigma > |z|^\rho$, $\alpha_1 \in (-3\pi/2, -\pi/2)$, and $\alpha_2 \in (\pi/2, 3\pi/2)$ the following relation holds*:

$$E_\rho(z;\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{\zeta^{1/\rho-\mu} e^\zeta \, d\zeta}{\zeta^{1/\rho} - z}. \tag{1.1.1}$$

Remark 1.1.1. If $\operatorname{Re}\mu > 0$, then, by the convergence of the corresponding improper integrals, formula (1.1.1) is valid also for $\alpha_1 = -\pi/2$, $\alpha_1 = -3\pi/2$, $\alpha_2 = \pi/2$, $\alpha_2 = 3\pi/2$.

Before proving the theorem, we recall the following Hankel lemma.

Lemma 1.1.1. *For any $\sigma > 0$, $\alpha_1 \in (-3\pi/2, -\pi/2)$, and $\alpha_2 \in (\pi/2, 3\pi/2)$, the following identity holds*:

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \zeta^{-s} e^\zeta \, d\zeta, \quad s \in \mathbb{C}. \tag{1.1.2}$$

Proof. First, we note that integral (1.1.2) is independent of the numbers σ, α_1, and α_2 in the boundaries specified in the lemma since there are no singularities of the integrand $\zeta^{-s}e^\zeta$ on the Riemann surface $\mathfrak{P}$ (on the "ordinary" complex plane, for the principal branch of the argument, it is holomorphic on

$\mathbb{C}\setminus(-\infty,0]$) and the exponent decreases sufficiently rapidly in the closed angles lying inside the open domains

$$\left\{\zeta = re^{i\varphi}\;\middle|\; r>0,\ \frac{\pi}{2}<|\varphi|<\frac{3\pi}{2}\right\}.$$

Here and in the sequel, $\ln(re^{i\varphi})$, $r>0$, means $\ln r + i\varphi$; following this, we define noninteger powers of ζ.

We also note that the integral

$$\int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \zeta^{-s} e^{\zeta}\, d\zeta$$

is an entire function of the complex variable s. Indeed, it exists for all $s \in \mathbb{C}$ and has the derivative

$$\int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{d}{ds}(\zeta^{-s}e^{\zeta})\, d\zeta = -\int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \zeta^{-s}e^{\zeta}\ln\zeta\, d\zeta. \tag{1.1.3}$$

The last assertions is valid due to the uniform convergence with respect to s on any compact in $\mathbb{C}$ of integral (1.1.3). Therefore, by the uniqueness theorem, is suffices to prove identity (1.1.2) only for $s\in(0,1)$.

We prove (1.1.2) for $s\in(0,1)$, $\alpha_1=-\pi$, and $\alpha_2=\pi$. If $s\in(0,1)$, we can pass to $\sigma=0$ since

$$\int\limits_{|\zeta|=\sigma} \zeta^{-s}e^{\zeta}\, d\zeta = O\left(\int\limits_{|\zeta|=\sigma} |\zeta|^{-s}|\, d\zeta|\right) = O(\sigma^{1-s}) = o(1) \quad (\sigma\to 0+).$$

Therefore, for $s\in(0,1)$, the integral

$$I(s) = \frac{1}{2\pi i}\int\limits_{\gamma(\sigma,-\pi,\pi)} \zeta^{-s}e^{\zeta}\, d\zeta$$

is equal to the difference of the integrals of $\zeta^{-s}e^{\zeta}$ along the lower and upper sides of the cut $(-\infty,0]$, where integration is performed from $-\infty$ to 0. We obtain

$$\begin{aligned} I(s) &= \frac{1}{2\pi i}\left(\int_{+\infty}^{0} e^{-r}(re^{-\pi i})^{-s} d(-r) - \int_{+\infty}^{0} e^{-r}(re^{-\pi i})^{-s} d(-r)\right)\\ &= \frac{1}{2\pi i}\left(\int_{0}^{+\infty} r^{-s}e^{-r}e^{-\pi i s}dr - \int_{0}^{+\infty} r^{-s}e^{-r}e^{-\pi i s}dr\right)\\ &= \frac{1}{\pi}\int_{0}^{+\infty} r^{-s}e^{-r}\left(\frac{e^{\pi i s}-e^{-\pi i s}}{2i}\right)dr = \frac{\sin(\pi s)}{\pi}\int_{0}^{+\infty} r^{-s}e^{-r}\,dr = \frac{\sin(\pi s)}{\pi}\Gamma(1-s). \end{aligned} \tag{1.1.4}$$

Applying the complement formula

$$\frac{\sin(\pi s)}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)}, \tag{1.1.5}$$

from (1.1.4) we obtain (1.1.2). The lemma is proved. □

Proof of Theoremv 1.1.1. By Lemma 1.1.1, for any integer k we have the equality

$$\frac{1}{\Gamma(\mu+k/\rho)} = \frac{1}{2\pi i}\int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{e^{\zeta}\, d\zeta}{\zeta^{\mu+k/\rho}}. \tag{1.1.6}$$

Multiplying relations (1.1.6) by z^k and summing over the index k from 0 to N, where N is an arbitrary natural number, we obtain the relation

$$\sum_{k=0}^{N} \frac{z^k}{\Gamma(\mu + k/\rho)} = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{e^\zeta}{\zeta^\mu} \left(\sum_{k=0}^{N} \left(\frac{z}{\zeta^{1/\rho}} \right)^k \right) d\zeta. \tag{1.1.7}$$

Now in (1.1.7) we pass to the limit as $N \to \infty$. The left-hand side of (1.1.7) converges to $E_\rho(z;\mu)$. For any $\zeta \in \gamma(\sigma, \alpha_1, \alpha_2)$, the sequence of sums under the sign of integral converges to

$$(1 - z\zeta^{-1/\rho})^{-1}$$

since $|z| < \sigma^{1/\rho} \le |\zeta|^{1/\rho}$, and hence the modulus of the common ratio of the geometric progression $(z\zeta^{-1/\rho})^k$ is less than 1. We prove that the following representation holds:

$$E_\rho(z;\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{e^\zeta \, d\zeta}{\zeta^\mu (1 - z\zeta^{-1/\rho})} = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{\zeta^{1/\rho-\mu} e^\zeta \, d\zeta}{\zeta^{1/\rho} - z},$$

This passage must be justified. As is known, the pointwise (and even uniform) convergence of a sequence of entire function $f_n(\zeta)$ to $f(\zeta)$ on a ray $\ell \subset \mathbb{C}$ does not imply the limit relation

$$\lim_{n\to\infty} \int\limits_{\ell} f_n(\zeta)\, d\zeta = \int\limits_{\ell} f(\zeta)\, d\zeta, \tag{1.1.8}$$

even in the case where the function f is summable on ℓ. By the Lebesgue theorem, relation (1.1.8) is implied by the existence of a common majorant of the sequence $\{|f_n(\zeta)|\}_{n\in\mathbb{N}}$, which is summable on ℓ (see, e.g., [39, Chap. 1]). In our case, we can take the majorant in the following form (here $\mu = \mu_1 + i\mu_2$, $\mu_1, \mu_2 \in \mathbb{R}$, and $\varphi(\zeta) = \operatorname{Arg}\zeta$):

$$\begin{aligned}\frac{\exp(\operatorname{Re}\zeta)}{|\zeta^\mu|} \sum_{k=0}^{\infty} \left| \frac{z}{\zeta^{1/\rho}} \right|^k &= \frac{\exp(\operatorname{Re}\zeta)}{|\zeta|^{\mu_1} \exp(-\mu_2 \varphi(\zeta))} \sum_{k=0}^{\infty} \left(\frac{|z|}{|\zeta|^{1/\rho}} \right)^k \\ &= \frac{\exp(\operatorname{Re}\zeta + \mu_2\varphi(\zeta))}{|\zeta|^{\mu_1}(1 - |z|\,|\zeta|^{-1/\rho})} \le \frac{|\zeta|^{1/\rho-\mu_1} \exp(\operatorname{Re}\zeta + 3\pi\mu_2/2)}{|\zeta|^{1/\rho} - |z|}. \end{aligned} \tag{1.1.9}$$

It is easy to prove that the majorant on the right-hand side of (1.1.9) is summable on the contour γ since the exponent of $\operatorname{Re}\zeta$ decreases on the rays this contour faster than any negative power of $|\zeta|$ and the denominator of the last fraction in (1.1.9) is separated from zero due to the condition $|z|^\rho < \sigma \le |\zeta|$. Therefore, the limit in (1.1.7) is valid and the theorem is completely proved. □

For some values of the parameters ρ and μ, we need special integral representations for proofs of "nonasymptotic" theorems on zeros of $E_\rho(z;\mu)$.

For $\rho > 1$ and $\mu < 1 + 1/\rho$, we set

$$g_\rho(z;t,\mu) = \frac{t \sin \pi\mu - z \sin \pi(\mu - 1/\rho)}{t^2 - 2tz\cos(\pi/\rho) + z^2}, \tag{1.1.10}$$

$$f_\rho(z;t,\mu) = \frac{e^{-\pi i \mu}}{z e^{-\pi i/\rho} - t} - \frac{e^{\pi i \mu}}{z e^{\pi i/\rho} - t}, \tag{1.1.11}$$

$$F_\rho(z;\mu) = \frac{\rho}{\pi} \int\limits_0^{+\infty} g_\rho(z,t,\mu) t^{\rho(1-\mu)} \exp(-t^\rho) dt. \tag{1.1.12}$$

It is easy to prove the identity

$$f_\rho(z;t,\mu) = 2i g_\rho(z;t,\mu).$$

Therefore, in addition to (1.1.12), we have the relation

$$F_\rho(z;\mu) = \frac{\rho}{2\pi i}\int\limits_0^{+\infty} f_\rho(z,t,\mu)t^{\rho(1-\mu)}\exp(-t^\rho)dt. \tag{1.1.13}$$

If $\arg z = \pm\pi/\rho$, then integrals (1.1.12) and (1.1.13) are understood in the sense of principal value.

Theorem 1.1.2. *For $\rho > 1$ and $\mu < 1 + 1/\rho$, the following representations hold*:

$$E_\rho(z;\mu) = F_\rho(z;\mu), \qquad \frac{\pi}{\rho} < |\arg z| \le \pi, \tag{1.1.14}$$

$$E_\rho(z;\mu) = F_\rho(z;\mu) + \frac{\rho}{2}z^{\rho(1-\mu)}\exp(z^\rho), \quad \arg z = \pm\frac{\pi}{\rho}, \tag{1.1.15}$$

$$E_\rho(z;\mu) = F_\rho(z;\mu) + \rho z^{\rho(1-\mu)}\exp(z^\rho), \quad |\arg z| < \frac{\pi}{\rho}. \tag{1.1.16}$$

Proof. Setting in formula (1.1.1) $\alpha_1 = -\pi$ and $\alpha_2 = \pi$ and changing the variable by the formula $w = \zeta^{1/\rho}$ we obtain the relation

$$E_\rho(z;\mu) = \frac{\rho}{2\pi i}\int\limits_{\gamma_R} \frac{w^{\rho(1-\mu)}\exp(w^\rho)dw}{w-z}, \tag{1.1.17}$$

where $\gamma_R = \gamma(R, -\pi/\rho, \pi/\rho)$, $R > |z|$. If we deform the contour γ_R by letting R tend to zero, then for $|\arg z| > \pi/\rho$, the singularity of the integrand lies to the left of the contour. Therefore, representation (1.1.17) is valid for any $R > 0$, not only for $|z| < R$, if $|\arg z| > \pi/\rho$. The transition in (1.1.17) to the value $R = 0$ is possible only if the integrand is summable on the rays $\arg w = \pm\pi/\rho$. This holds if $\rho(1-\mu) > -1$, i.e., if $\mu < 1 + 1/\rho$. If $|\arg z| < \pi/\rho$, then, deforming the contour, we meet a pole at the point $w = z$ and, having bypassed it, we must add the residue, which is equal to $\rho z^{\rho(1-\mu)}\exp(z^\rho)$. Thus, we obtain the representation

$$E_\rho(z;\mu) = \begin{cases} \dfrac{\rho}{2\pi i}\displaystyle\int\limits_{\gamma_0} \frac{w^{\rho(1-\mu)}\exp(w^\rho)dw}{w-z}, & |\arg z| > \dfrac{\pi}{\rho}, \\ \dfrac{\rho}{2\pi i}\displaystyle\int\limits_{\gamma_0} \frac{w^{\rho(1-\mu)}\exp(w^\rho)dw}{w-z} + \rho z^{\rho(1-\mu)}\exp(z^\rho), & |\arg z| < \dfrac{\pi}{\rho}. \end{cases} \tag{1.1.18}$$

If $\arg z = \pm\dfrac{\pi}{\rho}$, then after deformation of the contour the pole falls on γ_0 and we must add half of the residue to the integral by the Sochocki formula.

To complete the proof of theorem 1.1.2 (see (1.1.13)), we need to verify the relation

$$\int\limits_0^{+\infty} f_\rho(z;t,\mu)t^{\rho(1-\mu)}\exp(-t^\rho)dt = \int\limits_{\gamma_0} \frac{w^{\rho(1-\mu)}\exp(w^\rho)dw}{w-z}.$$

The contour γ_0 is the union of the two rays $w = t\exp\left(\pm\frac{\pi i}{\rho}\right)$, $0 < t < +\infty$. Therefore,

$$\int_{\gamma_0} \frac{w^{\rho(1-\mu)}\exp(w^\rho)dw}{w-z} = \int_{+\infty}^{0} \frac{t^{\rho(1-\mu)}\exp\left(-\pi i(1-\mu)\right)dt\exp\left(-\frac{\pi i}{\rho}\right)}{t\exp\left(-\frac{\pi i}{\rho}\right)-z}$$

$$+\int_{0}^{+\infty} \frac{t^{\rho(1-\mu)}\exp\left(\pi i(1-\mu)\right)dt\exp\left(\frac{\pi i}{\rho}\right)}{t\exp\left(\frac{\pi i}{\rho}\right)-z}$$

$$=\int_{0}^{+\infty} t^{\rho(1-\mu)}\exp\left(-t^\rho\right)\left(\frac{e^{\pi i(1-\mu+1/\rho)}}{t\exp\left(\frac{\pi i}{\rho}\right)-z}-\frac{e^{\pi i(\mu-1-1/\rho)}}{t\exp\left(-\frac{\pi i}{\rho}\right)-z}\right)dt$$

$$=\int_{0}^{+\infty} t^{\rho(1-\mu)}\exp\left(-t^\rho\right)\left(\frac{e^{-\pi i\mu}}{ze^{-\pi i/\rho}-t}-\frac{e^{\pi i\mu}}{ze^{\pi i/\rho}-t}\right)dt.$$

Theorem 1.1.2 is proved. □

To examine the behavior of the Mittag-Leffler functions in a neighborhood of the negative part of the real axis, we need a special formula for $E_\rho(z;\mu)$, which is useful for $\rho \in [2/5, 2/3]$, but we prove it for $\rho \in (1/3, 1)$.

Theorem 1.1.3. *For all $\rho \in (1/3,1)$, $\mu \in \mathbb{R}$, and $m \in \mathbb{Z}_+$, $m \geq \rho\mu - 1$, the following identity holds on the right-hand half-plane* $\operatorname{Re} z > 0$:

$$E_\rho(-z^{1/\rho};\mu) = 2\rho z^{1-\mu}\exp\left(z\cos(\pi\rho)\right)\cos\left(z\sin(\pi\rho)-\pi\rho(\mu-1)\right) + \sum_{k=1}^{m}\frac{(-1)^{k-1}z^{-k/\rho}}{\Gamma(\mu-k/\rho)} + \Omega_m(z;\rho,\mu), \quad (1.1.19)$$

where

$$\Omega_m(z;\rho,\mu) = \frac{(-1)^m z^{1-\mu}}{\pi} \times\left[I_{1,m}(z;\rho,\mu)\sin\left(\pi\left(\mu-\frac{m+1}{\rho}\right)\right)+I_{2,m}(z;\rho,\mu)\sin\left(\pi\left(\mu-\frac{m}{\rho}\right)\right)\right], \quad (1.1.20)$$

$$I_{1,m}(z;\rho,\mu) = \int_0^{+\infty}\frac{t^{(m+1)/\rho-\mu}e^{-zt}\,dt}{t^{2/\rho}+2t^{1/\rho}\cos(\pi/\rho)+1},$$

$$I_{2,m}(z;\rho,\mu) = \int_0^{+\infty}\frac{t^{(m+1)/\rho-\mu}e^{-zt}\,dt}{t^{1/\rho}+2\cos(\pi/\rho)+t^{-1/\rho}}.$$

Proof. It is easy to see that it suffices to verify representation (1.1.19) only for $z = x > 0$. Indeed, on both sides of (1.1.19), we have functions that are holomorphic in the open right half-plane. Therefore, by the uniqueness theorem, they coincide for $\operatorname{Re} z > 0$ if and only if they coincide for $(0, +\infty)$.

We represent the function $E_\rho(-x^{1/\rho};\mu)$ by formula (1.1.1) setting $\alpha_1 = -\pi$ and $\alpha_2 = \pi$. We obtain the relation

$$E_\rho(-x^{1/\rho};\mu) = \frac{1}{2\pi i}\int_{\gamma(\sigma,-\pi,\pi)}\frac{\zeta^{1/\rho-\mu}e^{\zeta}\,d\zeta}{\zeta^{1/\rho}+x^{1/\rho}}, \quad x < \sigma.$$

Inside the Hankel loops $\gamma(\sigma,-\pi,\pi)$, $x<\sigma$, the integrand has exactly two simple poles $\zeta_\pm = x\exp(\pm\pi i\rho)$. All other poles lying on the Riemann surface of argument, namely, $\{x\exp(\pm(2k+1)\pi i\rho)\}_{k\in\mathbb{N}}$, lie strictly outside the loop since $\rho>1/3$. The sum of the residues of the integrand at the points $\zeta_\pm$ is equal to

$$\begin{aligned}
&\lim_{\zeta\to x\exp(\pi i\rho)} \frac{\zeta^{1/\rho-\mu}e^\zeta(\zeta-x\exp(\pi i\rho))}{\zeta^{1/\rho}+x^{1/\rho}} + \lim_{\zeta\to x\exp(-\pi i\rho)} \frac{\zeta^{1/\rho-\mu}e^\zeta(\zeta-x\exp(-\pi i\rho))}{\zeta^{1/\rho}+x^{1/\rho}}\\
&\quad= \zeta^{1/\rho-\mu}e^\zeta\Big|_{\zeta=x\exp(\pi i\rho)} \cdot \lim_{\zeta\to x\exp(\pi i\rho)} \frac{\zeta-x\exp(\pi i\rho)}{\zeta^{1/\rho}+x^{1/\rho}}\\
&\quad+ \zeta^{1/\rho-\mu}e^\zeta\Big|_{\zeta=x\exp(-\pi i\rho)} \cdot \lim_{\zeta\to x\exp(-\pi i\rho)} \frac{\zeta-x\exp(-\pi i\rho)}{\zeta^{1/\rho}+x^{1/\rho}}\\
&\quad= \rho\zeta^{1-\mu}e^\zeta\Big|_{\zeta=x\exp(\pi i\rho)} + \rho\zeta^{1-\mu}e^\zeta\Big|_{\zeta=x\exp(-\pi i\rho)} = 2\rho\,\mathrm{Re}(\zeta^{1-\mu}e^\zeta)\Big|_{\zeta=x\exp(\pi i\rho)}\\
&\quad= 2\rho\,\mathrm{Re}\exp\Big(xe^{\pi i\rho}+(1-\mu)\big(\ln x+\pi i\rho\big)\Big)\\
&\quad= 2\rho x^{1-\mu}\exp\big(x\cos(\pi\rho)\big)\cos\big(x\sin(\pi\rho)+(1-\mu)\pi\rho\big). \qquad (1.1.21)
\end{aligned}$$

Now we integrate along the contour $\gamma(\sigma',-\pi,\pi)$, $0<\sigma'<x$, having added function (1.1.21) to the integral:

$$\begin{aligned}
E_\rho(-x^{1/\rho};\mu) &= 2\rho x^{1-\mu}\exp(x\cos(\pi\rho))\cos(x\sin(\pi\rho)+(1-\mu)\pi\rho)\\
&\quad+ \frac{1}{2\pi i}\int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{1/\rho-\mu}e^\zeta\,d\zeta}{\zeta^{1/\rho}+x^{1/\rho}}. \qquad (1.1.22)
\end{aligned}$$

We transform the integrand in (1.1.22) by the identity

$$\frac{q}{1+q} = \sum_{k=1}^{m}(-1)^{k-1}q^k + \frac{(-1)^m q^{m+1}}{1+q}, \quad q\in\mathbb{C}, \quad q\neq -1, \quad m\in\mathbb{N}. \qquad (1.1.23)$$

We see that for any $m\in\mathbb{N}$, the fraction

$$\frac{\zeta^{1/\rho}}{\zeta^{1/\rho}+x^{1/\rho}} = \frac{(\zeta/x)^{1/\rho}}{1+(\zeta/x)^{1/\rho}}$$

can be written in the form

$$\frac{\zeta^{1/\rho}}{\zeta^{1/\rho}+x^{1/\rho}} = \sum_{k=1}^{m}\frac{(-1)^{k-1}\zeta^{k/\rho}}{x^{k/\rho}} + \frac{(-1)^m\zeta^{\frac{m+1}{\rho}}}{(1+(\zeta/x)^{1/\rho})x^{\frac{m+1}{\rho}}}. \qquad (1.1.24)$$

By (1.1.24) we have

$$\frac{1}{2\pi i}\int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{1/\rho-\mu}e^\zeta\,d\zeta}{\zeta^{1/\rho}+x^{1/\rho}} = \sum_{k=1}^{m}\frac{(-1)^{k-1}x^{-k/\rho}}{2\pi i}\int_{\gamma(\sigma',-\pi,\pi)} \zeta^{k/\rho-\mu}e^\zeta\,d\zeta + \Omega_m(x,\rho,\mu), \qquad (1.1.25)$$

where

$$\Omega_m(x;\rho,\mu) = \frac{(-1)^m x^{-\frac{m+1}{\rho}}}{2\pi i}\int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{\frac{m+1}{\rho}-\mu}e^\zeta\,d\zeta}{1+(\zeta/x)^{1/\rho}}. \qquad (1.1.26)$$

The power of ζ in the numerator of the fraction in integral (1.1.26) is nonnegative owing to the restriction $m\geq\rho\mu-1$ imposed in the theorem. Therefore, we can pass to the value $\sigma'=0$. Then the Hankel loop becomes the negative part of the real axis traversed first from $-\infty$ to 0 along the lower side of the cut ($\zeta = re^{-\pi i}$) and then from 0 to $-\infty$ along the upper side ($\zeta = re^{\pi i}$). Changing the variables in the integrals we obtain

$$\Omega_m(x;\rho,\mu) = \frac{(-1)^m x^{-\frac{m+1}{\rho}}}{2\pi i} \times \int_0^{+\infty} r^{\frac{m+1}{\rho}-\mu} e^{-r} \left(\frac{\exp\left(\pi i\left(\mu - \frac{m+1}{\rho}\right)\right)}{1+\left(\frac{r}{x}\right)^{1/\rho}\exp\left(-\frac{\pi i}{\rho}\right)} - \frac{\exp\left(-\pi i\left(\mu - \frac{m+1}{\rho}\right)\right)}{1+\left(\frac{r}{x}\right)^{1/\rho}\exp\left(\frac{\pi i}{\rho}\right)} \right) dr. \quad (1.1.27)$$

Further, setting $r = xt$, we obtain the relation

$$\Omega_m(x;\rho,\mu) = \frac{(-1)^m x^{1-\mu}}{\pi} \int_0^{+\infty} t^{\frac{m+1}{\rho}-\mu} e^{-tx} U_m(t;\rho,\mu)\,dt,$$

where

$$U_m(t;\rho,\mu) = \frac{1}{2i}\,\frac{(1+t^{1/\rho}\exp(\frac{\pi i}{\rho}))\exp\left(\pi i\left(\mu-\frac{m+1}{\rho}\right)\right) - (1+t^{1/\rho}\exp(-\pi i\rho))\exp\left(\pi i\left(\frac{m+1}{\rho}-\mu\right)\right)}{1+2t^{1/\rho}\cos(\pi/\rho)+t^{2/\rho}}$$
$$= \frac{\sin\left(\pi\left(\mu-\frac{m+1}{\rho}\right)\right) + t^{1/\rho}\sin\left(\pi\left(\mu-\frac{m}{\rho}\right)\right)}{1+2t^{1/\rho}\cos(\pi/\rho)+t^{2/\rho}}. \quad (1.1.28)$$

This immediately implies the representation of the remainder $\Omega_m(x;\rho,\mu)$ in the form proposed in Theorem 1.1.3. Relations (1.1.22) and (1.1.25) together with Lemma 1.1.1 yield the form of the principal part of the representation of $E_\rho(-z^{1/\rho};\mu)$. The theorem is proved. ☐

1.2. Basic Theorems on Asymptotic Behavior of Mittag-Leffler Functions

In this section, we present basic results on the asymptotic behavior of $E_\rho(z;\mu)$ as $z \to \infty$. We omit proofs; all assertions of this section are consequences of results obtained in Secs. 1.3–1.5 below.

First, we describe the growth of the modulus maximum of Mittag-Leffler functions in terms of order and type. Recall that the *order* of an entire function $f(z)$ is defined as the supremum limit

$$\limsup_{R\to\infty} \frac{\ln\ln\left(\max\limits_{|z|\le R}|f(z)|\right)}{\ln R}. \quad (1.2.1)$$

Proposition 1.2.1. *The order of the function $E_\rho(z;\mu)$ is equal to ρ. Moreover, the supremum limit in* (1.2.1) *can be replaced by the ordinary limit.*

A more subtle characteristic of the growth is the *type* under the order ρ:

$$\limsup_{R\to\infty} \frac{\ln\left(\max\limits_{|z|\le R}|f(z)|\right)}{R^\rho}. \quad (1.2.2)$$

Proposition 1.2.2. *The type under the order ρ of the function $E_\rho(z;\mu)$ is equal to* 1 *for any $\mu \in \mathbb{C}$, and the supremum limit in* (1.2.2) *can be replaced by the ordinary limit.*

Propositions 1.2.1 and 1.2.2 are trivial consequences of formulas that express the order and type of an entire function by the coefficients of its Taylor series (see [14, Chap. 7, Sec. 12]).

We formulate a less obvious result.

Proposition 1.2.3. *For any $\rho > 0$ and $\mu \in \mathbb{C}$, the following asymptotics holds:*

$$\ln\left(\max_{|z|\le R} |E_\rho(z;\mu)|\right) = R^\rho + O(\ln R).$$

The constant in O depends on ρ and μ.

The behavior of an ρ-order entire function of normal type[1] on rays in the complex plane is characterized by the growth indicatrix:

$$H_\rho(f,\theta) = \lim_{R\to+\infty} \frac{\ln|f(Re^{i\theta})|}{R^\rho}, \quad -\pi < \theta \le \pi.$$

Proposition 1.2.4. *For any $\mu \in \mathbb{C}$, the following relations hold:*

$$H_\rho(E_\rho(z;\mu),\theta) = \cos(\rho\theta), \quad -\pi < \theta \le \pi, \quad 0 < \rho \le \frac{1}{2},$$

$$H_\rho(E_\rho(z;\mu),\theta) = \begin{cases} \cos(\rho\theta), & |\theta| \le \dfrac{\pi}{2\rho}, \\ 0, & \dfrac{\pi}{2\rho} < |\theta| \le \pi, \end{cases}, \qquad \frac{1}{2} < \rho.$$

The exception in the last relation is the pairs of parameters $\rho = 1$, $\mu \in \mathbb{Z}$, $\mu \le 1$. Then

$$E_1(z;\mu) = z^{1-\mu}e^z, \quad H(\theta) = \cos\theta, \quad -\pi < \theta \le \pi.$$

Propositions 1.2.3 and 1.2.4 can be easily obtained from the following two theorems on the complete asymptotic expansion of Mittag-Leffler functions. In the sequel, in the definition of noninteger powers of z in the complex plane (not on the Riemann surface $\mathfrak{P}$!) we choose the principal branch of the argument: $\arg(re^{i\theta}) = \theta$, $-\pi < \theta \le \pi$, $r > 0$.

Theorem 1.2.1. *For any $\rho > 1/2$, $\mu \in \mathbb{C}$, and $m \in \mathbb{N}$, the following asymptotics hold.*

If $|\arg z| \le \min(\pi, \frac{\pi}{\rho})$, then

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)}\exp(z^\rho) - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}) \tag{1.2.3}$$

as $z \to \infty$.

If $\rho > 1$ and $\frac{\pi}{\rho} \le |\arg z| \le \pi$, then

$$E_\rho(z;\mu) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}) \tag{1.2.4}$$

as $z \to \infty$.

Remark 1.2.1. For $\rho \in (1/2, 1]$, formula (1.2.3) is valid for $|\arg z| \le \pi$, i.e., on the whole plane. The first term has a discontinuity on the ray $(-\infty, 0)$m but this does not lead to a contradiction since in any angle $\alpha \le |\arg z| \le \pi$, $\alpha > \frac{\pi}{2\rho}$, the modulus of the function $\rho z^{\rho(1-\mu)}\exp(z^\rho)$ tends to zero (as $r \to +\infty$) faster than any negative power of r. In particular, on the negative part of the real axis, for any $\rho > 1/2$ and $\mu \in \mathbb{C}$, we have the relation

$$E_\rho(z;\mu) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}), \quad z \in \mathbb{R}, \quad z \to -\infty. \tag{1.2.5}$$

[1] This means "of finite and positive type under the order ρ."

Theorem 1.2.2. *For any $\rho \le 1/2$, $\mu \in \mathbb{C}$, and $m \in \mathbb{N}$, the following asymptotics holds*:

$$E_\rho(z;\mu) = \rho \sum_{|\arg z + 2\pi n| \le \frac{3\pi}{4\rho}} (z^\rho e^{2\pi i n\rho})^{1-\mu} \exp(z^\rho e^{2\pi i n\rho}) - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}), \quad z \to \infty. \tag{1.2.6}$$

We discuss Theorems 1.2.1 and 1.2.2.

Theorem 1.2.1 is an improvement of a well-known result of Dzhrbashyan (see [6, Chap. 3, p. 134]). Dzhrbashyan proved asymptotics (1.2.3) in the angles $|\arg z| \le \alpha$, where α is an arbitrary number from the interval $(\pi/(2\rho), \min(\pi/\rho, \pi))$; in the complements of these angles, asymptotics (1.2.4) is valid. It is unclear why Dzhrbashyan did not consider the value $\alpha = \pi/\rho$. We set

$$\alpha = \min\left(\pi, \frac{\pi}{\rho}\right) \tag{1.2.7}$$

and thus exclude from the statement of the theorem an unnecessary parameter. However, a thoughtful reader can ask why the value of α chosen in (1.2.7) is better than, say, $\alpha_1 = \min(\pi, 3\pi/(4\rho))$. Only the remainder of the asymptotics depends on the choice of α. In Theorems 1.4.1–1.5.1, we give an explicit estimate of this remainder (without O-symbols). If anyone, having taken a different setting of the parameter α, gets a better estimate, we accept his/her choice of α as more successful. Meanwhile, we are of the opinion that formula (1.2.7) is the most reasonable.

Theorem 1.2.2 is a correction of an erroneous result of Nersesyan [6, Lemma 3.6, p. 137]. This result is as follows: *If $0 < \rho \le 1/2$, then for any integer $m \ge 1$, the following asymptotic formula as $|z| \to \infty$ holds*:

$$E_\rho(z;\mu) = \rho \sum_{|\arg z + 2\pi n| \le \frac{\pi}{2\rho}} (z^\rho e^{2\pi i n\rho})^{1-\mu} \exp(z^\rho e^{2\pi i \rho}) - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}); \tag{1.2.8}$$

the first sum is taken over the values of $n = 0, \pm1, \pm2, \dots$, for which $|\arg z + 2\pi n| \le \pi/(2\rho)$.

Is not easy to detect an error in this statement. Formulas (1.2.6) and (1.2.8) differ only by a small detail. In (1.2.6), the sum is taken over the values of $n \in \mathbb{Z}$ for which $|\arg z + 2\pi n| \le 3\pi/(4\rho)$ (not $\pi/(2\rho)$). Thus, in (1.2.8), compared with (1.2.6), several terms are missed, and hence asymptotics (1.2.8) is valid on any ray in $\mathbb{C}$ emanating from the point $z = 0$, but it is invalid as $z \to \infty$!

We prove this by a simple example. Consider the function

$$E_{1/4}(w^4, 1) = \sum_{k=0}^{\infty} \frac{w^4}{(4k)!} = \frac{1}{4}(e^w + e^{-w} + e^{iw} + e^{-iw}). \tag{1.2.9}$$

If we accept formula (1.2.8), then, taking into account the relation $\exp(\pi i n/2) = i^n$, we have

$$E_{1/4}(z, 1) = \frac{1}{4} \sum_{|\arg z + 2\pi n| \le 2\pi} \exp(\sqrt[4]{z}\, i^n) + O\left(\frac{1}{|z|}\right), \quad z \to \infty. \tag{1.2.10}$$

Now we consider the curve

$$K = \{z \in \mathbb{C} \mid z = (x - i)^4,\ x \ge 3\}.$$

It easy to see that the curve K lies inside the angle $-\pi/2 < \arg z < 0$. Therefore, in (1.2.10) we must take the values $n = 0$ and $n = 1$ for $z \in K$. Hence from (1.2.10) we obtain[2]

$$E_{1/4}\big((x-i)^4, 1\big) = \frac{1}{4}\big(e^{x-i} + e^{i(x-i)}\big) + O(x^{-4}), \quad x \to +\infty. \tag{1.2.11}$$

Subtracting asymptotics (1.2.11) from (1.2.9) for $w = x - i$ we obtain the relation

$$\frac{1}{4}\big(e^{-x+i} + e^{-i(x-i)}\big) = O(x^{-4}), \quad x \to +\infty,$$

and hence

$$e^{-1-ix} = O(x^{-4} + e^{-x}) = o(1), \quad x \to +\infty.$$

The last relation is contradictory. This means that formula (1.2.10) (and hence (1.2.8)) is invalid.

Coverage of this by Evgrafov (see [7, Chap. 4, Sec.4]) is incomplete. He considered only the classical Mittag-Leffler function ($\mu = 1$) and presented an analog of formula (1.2.6) with the condition $|\arg z + 2\pi n| \le \pi/(2\rho) + \eta$ and a restriction of the range of the summation index $|n| \le [1/\rho]$. However, Evgrafov wrote nothing about η except that it is positive and did not present a proof of asymptotics.

By virtue of the presence of such defects in the presentation of asymptotics of Mittag-Leffler functions of order $\rho < 1/2$, even in works of well-known scientists, we found ourselves obliged to give the correct result with the maximal detailed proof (see below Secs. 1.3 and 1.5) and the explicit estimate of remainders without O-symbols.

In concluding this section, we note another feature of our approach. We prove Theorem 1.2.1 only for $\rho \ge 3/4$ and Theorem 1.2.2 for $\rho \le 3/4$. It turns out that Theorem 1.2.2 is valid not only for $\rho \le 1/2$, but also Theorems 1.2.1 and 1.2.2 are equivalent for $\rho \in (1/2, 3/4]$. The reason lies in the absence of uniformity of asymptotics (1.2.4) with respect to the parameter $\rho \in (1/2, 3/4]$. Adding, by formula (1.2.6) another term into the asymptotics of $E_\rho(z;\mu)$ in the angle $|\arg(-z)| \le 3\pi/(4\rho) - \pi$, we restore the uniformity of the asymptotics with respect to ρ. This improvement will play a crucial role in the study of real zeros of $E_\rho(z;\mu)$ for $\rho \in (1/2, 2/3]$ and $\mu \in \mathbb{R}$.

1.3. Complete Asymptotic Expansion of One Integral

This section is of " technical nature": we obtain an asymptotics of the integral

$$J_\rho(z, \mu, \alpha_1, \alpha_2) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,\alpha_1,\alpha_2)} \frac{\zeta^{1/\rho-\mu} e^{\zeta}\, d\zeta}{\zeta^{1/\rho} - z}, \tag{1.3.1}$$

where $z, \mu \in \mathbb{C}$, $\rho > 0$, $0 < \sigma < |z|^\rho$, $\alpha_1 \in (-3\pi/2, -\pi/2)$, and $\alpha_2 \in (\pi/2, 3\pi/2)$. This integral naturally arises in the theory of Mittag-Leffler functions since it differs from their representing integral (see Theorem 1.1.1) by the value of the parameter σ of the Hankel loop by which integration is carried out; here σ is less (and not greater) than $|z|^\rho$, as in (1.1.1). We assume that z does not lie on the rays

$$\ell_j = \{r \exp(i\alpha_j/\rho) \mid r \ge 0\}, \quad j = 1, 2, \tag{1.3.2}$$

and in this case, integral (1.3.1) is independent of the choice of $\sigma \in (0, |z|^\rho)$. The function $J_\rho(z, \mu, \alpha_1, \alpha_2)$ is an analytic function of the variable z in open angles that are formed after the removal from $\mathbb{C}$ of rays (1.3.2). We denote by $\ell(z)$ the ray emanating from the origin 0 and passing

[2]By the choice of the branch of argument in the z-plane (see p. 219 before the statement of Theorem 1.2.1), we have the relation $\sqrt[4]{z^4} = z$ if $|\arg z| < \pi/4$.

through the point z and by $\delta(z)$ the value of the minimal angle between the rays $\ell(z)$ and ℓ_j. Finally, we set

$$\varkappa(z) = \begin{cases} \sin^{-1}\delta(z), & 0 < \delta(z) \le \pi/2, \\ 1, & \pi/2 < \delta(z), \end{cases}$$

$$A = \max\left(\left|\frac{1}{\cos\alpha_1}\right|, \left|\frac{1}{\cos\alpha_2}\right|\right), \quad \alpha = \max(|\alpha_1|, \alpha_2).$$

The following lemma on the complete asymptotic expansion of the integral J_ρ in the domain $\mathbb{C} \setminus (\ell_1 \cup \ell_2)$ is the central result of this section.

Lemma 1.3.1. *For any $\rho > 0$, $\mu \in \mathbb{C}$, $\alpha_1 \in (-3\pi/2, -\pi/2)$, $\alpha_2 \in (\pi/2, 3\pi/2)$, $m \in \mathbb{N}$, and $z \in \mathbb{C} \setminus (\ell_1 \cup \ell_2)$, the following relation holds:*

$$J_\rho(z, \mu, \alpha_1, \alpha_2) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + R_m(z, \rho, \mu, \alpha_1, \alpha_2),$$

where the remainder admits the estimate

$$|R_m(z, \rho, \mu, \alpha_1, \alpha_2)| \le \frac{\varkappa(z) A^{b+1} \Gamma(b+1)(e^{\alpha_1 \operatorname{Im}\mu} + e^{\alpha_2 \operatorname{Im}\mu})}{2\pi |z|^{m+1}} \tag{1.3.3}$$

if $b = \dfrac{m+1}{\rho} - \operatorname{Re}\mu \ge 0$. If $b < 0$, then for $|z| \ge 2$ we have the relation

$$|R_m(z, \rho, \mu, \alpha_1, \alpha_2)| \le \frac{(e^{\alpha_1 \operatorname{Im}\mu} + e^{\alpha_2 \operatorname{Im}\mu})}{\pi |z|^{m+1}} \left(\frac{\varkappa(z) A}{2} + \int_{\alpha_1}^{\alpha_2} e^{\cos\varphi}\, d\varphi\right). \tag{1.3.4}$$

Remark 1.3.1. Note that estimate (1.3.3) is valid not only for "sufficiently large" $|z|$, but also for all $z \in \mathbb{C} \setminus (\ell_1 \cup \ell_2)$. Thus, we have obtained not only an asymptotics of $J_\rho(z, \mu, \alpha_1, \alpha_2)$ as $z \to \infty$, but also two-sided estimates of these integrals. Clearly, these estimates are applicable for $|z| \ge 1$, and the greater $|z|$, the more exact these estimates.

Remark 1.3.2. It is easy to see that for $\mu \in \mathbb{R}$, inequalities (1.3.3) and (1.3.4) are simplified. For example,

$$|R_m(z, \rho, \mu, \alpha_1, \alpha_2)| \le \frac{\varkappa(z) A^{b+1} \Gamma(b+1)}{\pi |z|^{m+1}}, \quad \text{if} \quad b = \frac{m+1}{\rho} - \mu \ge 0. \tag{1.3.5}$$

Proof of Lemma 1.3.1. Similarly to (1.1.23), we have the relation

$$\frac{1}{1-q} = \sum_{k=1}^{m} q^k + \frac{q^m}{1-q}, \quad q \in \mathbb{C}, \quad q \ne 1, \quad m \in \mathbb{N}. \tag{1.3.6}$$

Setting in (1.3.6) $q = \zeta^{1/\rho}/z$ ($q \ne 1$ due to the condition $z \notin (\ell_1 \cup \ell_2)$), we obtain

$$\frac{1}{z} \cdot \frac{1}{1 - \zeta^{1/\rho}/z} = \frac{1}{z} \left(\sum_{k=0}^{m-1} \left(\frac{\zeta^{1/\rho}}{z}\right)^k + \frac{\zeta^{m/\rho}}{z^m (1 - \zeta^{1/\rho}/z)}\right)$$

or, equivalently,

$$\frac{1}{\zeta^{1/\rho} - z} = -\sum_{k=1}^{m} \frac{\zeta^{(k-1)/\rho}}{z^k} + \frac{\zeta^{m/\rho}}{z^m (\zeta^{1/\rho} - z)}. \tag{1.3.7}$$

From (1.3.7) we obtain the representation (for brevity, we denote the contour $\gamma(\sigma, \alpha_1, \alpha_2)$ by γ)

$$\int_\gamma \frac{\zeta^{1/\rho-\mu}e^\zeta}{\zeta^{1/\rho}-z}\,d\zeta = -\sum_{k=1}^{m} z^{-k}\int_\gamma \zeta^{k/\rho-\mu}e^\zeta\,d\zeta + z^{-m}\int_\gamma \frac{\zeta^{\frac{m+1}{\rho}-\mu}e^\zeta}{\zeta^{1/\rho}-z}\,d\zeta.$$

By Lemma 1.1.1, we have

$$\frac{1}{2\pi i}\int_\gamma \zeta^{k/\rho-\mu}e^\zeta\,d\zeta = \frac{1}{\Gamma(\mu-k/\rho)}.$$

This and (1.3.1) imply the following expression for the integral $J_\rho(z,\mu,\alpha_1,\alpha_2)$:

$$J_\rho(z,\mu,\alpha_1,\alpha_2) = -\sum_{k=1}^{m}\frac{z^{-k}}{\Gamma(\mu-k/\rho)} + \frac{z^{-m}}{2\pi i}\int_\gamma \frac{\zeta^{\frac{m+1}{\rho}-\mu}e^\zeta}{\zeta^{1/\rho}-z}\,d\zeta. \tag{1.3.8}$$

The principal part of the asymptotic expansion is found. Now we estimate the remainder

$$R_m(z,\rho,\mu,\alpha_1,\alpha_2) = \frac{z^{-m}}{2\pi i}\int_\gamma \frac{\zeta^{\frac{m+1}{\rho}-\mu}e^\zeta}{\zeta^{1/\rho}-z}\,d\zeta.$$

We estimate from above the modulus of the integrand on the three parts of the contour $\gamma = \gamma(\sigma,\alpha_1,\alpha_2)$: the two rays and the arc. For all $\zeta \in \mathfrak{P}$ and $s \in \mathbb{C}$, the following relation holds:

$$|\zeta^s| = |\zeta|^{\operatorname{Re} s}\exp(-(\operatorname{Arg}\zeta)I\mu s).$$

Therefore, taking into account the notation $b = \dfrac{m+1}{\rho} - \operatorname{Re}\mu$ introduced in the lemma, we obtain

$$\left|\zeta^{\frac{m+1}{\rho}-\mu}\right| = |\zeta|^b\exp(\varphi\operatorname{Im}\mu), \quad \varphi = \operatorname{Arg}\zeta. \tag{1.3.9}$$

It is also easy to verify that for any $x > 0$ we have

$$\min_{r\geq 0}|re^{\pm i\delta} - x| = \begin{cases} \dfrac{x}{\sin\delta}, & 0 < \delta \leq \pi/2, \\ x, & \pi/2 < \delta \leq \pi. \end{cases}$$

This and the definition of $\varkappa(z)$ stated before the formulation of the lemma immediately imply the estimate

$$|\zeta^{1/\rho} - z|^{-1} \leq \frac{\varkappa(z)}{|z|}, \quad \operatorname{Arg}\zeta = \alpha_1, \quad \operatorname{Arg}\zeta = \alpha_2. \tag{1.3.10}$$

The inequality

$$|\zeta^{1/\rho} - z|^{-1} \leq (|z| - \sigma^{1/\rho})^{-1}, \quad |\zeta| = \sigma, \quad 0 < \sigma < |z|^\rho, \tag{1.3.11}$$

is obvious. From (1.3.9) and (1.3.11) we obtain that for $\sigma > 0$, the integrand on the arc

$$\{\zeta \in \mathfrak{P} \mid |\zeta| = \sigma,\ \alpha_1 \leq \operatorname{Arg}\zeta \leq \zeta_2\}$$

is $O(\sigma^b)$ for any fixed $z \in \mathbb{C}(\ell_1 \cup \ell_2)$ (the constant in O depends on z, ρ, μ, α_1, and α_2). Therefore, the integral over this arc is $O(\sigma^{b+1}) = o(1)$ $(\sigma \to 0)$ for $b > -1$. By the arbitrariness of $\sigma \in (0, |z|^\rho)$, we can pass (in the case where $b > -1$) to the limit value $\sigma = 0$ and integrate in (1.3.1) over the union of the rays $\{\operatorname{Arg}\zeta = \alpha_1\} \cup \{\operatorname{Arg}\zeta = \alpha_2\}$. We perform this for $b \geq 0$.

Thus, for $b \geq 0$ we have the relation

$$R_m(z,\rho,\mu,\alpha_1,\alpha_2) = \frac{z^{-m}}{2\pi i}\int_{\gamma(0,\alpha_1,\alpha_2)} \frac{\zeta^{\frac{m+1}{\rho}-\mu}e^\zeta}{\zeta^{1/\rho}-z}\,d\zeta. \tag{1.3.12}$$

From (1.3.12), (1.3.9), and (1.3.10) we obtain

$$|R_m(z,\rho,\mu,\alpha_1,\alpha_2) \le \frac{|z|^{-m}}{2\pi} \int\limits_{\gamma(0,\alpha_1,\alpha_2)} \frac{|\zeta^{\frac{m+1}{\rho}-\mu}| \exp(\operatorname{Re}\zeta)|d\zeta|}{|\zeta^{1/\rho}-z|}\, d\zeta$$

$$\le \frac{\varkappa(z)}{2\pi|z|^{m+1}} \int\limits_{\gamma(0,\alpha_1,\alpha_2)} |\zeta^{\frac{m+1}{\rho}-\mu}| \exp(\operatorname{Re}\zeta)\, |d\zeta|$$

$$= \frac{\varkappa(z)}{2\pi|z|^{m+1}} \left(\int\limits_0^{+\infty} e^{\alpha_1 \operatorname{Im}\mu} r^b e^{r\cos\alpha_1}\, dr + \int\limits_0^{+\infty} e^{\alpha_2 \operatorname{Im}\mu} r^b e^{r\cos\alpha_2}\, dr \right). \quad (1.3.13)$$

By the inclusions $\alpha_1 \in (-3\pi/2, -\pi/2)$ and $\alpha_2 \in (\pi/2, 3\pi/2)$, the numbers $\cos\alpha_1$ and $\cos\alpha_2$ are negative. Therefore,

$$\int\limits_0^{+\infty} r^b e^{r\cos\alpha_j}\, dr = \frac{\Gamma(b+1)}{|\cos\alpha_j|^{b+1}} \le A^{b+1}\Gamma(b+1), \quad j=1,2 \quad (1.3.14)$$

(recall the notation $A = \max(|\cos\alpha_1|^{-1}, |\cos\alpha_2|^{-1})$. Relations (1.3.13) and (1.3.14) lead to estimate (1.3.3).

Now let $b < 0$. In this case, we consider only values $|z| \ge 2$ and we set $\sigma = 1$. On the contour $\gamma(1,\alpha_1,\alpha_2)$, the parameter r varies on the rays $\zeta = re^{i\alpha_j}$, $j = 1,2$, from 1 to $+\infty$, and the integral over the arc $\zeta = e^{i\varphi}$, $\alpha_1 \le \varphi_1 \le \alpha_2$, appears. Therefore, taking into account inequalities (1.3.9)–(1.3.11), we obtain the inequality

$$|R_m(z,\rho,\mu,\alpha_1,\alpha_2)| \le \frac{\varkappa(z)}{2\pi|z|^{m+1}} \left(e^{\alpha_1 \operatorname{Im}\mu} \int\limits_1^{+\infty} r^b e^{r\cos\alpha_1}\, dr \right.$$

$$\left. + e^{\alpha_2 \operatorname{Im}\mu} \int\limits_1^{+\infty} r^b e^{r\cos\alpha_2}\, dr \right) + \frac{1}{2\pi(|z|-1)|z|^m} \int\limits_{\alpha_1}^{\alpha_2} e^{\varphi \operatorname{Im}\mu + \cos\varphi}\, d\varphi.$$

Since $b < 0$, we bound r^b by unity from above for $r \ge 1$ and calculate the integrals

$$\int\limits_1^{+\infty} e^{r\cos\alpha_j}\, dr = \frac{e^{\cos\alpha_j}}{|\cos\alpha_j|} \le A, \quad j = 1,2.$$

Then we arrive at the inequality

$$|R_m(z,\rho,\mu,\alpha_1,\alpha_2)|$$

$$\le \frac{\varkappa(z)A(e^{\alpha_1 \operatorname{Im}\mu} + e^{\alpha_2 \operatorname{Im}\mu})}{2\pi|z|^{m+1}} + \frac{\max(e^{\alpha_1 \operatorname{Im}\mu}, e^{\alpha_2 \operatorname{Im}\mu}) \int_{\alpha_1}^{\alpha_2} \exp(\cos\varphi)\, d\varphi}{\pi|z|^{m+1}}$$

$$\le \frac{(e^{\alpha_1 \operatorname{Im}\mu} + e^{\alpha_2 \operatorname{Im}\mu})}{\pi|z|^{m+1}} \left(\frac{\varkappa(z)A}{2} + \int\limits_{\alpha_1}^{\alpha_2} e^{\cos\varphi}\, d\varphi \right).$$

Lemma 1.3.1 is completely proved. □

Remark 1.3.3. As a rule, the case $b \geq 0$ is more interesting than the case $b < 0$ since we can arrive at it setting $m \geq \rho\mu - 1$. In view of this fact, we estimated the remainder for $b \geq 0$ more carefully, and in the case $b < 0$ we made a slight roughening that allowed us to simplify the proof.

In conclusion of this section, we note that we need the integrals $J_\rho(z, \mu, \alpha_1, \alpha_2)$ since they appear in the study of Mittag-Leffler functions. In the sequel, we choose α_1 and α_2 appropriately depending on the parameter ρ and the argument of the variable z. As for the "abstract theoretic" problem on the asymptotic expansion of $J_\rho(z, \mu, \alpha_1, \alpha_2)$ for $z \to \infty$ in the domain $\mathbb{C} \setminus (\ell_1 \cup \ell_2)$, we can make the following remarks based on Lemma 1.3.1.

1. In any closed angle $\mathcal{L}$ lying inside any of open angles formed by the union of the rays $\ell_1 \cup \ell_2$, the function $\delta(z)$ is nonzero and hence

$$\sup_{z \in \mathcal{L}} \varkappa(z) = C(\mathcal{L}) < +\infty. \tag{1.3.15}$$

Then Lemma 1.3.1 implies the complete asymptotic expansion

$$J_\rho(z, \mu, \alpha_1, \alpha_2) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(\mu - k/\rho)}, \quad z \in \mathcal{L}, \quad z \to \infty, \tag{1.3.16}$$

in the sense of the formulas

$$J_\rho(z, \mu, \alpha_1, \alpha_2) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + O(|z|^{-m-1}), \quad z \in \mathcal{L}, \quad z \to \infty, \tag{1.3.17}$$

for any $m \in \mathbb{N}$.

2. In fact, Lemma 1.3.1 allows one to prove formulas (1.3.16) and (1.3.17) for a wider class of domains, for example, for "inner parts" of hyperbolas whose asymptotes are the rays ℓ_1 and ℓ_2 or domains bounded by curves each of whose point z is located at a distance not less than $C|z|^{-p}$ from $\ell_1 \cup \ell_2$ (here C and p are fixed positive constants). Then $\varkappa(z) = O(|z|^{p+1})$ ($z \to \infty$, $z \in G$) and, taking in the asymptotic series $m + p + 1$ terms, we obtain (1.3.17).

1.4. Asymptotic Expansions of Mittag-Leffler functions of Order $\rho \geq 3/4$

We denote by $\mathcal{L}_\rho$, $\rho > 1$, the angle

$$\mathcal{L}_\rho = \{z \in \mathbb{C} \mid z \neq 0, \ |\arg z| \leq \pi/\rho\}.$$

Theorem 1.4.1. *For any $\rho > 1$, $\mu \in \mathbb{C}$, and $m \in \mathbb{N}$, the following asymptotics hold.*

(1) *If $z \in \mathcal{L}_\rho$, then*

$$E_\rho(z; \mu) = \rho z^{\rho(1-\mu)} \exp(z^\rho) - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + R_m^{[1]}(z; \rho, \mu). \tag{1.4.1}$$

The remainder $R_m^{[1]}$ admits the estimate

$$|R_m^{[1]}(z; \rho, \mu)| \leq \begin{cases} \dfrac{2^{\frac{b+2}{2}} \Gamma(b+1) \rho \exp\left(\frac{5\pi}{4} |\operatorname{Im} \mu|\right)}{\pi |z|^{m+1}}, & \text{if } \ b = \dfrac{m+1}{\rho} - \operatorname{Re} \mu \geq 0, \\ \dfrac{(6 + 2\rho/\pi) \exp\left(\frac{5\pi}{4} |\operatorname{Im} \mu|\right)}{|z|^{m+1}}, & \text{if } \ b < 0. \end{cases} \tag{1.4.2}$$

The first of estimates (1.4.2) is valid for all $z \in \mathcal{L}_\rho$ and the second under the additional condition $|z| \geq 2$.

(2) *If* $z \in \mathbb{C} \setminus \mathcal{L}_\rho$, $z \neq 0$, *then*

$$E_\rho(z;\mu) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + R_m^{[2]}(z;\rho,\mu). \tag{1.4.3}$$

The remainder $R_m^{[2]}$ *admits the estimate*

$$|R_m^{[2]}(z;\rho,\mu)| \le \begin{cases} \dfrac{2^{\frac{b+2}{2}}\Gamma(b+1)\rho \exp\left(\frac{3\pi}{4}|\operatorname{Im}\mu|\right)}{\pi|z|^{m+1}}, & \text{if } b \ge 0, \\ \dfrac{(6+2\rho/\pi)\exp\left(\frac{3\pi}{4}|\operatorname{Im}\mu|\right)}{|z|^{m+1}}, & \text{if } b < 0. \end{cases} \tag{1.4.4}$$

The first of estimates (1.4.4) *is valid for all* $z \in \mathbb{C} \setminus \mathcal{L}_\rho$ *and the second under the additional condition* $|z| \ge 2$.

Proof. First, we consider the case where $z \in \mathcal{L}_\rho$ and $\operatorname{Im} z \ge 0$ (the case where $\operatorname{Im} z < 0$ is examined similarly). Apply Theorem 1.1.1 setting $\alpha_1 = -3\pi/4$ and $\alpha_2 = 5\pi/4$. On the Riemann surface $\mathfrak{P}$ of the variable ζ, the integrand has only simple poles

$$\zeta_n = z^\rho \exp(2\pi i n\rho), \quad n \in \mathbb{Z}. \tag{1.4.5}$$

The residues of the function $\zeta^{1/\rho-\mu}e^\zeta(\zeta^{1/\rho} - z)^{-1}$ at these points are equal to

$$\operatorname{Res}\left(\frac{\zeta^{1/\rho-\mu}e^\zeta}{\zeta^{1/\rho}-z}\right)\Bigg|_{\zeta=\zeta_n} = \lim_{\zeta\to\zeta_n}\frac{\zeta^{1/\rho-\mu}e^\zeta(\zeta-\zeta_n)}{\zeta^{1/\rho}-z} = \zeta_n^{1/\rho-\mu}e^{\zeta_n}\lim_{\zeta\to\zeta_n}\frac{\zeta-\zeta_n}{\zeta^{1/\rho}-z}$$

$$= \zeta_n^{1/\rho-\mu}e^{\zeta_n}\lim_{\zeta\to\zeta_n}\frac{1}{\frac{1}{\rho}\zeta^{1/\rho-1}} = \rho\zeta_n^{1-\mu}e^{\zeta_n}. \tag{1.4.6}$$

By the restriction

$$0 \le \arg z \le \pi/\rho \iff 0 \le \arg z^\rho \le \pi, \tag{1.4.7}$$

only one pole ζ_0 of the poles (1.4.5) lies on the part of the Riemann surface

$$\{\zeta \in \mathfrak{P} \mid -3\pi/4 \le \operatorname{Arg}\zeta \le 5\pi/4\}$$

containing the integration contour $\gamma(\sigma, -3\pi/4, 5\pi/4)$. Indeed, (1.4.7) implies that

$$\operatorname{Arg}\zeta_1 = 2\pi\rho + \arg z^\rho \ge 2\pi\rho \ge 2\pi, \quad \operatorname{Arg}\zeta_{-1} = -2\pi\rho + \arg z^\rho \le \pi - 2\pi\rho < \pi - 2\pi = -\pi.$$

Therefore, by Theorem 1.1.1 and the residue theorem, we have the relation

$$E_\rho(z;\mu) = \frac{1}{2\pi i}\int_{\gamma(\sigma,-\frac{3\pi}{4},\frac{5\pi}{4})}\frac{\zeta^{1/\rho-\mu}e^\zeta\,d\zeta}{\zeta^{\frac{1}{\rho}}-z}\operatorname{Res}\left(\frac{\zeta^{1/\rho-\mu}e^\zeta}{\zeta^{1/\rho}-z}\right)\Bigg|_{\zeta=z^\rho}$$

$$+\frac{1}{2\pi i}\int_{\gamma(\sigma',-\frac{3\pi}{4},\frac{5\pi}{4})}\frac{\zeta^{1/\rho-\mu}e^\zeta\,d\zeta}{\zeta^{1/\rho}-z}, \quad 0 < \sigma' < |z|^\rho. \tag{1.4.8}$$

From (1.4.8), (1.4.6), and (1.3.1) we obtain the representation

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)}\exp(z^\rho) + J_\rho\left(z;\mu,-\frac{3\pi}{4},\frac{5\pi}{4}\right).$$

The relation obtained and Lemma 1.3.1 immediately imply representation (1.4.1), in which we must estimate the remainder. The corresponding estimates have been proved in Lemma 1.3.1 in the general case. To apply them in this situation, we must calculate or estimate from above the constants A,

$$I = \int_{\alpha_1}^{\alpha_2} \exp(\cos\varphi)\, d\varphi = \int_{-3\pi/4}^{5\pi/4} \exp(\cos\varphi)\, d\varphi,$$

and

$$\varkappa = \sup\{\varkappa(z) \mid z \in \mathcal{L}_\rho,\ \operatorname{Im} z \geq 0\}.$$

We have

$$A = \max\left(|\cos^{-1}(-3\pi/4)|, \|\cos^{-1}(5\pi/4)|\right) = \sqrt{2}, \tag{1.4.9}$$

$$I = \int_{-\pi}^{\pi} \exp(\cos\varphi)\, d\varphi = 2\int_{-\pi/2}^{\pi/2} \cosh(\cos\varphi)\, d\varphi < 2\int_{-\pi/2}^{\pi/2} (1+\cos^2\varphi)\, d\varphi = 3\pi. \tag{1.4.10}$$

To calculate the constant $\varkappa$, we first find the minimum of the angular distance between $z \in \mathcal{L}_\rho$, $\operatorname{Im} z \geq 0$, and the rays

$$\ell_1 = \left\{r\exp\Big(-\frac{3\pi i}{4\rho}\Big),\ r>0\right\}, \quad \ell_2 = \left\{r\exp\Big(\frac{5\pi i}{4\rho}\Big),\ r>0\right\}.$$

Among the intersection points of the angle $\mathcal{L}_\rho$ with the upper half-plane, the closest to the ray ℓ_1 (in the sense of the angular distance) can be either the points $\mathbb{R}_+$, for which the angle is equal to $3\pi/(4\rho)$, or the points of the ray

$$\left\{r\exp\left(\frac{\pi i}{\rho}\right),\ r>0\right\},$$

for which the angle is equal to

$$\left(\pi - \frac{\pi}{\rho}\right) + \left(\pi - \frac{3\pi}{4\rho}\right) = 2\pi - \frac{7\pi}{4\rho} > \frac{\pi}{4}.$$

Similar angular distances for the ray ℓ_2 are equal to

$$2\pi - \frac{5\pi}{4\rho} > 2\pi - \frac{5\pi}{4} \geq \frac{3\pi}{4\rho}, \quad \frac{5\pi}{4\rho} - \frac{\pi}{\rho} = \frac{\pi}{4\rho},$$

respectively. Therefore, for all $z \in \mathcal{L}_\rho \cap \{\operatorname{Im} z \geq 0\}$, the angle $\delta(z)$ is not less than $\pi/(4\rho)$ and

$$\varkappa \leq \frac{1}{\sin(\frac{\pi}{4\rho})} = \frac{4\rho}{\pi}\cdot\frac{t}{\sin t},$$

where $t = \pi/(4\rho)$. Since the function $t/\sin t$ increases on $(0, \pi/4]$ and is equal to $\pi/\sqrt{8}$ at the point $t = \pi/4$, we have

$$\varkappa \leq \rho\sqrt{2}. \tag{1.4.11}$$

From (1.4.9) and (1.4.11) and the estimates of the remainder from Lemma 1.1.1 we immediately obtain inequalities (1.4.2). For the case $\operatorname{Im} z < 0$, we must set $\alpha_1 = -5\pi/4$ and $\alpha_2 = 3\pi/4$ and use the same arguments. The first part of Theorem 1.4.1 is proved.

Now we prove the second part of the theorem. Now

$$z = re^{i\theta}, \quad r > 0, \quad \frac{\pi}{\rho} \leq |\theta| \leq \pi.$$

We apply the integral representation (1.1.1) with $\alpha_1 = -3\pi/4$ and $\alpha_2 = 3\pi/4$. For the considered values of z, the angle

$$-\frac{3\pi}{4} \le \operatorname{Arg}\zeta \le \frac{3\pi}{4} \tag{1.4.12}$$

does not contain poles of the integrand since under condition (1.4.12), the variable $\zeta^{1/\rho}$ takes its values in the angle $\mathcal{L}_{3\rho/4} \subset \mathcal{L}_\rho$. Therefore, we can arbitrarily change the parameter σ of the Hankel loop $\gamma(\sigma, -3\pi/4, 3\pi/4)$ in integral (1.1.1). Taking $\sigma \in (0, |z|^\rho)$, we obtain the representation

$$E_\rho(z;\mu) = J_\rho\left(z;\mu,-\frac{3\pi}{4},\frac{3\pi}{4}\right), \quad z \in \mathbb{C}\setminus\mathcal{L}_\rho, \quad z \neq 0.$$

The asymptotic formula for the integral J_ρ proved in Lemma 1.3.1 yields expansion (1.4.3) and estimates (1.4.4) since here, as in the first part of the theorem, $A = \sqrt{2}$ and $\varkappa \le \rho\sqrt{2}$. The last inequality follows from the fact that

$$\sup\left\{\delta(z) \;\middle|\; z \in \mathbb{C}\setminus\mathcal{L}_\rho,\ z \neq 0\right\} = \frac{\pi}{4\rho}$$

(the definition of $\delta(z)$ is stated before Lemma 1.3.1). An improvement in inequality (1.4.4) compared with (1.4.2) is due to the exact relation

$$\exp\left(-\frac{3\pi}{4}\operatorname{Im}\mu\right) + \exp\left(\frac{3\pi}{4}\operatorname{Im}\mu\right) = 2\cosh\left(\frac{5\pi}{4}\operatorname{Im}\mu\right),$$

which we used instead of the estimate

$$\exp\left(-\frac{3\pi}{4}\operatorname{Im}\mu\right) + \exp\left(\frac{5\pi}{4}\operatorname{Im}\mu\right) \le 2\exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right)$$

in the first part of the theorem. Theorem 1.4.1 is completely proved. □

Theorem 1.4.2. *For any $\rho \in [3/4, 1]$, $\mu \in \mathbb{C}$, $m \in \mathbb{N}$, and $z \in \mathbb{C}$, $z \neq 0$, the following relation holds:*

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)}\exp(z^\rho) - \sum_{k=1}^{m}\frac{z^{-k}}{\Gamma(\mu - k/\rho)} + R_m(z;\rho,\mu), \tag{1.4.13}$$

in which the remainder R_m admits the estimate

$$|R_m(z;\rho,\mu)| \le \begin{cases} \dfrac{2^{\frac{b+2}{2}}\Gamma(b+1)\exp(\frac{5\pi}{4}|\operatorname{Im}\mu|)}{|z|^{m+1}}, & \text{if } b = \dfrac{m+1}{\rho} - \operatorname{Re}\mu \ge 0, \\ 8|z|^{-m-1}\exp\left(\dfrac{5\pi}{4}|\operatorname{Im}\mu|\right), & \text{if } b < 0. \end{cases} \tag{1.4.14}$$

The second estimate in (1.4.14) *is valid under the additional condition* $|z| \ge 2$.

Remark 1.4.1. The first term of asymptotics (1.4.13) is discontinuous on the ray $(-\infty, 0)$, but this does not leads to a contradiction since in the angle $5\pi/6 \le |\arg z| \le \pi$ it tends to zero as $|z| \to +\infty$ faster than any negative power of $|z|$ and goes to the remainder.

Remark 1.4.2. Representation (1.4.13) with the remainder

$$R_m(z;\rho,\mu) \le C(\rho,\mu,m)(|z|^{-m-1}), \quad |z| \to \infty,$$

is valid for all $\rho \in (1/2, 1]$; this was proved by Dzhrbashyan (see [6]), who was not interested in the dependence of the function $C(\rho,\mu,m)$ on the parameters ρ and μ. We restrict ourselves to the case $\rho \in [3/4, 1]$ since for $C(\rho,\mu,m)$ we cannot obtain an upper estimate better than $\exp(C_1/(\rho - 0.5))$, $C_1 > 0$, when ρ tends to 1/2. For $\rho < 3/4$, it is better to use another asymptotics, which will be introduced in the following section.

Proof of Theorem 1.4.2. We prove Theorem 1.4.2 for z lying in the upper half-plane. We use the integral representation (1.1.1) of the Mittag-Leffler function taking $\alpha_1 = -2\pi/3$ and $\alpha_2 = 5\pi/4$. If $\operatorname{Im} z < 0$, then we take $\alpha_1 = -5\pi/4$ and $\alpha_2 = 2\pi/3$; in other respects, we operate similarly, up to the symmetry with respect to $\mathbb{R}$.

It is easy to verify that in the angle on the Riemann surface $\mathfrak{P}$ made by the rays of the chosen loop, there are no poles of the integrand except for $\zeta_0 = z^\rho$. Indeed,

$$\operatorname{Arg}\zeta_1 = \operatorname{Arg} z^\rho + 2\pi\rho \geq 2\pi\rho \geq \frac{2\pi\cdot 3}{4} = \frac{3\pi}{2} > \frac{5\pi}{4},$$

$$\operatorname{Arg}\zeta_{-1} = \operatorname{Arg} z^\rho - 2\pi\rho \leq \pi\rho - 2\pi\rho = -\pi\rho \leq -\frac{3\pi}{4} < -\frac{2\pi}{3}.$$

Therefore, adding the residue at the point ζ_0 (we have calculated this residue in the proof of the previous theorem), we pass to integration along the loops $\gamma(\sigma, -2\pi/3, 5\pi/4)$, where $0 < \sigma < |z|^\rho$. Therefore, we obtain the representation

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)}\exp(z^\rho) + J_\rho\left(z, \mu, -\frac{2\pi}{3}, \frac{5\pi}{4}\right). \tag{1.4.15}$$

Relation (1.4.15) and Lemma 1.3.1 yield (1.4.13), and estimate (1.4.14) is obtained after calculating A and $\varkappa = \sup\{\varkappa(z) \mid \operatorname{Im} z \geq 0\}$. In this case,

$$A = \max\left(|\cos(-2\pi/3)|^{-1}, |\cos(5\pi/4)|^{-1}\right) = 2. \tag{1.4.16}$$

The minimal angle between the points of the upper half-plane and the rays

$$\left\{r\exp\left(\frac{5\pi i}{4\rho}\right),\ r > 0\right\}, \quad \left\{r\exp\left(-\frac{2\pi i}{3\rho}\right),\ r > 0\right\}$$

is equal to

$$\min\left(\frac{5\pi}{4\rho} - \pi,\ 2\pi - \frac{5\pi}{4\rho},\ \pi - \frac{2\pi}{3\rho}\right). \tag{1.4.17}$$

Function (1.4.17) on the segment $3/4 \leq \rho \leq 1$ attains the minimal value $\pi/9$. Therefore,

$$\varkappa \leq \frac{1}{\sin(\pi/9)} < \pi. \tag{1.4.18}$$

From (1.4.18), (1.4.16), (1.3.3), and (1.3.4) we obtain inequality (1.4.14). Theorem 1.4.2 is proved.. □

Of special interest are Mittag-Leffler functions of order $\rho = 1$:

$$E_1(z;\mu) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k)} = \frac{1}{\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{z^k}{(\mu)_k}, \tag{1.4.19}$$

where

$$(\mu)_k = \prod_{j=0}^{k-1}(\mu+j), \quad k\in\mathbb{N}, \qquad (\mu)_0 = 1.$$

They are not generalized hypergeometric functions

$${}_pF_q(z;\alpha_1,\alpha_2,\dots,\alpha_p;\beta_1,\beta_2,\dots,\beta_q) = \sum_{k=0}^{\infty}\frac{z^k}{k!}\cdot\frac{\prod\limits_{j=1}^{p}(\alpha_j)_k}{\prod\limits_{j=1}^{p}(\beta_j)_k}. \tag{1.4.20}$$

Indeed, from (1.4.19) and (1.4.20) we see that

$$E_1(z;\mu) = \frac{1}{\Gamma(\mu)}\,{}_1F_1(z;1;\mu).$$

Mittag-Leffler functions of order 1 are closely related to the incomplete gamma-function

$$\gamma(\lambda, x) = \int_0^x t^{\lambda-1} e^{-t}\, dt$$

due to the identity

$$E_1(z;\mu) = \frac{1}{\Gamma(\mu-1)} \int_0^1 (1-t)^{\mu-2} e^{zt}\, dt, \quad \operatorname{Re}\mu > 1, \quad z \in \mathbb{C}, \tag{1.4.21}$$

which can be easily proved:

$$\begin{aligned}
\frac{1}{\Gamma(\mu-1)} \int_0^1 (1-t)^{\mu-2} e^{zt}\, dt &= \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu-1)} \int_0^1 (1-t)^{\mu-2} t^k\, dt \\
&= \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu-1)} B(\mu-1, k+1) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu-1)} \frac{\Gamma(\mu-1)\Gamma(k+1)}{\Gamma(\mu+k)} \\
&= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu+k)} = E_1(z,\mu).
\end{aligned}$$

From (1.4.21) we obtain

$$\begin{aligned}
E_1(z;\mu+1) &= \frac{1}{\Gamma(\mu)} \int_0^1 (1-t)^{\mu-1} e^{zt}\, dt = \frac{1}{\Gamma(\mu)} \int_0^1 \tau^{\mu-1} e^{z(1-\tau)}\, d\tau \\
&= \frac{e^z}{\Gamma(\mu)} \int_0^1 \tau^{\mu-1} e^{-\tau z}\, dz = \frac{e^z}{z^\mu \Gamma(\mu)} \int_0^z u^{\mu-1} e^{-u}\, du,
\end{aligned}$$

i.e.,

$$\gamma(\mu, z) = z^\mu \Gamma(\mu) e^{-z} E_1(z;\mu+1). \tag{1.4.22}$$

Thus, the question on zeros of the incomplete gamma-function in the complex plane is equivalent to the question on zeros of a Mittag-Leffler function of order 1.

Note that the functions $E_1(z;m)$, $m \in \mathbb{Z}$, are elementary:

$$E_1(z;m) = z^{1-m} e^z, \qquad m \le 1,\ m \in \mathbb{Z}, \tag{1.4.23}$$

$$E_1(z;m) = z^{1-m} e^z - \sum_{k=1}^{m-1} \frac{z^{-k}}{\Gamma(m-k)}, \qquad m \ge 2,\ m \in \mathbb{Z}. \tag{1.4.24}$$

Formula (1.4.24) seems invalid since its right-hand side has a singularity at the point $z = 0$. However, this is a removable singularity; for example,

$$E_1(z;2) = \frac{e^z - 1}{z}, \quad E_1(z;3) = \frac{e^z - 1 - z}{z^2}, \quad E_1(z;4) = \frac{e^z - 1 - z - z^2/2}{z^3}, \quad \text{etc.}$$

Identities (1.4.23) and (1.4.24) can be proved by calculating the coefficients of power series for both sides.

In the following theorem, we present an asymptotics of $E_1(z;\mu)$ for $\rho = 1$ outside the negative part of the real axis with a substantially more exact (outside a small neighborhood of $(-\infty, 0]$) estimate of the remainder than in (1.4.14). It is especially useful if the parameter μ is sufficiently close to an integer number. Further, we show that this estimate is very close to unimprovable; moreover, this estimate is closer to the best possible the farther the point z is from $(-\infty, 0]$.

We denote by $d(z)$ the distance from the point z to the ray $(-\infty, 0]$:

$$d(z) = \begin{cases} |\operatorname{Im} z|, & \operatorname{Re} z \le 0, \\ |z|, & \operatorname{Re} z > 0. \end{cases}$$

Theorem 1.4.3. *For any $\mu \in \mathbb{C}$, $m \in \mathbb{N}$, $m \ge \operatorname{Re}\mu - 2$, and $z \notin (-\infty, 0]$, the following representation holds*:

$$E_1(z;\mu) = z^{1-\mu} e^z - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k)} + R_m(z;\mu); \tag{1.4.25}$$

here the remainder has the form

$$R_m(z;\mu) = \frac{(-1)^m \sin(\pi\mu)}{\pi z^{m+1}} \left(\Gamma(m+2-\mu) - \int_0^{+\infty} \frac{r^{m+2-\mu} e^{-r}}{r+z}\, dr \right) \tag{1.4.26}$$

and admits the following estimate:

$$|R_m(z;\mu)| \le \frac{|\sin \pi\mu| \Gamma(b+1)}{\pi |z|^{m+1}} \left(1 + \frac{b+1}{d(z)} \right), \tag{1.4.27}$$

where $b = m + 1 - \operatorname{Re}\mu$.

Proof. By Theorem 1.1.1,

$$E_1(z;\mu) = \frac{1}{2\pi i} \int_{\gamma(\sigma,-\pi,\pi)} \frac{\zeta^{1-\mu} e^{\zeta}\, d\zeta}{\zeta - z}, \quad |z| < \sigma.$$

Now we can forget about the Riemann surface $\mathfrak{P}$ and assume that the integration variable ζ belongs to the complex plane with the cut $\mathbb{C} \setminus (-\infty, 0]$. The integrand has a unique singularity (encompassed by the contour) on $\mathbb{C} \setminus (-\infty, 0]$, namely, a simple pole at the point $\zeta = z$, with residue equal to $z^{1-\mu} e^z$. Therefore, we can bypass this pole by taking the integral over the loop $\gamma(\sigma', -\pi, \pi)$ with parameter $\sigma' < |z|$ and, adding the residue, obtain the representation

$$E_1(z;\mu) = z^{1-\mu} e^z + \frac{1}{2\pi i} \int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{1-\mu} e^{\zeta}\, d\zeta}{\zeta - z} \tag{1.4.28}$$

(for definiteness, we can set $\sigma' = |z|/2$). Further, we use the identity

$$\frac{1}{\zeta - z} = -\sum_{k=1}^{m+1} \frac{\zeta^{k-1}}{z^k} + \frac{\zeta^{m+1}}{z^{m+1}(\zeta - z)}. \tag{1.4.29}$$

From (1.4.28) and (1.4.29) we obtain the relation

$$E_1(z;\mu) = z^{1-\mu} e^z - \sum_{k=1}^{m+1} \frac{z^{-k}}{2\pi i} \int_{\gamma(\sigma',-\pi,\pi)} \zeta^{k-\mu} e^{\zeta}\, d\zeta + \frac{z^{-m-1}}{2\pi i} \int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{m+2-\mu} e^{\zeta}\, d\zeta}{\zeta - z},$$

which, together with Lemma 1.1.1 yields

$$E_1(z;\mu) = z^{1-\mu} e^z - \sum_{k=1}^{m+1} \frac{z^{-k}}{\Gamma(\mu - k)} + \frac{z^{-m-1}}{2\pi i} \int_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{m+2-\mu} e^{\zeta}}{\zeta - z}\, d\zeta.$$

Introducing the notation

$$R_m(z;\mu) = \frac{z^{-m-1}}{\Gamma(\mu-m-1)} + \frac{z^{-m-1}}{2\pi i} \int\limits_{\gamma(\sigma',-\pi,\pi)} \frac{\zeta^{m+2-\mu} e^{\zeta}\, d\zeta}{\zeta - z}\, d\zeta,$$

we can rewrite the last relation in the following form:

$$E_1(z;\mu) = z^{1-\mu} e^z - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu-k)} + R_m(z;\mu).$$

Thus, representation (1.4.25) is proved. Now we show that the remainder $R_m(z;\mu)$ has the form (1.4.26) and admits estimate (1.4.27).

We transform $R_m(z;\mu)$. First, we apply identity (1.1.5):

$$\frac{1}{\Gamma(\mu-m-1)} = \frac{\sin\pi(\mu-m-1)}{\pi}\Gamma(m+2-\mu).$$

Further, we pass from integration over $\gamma(\sigma,-\pi,\pi)$ to integration over $\gamma(0,-\pi,\pi)$, i.e., over the union of the lower and upper sides of the cut $(-\infty,0]$ (this is possible since $m+2-\operatorname{Re}\mu \geq 0$). We obtain the relation

$$R_m(z;\mu) = \frac{(-1)^m \sin(\pi\mu)\Gamma(m+2-\mu)}{\pi z^{m+1}}$$
$$+ \frac{z^{-m-1}}{2\pi i}\left(\int\limits_0^{+\infty} \frac{(-r)^{m+2}(re^{-\pi i})^{-\mu} e^{-r}\, d(-r)}{-r-z} + \int\limits_0^{+\infty} \frac{(-r)^{m+2}(re^{\pi i})^{-\mu} e^{-r}\, d(-r)}{-r-z} \right)$$
$$= \frac{(-1)^m \sin(\pi\mu)\Gamma(m+2-\mu)}{\pi z^{m+1}} - \frac{z^{-m-1}(-1)^m}{2\pi i}\left(\int\limits_0^{+\infty} \frac{r^{m+2-\mu} e^{\pi i\mu - r}\, dr}{r+z} - \frac{r^{m+2-\mu} e^{-\pi i\mu - r}\, dr}{r+z} - \right)$$
$$= \frac{(-1)^m \sin(\pi\mu)}{\pi z^{m+1}} \left(\Gamma(m+2) - \int\limits_0^{+\infty} \frac{r^{m+2-\mu} e^{-r}\, dr}{r+z} \right). \quad (1.4.30)$$

Relation (1.4.26) is proved. Now we obtain estimate (1.4.27). Since $|r+z| \geq d(z)$ for all $r > 0$, we have the relation

$$\left| \int\limits_0^{+\infty} \frac{r^{m+2-\mu} e^{-r}\, dr}{r+z} \right| \leq \frac{1}{d(z)} \int\limits_0^{+\infty} r^{m+2-\operatorname{Re}\mu} e^{-r} dr = \frac{\Gamma(m+3-\operatorname{Re}\mu)}{d(z)}. \quad (1.4.31)$$

Recalling the notation $b = m+1-Re\,\mu$, we obtain from (1.4.30) and (1.4.31)[3]

$$|R_m(z;\mu)| \leq \frac{|\sin(\pi\mu)|}{\pi |z|^{m+1}} \left(\Gamma(b+1) + \frac{1}{d(z)}\Gamma(b+2) \right) = \frac{|\sin(\pi\mu)|\Gamma(b+1)}{\pi |z|^{m+1}} \left(1 + \frac{b+1}{d(z)} \right).$$

The theorem is proved.

Estimate (1.4.27) is quite exact. We show that if it can be improved for real μ, this is only due to the factor $1 + \dfrac{b+1}{d(z)}$. Indeed, the asymptotic expansion of the function $E_1(z;\mu)$ as $z \to \infty$ proved in Theorem 1.4.3 implies

$$R_m(z;\mu) \sim \frac{1}{\Gamma(\mu-m-1)z^{m+1}}.$$

[3]As in (1.4.31), we use the inequality $|\Gamma(w)| \leq \Gamma(\operatorname{Re} w)$, $\operatorname{Re} w > 0$, for an upper estimate of the modulus of the Γ-function.

This equivalence is valid if μ is not an integer not greater than $m+1$. In the opposite case, by (1.4.24), we have the relation $R_m(z,\mu)=0$. Since

$$\frac{1}{\Gamma(\mu-m-1)}=\frac{\sin(\pi\mu)(-1)^m\Gamma(m+2-\mu)}{\pi},$$

we obtain for $\mu\in\mathbb{R}$

$$R_m(z;\mu)\sim\frac{(-1)^m\sin(\pi\mu)\Gamma(b+1)}{\pi z^{m+1}},\quad z\to\infty. \tag{1.4.32}$$

Therefore, if we want to replace equivalence (1.4.32) by an estimate valid for all $z\notin(-\infty,0]$, we get the factor $1+\dfrac{b+1}{d(z)}$ in the estimate. $\square$

1.5. Asymptotic Expansion of Mittag-Leffler Functions of Order $\rho\le 3/4$

Theorem 1.5.1. *For any $\rho\le 3/4$, $\mu\in\mathbb{C}$, and $m\in\mathbb{N}$, the following representation holds*:

$$E_\rho(z;\mu)=\rho\sum_{|\arg z+2\pi n|\le\frac{3\pi}{4\rho}}(z^\rho e^{2\pi in\rho})^{1-\mu}\exp(z^\rho e^{2\pi in\rho})-\sum_{k=1}^{m}\frac{z^{-k}}{\Gamma(\mu-k/\rho)}+R_m(z,\rho,\mu),\quad z\ne 0. \tag{1.5.1}$$

The remainder admits the estimate

$$|R_m(z;\rho,\mu)|\le\begin{cases}\dfrac{2^{\frac{b+2}{2}}\Gamma(b+3/2)\exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right)}{|z|^{m+1}}, & \text{if } b=\dfrac{m+1}{\rho}-\operatorname{Re}\mu\ge 0,\\ 7.1|z|^{-m-1}\exp\left(\dfrac{5\pi}{4}|\operatorname{Im}\mu|\right), & \text{if } b<0.\end{cases} \tag{1.5.2}$$

The first of estimates (1.5.2) *is valid for all $z\ne 0$ and the second for $|z|\ge 2$.*

Proof. We choose the parameters α_1 and α_2 defining the Hankel loop in the integral representation (1.1.1) depending on $\theta=\arg z$. The problem is to find numbers

$$\alpha_1\in\left[-\frac{5\pi}{4},-\frac{3\pi}{4}\right],\quad \alpha_2\in\left[\frac{3\pi}{4},\frac{5\pi}{4}\right] \tag{1.5.3}$$

such that the angles between the rays $\{r\exp(i\theta)\mid r>0\}$ and $\{r\exp(i\alpha_1/\rho)\mid r>0\}$ and between the rays $\{r\exp(i\theta)\mid r>0\}$ and $\{r\exp(i\alpha_2/\rho)\mid r>0\}$ will be as large as possible. These angles are equal to

$$\min_{p\in\mathbb{Z}}\left|\theta+2\pi p-\frac{\alpha_j}{\rho}\right|,\quad j=1,2. \tag{1.5.4}$$

To solve this problem, we use the following simple fact. If the β runs over a segment ℓ of the real axis, $|\ell|\le 2\pi$, then

$$\max_{\beta\in\ell}\min_{p\in\mathbb{Z}}|2\pi p-\beta|=\frac{|\ell|}{2} \tag{1.5.5}$$

(we denote by $|\ell|$ the length of the segment ℓ). The proof of relation (1.5.5) is left to the reader. Since in Eqs. (1.5.4) the variable $\theta-\alpha_j/\rho$, due to (1.5.3), runs [4] over a segment of length $\dfrac{\pi}{2\rho}\ge\dfrac{2\pi}{3}$ for $0<\rho\le 3/4$, we see that there exist numbers $\alpha_1(\theta)$ and $\alpha_2(\theta)$ satisfying inclusions (1.5.3) such that angles (1.5.4) are not less than $\pi/3$. Thus, we conclude that for the chosen Hankel loop $\gamma(\sigma,\alpha_1(\theta),\alpha_2(\theta))$, we have the following inequality for $\varkappa(z)$ (see its definition before Lemma 1.3.1):

$$\varkappa(z)\le\frac{1}{\sin(\pi/3)}=\frac{2}{\sqrt{3}}. \tag{1.5.6}$$

[4] Here θ and p are fixed and α_j vary within the limits indicated in (1.5.3).

From (1.5.3) we also obtain that

$$A = \max(|\cos\alpha_1|^{-1}, |\cos\alpha_2|^{-1}) \le \sqrt{2}. \tag{1.5.7}$$

Here the integrand in the angle of the Riemann surface of the argument $\alpha_1 \le \operatorname{Arg}\zeta \le \alpha_2$ may have sufficiently many poles $\{\zeta_n\}_{n\in\mathbb{N}}$ that were indicated in (1.4.4). We denote by $\mathcal{A}(z,\rho)$ the set of all integers n for which the inclusion $\operatorname{Arg}\zeta_n \in [\alpha_1, \alpha_2]$ holds. Obviously, this set is finite and

$$n \in \mathcal{A}(z,\rho) \quad \Longleftrightarrow \quad \frac{\alpha_1(\theta)}{\rho} \le \theta + 2\pi n \le \frac{\alpha_2(\theta)}{\rho}. \tag{1.5.8}$$

We transform integral (1.1.1) moving the contour behind the poles and adding the residues at these poles (they were calculated in (1.4.6). We obtain the relation

$$E_\rho(z;\mu) = \rho \sum_{n\in\mathcal{A}(z,\rho)} (z^\rho e^{2\pi in\rho})^{1-\mu} \exp(z^\rho e^{2\pi in\rho}) + J_\rho(z,\mu,\alpha_1(\theta),\alpha_2(\theta)). \tag{1.5.9}$$

By Lemma 1.3.1 and estimates (1.5.3), (1.5.6), and (1.5.7), we have

$$J_\rho(z;\mu,\alpha_1(\theta),\alpha_2(\theta)) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - k/\rho)} + U_m(z;\rho,\mu), \tag{1.5.10}$$

where

$$|U_m(z;\rho,\mu)| \le \begin{cases} \dfrac{2^{\frac{b+2}{2}}\Gamma(b+1)\exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right)}{\pi|z|^{m+1}}, & \text{if } b = \dfrac{m+1}{\rho} - \operatorname{Re}\mu \ge 0, \\ 6|z|^{-m-1}\exp\left(\dfrac{5\pi}{4}|\operatorname{Im}\mu|\right), & \text{if } b < 0. \end{cases} \tag{1.5.11}$$

We explain the appearance of the factor 6 in the second estimate (1.5.11). We apply Lemma 1.3.1 and the inequality

$$e^{\alpha_1 \operatorname{Im}\mu} + e^{\alpha_2 \operatorname{Im}\mu} \le e^{\alpha|\operatorname{Im}\mu|}, \quad \alpha = \max(|\alpha_1|, |\alpha_2|).$$

For $b < 0$, we obtain the estimate

$$|U_m(z;\rho,\mu)| \le \frac{2}{\pi}\left(\frac{\varkappa(z)A}{2} + \int_{\alpha_1}^{\alpha_2} e^{\cos\theta}\,d\theta\right)|z|^{-m-1}e^{\alpha|\operatorname{Im}\mu|} \le C|z|^{-m-1}e^{\alpha|IM\mu|}. \tag{1.5.12}$$

In our case, $\alpha \le 5\pi/4$, by (1.5.6) and (1.5.7), we have the inequality $\varkappa(z)A/2 \le \sqrt{2/3}$. Therefore, inequality (1.5.12) is valid if we take

$$C = \frac{2}{\pi}\left(\sqrt{\frac{2}{3}} + \int_{-\frac{5\pi}{4}}^{\frac{5\pi}{4}} e^{\cos\theta}\,d\theta\right) < 6.$$

To complete the proof of the theorem, we pass from the sum $\sum\limits_{n\in\mathcal{A}(z,\rho)}$ in (1.5.9) to the sum $\sum\limits_{|\theta+2\pi n|\le\frac{3\pi}{4\rho}}$.

By (1.5.3) and (1.5.8), the set $\mathcal{A}(z,\rho)$ contains all values of $n \in \mathbb{N}$ for which $|\theta + 2\pi n| \le 3\pi/(4\rho)$ and can also contain n satisfying the inequality

$$\frac{3\pi}{4\rho} < |\theta + 2\pi n| \le \frac{5\pi}{4\rho}. \tag{1.5.13}$$

Therefore, if we pass from one sum to the other, we see that the error does not exceed

$$V_m(z;\rho,\mu) = \rho \sum_{\frac{3\pi}{4\rho}<|\theta+2\pi n|\le\frac{5\pi}{4\rho}} \left|\left(z^\rho e^{2\pi in\rho}\right)^{1-\mu}\exp\left(z^\rho e^{2\pi in\rho}\right)\right| \tag{1.5.14}$$

and hence the remainder $R_m(z;\rho,\mu)$ in (1.5.1) admits the estimate

$$|R_m(z;\rho,\mu)| \le |U_m(z;\rho,\mu)| + V_m(z;\rho,\mu). \tag{1.5.15}$$

Now we estimate $V_m R$ from above. The number of terms in sum (1.5.14) (i.e., the number of integers n satisfying (1.5.13)) does not exceed

$$2\left(\frac{1}{2\pi}\left[\frac{5\pi}{4\rho} - \frac{3\pi}{4\rho}\right] + 1\right) \le 2\left(\frac{1}{4\rho} + 1\right) \le 2 + \frac{1}{2\rho}.$$

Therefore, taking into account the relation $z^\rho = |z|^\rho \exp(i\theta\rho)$, we obtain the inequality

$$V_m \le (0.5 + 2\rho)|z|^{\rho(1-\operatorname{Re}\mu)} \max_{\frac{3\pi}{4\rho} < |\theta + 2\pi n| \le \frac{5\pi}{4\rho}} \left| \exp\left((2\pi n + \theta)i\rho(1-\mu) + |z|^\rho e^{(2\pi n+\theta)i\rho}\right)\right|. \tag{1.5.16}$$

Using the obvious relations

$$|\exp w| = \exp(\operatorname{Re}\, w), \quad w \in \mathbb{C}, \quad 0.5 + 2\rho \le 2, \quad 0 < \rho \le 0.75,$$

we obtain from (1.5.16)

$$V_m(z,\rho,\mu) \le 2|z|^{\rho(1-\operatorname{Re}\mu)} \max_{\frac{3\pi}{4\rho} < |\theta + 2\pi n| \le \frac{5\pi}{4\rho}} \left| \exp\left((2\pi n + \theta)\rho \operatorname{Im}\mu + |z|^\rho \cos((2\pi n + \theta)\rho)\right)\right|. \tag{1.5.17}$$

Owing to (1.5.13) we have the inequality

$$\cos(2\pi n + \theta)\rho \le -1/\sqrt{2}, \quad (2\pi n + \theta)\rho \operatorname{Im}\mu \le \frac{5\pi}{4}|\operatorname{Im}\mu|.$$

Therefore, from (1.5.17) we obtain

$$\begin{aligned} V_m(z;\rho,\mu) &\le 2|z|^{\rho(1-\operatorname{Re}\mu)} \exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu| - \frac{|z|^\rho}{\sqrt{2}}\right) \\ &= 2|z|^{-m-1} \exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right) \cdot |z|^\rho \left(|z|^{m+1-\rho\operatorname{Re}\mu} \exp\left(-\frac{|z|^\rho}{\sqrt{2}}\right)\right) \\ &= 2|z|^{-m-1} \exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right) t^{b+1} \exp\left(-\frac{t}{\sqrt{2}}\right), \end{aligned} \tag{1.5.18}$$

where

$$t = |z|^\rho, \quad b = \frac{m+1}{\rho} - \operatorname{Re}\mu.$$

For $b \ge 0$, Eq. (1.5.18) implies

$$V_m(z,\rho,\mu) \le \frac{2M(b)\exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right)}{|z|^{m+1}}, \tag{1.5.19}$$

where

$$M(b) = \max_{t\ge 0} t^{b+1} \exp\left(-\frac{t}{\sqrt{2}}\right) = 2^{\frac{b+1}{2}}(b+1)^{b+1}e^{-b-1}.$$

By the inequality[5]

$$a^a e^{-a} \le \frac{2\Gamma(a+0.5)}{\sqrt{\pi}\, e} < 0.45\,\Gamma(a+0.5) \quad \forall a \ge 1$$

we have the inequality

$$2M(b) \le 0.9 \cdot 2^{\frac{b+1}{2}}\Gamma(b+3/2). \tag{1.5.20}$$

[5]The inequality

$$a^a e^{-a}\sqrt{\pi}\, e \le 2\Gamma(a+0.5) \quad \Longleftrightarrow \quad H(a) \equiv a\ln a - a - \ln\Gamma(a+0.5) + \ln(\sqrt{\pi}\, e/2) \le 0$$

is proved as follows: $H(1) = 0$, $H'(a) = \ln a - \Psi(a+0.5) < 0$ (see Chap. 3, Sec. 3.4 below).

From (1.5.19) and (1.5.20) we finally obtain the inequality

$$V_m(z;\rho,\mu) \le \frac{0.9\Gamma(b+3/2)2^{\frac{b+1}{2}}\exp\left(\frac{5\pi}{4}|\operatorname{Im}\mu|\right)}{|z|^{m+1}}, \quad b \ge 0. \tag{1.5.21}$$

For $b < 0$, we consider only values of z for which $|z| \ge 2$ and hence $t \ge 1$. Therefore, $t^b \le 1$ and (1.5.18) implies

$$\begin{aligned} V_m(z;\rho,\mu) &\le 2|z|^{-m-1}\exp\left(\frac{5\pi|\operatorname{Im}\mu|}{4}\right)\cdot\max_{t\ge 1} t\ \exp\left(-\frac{t}{\sqrt{2}}\right) \\ &= \frac{2\sqrt{2}}{e}|z|^{-m-1}\exp\left(\frac{5\pi|\operatorname{Im}\mu|}{4}\right) < 1.1|z|^{-m-1}\exp\left(\frac{5\pi|\operatorname{Im}\mu|}{4}\right). \end{aligned} \tag{1.5.22}$$

From (1.5.22), the second inequality (1.5.11), and (1.5.15) we immediately obtain the second inequality (1.5.2).

To deduce the first inequality (1.5.2) from (1.5.15), (1.5.11), and (1.5.21), it suffices to verify the relation

$$\frac{\Gamma(b+1)}{\pi} + \frac{0.9\Gamma(b+3/2)}{\sqrt{2}} \le \Gamma(b+3/2) \quad \forall b \ge 0,$$

which can be rewritten in the following equivalent form:

$$\frac{\Gamma(b+1)}{\Gamma(b+3/2)}\cdot\frac{1}{\pi} + \frac{0.9}{\sqrt{2}} \le 1 \quad \forall b \ge 0. \tag{1.5.23}$$

Since the logarithmic derivative of the gamma-function increases, the ratio $\Gamma(b+1)/\Gamma(b+3/2)$ decreases and hence

$$\Gamma(b+1)/\Gamma(b+3/2) \le \Gamma(1)/\Gamma(3/2) = 2/\sqrt{\pi} \quad \forall b \ge 0.$$

This shows that to complete the proof of (1.5.23), it remains to verify the numeric inequality $2\pi^{-3/2} + 0.9\cdot 2^{-1/2} < 1$. Theorem 1.5.1 is proved. □

In the following theorem, we obtain a substantially more exact estimate than in Theorem 1.5.1, of the remainder in the asymptotic representation of the Mittag-Leffler function in the left half-plane in the case where the parameter is close to 1/2 and the ratio $|z/x|$ is close to 1.

Theorem 1.5.2. *For any* $\rho \in [2/5, 2/3]$, $\mu \in \mathbb{R}$, $m \in \mathbb{Z}_+$, $m \ge \rho\mu - 1$, *and* $z \in \mathbb{C}$, $\operatorname{Re} z > 0$, *the following representation holds*:

$$E_\rho(-z^{1/\rho};\mu) = 2\rho z^{1-\mu}\exp(z\cos\pi\rho)\cos\big(z\sin(\pi\rho) - \pi\rho(\mu-1)\big) + \sum_{k=1}^{m}\frac{(-1)^{k-1}z^{-k/\rho}}{\Gamma(\mu-k/\rho)} + \Omega_m(z;\rho,\mu), \tag{1.5.24}$$

in which the remainder admits the following estimate ($x = \operatorname{Re} z$):

$$|\Omega_m(z;\rho,\mu)| \le \frac{(|z|/x)^{1-\mu}\Gamma\big(1-\mu+\frac{m+1}{\rho}\big)}{\pi x^{\frac{m+1}{\rho}}}\left(\left|\sin\left(\pi(\mu-\tfrac{m+1}{\rho})\right)\right| + \frac{|\sin(\pi(\mu-\frac{m}{\rho}))|}{4\cos^2(\frac{\pi}{2\rho})}\right). \tag{1.5.25}$$

If the numbers $\sin\left(\pi\left(\mu - \dfrac{m+1}{\rho}\right)\right)$ *and* $\sin\left(\pi\left(\mu-\dfrac{m}{\rho}\right)\right)$ *have opposite signs, then for* $z = x > 0$ *the following inequality holds*:

$$|\Omega_m(z;\rho,\mu)| \le \frac{\Gamma\big(1-\mu+\frac{m+1}{\rho}\big)}{\pi x^{\frac{m+1}{\rho}}}\max\left(\left|\sin\left(\pi(\mu-\tfrac{m+1}{\rho})\right)\right|, \frac{\left|\sin\left(\pi(\mu-\frac{m}{\rho})\right)\right|}{4\cos^2(\frac{\pi}{2\rho})}\right). \tag{1.5.26}$$

Proof. Representation (1.5.24) was obtained in Theorem 1.1.3, where we have obtained a formula for the remainder Ω_m through the integrals $I_{1,m}$ and $I_{2,m}$. To obtain an upper estimate for the moduli of these integrals, we first deduce lower estimates for the denominators (note that they are positive) of the integrands.

We have the inequalities

$$\begin{gathered} t^{1/\rho} + 2\cos\frac{\pi}{\rho} + t^{-1/\rho} \geq 2 + 2\cos\frac{\pi}{\rho} = 4\cos^2\frac{\pi}{2\rho}, \\ 1 + 2t^{1/\rho}\cos\frac{\pi}{\rho} + 2t^{2/\rho} \geq 1. \end{gathered} \tag{1.5.27}$$

In the last inequality, we have used the nonnegativeness of $\cos(\pi/\rho)$ for $\rho \in [2/5, 2/3]$. From (1.5.27) we obtain

$$|I_{1,m}(z;\rho,\mu)| \leq \int_0^{+\infty} |t^{\frac{m+1}{\rho}-\mu} e^{zt}|\,dt = \int_0^{+\infty} t^{\frac{m+1}{\rho}-\mu} e^{-xt}\,dt = x^{\mu-1-\frac{m+1}{\rho}}\,\Gamma\left(1-\mu+\frac{m+1}{\rho}\right) \tag{1.5.28}$$

and

$$\begin{aligned} |I_{2,m}(z;\rho,\mu)| &\leq \frac{1}{4}\cos^{-2}\left(\frac{\pi}{2\rho}\right)\int_0^{+\infty} t^{\frac{m+1}{\rho}-\mu} e^{-xt}\,dt \\ &= \frac{1}{4}\cos^{-2}\left(\frac{\pi}{2\rho}\right) x^{\mu-1-\frac{m+1}{\rho}}\,\Gamma\left(1-\mu+\frac{m+1}{\rho}\right). \end{aligned} \tag{1.5.29}$$

From (1.5.28), (1.5.29), and (1.1.20) we immediately obtain (1.5.25). To prove inequality (1.5.26), we note that the integrals $I_{j,m}(z,\rho,\mu)$ are positive if $x > 0$ and for the sum $as_1 + bs_2$, where $a > 0$, $b > 0$, and $\operatorname{sgn} s_1 = -\operatorname{sgn} s_2$, the following estimate is valid:

$$|as_1 + bs_2| \leq \max(a|s_1|, b|s_2|). \tag{1.5.30}$$

From (1.5.28)–(1.5.30) we obtain (1.5.26). The theorem is proved. □

Theorem 1.5.3. *For any $\rho \in [2/5, 2/3]$, $\mu \leq 1/\rho$, and $x > 0$, the following representation holds:*

$$x^{\mu-1}E_\rho(-x^{1/\rho};\mu) = 2\rho\exp\left(x\cos(\pi\rho)\right)\cos(z\sin(\pi\rho) - \pi\rho(\mu-1)) + \omega(x;\rho,\mu), \tag{1.5.31}$$

where the remainder ω admits the estimates

$$|\omega(x;\rho,\mu)| < 0.48\,x^{-s}\Gamma(s), \quad s = 1 + \frac{1}{\rho} - \mu, \tag{1.5.32}$$

$$|\omega(x;\rho,1)| < \frac{\Gamma(1/\rho)}{\pi x^{1/\rho}}, \quad \operatorname{sgn}\omega(x;\rho,1) = \begin{cases} 1, & \frac{2}{5} \leq \rho < \frac{1}{2}, \\ -1, & \frac{1}{2} < \rho \leq \frac{2}{3}, \end{cases} \tag{1.5.33}$$

$$|\omega(x;\rho,\mu)| < \left(\frac{3}{2}\min(\mu, 1-\mu) + \frac{1}{\rho} - 2\right) x^{-s}\Gamma(s), \quad 0 < \mu < 1, \quad \frac{2}{5} \leq \rho < \frac{1}{2}, \tag{1.5.34}$$

$$|\omega(x;\rho,\mu)| < \left(\frac{1}{\rho} - 2\right) x^{-2}, \quad x \geq \frac{1}{\rho}, \quad \frac{2}{5} \leq \rho < \frac{1}{2}, \quad 1 < \mu \leq \frac{1}{\rho} - 1, \tag{1.5.35}$$

$$\left|\omega\left(x;\rho,\frac{1}{\rho}\right)\right| \leq \frac{|\sin(\pi/\rho)|}{2\pi x}. \tag{1.5.36}$$

Proof. We set $m = 0$ in Theorem 1.5.2 and use the fact that

$$4\cos^2\left(\frac{\pi}{2\rho}\right) \geq 2.$$

From (1.5.25) we obtain the inequality

$$|\omega(x;\rho,\mu)| < \frac{\Gamma(s)}{\pi x^s}\left(\left|\sin\pi\left(\mu-\frac{1}{\rho}\right)\right| + \frac{|\sin(\pi\mu)|}{2}\right). \tag{1.5.37}$$

From (1.5.37) we immediately obtain (1.5.36) since $s=1$ in the case where $\mu=1/\rho$. Setting $\mu=1$ in (1.5.37), we obtain (1.5.33). From (1.1.20) we also see that

$$\operatorname{sgn}\omega(x;\rho,1) = \operatorname{sgn}\sin(\pi/\rho).$$

Since the moduli of sines are not greater than 1, we have from (1.5.37) the inequality

$$|\omega(x;\rho,\mu)| \le (1.5/\pi)x^{-s}\Gamma(s) < 0.48\,x^{-s}\Gamma(s),$$

which yields (1.5.32). Applying the inequality

$$\left|\sin\pi\left(\mu-\frac{1}{\rho}\right)\right| \le |\sin(\pi\mu)| + \left|\sin\left(\frac{\pi}{\rho}\right)\right|,$$

we obtain from (1.5.37) for $0<\mu<1$ and $0.4<\rho<0.5$ the inequality

$$|\omega(x;\rho,\mu)| < \frac{\Gamma(s)}{\pi x^s}\left(\frac{3}{2}\sin\pi\mu + \sin\frac{\pi}{\rho}\right) < \frac{\Gamma(s)}{x^s}\left(\frac{3}{2}\min(\mu,1-\mu) + \frac{1}{\rho} - 2\right).$$

Inequality (1.5.34) is proved.

In the case where

$$\frac{2}{5} \le \rho < \frac{1}{2}, \quad 1 < \mu \le \frac{1}{\rho} - 1,$$

the numbers $\sin(\pi\mu)$ and $\sin\pi\left(\mu - 1/\rho\right)$ have opposite signs:

$$\sin(\pi\mu) < 0, \quad -\frac{3}{2} \le 1 - \frac{1}{\rho} < \mu - \frac{1}{\rho} \le -1;$$

this implies

$$\sin\pi\left(\mu-\frac{1}{\rho}\right) \ge 0,$$

and the modulus of each of these sines is not greater than $\sin\pi(1/\rho-2)$. Therefore, by (1.5.26), we have the inequality

$$|\omega(x;\rho,\mu)| \le \frac{\Gamma(s)}{\pi x^s}\sin\left(\pi\left(\frac{1}{\rho}-2\right)\right) < \left(\frac{1}{\rho}-2\right)x^{-s}\Gamma(s).$$

Now we note that $2 \le s \le 1/\rho$ and the function $g(s) = x^{-s}\Gamma(s)$ decreases on this segment for $x \ge 1/\rho$ since $(\ln g(s))' = \psi(s) - \ln x < 0$ (recall that $\psi(s) < \ln s$ for $s>0$). Therefore, $g(s) \le g(2)$, and we arrive at (1.5.35). The theorem is completely proved. □

In the final theorem of Chap. 1 we obtain an estimate of the remainder in the asymptotic expansion (1.5.1) for $z\in\mathbb{R}$, $z<0$, in the particular case where $m=0$, $0<\rho\le 2/5$, and $0<\mu\le 1/\rho$. The obtained estimate of the remainder is worse than in Theorem 1.5.1 for large $|z|$ but is better for "non-large" values of $|z|$, which is convenient in some applications.

Theorem 1.5.4. *For any $\rho\in(0,2/5]$, $\mu\in(0,1/\rho]$, and $x>0$, the following representation holds*:

$$E_\rho(-x^{1/\rho};\mu) = S_\rho(x,\mu) + \omega_\rho(x,\mu), \tag{1.5.38}$$

where

$$S_\rho(x,\mu) = 2\rho x^{1-\mu}\sum_{k=1}^{N(\rho)} \exp\Big(x\cos\big((2k-1)\pi\rho\big)\Big)\cos\Big(x\sin\big((2k-1)\pi\rho\big) + (2k-1)\pi\rho(1-\mu)\Big), \tag{1.5.39}$$

and

$$N(\rho) = \left[\frac{1}{2\rho}\right], \quad |\omega_\rho(x,\mu)| \le 0.74\, x^{-\mu} \quad \forall x > 0.$$

Proof. By (1.1.1), for $z = -x^{1/\rho}$ we have

$$E_\rho(-x^{1/\rho};\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,-\alpha,\alpha)} \frac{\zeta^{1/\rho-\mu} e^\zeta\, d\zeta}{\zeta^{1/\rho} + x^{1/\rho}}, \tag{1.5.40}$$

where $\sigma > x$ and α is an arbitrary number from the interval $(\pi/2, 3\pi/2)$. We take

$$\alpha = \alpha(\rho) = \begin{cases} \dfrac{5\pi\rho}{2}, & \dfrac{4}{15} \le \rho < \dfrac{2}{5}, \\ \dfrac{8\pi\rho}{3}, & \dfrac{1}{4} < \rho < \dfrac{4}{15}, \\ 2\pi\rho N(\rho), & 0 < \rho \le \dfrac{1}{4}. \end{cases} \tag{1.5.41}$$

Then on the rays $\{\zeta = r\exp(\pm i\alpha) \mid r > 0\}$ the integrand

$$F_\rho(z,\zeta,\mu) = \frac{\zeta^{1/\rho-\mu} e^\zeta}{\zeta^{1/\rho} + x^{1/\rho}} \tag{1.5.42}$$

does not have singularities and α is contained in the semi-interval $(\pi/2, \pi]$. Taking into account the definition of $N(\rho)$ and (1.5.41), we easily verify the inequality

$$(2N(\rho)-1)\pi\rho < \alpha(\rho) < (2N(\rho)+1)\pi\rho.$$

Therefore, in the angle $|\arg\zeta| < \alpha(\rho)$, the function $F(\zeta)$ has as singularities only simple poles at the points

$$x_k = x\exp\big((2k-1)\pi i\rho\big), \quad \bar{x}_k = x\exp\big(-(2k-1)\pi i\rho\big), \quad k \in \mathbb{N}, \quad k \le N(\rho). \tag{1.5.43}$$

We transform integral (1.5.40) passing to integration over the loop $\gamma(\sigma', -\alpha, \alpha)$, $0 < \sigma' < x$, and adding the residues at the points (1.5.43):

$$E_\rho(-x^{1/\rho};\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma,-\alpha,\alpha)} \frac{\zeta^{1/\rho-\mu} e^\zeta\, d\zeta}{\zeta^{1/\rho} + x^{1/\rho}} + \sum_{k=1}^{N(\rho)} \left(\operatorname{Res} F_\rho(x,\zeta,\mu)|_{\zeta=x_k} + \operatorname{Res} F_\rho(x,\zeta,\mu)|_{\zeta=\bar{x}_k}\right). \tag{1.5.44}$$

Since, due to the inclusion $\mu \in \mathbb{R}$, the function F is such that

$$F_\rho(x,\bar{\zeta},\mu) = \bar{F}_\rho(x,\zeta,\mu) \quad \forall \zeta \in \mathbb{C} \setminus (-\infty, 0], \tag{1.5.45}$$

the sum of its residues at two complex conjugate points $x\exp(\pm(2k-1)\pi i\rho)$ is equal to twice the real part of the residue at one of these points. Therefore, we obtain

$$\begin{aligned} \operatorname{Res} F(\zeta)|_{\zeta=x_k} + \operatorname{Res} F(\zeta)|_{\zeta=\bar{x}_k} &= 2\operatorname{Re} \lim_{\zeta\to x_k} \frac{\zeta^{1/\rho-\mu} e^\zeta(\zeta - x\exp((2k-1)\pi i\rho))}{\zeta^{1/\rho} + x^{1/\rho}} \\ = 3\rho \operatorname{Re}(\zeta^{1-\mu} e^\zeta)|_{\zeta = x\exp((2k-1)\pi i\rho)} &= 2\rho \operatorname{Re}\Big(x^{1-\mu}\exp((2k-1)\pi i\rho(1-\mu))\exp(x\exp(2k-1)\pi i\rho)\Big) \\ &= 2\rho x^{1-\mu} \operatorname{Re}\exp\Big(xe^{(2k-1)\pi i\rho} + \pi i\rho(1-\mu)(2k-1)\Big) \\ = 2\rho x^{1-\mu} \exp\Big(x\cos((2k-1)\pi\rho)\Big) &\cos\Big(x\sin((2k-1)\pi\rho) + (2k-1)\pi\rho(1-\mu)\Big). \end{aligned} \tag{1.5.46}$$

(in the calculation of the limit we apply the L'Hôpital–Bernoulli rule).

Substituting the values of the sums of two conjugate residues (1.5.45) in (1.5.44), we obtain

$$E_\rho(-x^{1/\rho};\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma',-\alpha,\alpha)} F_\rho(x,\zeta,\mu)\, d\zeta + S_\rho(x,\mu),$$

where the function $S_\rho(x,\mu)$ is defined in (1.5.39) and the function $F_\rho(x,\zeta,\mu)$ is defined in (1.5.42). We have obtained representation (1.5.38) in which

$$\omega_\rho(x,\mu) = \frac{1}{2\pi i} \int\limits_{\gamma(\sigma',-\alpha,\alpha)} F_\rho(x,\zeta,\mu)\, d\zeta, \quad 0 < \sigma' < x. \tag{1.5.47}$$

Since the function $F_\rho(x,\zeta,\mu)$ is bounded with respect to the variable ζ in the disk with cut

$$\left\{\zeta \in \mathbb{C} \;\middle|\; 0 < |\zeta| < \frac{x}{2}, \quad -\pi < \arg\zeta < \pi\right\}$$

(we have used the condition $0 < \mu \le 1/\rho$), we can pass to the value $\sigma' = 0$ in (1.5.47). The loop $\gamma(0,-\alpha,\alpha)$ is the union of the rays $\ell_{\pm\alpha}$ ($\ell_\theta = \{re^{i\theta} \mid r > 0\}$). Using this fact and property (1.5.45), we find

$$|\omega_\rho(x,\mu)| \le \frac{1}{\pi} \int\limits_{\ell_\alpha} |F_\rho(x,\zeta,\mu)|\, |d\zeta| = \frac{1}{\pi} \int\limits_0^{+\infty} \frac{r^{1/\rho-\mu} e^{r\cos\alpha}\, dr}{|r^{1/\rho} e^{i\alpha/\rho} + x^{1/\rho}|}.$$

Performing the change of variable $r = xt$ in the integral, we obtain the inequality

$$|\omega_\rho(x,\mu)| \le \frac{x^{1-\mu} I_\rho(x,\mu)}{\pi}, \tag{1.5.48}$$

where

$$I_\rho(x,\mu) = \int\limits_0^{+\infty} \frac{t^{1/\rho-\mu} \exp(xt\cos\alpha)\, dt}{|1 + t^{1/\rho} e^{i\alpha/\rho}|}.$$

In correspondence with (1.5.41), we consider the following three intervals of values of ρ.

1. $\dfrac{4}{15} \le \rho < \dfrac{2}{5}$. Then

$$\alpha = \frac{5\pi\rho}{2}, \quad \cos\alpha \le -\frac{1}{2}, \quad \exp\frac{i\alpha}{\rho} = i.$$

Therefore,

$$I_\rho(x,\mu) = \int\limits_0^{+\infty} \frac{t^{1/\rho-\mu} \exp(xt\cos\alpha)}{|1 + it^{1/\rho}|}\, dt \le \int\limits_0^{+\infty} \frac{t^{1/\rho-\mu} \exp(-xt/2)}{|1 + it^{1/\rho}|} dt. \tag{1.5.49}$$

By the inequality

$$\frac{1}{|1 + it^{1/\rho}|} \le \begin{cases} 1, & 0 < t \le 1, \\ t^{-1/\rho}, & 1 \le t, \end{cases}$$

from (1.5.49) we obtain the inequality

$$I_\rho(x,\mu) \le \int\limits_0^1 t^{1/\rho-\mu} \exp\left(-\frac{xt}{2}\right) dt + \int\limits_1^{+\infty} t^{-\mu} \exp\left(-\frac{xt}{2}\right) dt.$$

Since $0 < \mu \le 1/\rho$, in each of the integrals, the power of t before the exponent is not greater than 1. This implies the estimate

$$I_\rho(x,\mu) \le \int_0^1 \exp\left(-\frac{xt}{2}\right)\,dt + \int_1^{+\infty} \exp\left(-\frac{xt}{2}\right)\,dt = \int_1^{+\infty} \exp\left(-\frac{xt}{2}\right)\,dt = \frac{2}{x}. \tag{1.5.50}$$

From (1.5.50) and (1.5.48) we obtain the inequality

$$|\omega_\rho(x,\mu)| \le \frac{2}{\pi}x^{-\mu} < 0.65x^{-\mu} \quad \forall\rho \in \left[\frac{4}{15},\frac{2}{5}\right], \quad \forall x > 0. \tag{1.5.51}$$

2. $\dfrac{1}{4} < \rho < \dfrac{4}{15}$. In this case,

$$\alpha = \frac{8\pi\rho}{3}, \quad \cos\alpha \le -\frac{1}{2}, \quad \exp\frac{i\alpha}{\rho} = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}, \quad \left|1 + t^{1/\rho}\exp\frac{i\alpha}{r}\right| = \sqrt{1 - t^{1/\rho} + t^{2/\rho}}.$$

Therefore,

$$I_\rho(x,\mu) = \int_0^{+\infty} \frac{t^{1/\rho-\mu}\exp(xt\cos\alpha)dt}{\sqrt{1-t^{1/\rho}+t^{2/\rho}}} \le \int_0^{+\infty} \frac{t^{1/\rho-\mu}\exp(-xt/2)dt}{\sqrt{1-t^{1/\rho}+t^{2/\rho}}}. \tag{1.5.52}$$

By the inequality

$$1 - t^{1/\rho} + t^{2/\rho} \le \begin{cases} 2/\sqrt{3}, & 0 < t < 1, \\ (2/\sqrt{3})t^{-1/\rho}, & t \ge 1, \end{cases}$$

from (1.5.52) we obtain the inequality

$$\begin{aligned} I_\rho(x,\mu) &\le \frac{2}{\sqrt{3}}\int_0^1 t^{1/\rho-\mu}\exp\left(-\frac{xt}{2}\right)\,dt + \frac{2}{\sqrt{3}}\int_0^1 t^{-\mu}\exp\left(-\frac{xt}{2}\right)\,dt \\ &\le \frac{2}{\sqrt{3}}\int_0^1 \exp\left(-\frac{xt}{2}\right)\,dt + \frac{2}{\sqrt{3}}\int_1^{+\infty} \exp\left(-\frac{xt}{2}\right)\,dt = \frac{2}{\sqrt{3}}\int_0^{+\infty} \exp\left(-\frac{xt}{2}\right)\,dt = \frac{4}{x\sqrt{3}}. \end{aligned} \tag{1.5.53}$$

From (1.5.53) and (1.5.48) we obtain the inequality

$$\left|\omega_\rho(x,\mu)\right| \le \frac{4x^{-\mu}}{\pi\sqrt{3}} < 0.74\,x^{-\mu}, \quad x > 0, \quad \rho \in \left(\frac{1}{4},\frac{4}{15}\right). \tag{1.5.54}$$

3. $0 < \rho \le \dfrac{1}{4}$. In this case

$$\alpha(\rho) = 2\pi\rho\left[\frac{1}{2\rho}\right] = \pi - 2\pi\rho\left\{\frac{1}{2\rho}\right\} \in \left(\frac{2\pi}{3},\pi\right],$$

and hence $\cos\alpha < -1/2$ (it is easy to verify that the function $t\{1/t\}$ for $t \in (0,1/2]$ takes values on the semi-interval $[0,1/3)$). By the choice of $\alpha(\rho)$, we have the relation $\exp(i\alpha/\rho) = 1$. Therefore,

$$\begin{aligned} I_\rho(x,\mu) \le \int_0^{+\infty} \frac{t^{1/\rho-\mu}\exp\left(-\frac{xt}{2}\right)\,dt}{1+t^{1/\rho}} &\le \int_0^1 t^{1/\rho-\mu}\exp\left(-\frac{xt}{2}\right)\,dt \\ &\int_1^{+\infty} t^{-\mu}\exp\left(-\frac{xt}{2}\right)\,dt \le \int_0^{+\infty} \exp\left(-\frac{xt}{2}\right)\,dt = \frac{2}{x}. \end{aligned} \tag{1.5.55}$$

From (1.5.55) and (1.5.48) we obtain the inequality

$$\left|\omega_\rho(x,\mu)\right| \le \frac{2}{\pi}x^{-\mu}, \quad x>0, \quad 0<\rho\le\frac{1}{4}. \tag{1.5.56}$$

From (1.5.51), (1.5.54), and (1.5.56) we conclude that in all the cases considered, the estimate

$$\left|\omega_\rho(x,\mu)\right| < 0.74x^{-\mu}$$

is valid. The theorem is proved. □

Chapter 2

ASYMPTOTIC PROPERTIES OF ZEROS OF MITTAG-LEFFLER FUNCTIONS

2.1. Asymptotic Formulas for Zeros

The definition

$$E_\rho(z;\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu+n/\rho)}, \quad \rho>0, \quad \mu\in\mathbb{C},$$

of the function $E_\rho(z;\mu)$ implies the following formulas:

$$E_1(z;1) = e^z, \quad E_1(z;-m) = z^{m+1}e^z, \quad m\in\mathbb{Z}_+, \tag{2.1.1}$$

$$E_{1/2}(z;1) = \cosh\sqrt{z}, \quad E_{1/2}(z;2) = \frac{\sinh\sqrt{z}}{\sqrt{z}}, \quad E_{1/2}(z;3) = \frac{\cosh\sqrt{z}-1}{\sqrt{z}}, \tag{2.1.2}$$

$$E_{1/2}(z;-2m) = z^{m+1/2}\sinh\sqrt{z},\ m\in\mathbb{Z}_+; \quad E_{1/2}(z;-(2m-1)) = z^m\cosh\sqrt{z},\ m\in\mathbb{N}. \tag{2.1.3}$$

Therefore, the function $E_1(z;-m)$, $m\in\mathbb{Z}_+$, has a unique zero $z=0$ of multiplicity $m+1$, and the function $E_1(z;1)$ does not have zeros. In the sequel, we see that, except for these two cases, the function $E_\rho(z;\mu)$ has an infinite set of zeros, and zeros of the function $E_{1/2}(z;3)$ are described by the formula

$$z_n = -(2\pi n)^2, \quad n\in\mathbb{N}, \tag{2.1.4}$$

and have multiplicity 2. This is the unique case of an infinite number of multiple zeros of the function $E_\rho(z;\mu)$.

We introduce the following constants:

$$c_\mu = \frac{1}{\rho\Gamma(\mu-1/\rho)}, \quad d_\mu = \frac{1}{\rho\Gamma(\mu-2/\rho)}, \quad \tau_\mu = 1+\rho(1-\mu), \quad \text{if} \quad \mu\ne\frac{1}{\rho}-l, \quad l\in\mathbb{Z}_+, \tag{2.1.5}$$

$$c_\mu = \frac{1}{\rho\Gamma(\mu-2/\rho)}, \quad d_\mu = \frac{1}{\rho\Gamma(\mu-3/\rho)}, \quad \tau_\mu = 2+\rho(1-\mu), \quad \text{if} \quad \mu=\frac{1}{\rho}-l, \quad l\in\mathbb{Z}_+, \quad \frac{1}{\rho}\notin\mathbb{N}. \tag{2.1.6}$$

By construction, $c_\mu\ne 0$ for the considered values of ρ and μ. Pairs of the parameters ρ and μ missing in (2.1.5) and (2.1.6) are said to be *exceptional*. They are as follows:

$$\frac{1}{\rho}\in\mathbb{N}, \quad \mu=\frac{1}{\rho}-l, \quad l\in\mathbb{Z}_+.$$

Formulas (2.1.1)–(2.1.3) yield the complete set of functions $E_\rho(z;\mu)$ corresponding to the exceptional values of the parameters ρ and μ, where $\rho=1$ and $\rho=1/2$. Since zeros of the functions (2.1.1)–(2.1.3) can be written in the explicit form, we cannot consider such pairs of parameters.

Unless otherwise stated, for the power and logarithmic functions we choose their principal values in the half-plane with the cut along the negative real semi-axis.

In this section, we obtain asymptotic formulas for zeros z_n of the function $E_\rho(z;\mu)$. The cases

$$\rho > \frac{1}{2}, \quad \rho = \frac{1}{2},\ \operatorname{Re}\mu > 3, \quad \rho = \frac{1}{2},\ \operatorname{Re}\mu < 3, \quad \rho = \frac{1}{2},\ \operatorname{Re}\mu = 3, \quad \rho < \frac{1}{2}$$

are considered separately.

1. By formula (2.1.1), considering the case $\rho = 1$, we exclude the values $\mu = 1, 0, -1, -2, \ldots$.

Theorem 2.1.1. *Let*

(1) $\rho > 1/2$, *and* $\mu \in \mathbb{C}$, *where* $\mu \neq 1, 0, -1, -2, \ldots$ *for* $\rho = 1$, *or*
(2) $\rho = 1/2$ *and* $\operatorname{Re}\mu > 3$.

Then all sufficiently large (in modulus) zeros z_n of the function $E_\rho(z;\mu)$ are simple and the following asymptotic formula holds:

$$z_n^\rho = 2\pi in - \frac{\tau_\mu}{\rho}\ln 2\pi in + \ln c_\mu + \frac{d_\mu/c_\mu}{(2\pi in)^{1/\rho}} + \left(\frac{\tau_\mu}{\rho}\right)^2 \frac{\ln 2\pi in}{2\pi in} - \frac{\tau_\mu}{\rho}\frac{\ln c_\mu}{2\pi in} + \alpha_n, \tag{2.1.7}$$

as $n \to \pm\infty$, *where*

$$\alpha_n = O\left(\frac{\ln|n|}{|n|^{1+1/\rho}}\right) + O\left(\frac{1}{|n|^{2/\rho}}\right) + O\left(\frac{\ln^2|n|}{n^2}\right), \quad \textit{if}\ \ \rho > 1/2, \tag{2.1.8}$$

$$\alpha_n = \frac{e^{\pm i\pi\mu}}{c_\mu^2(2\pi in)^{-4\tau_\mu}} + O\left(\frac{1}{|n|^{-8\tau_\mu}}\right) + O\left(\frac{\ln|n|}{|n|^{1-4\tau_\mu}}\right) + O\left(\frac{\ln^2|n|}{n^2}\right), \quad \textit{if}\ \ \rho = 1/2. \tag{2.1.9}$$

For fixed $\delta \in (0, \pi/2)$ and $R > 0$, we introduce the sets

$$Z = Z_{\delta,R} = (z : |\arg z| < \pi - \delta,\ |z| > R),$$
$$W = W_{\delta,R} = (w : |\arg w| < \pi - 2\delta|w| > 2R).$$

Lemma 2.1.1. *Let $A \in \mathbb{C}$ and $\delta \in (0, \pi/2)$. If $R > 0$ is sufficiently large, then the equation*

$$z - A\ln z = w, \quad w \in W,$$

has a unique root $z = z(w)$ on the set Z. This root is simple and admits the asymptotic formula

$$z = w + A\ln w + A^2\frac{\ln w}{w} + O\left(\frac{\ln^2 w}{w^2}\right), \quad w \to \infty.$$

Proof. First, we prove that the function

$$w = z - A\ln z \tag{2.1.10}$$

for all sufficiently large R defines a univalent mapping of the "sector"

$$S_R = (z : |z| > R,\ -\pi < \alpha < \arg z < \beta < \pi,\ \beta - \alpha < \pi).$$

Assume the contrary, i.e., assume that there exist arbitrarily large R and points $z_1, z_2 \in S_R$ such that

$$w_1 = w(z_1) = w(z_2) = w_2.$$

Since R can be arbitrarily large, we can assume that the segment $[z_1, z_2]$ lies in S_R. Then

$$z_2 - z_1 = A(\ln z_2 - \ln z_1) = A\int_{z_1}^{z_2}\frac{dz}{z},$$

$$|z_2 - z_1| \le |A||z_2 - z_1|\frac{1}{\min(|z_1|, |z_2|)} \le \frac{A}{R}|z_2 - z_1|,$$

and taking $R > |A|$, we arrive at a contradiction. We have proved that the mapping of the "sector" is univalent.

If $w_1 = w_2$, then, in particular,

$$\arg w_1 = \arg w_2. \tag{2.1.11}$$

Let R be such that

$$|A|\left|\frac{\ln z}{z}\right| < \frac{1}{2}, \quad |z| > R.$$

Then

$$-\frac{\pi}{6} < \arg\left(1 - A\frac{\ln z}{z}\right) < \frac{\pi}{6}$$

and, therefore, if $|z_1|, |z_2| > R$, then

$$\arg w_j = \arg z_j + \arg\left(1 - A\frac{\ln z_j}{z_j}\right) \in \left(\arg z_j - \frac{\pi}{6}, \arg z_j + \frac{\pi}{6}\right), \quad j = 1, 2.$$

By (2.1.11) this implies that

$$|\arg z_1 - \arg z_2| < \frac{\pi}{3},$$

i.e., the points z_1 and z_2 lie in the "sector" of aperture less than $\pi/3$. As we have proved on the first step, the mapping of the sector is univalent and the relation $w_1 = w_2$ is impossible.

Thus, we have proved that for sufficiently large $R > 0$, function (2.1.10) defines a univalent mapping of the set Z onto its image $w(Z)$. Therefore, the inverse function $z = z(w)$ for (2.1.10) with values in the set Z is defined on $w(Z)$, i.e., the equation

$$z - A\ln z = w, \quad w \in w(Z),$$

has a unique root $z = z(w)$ on the set Z. Since

$$w'(z) \neq 0 \quad \text{for } R > |A|,$$

the function $z(w)$ is differentiable on $w(Z)$ and

$$z'(w) = \frac{1}{w'(z)} \neq 0, \quad w \in w(Z).$$

In particular, this implies that the root is simple.

The assertion on the uniqueness and simpleness of the root $z(w)$ becomes valid if we (for simplicity) replace the set $w(Z)$ by an appropriate subset of it. Obviously, $W \subset w(Z)$ for sufficiently large $R > 0$, and it remains to prove the asymptotic formula for $z(w)$.

Substituting

$$z = w + r(w)$$

in Eq. (2.1.10) and noting that $r(w) = o(w)$, $w \to \infty$, we obtain

$$r(w) = A\ln w + A\ln\left(1 + \frac{r(w)}{w}\right).$$

Therefore,

$$r(w) = O(\ln w),\ w \to \infty,$$

and by the Taylor formula we have

$$r(w) = \left(A\ln w + O\left(\frac{\ln^2 w}{w^2}\right)\right)\left(1 - \frac{A}{w}\right)^{-1}.$$

Expanding the function $(1 - A/w)^{-1}$ in the series by negative powers of w, we obtain the required asymptotics. The lemma is proved. □

Proof of Theorem 2.1.1. In the case (1) the proof is based on Theorem 1.2.1:

(1) if $\rho > 1/2$, then for any $s \in \mathbb{N}$

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)} \exp z^\rho - \sum_{k=1}^{s} \frac{1}{z^k \Gamma(\mu - k/\rho)} + O\left(\frac{1}{z^{s+1}}\right), \quad |\arg z| \le \min\left(\pi, \frac{\pi}{\rho}\right); \tag{2.1.12}$$

(2) if $\rho > 1$, then for any $s \in \mathbb{N}$

$$E_\rho(z;\mu) = -\sum_{k=1}^{s} \frac{1}{z^k \Gamma(\mu - k/\rho)} + O\left(\frac{1}{z^{s+1}}\right), \quad \frac{\pi}{\rho} \le |\arg z| \le \pi. \tag{2.1.13}$$

In formulas (2.1.12) and (2.1.13) we set $s = 1$. If the value of μ is the same as in formula (2.1.5) and $s = 2$ for values of μ from formula (2.1.6), then $c_\mu \ne 0$, and formulas (2.1.12) and (2.1.13) imply that for any $\varepsilon > 0$, all sufficiently large (in modulus) zeros z_n of the function $E_\rho(z;\mu)$ lie in the angle $|\arg z| < \pi/2\rho + \varepsilon$. For points of this angle, formula (2.1.12) becomes

$$\frac{1}{\rho} z^s E_\rho(z;\mu) = z^{\tau_\mu} \exp z^\rho - c_\mu - d_\mu/z + O(1/z^2), \quad |\arg z| < \pi/(2\rho) + \varepsilon.$$

Therefore, all zeros of the function $E_\rho(z;\mu)$ for $|z| > r_0$ can be found from the equation

$$\exp(z^\rho + \tau_\mu \ln z) = c_\mu + \frac{d_\mu}{z} + O\left(\frac{1}{z^2}\right). \tag{2.1.14}$$

We set

$$w = z^\rho + \frac{\tau_\mu}{\rho} \ln z^\rho. \tag{2.1.15}$$

Then the left-hand side in (2.1.14) is equal to e^w; to rewrite Eq. (2.1.14) with respect to the variable w, we must express z^ρ through w using (2.1.15). By Lemma 2.1.1 (we apply its roughened version)

$$z^\rho = w + O(\ln w).$$

Therefore,

$$\frac{1}{z} = \frac{1}{w^{1/\rho}} \left(1 + O\left(\frac{\ln w}{w}\right)\right) = \frac{1}{w^{1/\rho}} + O\left(\frac{\ln w}{w^{1+1/\rho}}\right).$$

Substituting this in (2.1.14), we obtain the equation

$$e^w = c_\mu + \frac{d_\mu}{w^{1/\rho}} + O\left(\frac{\ln w}{w^{1+1/\rho}}\right) + O\left(\frac{1}{w^{2/\rho}}\right). \tag{2.1.16}$$

In particular,

$$e^w = c_\mu + o(1), \quad w \to \infty. \tag{2.1.17}$$

Since the roots of the function $e^w - c$ are simple and are described by formula $2\pi i n + \ln c$, $n \in \mathbb{Z}$, by the Rouché theorem, sufficiently large (in modulus) roots w_n of Eq. (2.1.17) are simple and can be described by the formula

$$w_n = 2\pi i n + \ln c_\mu + \epsilon_n, \quad \epsilon_n \to 0, \quad n \to \pm\infty. \tag{2.1.18}$$

Therefore,

$$\frac{1}{w_n} = \frac{1}{2\pi i n}\left(1 + O\left(\frac{1}{n}\right)\right), \quad \ln w_n = \ln|n| + O(1), \quad n \to \pm\infty. \tag{2.1.19}$$

Now, setting $w = w_n$ in (2.1.16) and applying formulas (2.1.18) and (2.1.19), we obtain

$$c_\mu e^{\epsilon_n} = c_\mu + \frac{d_\mu}{(2\pi i n)^{1/\rho}} + O\left(\frac{\ln|n|}{|n|^{1+1/\rho}}\right) + O\left(\frac{1}{|n|^{2/\rho}}\right).$$

The left-hand side is equal to

$$c_\mu + c_\mu \epsilon_n + O(\epsilon_n^2);$$

therefore,

$$\epsilon_n = O(1/n^{1/\rho})$$

and hence

$$\epsilon_n = \frac{d_\mu/c_\mu}{(2\pi in)^{1/\rho}} + O\left(\frac{\ln|n|}{|\,n|^{1+1/\rho}}\right) + O\left(\frac{1}{|n|^{2/\rho}}\right).$$

Substituting this in (2.1.18), we have the relation

$$w_n = 2\pi in + \ln c_\mu + \frac{d_\mu/c_\mu}{(2\pi in)^{1/\rho}} + O\left(\frac{\ln|n|}{|n|^{1+1/\rho}}\right) + O\left(\frac{1}{|n|^{2/\rho}}\right), \quad n \to \pm\infty, \tag{2.1.20}$$

$$\ln w_n = \ln 2\pi in + \frac{\ln c_\mu}{2\pi in} + O\left(\frac{1}{|n|^{1+1/\rho}}\right), \quad n \to \pm\infty. \tag{2.1.21}$$

Now we return to the points z_n, which are the preimages of the points w_n under mapping (2.1.15). Since

$$|\arg z_n| < \pi/2\rho + \varepsilon =: \alpha$$

and $\alpha\rho < \pi$, for sufficiently large $|w|$ the condition of Lemma 2.1.1 holds. By this lemma, it follows from (2.1.15) that

$$z_n^\rho = w_n - \frac{\tau_\mu}{\rho}\ln w_n + \left(\frac{\tau_\mu}{\rho}\right)^2 \frac{\ln w_n}{w_n} + O\left(\frac{\ln^2 w_n}{w_n^2}\right). \tag{2.1.22}$$

Substituting on the right-hand side of this formula the expressions for w_n and $\ln w_n$ from (2.1.20) and (2.1.21), we obtain the required formulas (2.1.7) and (2.1.8).

By Lemma 2.1.1, the simpleness of sufficiently large (in modulus) zeros w_n of Eq. (2.1.16) implies the simpleness of all zeros z_n of the function $E_\rho(z;\mu)$ starting from a certain root. For the case (1) the theorem is proved.

In the case (2) we start from the following formula, which follows from Theorem 1.2.2 and is valid for all $\mu \in \mathbb{C}$: for any $s \in \mathbb{N}$,

$$E_{1/2}(z;\mu) = \frac{1}{2} z^{(1-\mu)/2}\left(e^{\sqrt{z}} + e^{\mp i\pi(1-\mu)}e^{-\sqrt{z}}\right) - \sum_{k=1}^{s}\frac{1}{z^k\Gamma(\mu-2k)} + O\left(\frac{1}{z^{s+1}}\right), \quad |z| \to \infty, \tag{2.1.23}$$

respectively for $0 \le \arg z \le \pi$ and $-\pi \le \arg z \le 0$. Setting in this formula $s = 1$, we obtain the following equation for sufficiently large (by the modulus) zeros z_n:

$$e^{\sqrt{z} + \tau_\mu \ln z} = c_\mu + \frac{d_\mu}{z} + O\left(\frac{1}{z^2}\right) + e^{\pm i\pi\mu}e^{-\sqrt{z}}z^{\tau_\mu}. \tag{2.1.24}$$

This equation differs from Eq. (2.1.14) for $\rho = 1/2$ only by the term

$$e^{\pm i\pi\mu}e^{-\sqrt{z}}z^{\tau_\mu}$$

Thus, as for Eq. (2.1.14), we rewrite this equation using the change (2.1.15) and setting $\rho = 1/2$. The additional term takes the form

$$e^{\pm i\pi\mu}e^{-w}w^{4\tau_\mu}\left(1 + O\left(\frac{\ln w}{w}\right)\right),$$

and instead of (2.1.16) we obtain the equation

$$e^w = c_\mu + \frac{d_\mu}{w^2} + O\left(\frac{\ln w}{w^3}\right) + e^{\pm i\pi\mu}e^{-w}w^{4\tau_\mu} + O\left(\frac{\ln w}{w^{1-4\tau_\mu}}e^{-w}\right). \tag{2.1.25}$$

Since $\operatorname{Re}\tau_\mu < 0$ and $\operatorname{Re}\sqrt{z} \geq 0$, the last term in (2.1.24) is $o(1)$ as $z \to \infty$. Therefore, the sum of the last two terms in (2.1.25) is $o(1)$ as $w \to \infty$, i.e., Eq. (2.1.25) has the form (2.1.14). Therefore, formulas (2.1.18) and (2.1.19) are valid. Substituting them in formula (2.1.25), we obtain the relation

$$c_\mu \epsilon_n (1 + O(\epsilon_n)) = \frac{d_\mu}{(2\pi i n)^2} + O\left(\frac{\ln |n|}{|n|^3}\right) + \frac{e^{\pm i\pi\mu}}{c_\mu (2\pi i n)^{-4\tau_\mu}} (1 - \epsilon_n(1 + O(\epsilon_n))) + O\left(\frac{\ln |n|}{|n|^{1-4\tau_\mu}}\right),$$

i.e.,

$$c_\mu \epsilon_n \left(1 + O\left(\frac{1}{|n|^{-4\tau_\mu}}\right)\right) + O(\epsilon_n^2) = \frac{d_\mu}{(2\pi i n)^2} + \frac{e^{\pm i\pi\mu}}{c_\mu (2\pi i n)^{-4\tau_\mu}} + O\left(\frac{\ln |n|}{|n|^3}\right) + O\left(\frac{\ln |n|}{|n|^{1-4\tau_\mu}}\right).$$

In particular, this implies that

$$\epsilon_n = O\left(\frac{1}{|n|^2}\right) + O\left(\frac{1}{|n|^{-4\tau_\mu}}\right),$$

and the previous formula yields

$$\epsilon_n = \frac{d_\mu / c_\mu}{(2\pi i n)^2} + \frac{e^{\pm i\pi\mu}}{c_\mu^2 (2\pi i n)^{-4\tau_\mu}} + O\left(\frac{\ln |n|}{|n|^3}\right) + O\left(\frac{1}{|n|^{-8\tau_\mu}}\right) + O\left(\frac{\ln |n|}{|n|^{1-4\tau_\mu}}\right).$$

Substituting this in (2.1.18), we obtain the relation

$$w_n = 2\pi i n + \ln c_\mu + \frac{d_\mu / c_\mu}{(2\pi i n)^2} + \frac{e^{\pm i\pi\mu}}{c_\mu^2 (2\pi i n)^{-4\tau_\mu}} + O\left(\frac{\ln |n|}{|n|^3}\right) + O\left(\frac{1}{|n|^{-8\tau_\mu}}\right) + O\left(\frac{\ln |n|}{|n|^{1-4\tau_\mu}}\right), \tag{2.1.26}$$

$$\ln w_n = \ln 2\pi i n + \frac{\ln c_\mu}{2\pi i n} + O\left(\frac{1}{|n|^3}\right) + O\left(\frac{1}{|n|^{1-4\tau_\mu}}\right). \tag{2.1.27}$$

Now setting in (2.1.22) $\rho = 1/2$, $w = w_n$, and $z = z_n$ and applying formulas (2.1.26) and (2.1.27), we obtain the required formulas (2.1.7) and (2.1.9). The simpleness of all sufficiently large (in modulus) zeros follows from the same arguments as those in the case $\rho > 1/2$. The theorem is proved. □

Now we consider the case $\rho = 1/2$, $\operatorname{Re}\mu < 3$, which has the simplest proof.

Theorem 2.1.2. *Let* $\operatorname{Re}\mu < 3$, $\mu \neq 2 - l$, $l \in \mathbb{Z}_+$. *Then all sufficiently large* (*in modulus*) *zeros* z_n *of the function* $E_{1/2}(z;\mu)$ *are simple and the following asymptotic formula holds*:

$$\sqrt{z_n} = i\pi\left(n - 1 + \frac{\mu}{2}\right) + (-1)^n \frac{c_\mu e^{-i(\pi/2)\mu}}{2(i\pi n)^{2\tau_\mu}} + O\left(\frac{1}{n^{6\operatorname{Re}\tau_\mu}}\right) + O\left(\frac{1}{n^{1+2\operatorname{Re}\tau_\mu}}\right), \quad n \to +\infty, \tag{2.1.28}$$

where the branch of the function $\sqrt{z}$ *is defined by the condition* $0 \leq \arg z < 2\pi$. *If* μ *is real, then all sufficiently large* (*in modulus*) *zeros are real.*

Proof. We can immediately verify that the function

$$z^{(1-\mu)/2}(e^{\sqrt{z}} + e^{-i\pi(1-\mu)} e^{-\sqrt{z}}), \quad 0 < \arg z < 2\pi,$$

for $\pi \leq \arg z < 2\pi$ coincides with the function

$$z^{(1-\mu)/2}(e^{\sqrt{z}} + e^{i\pi(1-\mu)} e^{-\sqrt{z}}), \quad -\pi \leq \arg z < 0.$$

Therefore, formula (2.1.23) can be written in the form

$$E_{1/2}(z;\mu) = \frac{1}{2} z^{(1-\mu)/2}(e^{\sqrt{z}} - e^{i\pi\mu} e^{-\sqrt{z}}) - \sum_{k=1}^{s} \frac{1}{z^k \Gamma(\mu - 2k)} + O\left(\frac{1}{z^{s+1}}\right), \quad |z| \to \infty,$$

where $0 \le \arg z < 2\pi$. Setting here $s = 1$, we obtain the relation

$$e^{-i\pi\mu/2} z^{-(1-\mu)/2} E_{1/2}(z;\mu) = \sinh(\sqrt{z} - i\frac{\pi}{2}\mu) - \frac{c_\mu e^{-i\pi\mu/2}}{2z^{\tau_\mu}} + O\left(\frac{1}{z^{1+\operatorname{Re}\tau_\mu}}\right), \quad |z| \to \infty, \tag{2.1.29}$$

where $0 \le \arg z < 2\pi$. Since $\operatorname{Re}\tau_\mu > 0$ and $0 \le \arg\sqrt{z} < \pi$, by the Rouché theorem we obtain that all sufficiently large (in modulus) zeros of the function $E_{1/2}(z;\mu)$ are simple and

$$\sqrt{z}_n - i\frac{\pi}{2}\mu = i\pi(n-1) + \epsilon_n, \quad \epsilon_n \to 0, \quad n \to +\infty. \tag{2.1.30}$$

Perhaps this formula would look more natural if it contains n except for $n-1$. However, in Sec. 2.2, the presence of the term "-1" makes the numeration of zeros for $\rho \le 1/2$ more uniform and convenient. Substituting (2.1.30) on the right-hand side of (2.1.29), we obtain

$$(-1)^n \sinh\epsilon_n = \frac{c_\mu e^{-i\pi\mu/2}}{2(\pi i n)^{2\tau_\mu}} + O\left(\frac{1}{n^{1+2\operatorname{Re}\tau_\mu}}\right). \tag{2.1.31}$$

Therefore,

$$\epsilon_n = O\left(\frac{1}{n^{2\operatorname{Re}\tau_\mu}}\right);$$

since

$$\sinh\epsilon_n = \epsilon_n + O(\epsilon_n^3),$$

Eq. (2.1.31) implies that

$$\epsilon_n = (-1)^n \frac{c_\mu e^{-i\pi\mu/2}}{2(\pi i n)^{2\tau_\mu}} + O\left(\frac{1}{n^{1+2\operatorname{Re}\tau_\mu}}\right) + O\left(\frac{1}{n^{6\operatorname{Re}\tau_\mu}}\right).$$

This and (2.1.30) yield the required formula (2.1.28).

It remains to prove the fact that zeros are real. We have proved above that in any disk K_n of radius $c/n^{2\operatorname{Re}\tau_\mu}$ centered at the point $i\pi(n+\mu/2)$, where $n > n_1$, there exists a unique point of the form $\sqrt{z}_n$. If μ is real, then zeros of the function $E_{1/2}(z;\mu)$ as a power series with real coefficients are symmetric with respect to the real axis. Therefore, if a point z_k is not real, then the disk K_n contains at least two points of the form $\sqrt{z}_k$, a contradiction. Theorem 2.1.2 is proved. □

2. The case where $\rho = 1/2$ and $\operatorname{Re}\mu = 3$ is most peculiar. Zeros of the function $E_{1/2}(z;3)$ are defined by the explicit formula (2.1.4). Therefore, we examine the behavior of zeros of the function

$$E_{1/2}(z;3+i\beta), \quad 0 \ne \beta \in \mathbb{R}. \tag{2.1.32}$$

First, we recall some facts about the inverse Joukowsky (Zhukovsky) function

$$w = w(z) = z + \sqrt{z^2-1}. \tag{2.1.33}$$

In the domain $G = \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$, function (2.1.33) splits into two single-valued analytic branches $w^\pm(z)$ that conformally map the domain G on the upper and lower half-planes $\operatorname{Im} z \gtrless 0$, respectively. For $x \in \mathbb{R}$, $|x| > 1$, we set $w^+(x) = w^+(x+i0)$.

We choose a branch of the logarithm $\ln w$ such that $-\pi < \arg w \le \pi$. Since $w^+w^- = 1$, we have

$$\ln w^-(z) = -\ln w^+(z). \tag{2.1.34}$$

If $x \in \mathbb{R}$, $|x| > 1$, then $w^+(x+i0)w^+(x-i0) = 1$ and hence

$$\ln w^+(x) = \ln^+(x+i0) = -\ln w^+(x-i0), \quad x \in \mathbb{R}, \quad |x| > 1. \tag{2.1.35}$$

Consider the function inverse to the hyperbolic cosine:

$$\operatorname{Arch} z = \operatorname{Log}\left(z + \sqrt{z^2-1}\right) = \operatorname{Log} w(z), \quad z \in G.$$

By (2.1.34), any value of the multi-valued function $\operatorname{Arch} z$ in G has the form

$$\ln\left(z+\sqrt{z^2-1}\right)^+ + 2\pi i s^+ \quad \text{or} \quad -\ln\left(z+\sqrt{z^2-1}\right)^+ + 2\pi i s^-, \tag{2.1.36}$$

where $s^\pm \in \mathbb{Z}$. Here and in what follows,

$$\left(z+\sqrt{z^2-1}\right)^+ = w^+(z).$$

For brevity, we introduce the following notation for $0 \neq \beta \in \mathbb{R}$:

$$\gamma = \gamma(\beta) = \frac{1}{\Gamma(1+i\beta)}, \quad \rho_1 = \rho_1(\beta) = |\gamma| + \sqrt{|\gamma|^2-1}, \quad \rho_2 = \rho_2(\beta) = |\gamma| + \sqrt{|\gamma|^2+1}, \tag{2.1.37}$$

$$\delta_n = \delta_n(\beta) = \ln\left(\gamma e^{i\beta \ln 2\pi n} + \sqrt{(\gamma e^{i\beta \ln 2\pi n})^2 - 1}\right)^+; \tag{2.1.38}$$

in (2.1.37), we take the arithmetic value of root. By the complement formula

$$|\gamma|^2 = \frac{1}{\Gamma(1+i\beta)\Gamma(1-i\beta)} = \frac{1}{i\beta\Gamma(i\beta)\Gamma(1-i\beta)} = \frac{\sin \pi i\beta}{\pi i\beta} = \frac{\sinh \pi\beta}{\pi\beta} > 1; \tag{2.1.39}$$

in particular, this implies that $\rho_2 > \rho_1 > 1$. Further, since δ_n is a value of the function $\ln w^+$ at some point, we have

$$0 \leq \operatorname{Im} \delta_n \leq \pi.$$

We say that a sequence ζ_n *is asymptotically distributed in a semi-strip* $a \leq \operatorname{Re}\zeta \leq b$, $\operatorname{Im}\zeta > 0$, if for any $\epsilon > 0$, there exists $R > 0$ such that

$$\operatorname{Im}\zeta_n > R \quad \Longrightarrow \quad a - \epsilon \leq \operatorname{Re}\zeta_n \leq b + \epsilon.$$

In Theorem 2.1.3, we take the same single-valued branch of the function $\sqrt{z}$ as in Theorem 2.1.2.

Theorem 2.1.3. (1) *The set of multiple zeros of function* (2.1.32) *is no more than finite.*

(2) *Zeros* z_n *of function* (2.1.32) *form two sequences* z_n^+, $n > n^+$, *and* z_n^-, $n > n^-$, *such that*

$$\sqrt{z_n^\pm} = 2\pi i n - \frac{\pi\beta}{2} \pm \delta_n + O\left(\frac{1}{n}\right), \quad n \to +\infty. \tag{2.1.40}$$

(3) *The points* $\zeta_n = \sqrt{z_n}$ *are asymptotically distributed in the semi-strips*

$$\ln \rho_1 \leq \left|\operatorname{Re}\zeta + \frac{\pi\beta}{2}\right| \leq \ln \rho_2, \quad \operatorname{Im}\zeta > 0. \tag{2.1.41}$$

(4) *Each point of the segment* $[0,\pi]$ *is a limit point for the sequence* $\operatorname{Im}\delta_n$ *and each point of the segments* $\left[\ln\rho_1, \ln\rho_2\right]$ *and* $\left[\ln(1/\rho_2), \ln(1/\rho_1)\right]$ *is a limit point of the sequence* $\operatorname{Re}\delta_n$.

(5) *There exist* $R = R(\beta) > 0$ *and* $N = N(\beta) \in \mathbb{N}$ *such that the disks*

$$|\zeta - i\pi n| < R, \quad n > N,$$

do not contain the points $\zeta_k = \sqrt{z_k}$.

Lemma 2.1.2 (see [34, Sec. 2.3])**.** *Let a function* $g(z)$ *be analytic in some open rectangle* π *and continuous on its closure and do not vanish. Let*

$$0 < m \leq |g(z)| \leq M < +\infty, \quad z \in \partial\pi.$$

If $z_i \in \pi$ *and* $\operatorname{dist}(z_i, \partial\pi) \geq \delta > 0$, $i = 1, 2$, *then*

$$\left|\arg g(z_1) - \arg g(z_2)\right| \leq C(\delta, m, M).$$

Lemma 2.1.3 (see [34, Sec. 2.3])**.** *Let a function* $F(\zeta)$ *be analytic and bounded in a semi-strip* $|\operatorname{Re}\zeta| \leq H$, $\operatorname{Im}\zeta > 0$, *and let* $|F(\zeta)|$ *be separated from zero on some half-line* $\operatorname{Re}\zeta = h \in (0, H)$, $\operatorname{Im}\zeta > 0$. *Then the following assertions hold*:

(1) *The number of zeros of the function* $F(\zeta)$ *(counted up to their multiplicities) in the rectangle* $|\operatorname{Re}\zeta| \le h$, $t \le \operatorname{Im}\zeta \le t+1$ *is bounded for* $t > 1$;
(2) $|F(\zeta)|$ *is separated from zero outside small disks of the same radii centered at zeros of* $F(\zeta)$, $|\operatorname{Re}\zeta| \le h$, $\operatorname{Im}\zeta > 1$.

Proof of Theorem 2.1.3. First, we prove the following property of zeros z_n of function (2.1.32):

$$\sqrt{z_n} = i\pi n + \Delta_n, \quad \Delta_n = O(1), \quad n \to +\infty; \tag{2.1.42}$$

here, the sequence $\sqrt{z_n}$ is assumed to be simple, i.e., several elements of sequence (2.1.42) with different numbers correspond to any possible multiple zero of the function $E_{1/2}(z;\mu)$.

Since $\operatorname{Re}\mu = 3$, we have $\operatorname{Re}\tau_\mu = 0$ and hence $|z_n^{\tau_\mu}| \asymp 1$. Therefore, formula (2.1.29) shows that, having set $\zeta = \sqrt{z}$, we must examine roots $\zeta_n = \sqrt{z_n}$ of the equation

$$F(\zeta) := \sinh\left(\zeta - \frac{i\pi\mu}{2}\right) + O(1), \quad \operatorname{Im}\zeta > 0.$$

Obviously, for sufficiently large $h > 0$, the function $F(\zeta)$ satisfies the conditions of Lemma 2.1.3 with $H = h+1$, and all zeros ζ_n of the function $F(\zeta)$ from the upper half-plane lie in the semi-strip $|\operatorname{Re}\zeta| < h$, $\operatorname{Im}\zeta > 0$. We fix $a > 0$ such that the horizontal line $\operatorname{Im}\zeta = a$ does not contain zeros of the function $F(\zeta)$ and denote by $n(r)$ the number of zeros of the function $F(\zeta)$ in the rectangle

$$\pi(r) = \Big(\zeta : |\operatorname{Re}\zeta| < h, \ a < \operatorname{Im}\zeta < r\Big).$$

By Lemma 2.1.3, there exist a sequence $r_k \to +\infty$, $k \to +\infty$, and a number $\epsilon > 0$ such that

(1) $r_{k+1} - r_k = O(1)$,
(2) $n(r_{k+1}) - n(r_k) = O(1)$,
(3) $0 < m \le |F(\zeta)| \le M < +\infty$ in the rectangle $|\operatorname{Re}\zeta| \le h+1$, $|\operatorname{Im}\zeta - r_k| \le \epsilon$, where m and M are independent of k.

By Lemma 2.1.2, we obtain from (3) that on the segment $|\operatorname{Re}\zeta| \le h$, $\operatorname{Im}\zeta = r_k$ we have that the increment of the argument has an asymptotic

$$\Delta \arg F(\zeta) = O(1)$$

uniformly with respect to k. We prove that

$$\Delta \arg F(\zeta) = 2r_k + O(1), \quad k \to \infty, \quad \zeta \in \partial\pi(r_k), \tag{2.1.43}$$

when bypassing the rectangle $\pi(r_k)$.

Introduce the notation $d = -i\pi\mu/2$. For $\operatorname{Re}\zeta > 0$, we have the relation

$$2F(\zeta) = e^{\zeta+d}(1+\epsilon(\zeta)), \quad \epsilon(\zeta) = O(e^{-\xi}), \quad \zeta = \xi + i\eta;$$

therefore, $|\epsilon(\zeta)| < 1/2$ on the straight line $\operatorname{Re}\zeta = h$ for sufficiently large $h > 0$. Hence

$$\Delta \arg F(\zeta) = \Delta \arg e^{i\eta+d+h} + O(1) = r_k + O(1)$$

on the segment $\operatorname{Re}\zeta = h$, $a \le \operatorname{Im}\zeta \le r_k$. This relation is proved similarly for the increment of the argument of $F(\zeta)$ on the segment $\operatorname{Re}\zeta = -h$, $a \le \operatorname{Im}\zeta \le r_k$ bypassed downward. Since $\Delta \arg F(\zeta)$ on the segment $|\operatorname{Re}\zeta| \le h$, $\operatorname{Im}\zeta = a$ is constant and on the segment $|\operatorname{Re}\zeta| \le h$, $\operatorname{Im}\zeta = r_k$ is $O(1)$, we have proved (2.1.43).

By the principle of the argument, from (2.1.43) we have

$$n(r_k) = \frac{r_k}{\pi} + O(1), \quad r_k \to \infty.$$

Together with the properties (1) and (2) this yields the relation

$$n(r) = \frac{r}{\pi} + O(1), \quad r \to \infty. \tag{2.1.44}$$

Let n be a sufficiently large number. Setting $r = \operatorname{Im}\zeta_n$, we obtain from (2.1.44)

$$n = \frac{\operatorname{Im}\zeta_n}{\pi} + O(1) \quad\Longrightarrow\quad \operatorname{Im}\zeta_n = \pi n + O(1).$$

Since $|\operatorname{Re}\zeta_n| \le h < +\infty$ and $\zeta_n = \sqrt{z_n}$, we obtain the required relation (2.1.42).

(1) We use formula (2.1.29). Since now $\mu = 3 + i\beta$, $0 \ne \beta \in \mathbb{R}$, by formulas (2.1.5) and (2.1.6) we have

$$c_\mu = 2\gamma, \quad \tau_\mu = -\frac{i\beta}{2},$$

and for large (in modulus) zeros of function (2.1.32) we have the equation

$$\cosh\left(\sqrt{z} + \frac{\pi\beta}{2}\right) = \gamma z^{i\beta/2} e^{\pi\beta/2} + O\left(\frac{1}{z}\right), \quad z \to \infty. \tag{2.1.45}$$

Substitute $z = z_n$ in this relation. Since, by (2.1.42),

$$z_n^{i\beta/2} = (i\pi n + \Delta_n)^{i\beta} = e^{i\beta\ln(i\pi n + \Delta_n)} = e^{i\beta\ln i\pi n}\left(1 + O\left(\frac{1}{n}\right)\right) = e^{-\pi\beta/2} e^{i\beta\ln\pi n} + O\left(\frac{1}{n}\right), \tag{2.1.46}$$

we obtain

$$\cosh\left(\sqrt{z_n} + \frac{\pi\beta}{2}\right) = \gamma e^{i\beta\ln\pi n} + O\left(\frac{1}{n}\right), \quad n \to +\infty. \tag{2.1.47}$$

The formula

$$\rho E_\rho(z;\mu - 1) = \rho(\mu - 1)E_\rho(z;\mu) + zE'_\rho(z;\mu) \tag{2.1.48}$$

implies that a multiple zero of the function (2.1.32) is a zero of the function $E_{1/2}(z; 2 + i\beta)$. By Theorem 2.1.2, zeros of the function $E_{1/2}(z; 2 + i\beta)$ have an asymptotics

$$\sqrt{z_n} = i\pi\left(n + \frac{2 + i\beta}{2}\right) + O\left(\frac{1}{n}\right) = i\pi(n+1) - \frac{\pi\beta}{2} + O\left(\frac{1}{n}\right), \quad n \to +\infty. \tag{2.1.49}$$

Therefore, if the function (2.1.32) has an infinite number of multiple zeros, then some sequence of indices satisfies property (2.1.49). Then the corresponding limit point of the modulus of the left-hand side in (2.1.47) is equal to $\cos 0 = 1$, whereas the limit of the modulus of the right-hand side is equal to $|\gamma| > 1$ (see (2.1.39)); a contradiction. Assertion (1) is proved.

(2) Taking into account formula (2.1.42), we write the sequence of zeros of function (2.1.32) and the union of two sequences z_n^+, $n > n^+$, and z_n^-, $n > n^-$, and hence

$$\sqrt{z_n^\pm} = 2\pi i n + \Delta_n^\pm, \quad n > n^\pm, \tag{2.1.50}$$

and $\Delta_n^\pm = O(1)$. As in the case of (2.1.42), we assume that the sequences $\sqrt{z_n^\pm}$ are simple, i.e., to any possible multiple zero, several elements of (2.1.50) either with different numbers or with opposite signs correspond. Substituting the expression for $\sqrt{z} = \sqrt{z_n^\pm}$ from (2.1.50) in formula (2.1.45) and taking into account that now (similarly to (2.1.46))

$$(z_n^\pm)^{i\beta/2} = e^{-\pi\beta/2} e^{i\beta\ln 2\pi n} + O(1/n),$$

we have

$$\cosh(\Delta_n^\pm + \pi\beta/2) = \gamma e^{i\beta\ln 2\pi n} + O(1/n), \quad n \to +\infty.$$

Therefore,

$$\Delta_n^\pm + \frac{\pi\beta}{2} = \operatorname{Arch}\left(\gamma e^{i\beta\ln 2\pi n} + O\left(\frac{1}{n}\right)\right) = \operatorname{Log} w(\gamma e^{i\beta\ln 2\pi n}) + O\left(\frac{1}{n}\right).$$

Of all values of the multi-valued function on the right-hand side for fixed n, only two are appropriate for us. By (2.1.36), we have the relation

$$\Delta_n^{\pm} = -\frac{\pi\beta}{2} \pm \ln w^{+}(\gamma e^{i\beta \ln 2\pi n}) + 2\pi i s_n^{\pm} + O\left(\frac{1}{n}\right), \tag{2.1.51}$$

where s_n^{+} and s_n^{-} are integers. It remains to prove that $s_n^{\pm} = 0$.

To emphasize the dependence on β, we write

$$z_n^{\pm} = z_n^{\pm}(\beta), \quad \Delta_n^{\pm} = \Delta_n^{\pm}(\beta), \quad s_n^{\pm} = s_n^{\pm}(\beta).$$

Take $\varepsilon \in (0,\pi)$. The terms $O(1/z)$ and $O(1/n)$ in formulas (2.1.29) and (2.1.46) are uniform with respect to the parameters μ and β from bounded sets, respectively. If we fix $a > 0$, then there exists $N_1 \in \mathbb{N}$ such that the last term in formula (2.1.51) satisfies the

$$|O(1/n)| < \varepsilon/2, \quad n > N_1, \quad \beta \in [-a,a]. \tag{2.1.52}$$

We fix such N_1.

We also fix a sufficiently large $N_2 \in \mathbb{N}$ and introduce the notation

$$K(z_0;r) = \big(z : |z-z_0| < r\big), \quad P = \big(z : 2\pi N_1 + \pi < |z| < 2\pi N_2 - \pi,\ \operatorname{Im} z > 0\big).$$

Then

$$E_{1/2}(z^2; 3+i\beta) \to E_{1/2}(z^2;3) = \frac{\cosh z - 1}{z}, \quad \beta \to 0,$$

uniformly on P. Zeros of the right-hand sides lying in the semiring P have the form $2\pi i n$, $N_1 < n < N_2$, and all of them are double. By the Hurwitz theorem, there exists $\delta_1 > 0$ such that the disk $K(2\pi i n;\varepsilon)$ contains exactly two zeros of the function $E_{1/2}(z^2; 3+i\beta)$ under the condition

$$\beta \in U_{\delta_1}(0) = \big(\beta : 0 < |\beta| < \delta_1\big), \quad N_1 < n < N_2.$$

But zeros of the function $E_{1/2}(z^2; 3+i\beta)$ are the points (2.1.50).

Since $\ln w^{+}(z) \to 0$ as $z \to 1$, there exists $\delta_2 \in (0,\delta_1]$ such that

$$\left| -\frac{\pi\beta}{2} \pm \ln w^{+}\big(\gamma(\beta) e^{i\beta \ln 2\pi n}\big) \right| < \frac{\varepsilon}{2}, \quad \beta \in U_{\delta_2}(0), \quad N_1 < n < N_2. \tag{2.1.53}$$

Therefore, if in (2.1.51) we set $s_n^{\pm}(\beta) = 0$ for $\beta \in U_{\delta_2}(0)$, then by (2.1.50)–(2.1.53) we have

$$\sqrt{z_n^{\pm}(\beta)} \in K(2\pi i n; \varepsilon), \quad N_1 < n < N_2.$$

This means that for $\beta \in U_{\delta_2}(0)$, $N_1 < n < N_2$, formula (2.1.51) is valid for $s_n^{\pm} = 0$.

The convergence

$$E_\rho(z;\mu) \to E_\rho(z;\mu_0), \quad \mu \to \mu_0,$$

is uniform in any disk; hence by the Hurwitz theorem we deduce the continuous dependence of $\sqrt{z_n^{\pm}(\beta)}$ on β. By (2.1.50), the dependence $\Delta_n^{\pm}(\beta)$ on β is also continuous. The first two terms on the right-hand side of (2.1.51) are continuous functions of β; therefore, the sum of the last terms in (2.1.51)

$$2\pi i s_n^{\pm}(\beta) + O(1/n) =: A_n^{\pm}(\beta)$$

is a continuous function of β for fixed n. By the above and (2.1.52), there exists $\delta \in (0,\delta_2]$ such that

$$|A_n^{\pm}| < \frac{\varepsilon}{2} < \frac{\pi}{2}, \quad \beta \in U_\delta(0), \quad N_1 < n < N_2. \tag{2.1.54}$$

However, if $s_n^{\pm}(\beta) \neq 0$, i.e., $|s_n^{\pm}(\beta)| \geq 1$, then

$$|A_n^{\pm}(\beta)| > 2\pi - \varepsilon/2 > (3/2)\pi,$$

which contradicts (2.1.54) and the continuity of the function $A_n^{\pm}(\beta)$. Therefore, $s_n^{\pm}(\beta) = 0$, $-a \le \beta \le a$, $N_1 < n < N_2$.

On the real half-lines $(-\infty, -1)$ and $(1, +\infty)$, the function $\ln w^+(z)$ is continuous only by the set $\operatorname{Im} z \ge 0$. Therefore, we must clarify our reasoning related to the continuity if for some $0 \ne \beta_0 \in [-a, a]$, $N_1 < n < N_2$, the point $t_n(\beta_0)$, where

$$t_n(\beta) = \gamma(\beta) e^{i\beta \ln 2\pi n}, \tag{2.1.55}$$

lies on one of these half-lines.

Introduce the sets $U^{\pm}$ by the conditions

$$\beta \in U^+ \ (U^-) \quad \Longleftrightarrow \quad \operatorname{Im} t_n(\beta) \ge 0 \ (< 0).$$

Then the function

$$\begin{cases} \ln w^+(t_n(\beta)), & \beta \in U^+, \\ -\ln w^+(t_n(\beta)), & \beta \in U^-, \end{cases}$$

by formula (2.1.35), is continuous at the point β_0. Therefore (see (2.1.51)), if for $\beta \in U^-$ we change the roles of Δ_n^+ and Δ_n^- (and s_n^+ and s_n^-, respectively), then all the above reasoning based on the continuity remains valid.

Since we can take N_2 arbitrarily large, $s_n^{\pm}(\beta) = 0$, $n > N_1$, $\beta \in [-a, a]$. However, $a > 0$ in our reasoning is arbitrary. Therefore, formula (2.1.51) with $s_n^{\pm} = 0$ is valid for all $0 \ne \beta \in \mathbb{R}$. Taking into account notation (2.1.38) and formula (2.1.50), we see that assertion (2) is proved.

(3) Consider ellipses symmetric with respect to coordinate axes; we assume that the point of ellipses lying on the real axis belong to the upper semi-ellipses.

Let $\rho > 1$ and

$$a_\rho = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right), \quad b_\rho = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right). \tag{2.1.56}$$

The function $w^+(z)$ maps the upper (lower) semi-ellipse with semi-axes a_ρ and b_ρ on the semicircle $|w| = \rho$, $\operatorname{Im} w \ge 0$ (respectively, $|w| = 1/\rho$, $\operatorname{Im} w > 0$).

Let C_β be the circle $|z| = |\gamma(\beta)| > 1$ and C_β^+ and C_β^- be its upper and lower semi-circles. Let l_β and L_β be ellipses with semi-axes (2.1.56) such that l_β is inscribed in C_β and L_β is circumscribed about C_β. Let l_β^+ and l_β^- (respectively, L_β^+ and L_β^-) be the upper and lower semi-ellipses of the ellipse l_β (respectively, L_β). Let a be the first semi-axis of the ellipse l_β and b be the second semi-axis of the ellipse L_β, i.e., $a = b = |\gamma(\beta)|$. Then, taking into account (2.1.56), we conclude that the image of the upper semi-ellips l_β^+ (respectively, L_β^+) under the mapping $w^+(z)$ is the upper semi-circle $|w| = \rho_1 > 1$, $\operatorname{Im} w \ge 0$ (respectively, $|w| = \rho_2 > 1$, $\operatorname{Im} w \ge 0$), where

$$\frac{1}{2}\left(\rho_1 + \frac{1}{\rho_1}\right) = |\gamma(\beta)| = \frac{1}{2}\left(\rho_2 - \frac{1}{\rho_2}\right).$$

Solving these quadratic equations and taking into account the fact that $\rho_1, \rho_2 > 1$, we obtain formulas (2.1.37) for ρ_1 and ρ_2. Finally, the image of the semi-circle C_β^+ is a curve lying in the semi-ring $\rho_1 \le |w| \le \rho_2$, $\operatorname{Im} w \ge 0$, i.e.,

$$z \in C_\beta^+ \quad \Longrightarrow \quad \rho_1 \le |w^+(z)| \le \rho_2, \quad \operatorname{Im} w^+(z) \ge 0. \tag{2.1.57}$$

Arguing similarly for the lower semi-ellipses l_β^- and L_β^-, we obtain

$$z \in C_\beta^- \quad \Longrightarrow \quad \frac{1}{\rho_2} \le |w^+(z)| \le \frac{1}{\rho_1}, \quad \operatorname{Im} w^+(z) > 0. \tag{2.1.58}$$

By formula (2.1.38), we have $\delta_n = \ln w^+(z)$, where $z \in C_\beta$. Therefore, applying (2.1.57) and (2.1.58), we obtain the relation

$$|\operatorname{Re}\delta_n| = |\operatorname{Re}\ln w^+(z)| = |\ln|w^+(z)|| \in [\ln\rho_1, \ln\rho_2].$$

This and formula (2.1.40) prove assertion (3).

(4) We denote by K_β^+ the image of the semi-circle C_β^+ under the mapping $w^+(z)$. We know that the curve K_β^+ lies in the semi-ring $\rho_1 \le |w| \le \rho_2$, $\operatorname{Im} w \ge 0$, starts at the point ρ_1, passes through the point $i\rho_2$, and ends at the point $-\rho_1$. Therefore,

(1) for any $\theta \in (0, \pi)$, there exists a point $w = Re^{i\theta} \in K_\beta^+$;

(2) for any $\rho \in (\rho_1, \rho_2)$, there exists a point $w = \rho e^{i\varphi} \in K_\beta^+$.

Let $z \in C_\beta^+$ be the preimage of the point $w \in K_\beta^+$, $\operatorname{Im} w > 0$, $U(w)$ be a circular neighborhood of the point w, and $U(z)$ be the preimage of $U(w)$, i.e., $U(z)$ is a neighborhood of the point z. We assume that the neighborhood $U(w)$ is so small that it lies in the open upper half-plane. Then the neighborhood $U(z)$ also possesses this property.

Since $\ln 2\pi n \to +\infty$ as $n \to \infty$, the point $t_n \in C_\beta^+$ (see (2.1.55)) performs an infinite number of half-turns. Since $\ln 2\pi(n+1) - \ln 2\pi n \to 0$ as $n \to \infty$, we see that $U(z)$ contains infinitely many pairwise distinct points t_n of the semi-circle C_β^+. Therefore, $U(w)$ contains infinitely many pairwise distinct points $w_n^+ = w^+(t_n)$, $n \in T \subset \mathbb{N}$. From this point, we consider the sequences $\operatorname{Im}\delta_n$ and $\operatorname{Re}\delta_n$ separately.

Let $\theta \in (0, \pi)$ be fixed and $w = Re^{i\theta}$ be the corresponding point on K_β^+ (see (1)). We prove that the points $\arg w_n^+$, $n \in T$, are pairwise distinct. Indeed, if the points w_n^+ and w_m^+ have the same arguments, then they lie on the same ray in the upper half-plane ematating from the origin. The Joukowsky function maps this ray either in the ray $\arg z = \pi/2$ (if $\theta = \pi/2$) or in one of the branches (left or right) of a hyperbola symmetric with respect to the coordinate axes. But the semi-circle C_β^+ does not contain any pair or points that belong to one of these sets. We have proved that the points $\arg w_n^+$, $n \in T$, are pairwise distinct. Since the points w_n^+ lie in $U(w)$, the corresponding (real) neighborhood of the point θ contains an infinite set of points $\arg w_n^+$. By the arbitrariness of $U(w)$ this means that the point θ is a limit point for the sequence

$$\arg w_n^+ = \operatorname{Im}\ln w_n^+ = \operatorname{Im}\delta_n.$$

It remains to recall that θ is an arbitrary point from $(0, \pi)$ and that the set of limit points is closed. Assertion (4) for the sequence $\operatorname{Im}\delta_n$ is proved.

Let $\rho \in (\rho_1, \rho_2)$ be fixed and $w = \rho e^{i\varphi}$ be the corresponding point on K_β^+ (se (2)). We prove that any triple of points $|w_n^+|$, $n \in T$, contains a pair of distinct points. Indeed, if some three points w_n^+ have the same moduli, then these points lie on the same upper semi-circle centered at 0. The Joukowsky function maps this semi-circle on the upper semi-ellips (which does not coincide with the semi-circle). But on C_β^+ only two points can belong to such a semi-ellipse, i.e., our assumption is invalid. Therefore, the corresponding (real) neighborhood of the point ρ contains an infinite number of points $|w_n^+|$ and any neighborhood of the point $\ln\rho$ contains an infinite number of points

$$\ln|w_n^+| = \operatorname{Re}\ln w_n^+ = \operatorname{Re}\delta_n.$$

Since $\rho \in (\rho_1, \rho_2)$ is arbitrary and the set of limit points is closed, we have proved that any point of the segment $[\ln\rho_1, \ln\rho_2]$ is a limit point for the sequence $\operatorname{Re}\delta_n$. In our reasoning, we have used the upper semi-circle C_β^+ and its image K_β^+. Arguing similarly for the lower semi-circle C_β^- and its image K_β^- and taking into account (2.1.58), we obtain that any point of the segment $[\ln(1/\rho_2), \ln(1/\rho_1)]$ is also a limit point for the sequence $\operatorname{Re}\delta_n$. Assertion (4) is completely proved.

(5) We must prove that the sequence Δ_n is separated from zero (see (2.1.42)). Assume the contrary: let $\Delta_n \to 0$ for some sequence of indices $n = n_k \to \infty$. Passing to the limit as $n = n_k \to \infty$ in the equality for moduli in (2.1.47) and applying (2.1.39), we obtain

$$\cosh\frac{\pi\beta}{2} = |\gamma(\beta)| = \sqrt{\frac{\sinh \pi\beta}{\pi\beta}}.$$

However, for $\beta \neq 0$ this is impossible since for $t \neq 0$ the following relation holds:

$$\cosh^2\frac{t}{2} = \frac{1+\cosh t}{2} = 1 + \sum_{n=1}^{\infty}\frac{t^{2n}}{2(2n)!} > 1 + \sum_{n=1}^{\infty}\frac{t^{2n}}{(2n+1)!} = \frac{\sinh t}{t}.$$

We have proved assertion (5). Theorem 2.1.3 is completely proved. □

3. It remains to consider the case where $0 < \rho < 1/2$.

Lemma 2.1.4. *Let $\rho = 1/m$, $m \in \mathbb{N}$, $m \geq 2$. Then*

$$E_\rho(z;m) = E_{1/m}(z;m) = \rho z^{\rho-1}\sum_{n=0}^{m-1} e^{2\pi i n\rho}\exp(z^\rho e^{2\pi i n\rho}).$$

Proof. Since

$$mE^{1/m}(z;m) = \sum_{n=0}^{m-1} E_1(z^{1/m}e^{2\pi i n/m};m)$$

(see [6]) and

$$E_1(w;m) = \sum_{k=0}^{\infty}\frac{w^k}{(k+m-1)!} = \frac{1}{w^{m-1}}\left(e^w - \sum_{k=0}^{m-2}\frac{w^k}{k!}\right),$$

we have

$$mE_{1/m}(z;m) = z^{1/m-1}\sum_{n=0}^{m-1}\left(e^{2\pi i n(1/m-1)}\exp(z^{1/m}e^{2\pi i n/m}) - \sum_{k=0}^{m-2}\frac{z^{k/m}e^{2\pi i n(k+1-m)/m}}{k!}\right).$$

But the expression

$$\sum_{n=0}^{m-1}\sum_{k=0}^{m-2}\frac{z^{k/m}}{k!}e^{2\pi i n(k+1-m)/m} = \sum_{k=0}^{m-2}\frac{z^{k/m}}{k!}\sum_{n=0}^{m-1}e^{2\pi i n(k+1)/m}$$

vanishes since the last sum is zero. Substituting this into the last formula, we obtain the lemma. □

Lemma 2.1.5. *Let $\rho = 1/m$, $m \in \mathbb{N}$, $m \geq 2$, $s \in \mathbb{Z}_+$. Then*

$$E_\rho(z;m-s) = \rho z^{\rho-1+s\rho}\sum_{n=0}^{m-1} e^{2\pi i n\rho(1+s)}\exp(z^\rho e^{2\pi i n\rho}). \tag{2.1.59}$$

Proof. We prove the lemma by induction. For $s = 0$, formula (2.1.59) is the assertion of Lemma 2.1.4. We show that the inductive hypothesis for an index s implies its validity for the index $s+1$.

From (2.1.59) it follow that

$$zE'_\rho(z;m-s) = \rho(\rho-1+s\rho)z^{\rho-1+s\rho}\sum_{n=0}^{m-1} e^{2\pi i n\rho(1+s)}\exp(z^\rho e^{2\pi i n\rho})$$
$$+ \rho^2 z^{2\rho-1+s\rho}\sum_{n=0}^{m-1} e^{2\pi i n\rho(2+s)}\exp(z^\rho e^{2\pi i n\rho}).$$

We substitute this and (2.1.59) on the right-hand side of the formula

$$\rho E_\rho(z;\mu-1)=\rho(\mu-1)E_\rho(z;\mu)+zE'_\rho(z;\mu)$$

(see [6]) setting here $\mu=m-s$. Since $\rho=1/m$, we have proved the lemma. □

Remark 2.1.1. Formula (2.1.59) can be written in the form

$$E_\rho(z;\mu)=\rho z^{\rho(1-\mu)}\sum_{n=0}^{m-1}e^{2\pi in\rho(1-\mu)}\exp(z^\rho e^{2\pi in\rho}),\quad \rho=\frac{1}{m},\quad \mu=m-s. \tag{2.1.60}$$

It is valid for any branch of the function z^ρ since the right-hand side of (2.1.60) is invariant under the change of $\arg z$ by $\arg z+2\pi$.

Theorem 2.1.4. *Let $0<\rho<1/2$. Then all sufficiently large (in modulus) zeros z_n of the function $E_\rho(z;\mu)$ are simple and can be described by the asymptotic formula*

$$z_n=-\left(\frac{\pi}{\sin\pi\rho}\left(n-\frac{1}{2}+\rho(\mu-1)\right)+\alpha_n\right)^{1/\rho},\quad n\to+\infty, \tag{2.1.61}$$

where

(1) *in the case of exclusive pairs ρ,μ we have*

$$\alpha_n=O\big(e^{-\pi n(\cos\pi\rho-\cos 3\pi\rho)/\sin\pi\rho}\big);$$

(2) *if $1/4<\rho<1/2$, then*

$$\alpha_n=O\big(n^{-\operatorname{Re}\tau_\mu/\rho}e^{-\pi n\cot\pi\rho}\big);$$

(3) *if $\rho\le 1/4$, then*

$$\alpha_n=e^{-\pi n\cot\pi\rho}\Big(O\big(e^{\pi n\cos 3\pi\rho/\sin\pi\rho}\big)+O\big(n^{-\operatorname{Re}\tau_\mu/\rho}\big)\Big).$$

If μ is real, then all sufficiently large (in modulus) zeros are real.

Proof. First, we consider exclusive pairs ρ,μ, i.e.,

$$\rho=1/m,\quad m\in\mathbb{N},\quad m\ge 3,\quad \mu=m-s,\quad s\in\mathbb{Z}_+.$$

We set $w=z^\rho e^{-i\pi\rho}$; then to the whole plane $0\le\arg z<2\pi$ corresponds the sector $-\pi\rho\le\arg w<\pi\rho$, and by (2.1.60) we must examine the asymptotics of zeros of the quasi-polynomial

$$P(w)=\sum_{n=0}^{m-1}e^{2\pi in\rho(1-\mu)}\exp(we^{(2n+1)\pi i\rho}).$$

In this sum, we consider the terms with $n=0$ and $n=m-1$ (in the sector specified, they play the key role):

$$P(w)=\exp(we^{i\pi\rho})+e^{-i(1-\mu)2\pi\rho}\exp(we^{-i\pi\rho})+\alpha(w), \tag{2.1.62}$$

$$\alpha(w)=\sum_{n=1}^{m-2}d_n\exp(\lambda_n w),\quad \lambda_n=e^{(2n+1)i\pi\rho}.$$

We have the relation

$$\big|\alpha(w)\big|=\left|\sum_{n=1}^{m-2}d_n\exp\left(we^{i\pi\rho(2n+1)}\right)\right|=O\Big(\exp\big(\operatorname{Re}w\cos 3\pi\rho-\operatorname{Im}w\sin 3\pi\rho\big)\Big) \tag{2.1.63}$$

in the sector $|\arg w| \le \pi\rho$, since in this sector the growth of the quasi-polynomial $\alpha(w)$ is determined by the right-hand side of the points $\exp(i\pi\rho(2n+1))$, i.e., by the points with indices $n = 1, m-2$; for these points

$$\left|\exp\left(we^{i\pi\rho(2n+1)}\right)\right| = O\Big(\exp\left(\operatorname{Re} w\cos 3\pi\rho - \operatorname{Im} w \sin 3\pi\rho\right)\Big).$$

On the set $0 < \varepsilon \le \arg w \le \pi\rho$ ($-\pi\rho \le \arg w \le -\varepsilon$) in formula (2.1.62), the second (respectively, first) term dominates. It grows exponentially on this set, and hence the quasi-polynomial $P(w)$ grows exponentially in these sectors. Therefore, for arbitrarily small $\varepsilon > 0$, all zeros of the function $E_\rho(z;\mu)$ belonging to the set $|\arg w| \le \pi\rho$, perhaps, except for a finite number, lie in the sector

$$|\arg w| < \varepsilon. \tag{2.1.64}$$

From (2.1.62) and (2.1.63) we see that if $\theta = \arg w \in (-\varepsilon, \varepsilon)$, then

$$\begin{aligned} P(w) &= \exp\left(we^{i\pi\rho}\right) + e^{-i(1-\mu)2\pi\rho}\exp\left(we^{-i\pi\rho}\right) + O\Big(e^{\operatorname{Re} w\left(\cos 3\pi\rho - \tan\theta\sin 3\pi\rho\right)}\Big) \\ &= 2e^{w\cos\pi\rho - i\pi\rho(1-\mu)}\cos\left(w\sin\pi\rho + \pi\rho(1-\mu)\right) + O\Big(e^{\operatorname{Re} w\left(\cos 3\pi\rho - \tan\theta\sin 3\pi\rho\right)}\Big). \end{aligned}$$

If ε is sufficiently small, then

$$\cos\pi\rho - \cos 3\pi\rho - \tan\varepsilon|\sin 3\pi\rho| > 0;$$

therefore, for zeros w_n of the quasi-polynomial $P(w)$ in the sector (2.1.64) we have the equation

$$\cos\left(w\sin\pi\rho + \pi\rho(1-\mu)\right) = o(1), \quad \operatorname{Re} w \to +\infty, \tag{2.1.65}$$

where $o(1)$ is

$$O\Big(\exp\left(-\operatorname{Re} w(\cos\pi\rho - \cos 3\pi\rho + \tan\theta\sin 3\pi\rho)\right)\Big).$$

Equation (2.1.65) shows that $|\operatorname{Im} w_n| \le h < +\infty$ for all n. But then $\operatorname{Re} w \tan\theta = O(1)$ for $w = w_n$ and hence the term $o(1)$ in (3.7.4) is

$$O\Big(\exp\left(-\operatorname{Re} w(\cos\pi\rho - \cos 3\pi\rho)\right)\Big). \tag{2.1.66}$$

Setting $t = w\sin\pi\rho + \pi\rho(1-\mu)$, we rewrite Eq. (2.1.65) in the form

$$\cos t = O\left(\exp\left(-\frac{\cos\pi\rho - \cos 3\pi\rho}{\sin\pi\rho}\operatorname{Re} t\right)\right).$$

Obviously, all sufficiently large (in modulus) roots t_n of this equation in the right-hand half-plane are simple and are described by the asymptotic formula

$$t_n = \pi n + \frac{\pi}{2} + O\left(\exp\left(-\pi n\frac{\cos\pi\rho - \cos 3\pi\rho}{\sin\pi\rho}\right)\right), \quad n \to +\infty.$$

Zeros z_n of the function $E_\rho(z;\mu)$ are preimages of the points t_n under the mapping

$$t = z^\rho e^{-i\pi\rho}\sin\pi\rho + \pi\rho(1-\mu).$$

Together with the previous formula this yields formula (2.1.61) for exclusive pairs ρ, μ and the estimate of the remainder from assertion (1).

Now we consider the general (i.e., nonexclusive) case. We use Theorem 1.2.2: if $0 < \rho \le 1/2$, then

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)} \sum_{|\arg z + 2\pi n| \le 3\pi/4\rho} e^{2\pi i n\rho(1-\mu)}\exp(z^\rho e^{2\pi i n\rho}) + O\left(\frac{1}{z}\right), \quad |z| \to \infty. \tag{2.1.67}$$

Formula (2.1.67) is valid for any single-valued branch of the power function since the replacement of $\theta = \arg z$ by $\theta + 2\pi$ leads to the decreasing by 1 of the index n, i.e., to the renumbering of indices. It is convenient to assume that $0 \le \arg z < 2\pi$.

We set again $w = z^\rho e^{-i\pi\rho}$. Let w_n be the images of zeros z_n of the function $E_\rho(z;\mu)$ under this mapping. Then (2.1.67) shows that w_n are zeros of the function

$$P(w) + O\left(\frac{1}{w^{\operatorname{Re}\tau_\mu/\rho}}\right) \tag{2.1.68}$$

in the sector $-\pi\rho \le \arg w < \pi\rho$, where

$$P(w) = \sum_{|\arg w + (2n+1)\pi\rho| \le 3\pi/4} e^{2\pi i n\rho(1-\mu)} \exp(we^{(2n+1)\pi i\rho}). \tag{2.1.69}$$

Consider the sector (2.1.64) assuming that ε is sufficiently small. Since $\rho < 1/2$, formula (2.1.69) necessarily contains points w with indices $n = 0$ and $n = -1$.

We denote by N the maximal index in formula (2.1.69). We have the inequality

$$\frac{3\pi}{4} \ge \left|(2N+1)\pi\rho - \arg w\right| > (2N+1)\pi\rho - \varepsilon,$$

$$(2N+1)\pi\rho < \frac{3\pi}{4} + \varepsilon. \tag{2.1.70}$$

Therefore, points

$$\lambda_n = e^{(2n+1)i\pi\rho}$$

with nonnegative indices lie on the arc $|w| = 1$, $0 \le \arg w < 3\pi/4 + \varepsilon$. Similarly, points λ_n with negative indices lie on the arc $|w| = 1$, $-3\pi/4 - \varepsilon < \arg w \le 0$. Therefore, among the exponents λ_n of the quasi-polynomial

$$P(w) = \sum_n h_n e^{\lambda_n w}, \quad h_n = e^{2\pi i n\rho(1-\mu)} \ne 0, \quad \lambda_n = e^{(2n+1)i\pi\rho},$$

the points

$$\lambda_0 = e^{i\pi\rho}, \quad \lambda_{-1} = e^{-i\pi\rho},$$

have the maximal real parts.

From (2.1.70) follows that if $\rho > 1/4$ and ε is sufficiently small, then $N = 0$. By analogy, the minimal index n in formula (2.1.69) is $n = -1$, i.e., if $\rho > 1/4$, then formula (2.1.69) contains only indices $n = 0$ and $n = -1$.

We have denoted function (2.1.69) by the same symbol as function (2.1.62) not accidentally. In (2.1.69), in contrast to (2.1.62), the set of indices n varies depending on w, but the set of collections of such indices is finite when $|\arg w| \le \pi\rho$, and hence these function behave similarly.

Indeed, if $0 < \varepsilon \le \arg w \le \pi\rho$ ($-\pi\rho \le \arg w \le -\varepsilon$), then the term with index $n = -1$ (respectively, $n = 0$) dominates in (2.1.69). Therefore, on this set function (2.1.69) (and, therefore, (2.1.68)) grows exponentially. Hence zeros of function (2.1.68) lying in the sector $|\arg w| \le \pi\rho$, perhaps, except for a finite number, actually belong to the set (2.1.64).

After this remark, we can repeat the reasoning of the exclusive case. The unique change is as follows: If we write $P(w)$ in the form (2.1.62), then $\alpha(w)$ consists of the last expression from (2.1.68) and the terms of (2.1.69) corresponding to the most right-hand points $e^{\lambda_n w}$, i.e., to (possible) points with indices $n = 1$ and $n = -2$. Therefore, the form of the right-hand side $o(1)$ in Eq. (2.1.65) is slightly different from that in Eq. (2.1.66). However, formula (2.1.61) is valid in all cases, and we must only observe how this change affects the behavior of the remainder α_n.

Let $1/4 < \rho < 1/2$. Then, as we have seen, Eq. (2.1.69) contains only points with $n = 0$ and $n = -1$. Therefore, $\alpha(w)$ is reduced to the last expression in (2.1.68), and hence the right-hand side of (2.1.65) has the form

$$O\Big(|w|^{-\operatorname{Re}\tau_\mu/\rho} e^{-\operatorname{Re} w\cos\pi\rho}\Big).$$

Finally, we obtain the estimate of the remainder from assertion (2).

If $\rho \le 1/4$, then we must take into account points λ_n, $n = 1$, $n = -2$. Therefore, the right-hand side in (2.1.65) is the sum of expression (2.1.66) and

$$e^{-w\cos\pi\rho} O\big(w^{-\operatorname{Re}\tau_\mu/\rho}\big).$$

Then we obtain the estimate of the remainder from assertion (3).

In all cases, the simpleness of sufficiently large (in modulus) zeros of the function $E_\rho(z;\mu)$ follows from the simpleness of roots of Eq. (2.1.65) for $\operatorname{Re} w > R_0$.

The fact that all sufficiently large (in modulus) zeros are real for real μ is proved as in Theorem 2.1.2. Theorem 2.1.4 is completely proved. □

Apparently, Dzhrbashyan and Nersesyan first began to study asymptotics of zeros of Mittag-Leffler function (see [6]). Their results are restricted by various constraints imposed on the parameters ρ and μ, for example, $\rho \ge 1/2$. In [29], for all $\rho > 0$ and $\mu \in \mathbb{C}$, except for the case where $\rho = 1/2$ and $\operatorname{Re}\mu \ne 3$, asymptotic formulas for zeros were obtained with accuracy to an infinitesimal term whose order had, however, been estimated; in particular, Theorem 2.1.4 is contained in [29]. Improved formulas from Theorems 2.1.1 and 2.1.2, in which the first terms of asymptotics are presented, are published here for the first time. Theorem 2.1.3 relating to the exclusive case $\rho = 1/2$, $\operatorname{Re}\mu = 3$ was proved in [36].

2.2. Matching of Asymptotics and Numeration of Zeros

Formula for zeros z_n of the function $E_\rho(z;\mu)$ obtained in the previous section are asymptotic; they describe the behavior of zeros outside a disk of sufficiently large radius but do not yield information on the number of zeros inside this disk. Obviously, the problem on the number of such zeros is equivalent to the problem on matching of asymptotics and the numeration of all zeros.

Definition. Let $Z = (z_n)$ be the set of all zeros of an entire function $F(z)$ and let the following asymptotic formula hold:

$$z_n = \varphi(n) + o(1), \quad n \to \pm\infty \text{ (or } n \to +\infty), \tag{2.2.1}$$

where $\varphi(n) \to \infty$. We say that *asymptotics* (2.2.1) *is matched with the numeration* of all zeros of the function $F(z)$ by the index set T if there exists a bijection $T \leftrightarrow Z$ preserving asymptotics (2.2.1).

Note that we assume that to any zero λ of multiplicity m in Z, elements $z_{s+1} = \ldots = z_{s+m} = \lambda$ with different numbers correspond.

We explain this definition by the examples of the functions $F(z) = \sin z$ and $F(z) = (\sin z)/z$. Their zeros have the same asymptotics

$$z_n = \pi n + o(1), \quad n \to \pm\infty.$$

In the first case, this asymptotics is matched with the numeration of all zeros by the index set $\mathbb{Z}$, but in the second case by the index set $\mathbb{Z}\backslash\{0\}$.

In this section, we find index sets T used for matching of the asymptotic formulas of Theorem 2.1.1, 2.1.2, and 2.1.4 with the numeration of all zeros of the function $E_\rho(z;\mu)$. The following assertions hold.

Theorem 2.2.1. (1) *If*

$$\rho > \frac{1}{2},\ \mu \ne \frac{1}{\rho} - l,\ l \in \mathbb{Z}_+ \quad or \quad \rho = \frac{1}{2},\ \operatorname{Re}\mu > 3,$$

then the asymptotic formula (2.1.7) *is matched with the numeration of all zeros of the function* $E_\rho(z,\mu)$ *by the index set* $\mathbb{Z}\backslash\{0\}$.

(2) *If*

$$\rho > \frac{1}{2}, \ \rho \neq 1, \ \mu = \frac{1}{\rho} - l, \ l \in \mathbb{Z}_+,$$

then the asymptotic formula (2.1.7) *is matched with the numeration of all zeros of the function* $E_\rho(z,\mu)$ *by the index set* $\mathbb{Z}\backslash\{0\}\backslash\{1\}$.

Theorem 2.2.2. (1) *If*

$$\rho = \frac{1}{2}, \ \operatorname{Re}\mu < 3, \ \mu \neq 2 - l, \ l \in \mathbb{Z}_+,$$

then the asymptotic formula (2.1.28) *is matched with the numeration of all zeros of the function* $E_\rho(z;\mu)$ *by the index set* $\mathbb{N}$.

(2) *If*

$$\rho < \frac{1}{2},$$

then the asymptotic formula (2.1.61) *is matched with the numeration of all zeros of the function* $E_\rho(z;\mu)$ *by the index set* $\mathbb{N}$.

For example, consider the function

$$E_1(z;2) = \frac{e^z - 1}{z}.$$

Here $c_\mu = 1$ and $\tau_\mu = 0$ and by Theorem 2.1.1, the asymptotics of zeros of this function has the form

$$z_n = 2\pi ni + o(1), \quad n \to \pm\infty.$$

Theorem 2.2.1 asserts that the asymptotics is matched with the numeration of all zeros by the index set $\mathbb{Z}\backslash\{0\}$. On the other hand, we have exact expressions for zeros:

$$z_n = 2\pi ni, \quad n \in \mathbb{Z}\backslash\{0\}.$$

Proof. If a point $z = 0$ is a zero of the function $E_\rho(z,\mu)$ of multiplicity m, then we pass to the function

$$F(z) = \left(\frac{z-1}{z}\right)^m E_\rho(z;\mu), \quad F(0) \neq 0.$$

The proofs of Theorems 2.2.1 and 2.2.2 are based on the Jensen formula

$$\int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|F(re^{i\theta})| d\theta - \ln|F(0)|, \tag{2.2.2}$$

where $n(t)$ is the number of zeros of a function $F(z)$ in the disk $|z| < r$, which, obviously, coincides for $r > 1$ with the number of zeros of the function $E_\rho(z;\mu)$.

2.2.1. Estimate for the integral of the logarithm of the modulus.

Case $\rho > 1/2$. We use formulas (2.1.12) and (2.1.13), which are uniform in the corresponding sectors with respect to $\theta = \arg z$. In these formulas, we set $s = 1$ for $\mu \neq 1/\rho - l$, $k \in \mathbb{Z}_+$ and $s = 2$ for $\mu = 1/\rho - l$.

Let r_0 be sufficiently large. Then the set of points satisfying the conditions

$$\operatorname{Re}\left(z^\rho + \tau_\mu \ln z\right) - \ln|c_\mu| = 1, \quad |z| \geq r_0,$$

consists of two infinite curves starting at some points M_+ and N_+ of the circle $|z| = r_0$, and the union of this set with the right-hand arc M_+N_+ of the circle $|z| = r_0$ is an infinite curve, which we denote by γ_+. Similarly, we denote by γ_- the curve obtained as the union of points satisfying the condition

$$\operatorname{Re}\left(z^\rho + \tau_\mu \ln z\right) - \ln|c_\mu| = -1, \quad |z| \geq r_0,$$

with some left-hand arc M_-N_- of the circle $|z| = r_0$. Both curves $\gamma_\pm$ are asymptotically situated along the rays $\arg z = \pm\pi/(2\rho)$.

We denote by P_+ the set of points lying to the right of γ_+ and by P_- the set of points lying to the left of γ_-. Let P be the closure of the set of points lying between γ_+ and γ_- with the removed disk $|z| \le r_0$. The set P_+ (respectively, P_-) is a right-hand (respectively, left-hand) curvilinear half-plane and P is a curvilinear strip lying between them with the removed disk $|z| \le r_0$. We have

$$\mathbb{C} \setminus \big(z : |z| \le r_0\big) = P_+ \cup P_- \cup P$$

and the relations

$$\begin{aligned} P_- &= \Big(z : r = |z| > r_0,\ \operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big) - \ln|c_\mu| < -1\Big), \\ P_+ &= \Big(z : r > r_0,\ \operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big) - \ln|c_\mu| > 1\Big), \\ P &= \Big(z : r > r_0,\ -1 \le \operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big) - \ln|c_\mu| \le 1\Big). \end{aligned}$$

Let $\rho > 1$. If $z \in P_-$ and $\pi/\rho \le |\arg z| \le \pi$, then by formula (2.1.13)

$$z^s E_\rho(z;\mu) \rightrightarrows -\rho c_\mu, \quad r \to \infty, \quad \frac{\pi}{\rho} \le |\arg z| \le \pi. \tag{2.2.3}$$

If $z \in P_-$ and $|\arg z| \le \pi/\rho$, then the definition of the set P_- and formula (2.1.12) imply that

$$\left|\frac{1}{\rho} z^s E_\rho(z;\mu)\right| \le \frac{|c_\mu|}{e} + |c_\mu| + O\left(\frac{1}{r}\right) \le M < \infty, \quad r > r_0, \tag{2.2.4}$$

$$\left|\frac{1}{\rho} z^s E_\rho(z;\mu)\right| \ge |c_\mu| - \frac{|c_\mu|}{e} + O\left(\frac{1}{r}\right) \ge m > 0, \quad r > r_0. \tag{2.2.5}$$

From (2.2.3), (2.2.4), and (2.2.5) we obtain

$$|z^s E_\rho(z;\mu)| \asymp 1, \quad r > r_0, \quad z \in P_-. \tag{2.2.6}$$

If $1/2 < \rho \le 1$, then the reasoning related to formula (2.2.3) fails and formulas (2.2.4) and (2.2.5) are valid for $|\arg z| \le \pi$, and we again arrive at estimate (2.2.6).

Let $z \in P_+$. Then by formula (2.1.12)

$$\begin{aligned} &\exp\Big(\operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big)\Big) - |c_\mu| + O\left(\frac{1}{r}\right) \\ &\qquad \le \left|\frac{1}{\rho} z^s E_\rho(z;\mu)\right| \le \exp\Big(\operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big)\Big) + |c_\mu| + O\left(\frac{1}{r}\right). \end{aligned} \tag{2.2.7}$$

By the definition of the set P_+

$$|c_\mu| \le \frac{1}{e} \exp\Big(\operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big)\Big), \quad z \in P_+. \tag{2.2.8}$$

Substituting (2.2.8) in (2.2.7), we obtain the estimate

$$|z^s E_\rho(z;\mu)| \asymp \exp\Big(\operatorname{Re}\big(z^\rho + \tau_\mu \ln z\big)\Big), \quad r > r_0, \quad z \in P_+. \tag{2.2.9}$$

It remains to consider the set P. Under mapping (2.1.15) it transforms to a set, which, for large $|w|$, coincides with the vertical strip

$$\gamma_- \le \operatorname{Re} w \le \gamma_+, \quad \gamma\pm = \ln|c_\mu| \pm 1.$$

Obviously, for sufficiently large $|z|$, mapping (2.1.15) is univalent. We consider the image $e(w)$ of the function $(1/\rho) z^s E_\rho(z;\mu)$ under this mapping. By formula (2.1.12),

$$e(w) = e^w - c_\mu + o(1) =: \varphi(w) + o(1), \quad \gamma_- \le \operatorname{Re} w \le \gamma_+, \quad |\operatorname{Im} w| \to \infty. \tag{2.2.10}$$

The function $\varphi(w)$ is $2\pi i$-periodic and continuous. Therefore,

$$0 < c(\delta) \le |\varphi(w)| \le M < \infty$$

for $\gamma_- \le \operatorname{Re} w \le \gamma_+$, $|\operatorname{Im} w| > r_0$, and outside small disks of radius δ centered at zeros of the function $\varphi(w)$. This and relation (2.2.10) imply the estimate

$$0 < m \le |e(w)| \le M < \infty, \quad \gamma_- \le \operatorname{Re} w \le \gamma_+, \quad |\operatorname{Im} w| > r_0, \tag{2.2.11}$$

outside small disks of radius δ centered at zeros w_n of the function $e(w)$, which have the form

$$w_n = 2\pi n i + \ln c_\mu + o(1).$$

Taking sufficiently small δ and returning to the variable z, owing to (2.2.11) we conclude that there exist circles $|z| = r_k \to \infty$ such that on their intersection with the set P estimate (2.2.6) is valid, i.e.,

$$|z^s E_\rho(z;\mu)| \asymp 1, \quad z \in P, \quad |z| = r_k \to \infty. \tag{2.2.12}$$

Estimates (2.2.6), (2.2.9), and (2.2.12) show that on appropriate circles, the following estimates hold:

$$\ln\left|E_\rho(z;\mu)\right| = \operatorname{Re} z^\rho + (\operatorname{Re}\tau_\mu - s)\ln r + O(1), \quad z \in P_+, \qquad |z| = r_k \to \infty, \tag{2.2.13}$$

$$\ln\left|E_\rho(z;\mu)\right| = -s\ln r + O(1), \qquad z \in P_- \cup P, \quad |z| = r_k \to \infty. \tag{2.2.14}$$

Using these estimates, we examine the behavior of the integral

$$I(r) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \ln\left|E_\rho(re^{i\theta};\mu)\right| d\theta, \quad r = r_k \to \infty.$$

We denote by $\alpha_\pm(r)$ the arguments of points that restrict an arc of the circle $|z| = r$ lying in the set P_+, $\alpha_-(r) < 0 < \alpha_+(r)$. From the definition of the set P_+ it follows that both terms $\pm\alpha_\pm(r)$ have the form

$$\frac{\pi}{2\rho} + O\left(\frac{\ln r}{r^\rho}\right) =: \alpha(r). \tag{2.2.15}$$

We represent the integral $I(r)$ by the sum of the integrals I_j, $j = 1, 2, 3, 4$, which are taken over the intervals

$$(0, \alpha_+(r)),\ (\alpha_-(r), 0),\ (\alpha_+(r), \pi) \in (-\pi, \alpha_-(r)),$$

respectively. To estimate I_1 and I_2 we use formula (2.2.13), but for estimate I_3 and I_4 we have formula (2.2.14). Taking into account (2.2.15) we obtain the relation

$$I_3 = \frac{1}{2\pi}\int\limits_{\alpha(r)}^{\pi} (O(1) - s\ln r)d\theta = O(1) - \frac{s}{2\pi}(\pi - \alpha(r))\ln r = s\left(\frac{1}{4\rho} - \frac{1}{2}\right)\ln r + O(1).$$

A similar estimate is valid for I_4. Further, since by (2.2.15)

$$\sin\rho\alpha(r) = \cos\left(O\left(\frac{\ln r}{r^\rho}\right)\right) = 1 + O\left(\frac{\ln^2 r}{r^{2\rho}}\right),$$

we have

$$\begin{aligned} I_1 = \frac{1}{2\pi}\int\limits_0^{\alpha(r)} \Big(r^\rho\cos\rho\theta + (\operatorname{Re}\tau_\mu - s)\ln r + O(1)\Big)d\theta \\ = \frac{r^\rho}{2\pi\rho}\sin\rho\alpha(r) + \frac{\alpha(r)}{2\pi}(\operatorname{Re}\tau_\mu - s)\ln r + O(1) \\ = \frac{r^\rho}{2\pi\rho} + \frac{1}{4\rho}(\operatorname{Re}\tau_\mu - s)\ln r + O(1) \end{aligned}$$

and a similar estimate for I_2. Joining the estimate for I_j, we obtain an intermediate result for the case $\rho > 1/2$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|E_\rho(re^{i\theta};\mu)|d\theta = \frac{r^\rho}{\pi\rho} + \left(\frac{\operatorname{Re}\tau_\mu}{2\rho} - s\right)\ln r + O(1), \quad r = r_k \to \infty. \tag{2.2.16}$$

Case $\rho = 1/2$, $\operatorname{Re}\mu > 3$. The constants c_μ and τ_μ are calculated by formulas (2.1.5); we have the inequality $\operatorname{Re}\tau_\mu < 0$. Let r_0 be sufficiently large. Consider the sets

$$P_+ = \Big(z : r > r_0,\ \operatorname{Re}\left(\sqrt{z} + \tau_\mu \ln z\right) - \ln|c_\mu| \geq 1\Big),$$
$$P = \Big(z : r > r_0,\ \operatorname{Re}\left(\sqrt{z} + \tau_\mu \ln z\right) - \ln|c_\mu| < 1\Big).$$

Obviously,

$$\mathbb{C} \setminus (z : |z| \leq r_0) = P_+ \cup P.$$

Let $0 \leq \arg z \leq \pi$. By formula (2.1.23),

$$2zE_{1/2}(z;\mu) = \exp\left(\sqrt{z} + \tau_\mu \ln z\right) + e^{-i\pi(1-\mu)} z^{\tau_\mu} e^{-\sqrt{z}} - c_\mu + O\left(\frac{1}{r}\right)$$
$$= \exp(\sqrt{z} + \tau_\mu \ln z) - c_\mu + o(1), \quad r \to \infty. \tag{2.2.17}$$

If $0 \leq \arg z \leq \pi$, $z \in P_+$, then inequality (2.2.8) holds with $\rho = 1/2$. This inequality and (2.2.17) imply estimate (2.2.9) with $\rho = 1/2$ and $s = 1$.

Consider mapping (2.1.15) with $\rho = 1/2$. The image of the domain $P \cap (z : \operatorname{Im} z > 0)$ under this mapping for sufficiently large $|\operatorname{Im} w|$ coincides with an expanding curvilinear strip (we denote it by V), which is bounded by the vertical straight line

$$\operatorname{Re} w = 1 + \ln|c_\mu| =: b$$

from the right and by the curve

$$w = i\left(\sqrt{r} + (\operatorname{Im}\tau_\mu)\ln r\right) + (\operatorname{Re}\tau_\mu)\ln r + c, \quad r > r_0.$$

From (2.2.17) follows that the image $e(w)$ of the function $2zE_\rho(z;\mu)$ under the mapping considered has the form (2.2.10). We fix $a < 0$ such that

$$|e^w| = \exp(\operatorname{Re} w) \leq |c_\mu|/2 \quad \text{for } \operatorname{Re} w < a.$$

Introduce the semi-strip

$$V_+ = \Big(w : a < \operatorname{Re} w < b,\ \operatorname{Im} w > r_0\Big),$$

and let $V_- = V \setminus V_+$. By the choice of a and Eq. (2.2.10), we see that

$$|e(w)| \geq |c_\mu|/2 + o(1) \geq m > 0, \quad w \in V_-, \quad \operatorname{Im} w > r_0.$$

If $w \in V_+$, then, using the periodicity of the function e^w, we conclude that the lower estimate $|e(w)| \geq m > 0$ is also valid for points $w \in V_+$ lying outside small disks $K_n(\delta)$ of radius δ centered at zeros of the function $e(w)$. Thus,

$$|e(w)| \geq c(\delta) > 0, \quad w \in V, \quad w \notin \cup K_n(\delta), \quad r > r_0.$$

The upper estimate $|e(w)| \leq M < \infty$ for $w \in V$ immediately follows from (2.2.10) and the definition of the set V. Therefore,

$$|e(w)| \asymp 1, \quad w \in V, \quad w \notin \cup K_n(\delta), \quad r > r_0.$$

Taking sufficiently small δ and returning on the z-plane, we conclude that there exists a sequence $r_k \to \infty$ such that estimate (2.2.12) holds with $\rho = 1/2$ and $s = 1$.

Estimates (2.2.9) and (2.2.12) (with $\rho = 1/2$ and $s = 1$) are proved for $0 \le \arg z \le \pi$. They can also be similarly proved for $-\pi \le \arg z \le 0$. Therefore, these estimates with $s = 1$ are valid for the case $\rho = 1/2$, $\operatorname{Re}\mu > 3$ without restriction on $\arg z$. Finally, in the case where $\rho = 1/2$ and $\operatorname{Re}\mu > 3$ we arrive at estimates (2.2.13) and (2.2.14); moreover, in (2.2.14) the set P_- is absent now. Repeating the reasoning following (2.2.14), with formal replacement of ρ by 1/2 and s by 1, we obtain the following estimate of type (2.2.16):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln |E_{1/2}(re^{i\theta};\mu)|d\theta = \frac{2}{\pi}\sqrt{r} + (\operatorname{Re}\tau_\mu - 1)\ln r + O(1), \quad r = r_k \to \infty. \tag{2.2.18}$$

Cases $\rho = 1/2$, $\operatorname{Re}\mu < 3$ and $\rho < 1/2$. Now we apply formula (2.1.67) and Lemma 2.2.1 (see below) on the behavior of the quasi-polynomial

$$R(w) = \sum_{j=1}^{m} h_j e^{\gamma_j w}, \quad h_j \ne 0, \tag{2.2.19}$$

where γ_j are vertices of some convex polygon G numbered in order of its positive bypass. We set $\gamma_{m+1} = \gamma_1$. Denote by $k(\varphi)$ the support function of the polygon G, i.e.,

$$k(\varphi) = \sup\big(\operatorname{Re}(we^{-i\varphi}) : w \in G\big).$$

Lemma 2.2.1 (see [27]). (1) *The sequence Λ of zeros of quasi-polynomial* (2.2.19) *has the form*

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_m,$$

where the sequence Λ_j is asymptotically distributed along the ray

$$\arg w = \frac{\pi}{2} - \arg(\gamma_{j+1} - \gamma_j)$$

by the law of arithmetic progression, $j = 1, \ldots, m$.

(2) *Outside small disks of the same radii centered at zeros of the function $R(w)$, the following estimate holds*:

$$|R(w)| \asymp \exp(k(-\arg w)|w|).$$

We continue the proof of the theorem. Let $|\arg z| \le \pi$. Then sum (2.1.67) necessarily contains the index $n = 0$. Setting $w = z^\rho$, $|\arg w| \le \pi\rho$, we write this sum in the form

$$Q(w) = \sum_{|\arg w + 2\pi n\rho| \le 3\pi/4} h_n \exp(we^{2\pi i n\rho}), \quad h_n = e^{2\pi i n\rho(1-\mu)}. \tag{2.2.20}$$

According to the summing in (2.2.20), the sector $|\arg w| \le \pi\rho$ is divided into a finite number of sectors S_j without common interior points so that to any sector S_j in (2.2.20) its own index set I_j corresponds. For any j, the points $\exp(2\pi i n\rho)$, $n \in I_j$, are the vertices of some convex polygon G_j (it may degenerate into a segment) one of whose vertices is the point 1. Let Q_j be the quasi-polynomial corresponding to this polygon, i.e., the part of the sum in (2.2.20) corresponding to indices $n \in I_j$. Let $k_j(\varphi)$ be the support function of the polygon G_j. Then by Lemma 2.2.1

$$|Q_j(w)| \asymp \exp(k_j(-\arg w)|w|), \quad w \in S_j, \quad w \notin D_j(\delta), \tag{2.2.21}$$

where D_j in the union of small disks of radius δ centered at zeros of the function $Q_j(w)$. However, $Q_j(w) = Q(w)$ in the sector S_j. Further, the point 1 is a vertex of the polygon G_j; therefore, its support function $k_j(\varphi)$ in the sector $|\arg w| \le \pi\rho$ coincides with the support function of the point 1, i.e., $k_j(\varphi) = \cos\varphi$, $|\varphi| \le \pi\rho$, for all j. Since the set of numbers j is finite, relation (2.2.21) implies that outside $\bigcup D_j(\delta)$ we have

$$|Q(w)| \asymp \exp(|w|\cos\varphi), \quad |\varphi| = |\arg w| \le \pi\rho. \tag{2.2.22}$$

By Lemma 2.2.1, the sector $|\arg w| \le \pi\rho$ can contain only parts of those small disks from $\bigcup D_j$ whose centers are asymptotically distributed along a finite numbers of rays by the law of arithmetic progression. This implies that for sufficiently small δ, we can be sure that estimate (2.2.22) is valid on parts of appropriate circles $|w| = R_k \to \infty$ lying in the sector $|\arg w| \le \pi\rho$. Therefore, if $|\theta| = |\arg z| \le \pi$, then

$$|Q(z^\rho)| \asymp \exp(|z|^\rho \cos\rho\theta), \quad |z| = r_k = R_k^{1/\rho} \to \infty. \tag{2.2.23}$$

We write formula (2.1.67) in the form

$$z^{\rho(\mu-1)} E_\rho(z;\mu) = \rho Q(z^\rho) + O\left(\frac{1}{r^{1+\rho(1-\operatorname{Re}\mu)}}\right). \tag{2.2.24}$$

If $\rho < 1/2$, then $\cos\rho\theta \ge \cos\pi\rho > 0$ and (2.2.23) implies that on the right-hand side of (2.2.24) the first term dominates. If $\rho = 1/2$, then $\cos\rho\theta \ge 0$ and $|Q(z^\rho)| \ge m > 0$. But $\operatorname{Re}\mu < 3$ and hence the last term in (2.2.24) is $o(1)$ as $r \to \infty$; therefore, in the case where $\rho = 1/2$ and $\operatorname{Re}\mu < 3$, the first term on the right-hand side of (2.2.24) dominates. Finally, we have the relation

$$|E_\rho(z;\mu)| \asymp r^{\rho(1-\operatorname{Re}\mu)} \exp(r^\rho \cos\rho\,\theta), \quad |z| = r = r_k \to \infty.$$

From this we obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|E_\rho(re^{i\theta};\mu)|d\theta = \frac{r^\rho}{\pi\rho}\sin\pi\rho + \rho(1-\operatorname{Re}\mu)\ln r + O(1), \quad r = r_k \to \infty. \tag{2.2.25}$$

2.2.2. Estimate of the average density of a positive sequence.

Lemma 2.2.2. *Let a positive sequence $(z_n)_{n=m}^{+\infty}$ have the form*

$$z_n = (an + b\ln n + d + o(1))^{1/\rho}, \quad n \to +\infty, \tag{2.2.26}$$

where $a, \rho > 0$, $b, d \in \mathbb{R}$, and let $n(t)$ denote the number of points of this sequence in an interval $(0,t)$. Then, as $r \to +\infty$, the following relation holds:

$$N(r) := \int_0^r \frac{n(t)}{t}dt = \frac{r^\rho}{a\rho} - \frac{b\rho}{2a}\ln^2 r + \left(\frac{1}{2} - m - \frac{d}{a} + \frac{b}{a}\ln a\right)\ln r + o(\ln r).$$

Proof. We fix $\varepsilon > 0$. Introduce the sequences x_n^+ and x_n^- by the formulas

$$x_n^\pm = an + b\ln n + d \pm \varepsilon, \quad n = k, k+1, \dots,$$

where $k \in \mathbb{N}$ is chosen so large that the term $o(1)$ in (2.2.26) satisfies the inequality $|o(1)| < \varepsilon$, $n \ge k$, and the sequence x_n^- is positive and increasing. Let $n_\pm(t)$ be the number of points $x_n^\pm$ in the interval $(0,t)$. Since the number of points z_n in the interval $(0,t)$ is equal to the number of points z_n^ρ in the interval $(0,t^\rho)$, we have

$$k - m + n_+(t^\rho) \le n(t) \le k - m + n_-(t^\rho), \quad t > x_k^+,$$

and hence for all sufficiently large r, the following inequality holds:

$$M_1 + (k-m)\ln r + \int_{x_k^+}^r \frac{n_+(t^\rho)}{t}dt \le N(r) \le M_2 + (k-m)\ln r + \int_{x_k^-}^r \frac{n_-(t^\rho)}{t}dt. \tag{2.2.27}$$

Further, since

$$n_\pm(t) = n - k + 1, \quad t \in (x_n^\pm, x_{n+1}^\pm),$$

for

$$N = \max(n : (x_n^-)^{1/\rho} \le r)$$

we have the relation

$$\rho\int_{x_k^\pm}^{r}\frac{n_\pm(t^\rho)}{t}\,dt=\rho\sum_{n=k}^{N}(n-k+1)\int_{(x_n^\pm)^{1/\rho}}^{(x_{n+1}^\pm)^{1/\rho}}\frac{dt}{t}+O(1)$$

$$=\sum_{n=k}^{N}(n-k+1)\ln\frac{x_{n+1}^\pm}{x_n^\pm}+O(1)$$

$$=\sum_{n=k}^{N}(n-k+1)\ln\left(1+\frac{a+b\ln(1+1/n)}{an+b\ln n+d\pm\varepsilon}\right)+O(1)$$

$$=\sum_{n=k}^{N}(n-k+1)\left(\frac{a+b\ln(1+1/n)}{an+b\ln n+d\pm\varepsilon}-\frac{1}{2}\left(\frac{a+b\ln(1+1/n)}{an+b\ln n+d\pm\varepsilon}\right)^2\right)+O(1).$$

On the right-hand side, we replace $\ln(1+1/n)$ by $1/n$ and 0 in the first and second fractions, respectively. Taylor's formula shows that in this case we change the right-hand side by a quantity bounded with respect to N. Therefore,

$$\rho\int_{x_k^\pm}^{r}\frac{n_\pm(t^\rho)}{t}dt=\sum_{n=k}^{N}\left(\frac{an-a(k-1)+b}{an+b\ln n+d\pm\varepsilon}-\frac{1}{2}\frac{a^2n}{(an+b\ln n+d\pm\varepsilon)^2}\right)+O(1)$$

$$=\sum_{n=k}^{N}\left(1-\frac{b\ln n}{an+b\ln n+d\pm\varepsilon}+\frac{b-a(k-1)-d\mp\varepsilon}{an+b\ln n+d\pm\varepsilon}\right)-\frac{1}{2}\ln N+O(1)$$

$$=N-\frac{b}{2a}\ln^2 N+\left(\frac{b-a(k-1)-d\mp\varepsilon}{a}-\frac{1}{2}\right)\ln N+O(1).\quad(2.2.28)$$

Now we express N through r using the definition of N and the explicit form of the sequence x_n^-: N is the maximal index n for which $an+b\ln n+d-\varepsilon\le r^\rho$. Obviously,

$$N=\frac{r^\rho}{a}-\frac{b\rho}{a}\ln r+O(1),$$

and hence

$$\ln N=\rho\ln r-\ln a+o(1),$$
$$\ln^2 N=\rho^2\ln^2 r-2\rho\ln a\ln r+o(\ln r).$$

Substituting these expressions for N, $\ln N$, and $\ln^2 N$ in (2.2.28) and then the obtained expressions in (2.2.27), we see that the integral $N(r)$ is contained between the values

$$\frac{r^\rho}{a\rho}-\frac{b\rho}{2a}\ln^2 r+\left(\frac{1}{2}-m+\frac{\pm\varepsilon-d}{a}+\frac{b}{a}\ln a\right)\ln r+o(\ln r),\quad r\to\infty.$$

Since ε can be chosen arbitrarily small, this implies the required asymptotics. Lemma 2.2.2 is proved. □

Let the conditions of Theorem 2.2.1 hold. By Theorem 2.1.1, the sequence Z of zeros z_n of the function $F(z)$ defined in the beginning of the proof can be represented in the form

$$Z=Z_+\cup Z_-,\quad Z_+\cap Z_-=\emptyset,\quad Z_+=(z_n)_{n=m}^{+\infty},\quad Z_-=(z_n)_{n=-1}^{-\infty},$$

where m is an integer. If $|n|$ is sufficiently large, then $\operatorname{Im} z_n \gtrless 0$ respectively for $z_n \in Z_\pm$. Both sequences

$$|Z_+| = (|z_n|)_{n=m}^{+\infty}, \quad |Z_-| = (|z_n|)_{n=-1}^{-\infty}$$

are positive.

By (2.1.7)

$$\operatorname{Im} z_n^\rho = 2\pi n - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi|n| - \frac{\pi}{2\rho}(\operatorname{Re}\tau_\mu)\operatorname{sign} n + \arg c_\mu + o(1),$$
$$\operatorname{Re} z_n = a_1 \ln 2\pi|n| + a_2 \operatorname{sign} n + a_3 + o(1),$$

where a_j are some constants. Therefore,

$$|z_n^\rho| = |\operatorname{Im} z_n^\rho| + o(1),$$

and hence

$$|z_n|^\rho = \left|2\pi n - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln|n| + \arg c_\mu - \frac{\pi}{2\rho}(\operatorname{Re}\tau_\mu)\operatorname{sign} n - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi + o(1)\right|.$$

Thus, if $z_n \in Z_+$, then

$$|z_n| = \left(2\pi n - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln n + \arg c_\mu - \frac{\pi}{2\rho}\operatorname{Re}\tau_\mu - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi + o(1)\right)^{1/\rho}, \quad n \to +\infty,$$

and if $z_n \in Z_-$, then

$$|z_{-n}| = \left(2\pi n + \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln n - \arg c_\mu - \frac{\pi}{2\rho}\operatorname{Re}\tau_\mu + \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi + o(1)\right)^{1/\rho}, \quad n \to +\infty.$$

Therefore, the sequences $|Z_+|$ and $|Z_-|$ satisfy the conditions of Lemma 2.2.2, where in both cases $a = 2\pi$ and

$$b = -\frac{1}{\rho}\operatorname{Im}\tau_\mu, \quad d = \arg c_\mu - \frac{\pi}{2\rho}\operatorname{Re}\tau_\mu - \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi,$$
$$b = \frac{1}{\rho}\operatorname{Im}\tau_\mu, \quad d = -\arg c_\mu - \frac{\pi}{2\rho}\operatorname{Re}\tau_\mu + \frac{1}{\rho}(\operatorname{Im}\tau_\mu)\ln 2\pi$$

for the cases of the sequences $|Z_+|$ and $|Z_-|$, respectively.; note that in the second case one must set $m = 1$. Therefore, if we denote by $n_\pm(t)$ the number of points of the sequence $|Z_\pm|$ in an interval $(0, t)$, then by Lemma 2.2.2

$$\int_0^r \frac{n_+(t) + n_-(t)}{t}\,dt = \frac{r^\rho}{\pi\rho} + \left(\frac{\operatorname{Re}\tau_\mu}{2\rho} - m\right)\ln r + o(\ln r), \quad r \to \infty. \tag{2.2.29}$$

But $n_+(t) + n_-(t)$ is the number of zeros of the function $F(z)$ in the disk $|z| < t$. Therefore, in Jensen's formula (2.2.2) we have $n(t) = n_+(t) + n_-(t)$. Comparing estimates (2.2.29) and (2.2.16) with this formula, we see that if $\rho > 1/2$, then $m = s$. Recall that

$$s = 1 \quad \text{for} \quad \mu \neq \frac{1}{\rho} - l; \quad s = 2 \quad \text{for} \quad \mu = \frac{1}{\rho} - l, \quad l \in \mathbb{Z}_+.$$

Therefore, $m = 1$ and $m = 2$, respectively, and the case $\rho > 1/2$ has been examined.

If $\rho = 1/2$ and $\operatorname{Re}\mu > 3$, we apply formula (2.2.18) instead of formula (2.2.16). But (2.2.18) coincides with (2.2.16) for $\rho = 1/2$ and $s = 1$, and the case $\rho = 1/2$, $\operatorname{Re}\mu > 3$ has also been examined. Theorem 2.2.1 is proved.

Now we prove Theorem 2.2.2. By Theorems 2.1.2 and 2.1.4, the sequence Z of zeros z_n of the function $F(z)$ has the form $Z = (z_n)_{n=m}^{+\infty}$, where m is an integer, and

$$|z_n| = \left(\frac{\pi}{\sin \pi\rho}\left(n - \frac{1}{2} + \rho(\operatorname{Re}\mu - 1) + o(1)\right)\right)^{1/\rho}, \quad n \to +\infty. \tag{2.2.30}$$

Let $n(t)$ be the number of points z_n in a disk $|z| < t$ or, equivalently, the number of points $|z_n|$ in the interval $(0, t)$. Formula (2.2.30) shows that the sequence $|Z|$ satisfies the conditions of Lemma 2.2.2 with

$$a = \frac{\pi}{\sin \pi\rho}, \quad b = 0, \quad d = \frac{\pi}{\sin \pi\rho}\left(\rho(\operatorname{Re}\mu - 1) - \frac{1}{2}\right).$$

By Lemma 2.2.2,

$$\int_0^r \frac{n(t)}{t}dt = \frac{\sin \pi\rho}{\pi\rho} r^\rho - (m - 1 + \rho(\operatorname{Re}\mu - 1))\ln r + o(\ln r), \quad r \to \infty. \tag{2.2.31}$$

On the other hand by Jensen's formula (2.2.2) and estimate (2.2.25) we have the relation

$$\int_0^r \frac{n(t)}{t}dt = \frac{\sin \pi\rho}{\pi\rho} r^\rho - \rho(\operatorname{Re}\mu - 1)\ln r + O(1), \quad r = r_k \to \infty.$$

Comparing this with (2.2.31), we see that $m = 1$. Theorem 2.2.2 is proved. □

The material of this section is taken from [31].

Chapter 3

PROBLEM ON THE REALNESS OF ALL ZEROS OF THE MITTAG-LEFFLER FUNCTION OF ORDER LESS THAN 1/2

3.1. Main Results

In this chapter, we discuss the following problem, interesting from the theoretic standpoint and important in some applications: Are all zeros of the Mittag-Leffler function of order less than 1/2 real? Results of Chap. 2 shows that in the case $\rho \in (0, 1/2)$, $\mu \in \mathbb{R}$, all zeros of $E_\rho(z; \mu)$, perhaps except for a finite number, are real, negative, and simple. So is there an exceptional set of nonreal zeros or not?

The history of this question is more than a century old. It begins from Wiman's paper [44], which asserted that all zeros of the "classical" Mittag-Leffler function $E_\rho(z; 1)$ of order $\rho < 1/2$ are real, negative, simple, and, being ordered in the sequence $\{z_n = z_n(\rho, 1)\}_{n\in\mathbb{N}}$, satisfy the inequalities

$$-\left(\frac{\pi n}{\sin(\pi\rho)}\right)^{1/\rho} < z_n < -\left(\frac{\pi(n-1)}{\sin(\pi\rho)}\right)^{1/\rho} \quad \forall n \in \mathbb{N}. \tag{3.1.1}$$

However, [44] does not contain any proof; Wiman only gave some plausible arguments.

Probably, the absence of a proof in [44] motivated Pólya to publish the paper [21]. He proved that all zeros of $E_\rho(z; 1)$ are negative and simple, but only in the case where $\rho = 1/N$, $N \in \mathbb{N}$, $N \geq 2$. This fact suggests the complexity of the problem. If even such a competent analyst was unable to cope with the problem in full, then, probably, its solution requires a quite nontrivial approach. The localization problem was not studied by Pólya.

In [15], Mikusinski formulated a theorem on the alternation of zeros of the functions

$$E_{1/N}(z;p) \text{ and } E_{1/N}(z;q), \quad 1 \le p < q \le N, \quad p,q \in \mathbb{N}. \tag{3.1.2}$$

However, strangely enough, Mikusinski did not examine the problem on the realness of all zeros of $E_{1/N}(z;p)$. Thus, it is unclear whether he had generalized Pólya's theorem for function (3.1.2) or tried to prove the alternation of only real zeros (in our opinion, his proof is incomplete).

Then for a long time, the problem on the realness of all zeros of the Mittag-Leffler functions of order less than 1/2, to our knowledge, was not discussed. Ostrovskii and Peresyolkova have recently turned to it. In their collaboration [19], the negativeness and simpleness of all zeros of the functions $E_\rho(z;1)$ and $E_\rho(z;2)$ for all $\rho \in (0,1/2])$ were proved. It may appear that Wiman's brain teaser had been solved in [19], but this is deceptive: the localization problem for zeros and inequalities (3.1.1) was not even mentioned in [19].

One of the advantages of the paper [19], of course, is the statement of the following problem (here and elsewhere, we quote other people's works not literally, but in an equivalent and more convenient form).

Problem 1. For any $\rho \in (0,1/2]$, find the set $\mathcal{W}_\rho$ od all positive values of the parameter μ such that the function $E_\rho(z;\mu)$ has in $\mathbb{C}$ only negative and simply zeros.

Ostrovskii and Peresyolkova [19] had restricted themselves only to positive values of μ. However (see below), if we define the set $\mathcal{W}_\rho^-$ replacing in the definition of $\mathcal{W}_\rho$ the word "positive" by "negative," then for any $\rho < 1/2$ this set will be nonempty. Obviously, other problems can also be stated.

Problem 2. For any $\rho \in (0,1/2]$, find the set $\widetilde{\mathcal{W}}_\rho$ of all values of the parameter $\mu \in \mathbb{R}$ such that all zeros of $E_\rho(z;\mu)$ lie on $\mathbb{R}$.

Problem 3. For any $\rho \in (0,1/2]$ and $m \in \mathbb{N}$, find the set $\widetilde{\mathcal{W}}_\rho(m)$ of all $\mu \in \mathbb{R}$ such that the number of nonreal zeros of the function $E_\rho(z;\mu)$ is not greater than $2m$.

For Problem 1, Ostrovskii and Peresyolkova proved that in the case $\rho = 2^{-k}$, $k \in \mathbb{N}$, the interval $(0, 1+1/\rho)$ lies in $\mathcal{W}_\rho$ and quickly hypothesized that $\mathcal{W}_\rho = (0,1+1/\rho)$ for any $\rho \in (0,1/2]$. This is actually valid for $\rho = 1/2$ (for the proof, see [19]), but is invalid for $\rho < 1/2$: $\mathcal{W}_\rho$ is wider than the interval $(0,1+1/\rho)$.

We state the main result of this chapter.

Theorem 3.1.1. *For any $\rho \in (0,1/2)$ and $\mu \in (0,2/\rho - 1]$, all zeros of the function $E_\rho(z;\mu)$ in $\mathbb{C}$ lie on $(-\infty,0)$, are simple, and, being ordered into a sequence $\{z_n(\rho,\mu)\}_{n\in\mathbb{N}}$, satisfy the inequalities*

$$\begin{aligned} &-\xi_1^{1/\rho}(\rho,\mu) < z_1(\rho,\mu) < -\frac{\Gamma(\mu+1/\rho)}{\Gamma(\mu)}, \\ &-\xi_n^{1/\rho}(\rho,\mu) < z_n(\rho,\mu) < -\xi_{n-1}^{1/\rho}(\rho,\mu), \quad n \ge 2, \end{aligned} \tag{3.1.3}$$

where

$$\xi_n(\rho,\mu) = \frac{\pi(n+\rho(\mu-1))}{\sin(\pi\rho)}. \tag{3.1.4}$$

For $0 < \rho \le 1/4$, the negativeness and simpleness of all zeros of $E_\rho(z,\mu)$ and inequalities (3.1.3) *holds if $0 < \mu \le 2/\rho$.*

Thus, we prove the inclusions

$$(0,2/\rho-1] \subset \mathcal{W}_\rho \quad \forall \rho \in (1/4,1/2), \quad (0,2/\rho] \subset \mathcal{W}_\rho \quad \forall \rho \in (0,1/4],$$

that are stronger than those supposed in [19] (obviously, $1+\frac{1}{\rho}<\frac{2}{\rho}-1 \Longleftrightarrow 0<\rho<1/2$) and obtain two-sided estimates of zeros, which become (3.1.1) for $\mu=1$. We also add a nontrivial upper estimate of the first zero (Wiman had not obtained it even in the case $\mu=1$ examined by him).

In the case $\rho\in(0,1/6]$, for some first zeros we obtain more exact estimates. We set

$$R_n(\rho,\mu)=\frac{\Gamma(\mu+n/\rho)}{\Gamma(\mu+(n-1)/\rho)},\quad n\in\mathbb{N}. \tag{3.1.5}$$

Theorem 3.1.2. *Let $0<\rho\le 1/6$, $0<\mu\le 2/\rho$. Then for $1\le n\le[1/(3\rho)]-1$ we have the inequality*

$$-\sqrt{2}R_n(\rho,\mu)<z_n(\rho,\mu)<-R_n(\rho,\mu). \tag{3.1.6}$$

The following Lemma 3.1.1 (see below) shows that the inequalities from Theorem 3.1.2 are stronger than (3.1.3) for the values of n considered.

Lemma 3.1.1. *For $0<\rho\le 1/6$, $0<\mu\le 2/\rho$, and $1\le n\le[1/(3\rho)]-1$, the following inequalities hold*:

$$\xi_{n-1}^{1/\rho}(\rho,\mu)<R_n(\rho,\mu)<\sqrt{2}R_n(\rho,\mu)<\xi_n^{1/\rho}(\rho,\mu). \tag{3.1.7}$$

If $0<\mu\le 1/\rho$, then

$$2R_n(\rho,\mu)<\xi_n^{1/\rho}(\rho,\mu).$$

Indeed, inequalities (3.1.3) and (3.1.6) can be strengthened. In our opinion, efforts in this direction are useful in cases where such results have applications in other branches of mathematics. Therefore (see the Introduction), we are interested in Mittag-Leffler functions with values of the parameters $\rho=1/N$, $N\in\mathbb{N}$, and $\mu=1+1/\rho=N+1$. In this case, two-sided estimates of zeros more exact than (3.1.3) and (3.1.6) are obtained in [23].

Denote zeros of $E_{1/N}(z;N+1)$ by $z_n(N)=z_n(1/N,N+1)$, $n\in\mathbb{N}$, and introduce the notation

$$R_n(N)=\frac{((n+1)N)!}{(nN)!},\quad q_n(N)=\frac{3R_n(N)}{2R_{n+1}(N)}\min(1,Nn^{-2}). \tag{3.1.8}$$

We know (see Theorem 3.1.1) that $z_n(N)\in\mathbb{R}$.

Theorem 3.1.3. *For any $N\in\mathbb{N}$, $N\ge 3$, the following relations hold*:

$$z_n(N)=-\left[\left(\pi n+\frac{\pi}{2}+\alpha_n(N)\right)/\sin\left(\frac{\pi}{N}\right)\right]^N,\quad n\in\mathbb{N},\quad n\ge[N/3], \tag{3.1.9}$$

where $\alpha_n(N)\in\mathbb{R}$, $|\alpha_n(N)|\le x_n(N)$,

$$x_n(N)=\begin{cases}\exp\left(-\pi n\cot(\pi/N)\right), & 3\le N\le 6,\\ \exp\left(-2\pi n\sin(2\pi/N)\right), & 7\le N\le 1400,\\ 1.01\exp\left(-2\pi n\sin(2\pi/N)\right), & N>1400.\end{cases} \tag{3.1.10}$$

For $N\ge 6$, $1\le n\le[N/3]-1$, the following inequalities are valid:

$$-R_n(N)(1+q_n(N))<z_n(N)<-R_n(N). \tag{3.1.11}$$

Inequalities (3.1.11) are highly exact: it follows from (3.1.8) that

$$\max\left\{q_n(N)\;\middle|\;1\le n\le[N/3]-1,\ N\ge 6\right\}=q_1(6)<\frac{1}{15},$$
$$\lim_{N\to\infty}\max\left\{q_n(N)\;\middle|\;1\le n\le[N/3]-1\right\}=0.$$

As for two-sided estimates of zeros $z_n(N)$, $n \geq [N/3]$, expressed by relations (3.1.9) and (3.1.10), the ratio of the lower and upper estimates is equal to

$$\Delta_n(N) = \left(\frac{1 + \frac{x_n(N)}{\pi(n+1/2)}}{1 - \frac{x_n(N)}{\pi(n+1/2)}} \right)^N \leq \left(\Delta_{[N/3]} \right)^N \leq 1.01 \quad \forall N \geq 3.$$

It is interesting to obtain, for all values of the parameters $0 < \rho < 1/2$ and $0 < \mu < 2/\rho - 1$, two-sided estimates of zeros of the functions $E_\rho(z;\mu)$ such that the maximum of the ratio of the lower and upper estimates taken over all zeros of a fixed function tends to 1 as $\rho \to 0+$ and as $\rho \to 1/2 - 0$.

Finally, we note that the estimate

$$\alpha_n(N) = \begin{cases} O\Big(\exp\left(-\pi n \cot(\pi/N)\right) \Big), & 3 \leq N \leq 6, \\ O\Big(\exp\left(-2\pi n \sin(2\pi/N)\right) \Big), & 7 \leq N, \end{cases}$$

for $n \geq n_0(N)$ follows from Theorem 2.1.4. The fact that we succeed in this case in obtaining values of the constant in O equal to 1 or 1.01 and take $n_0(N) = [N/3]$ is nontrivial.

We complete the discussion of the realness problem for all zeros of the Mittag-Leffler function of order $<1/2$. By Theorem 3.1.1, for any $\mu \in [0, 2/\rho - 1]$, all zeros of $E_\rho(z;\mu)$, $0 < \rho < 1/2$, are real (if $\mu = 0$, then one zero of $E_\rho(z;0)$ is the point $z = 0$, and all other zeros are negative and simple; this follows from the identity $E_\rho(z;0) = zE_\rho(z;1/\rho)$). The answer is not completely known for large values of μ. Satisfactory results are obtained for ρ close to 1/2, but for "small" ρ they are far from the final.

Theorem 3.1.4. *For* $0 < \rho < 1/2$ *and* $\mu \geq 0.9 + (\rho^2 \ln 2)^{-1} - 1/\rho$, *the function* $E_\rho(z;\mu)$ *has zeros in* $\mathbb{C}\backslash\mathbb{R}$.

Theorem 3.1.5. *For any* $\rho \in (1/3, 1/2)$ *and* $\mu \in (1 - 1/\rho, -1)$ *and for* $\rho \in (0, 1/3]$, $\mu \in (-2, -1)$, *all zeros of the function* $E_\rho(z;\mu)$ *are negative and simple.*

3.2. Meaning of Proofs of Theorems 3.1.1–3.1.3

Proofs of Theorems 3.1.1–3.1.3 are quite long and cumbersome and contain a lot of calculations. Hence these proofs without explanation of their sense and indication of main obstructions can be difficult even for specialists in theory of functions.

The proof of Theorem 3.1.1 is based on the following simple reason. We take a sequence of points $w_n = -\xi_n^{1/\rho}(\rho,\mu)$ (see (3.1.4) above) and prove that at these points the Mittag-Leffler function changes sign:

$$\operatorname{sgn} E_\rho(w_n;\mu) = (-1)^n \quad \forall n \in \mathbb{N}. \tag{3.2.1}$$

Since in the case $\mu \in \mathbb{R}$ the function $E_\rho(z;\mu)$ is real-valued on $\mathbb{R}$ and

$$E_\rho(0;\mu) = 1/\Gamma(\mu) > 0 \quad \forall \mu > 0,$$

we conclude that for any natural number n, the function considered has no less than n distinct zeros on the interval $(w_n, 0) \subset \mathbb{R}$. On the other hand, by Theorems 2.1.4 and 2.2.2, for all sufficiently large n, the function $E_\rho(z;\mu)$ has exactly n zeros (with account of their multiplicity) in the disk $|z| \leq \xi_n^{1/\rho}(\rho,\mu)$. This means that the function has no other zeros except for real and simple roots lying in the intervals (w_n, w_{n-1}), $n \geq 2$, $(w_1, 0)$ (exactly one zero on each interval). Finally, Corollary 3.3.1 of Lemma 3.3.1 in Sec. 3.3 shows that the function $E_\rho(z;\mu)$, $\mu > 0$, has no zeros on the ray $[-\Gamma(\mu + 1/\rho)/\Gamma(\mu), +\infty)$. Thus, if Eqs. (3.2.1) and Lemma 3.3.1 are proved, then Theorem 3.1.1 will also be completely proved.

Wiman argued similarly in [44], but it is unclear whether he know the theorem on the number of zeros of the Mittag-Leffler function in a sufficiently large disk (though in the considered case $\mu = 1$)

and how he intended to prove Eqs. (3.2.1), since their proof is the main obstacle. For sufficiently large n, these equalities immediately follow from asymptotic formulas (1.5.24) and (1.5.39) (see also the note after Theorem 1.5.3). As $x \to +\infty$, we have the relations

$$\begin{aligned} E_\rho(-x^{1/\rho};\mu) &= 2\rho x^{1-\mu}\exp\left(x\cos\pi\rho\right)\cos\big(x\sin(\pi\rho)-\pi\rho(\mu-1)\big) \\ &\quad + O\left(x^{1-\mu}\exp\left(x\cos(3\pi\rho)\right)\right), \qquad 0<\rho<\frac{1}{6}, \\ E_{1/6}(-x^{-6};\mu) &= \frac{1}{3}x^{1-\mu}\exp\left(\frac{x\sqrt{3}}{2}\right)\cos\left(\frac{x}{2}-\pi\rho(\mu-1)\right)+O\left(x^{1-\mu}+x^{-6}\right), \\ E_\rho(-x^{1/\rho};\mu) &= 2\rho x^{1-\mu}\exp\left(x\cos\pi\rho\right)\cos\Big(x\sin(\pi\rho)-\pi\rho(\mu-1)\Big) \\ &\quad + O\big(x^{-1/\rho}\big), \qquad \frac{1}{6}<\rho<\frac{1}{2}. \end{aligned} \tag{3.2.2}$$

The point ξ_n are chosen in (3.1.4) so that

$$\cos\big(\xi_n\sin(\pi\rho)-\pi\rho(\mu-1)\big)=(-1)^n \quad \forall n\in\mathbb{N}. \tag{3.2.3}$$

From (3.2.2) and (3.2.3) we obtain

$$E_\rho(w_n;\mu)=2\rho\xi_n^{1-\mu}\exp(\xi_n\cos\pi\rho)[(-1)^n+o(1)], \quad n\to\infty.$$

This relation immediately implies (3.2.1) for all sufficiently large n.

As the previous reasoning show, Eqs. (3.2.1) must be proved precisely for all $n\in\mathbb{N}$, and this task is most difficult for $n=1$. For this, it is necessary to obtain an explicit estimate of remainders in (3.2.2) without the O-symbol for x and, which is especially important, this estimate must be uniform (in some sense) with respect to the parameters ρ and μ in the whole domain of their values. The corresponding results are contained in Chap. 11; they will used here.

We also note another important fact. Relations (3.2.2) show that the remainder in the asymptotics of the Mittag-Leffler functions on the negative part of $\mathbb{R}$ has a different behavior for $0<\rho<1/6$ and for $1/6<\rho<1/2$. Therefore, these cases must be considered separately in the proof of Theorem 3.1.1; the borderline value $\rho=1/6$ can be joined to any of them: this is not significantly in the problem considered. However, as ρ tends to 1/2, an unexpected difficulty arises. Since $\cos(\pi\rho)\approx 0$, the principal term of the asymptotics (3.2.2)

$$2\rho x^{1-\mu}\exp\big(\cos(\pi\rho)\big)\cos\big(x\sin(\pi\rho)-\pi\rho(\mu-1)\big)$$

at the points $\xi_n=\xi_n(\rho,\mu)$ for nonlarge values of n is close to $(-1)^n\xi_n^{1-\mu}$, and it is not known whether it dominates the remainder. Therefore, the orders $\rho\in[0.4,0.5)$ are considered in a separate section.

Another substantial obstruction appears when ρ is close to zero; it even leads to modifying the scheme of proof described at the beginning of this section. By Theorem 1.5.4 of Chap. 1, the first part of the remainder is equal to the product of $2\rho x^{1-\mu}$ and the sum

$$\sum_{k=2}^{[1/\rho]}\exp\big(x\cos(\pi(2k-1)\rho)\big)\cos\big(x\sin(\pi(2k-1)\rho)-\pi(2k-1)(\mu-1)\big).$$

This would seem to be unimportant: for fixed $x>0$, the majorants of moduli of these terms

$$\exp\big(x\cos(\pi(2k-1)\rho)\big) \tag{3.2.4}$$

decrease approximately as a geometric progression as the number k increases. Indeed, the ratio of the functions with numbers $k+1$ and k in (3.2.4) is equal to

$$\Delta_{k,\rho}(x)=\exp\Big[-2x\sin(\pi\rho)\sin(2\pi k\rho)\Big]. \tag{3.2.5}$$

If the parameter ρ is close to zero, then for any fixed $k \in \mathbb{N}$ we have

$$\lim_{\rho \to 0+} \Delta_{k,\rho}(x) = 1 \quad \text{uniformly with respect to } 0 \le x \le o(\rho^{-2}),$$

and we cannot obtain the required estimate. Thus, in the case where $0 < \rho \le 1/6$ and $0 < \mu \le 2/\rho$, Eqs. (3.2.1) can be proved only for $n \ge N_1(\rho) = [1/(3\rho)]$. If n is less than $N_1(\rho)$, one can prove the changes of sign of $E_\rho(z;\mu)$ for all $\rho \in (0, 1/6]$, $\mu \in (0, 2/\rho]$ in another sequence of points by another method. Namely, the following relations are proved (the sequence $R_n(\rho,\mu)$ is defined in (3.1.5)):

$$\operatorname{sgn} E_\rho(-R_n(\rho,\mu);\mu) = (-1)^{n-1}, \quad 1 \le n \le N_1(\rho), \tag{3.2.6}$$

$$\operatorname{sgn} E_\rho(-\sqrt{2}R_n(\rho,\mu);\mu) = (-1)^{n}, \quad 1 \le n < N_1(\rho). \tag{3.2.7}$$

These relations and inequalities (3.1.7) prove that on the segment $[-\xi_{N_1(\rho)}^{1/\rho}(\rho,\mu), 0]$, the function $E_\rho(z;\mu)$ has not less than $N_1(\rho)$ zeros. It is important that the signs of the function at the points $-R_{N_1(\rho)}$ and $-\xi_{N_1(\rho)}^{1/\rho}$ are distinct and $\xi_{N_1(\rho)-1} < R_{N_1(\rho)} < \xi_{N_1(\rho)}$ by Lemma 3.1.1. For completeness of the proof of Theorems 3.1.1 and 3.1.2, we must verify Eq. (3.2.6) for $n = N_1(\rho)$; for $n = N_1(\rho)$, we may omit the proof of Eq. (3.2.7).

Using the result on the number of zeros of $E_\rho(z;\mu)$ in a disk $|z| \le \xi_n^{1/\rho}(\rho,\mu)$ obtained above and relations (3.2.1), which were proved for any $n \ge N_1(\rho)$, we obtain that there is no other zeros except for real and simple zeros lying in the intervals

$$\Big(-\xi_n^{1/\rho}(\rho,\mu),\ -\xi_{n-1}^{1/\rho}(\rho,\mu)\Big), \quad n \ge N_1(\rho),$$

$$\Big(-\sqrt{2}R_n(\rho,\mu),\ -R_n(\rho,\mu)\Big) \subset \Big(-\xi_n^{1/\rho}(\rho,\mu),\ -\xi_{n-1}^{1/\rho}(\rho,\mu)\Big), \quad 1 \le n < N_1(\rho)$$

(we assume that $\xi_0 = 0$). Hence the proof of Theorem 3.1.1 for $0 < \rho \le 1/6$ is complete and, moreover, Theorem 3.1.2 is also proved.

We also consider the following problem. Are the proofs of Eqs. (3.2.1), (3.2.6), and (3.2.7) substantially easier in the case $\mu = 1$, which was considered by Wiman? For $1/6 < \rho < 1/2$, simplification is essential. We have almost no doubt that for these values of ρ, Wiman knew a complete proof of the theorem on the realness and simpleness of zeros of $E_\rho(z;1)$ and inequalities (3.1.1). A more difficult problem is to broaden the interval of values of the parameter μ. For $0 < \rho \le 1/6$, the elimination of gaps in Wiman's proof requires principally new ideas. Hence some simplification in the case $\mu = 1$ is also present, but it is more "technical" than fundamental: this particular case is less cumbersome.

The proof of Theorem 3.1.3 is based on the same ideas. Since the values of the parameters are more specific, one can obtain more exact two-sided estimates of all zeros than in Theorems 3.1.1 and 3.1.2.

3.3. Absence of Zeros of the Mittag-Leffler Function with Positive Parameter μ in a Neighborhood of the Point $z = 0$. Asymptotics of the First Zero of $E_\rho(z;\mu)$ by the Parameter $\rho \to 0+$ Uniform with Respect to $\mu \in (0, 2/\rho]$

Lemma 3.3.1. *Let $\{A_k\}_{k=0}^{\infty}$ be an arbitrary sequence of positive numbers satisfying the following condition*:

$$R_k = A_{k-1}/A_k, \quad k \in \mathbb{N}, \quad \textit{do not decrease.}$$

Then the function

$$F(z) = \sum_{k=0}^{\infty} A_k z^k$$

is holomorphic in the disk

$$|z| < R = \lim_{k\to\infty} R_k$$

(if $R = +\infty$, then F is an entire function) and is positive on the interval $(-R_1, R)$. If the sequence $\{R_k\}_{k=1}^{\infty}$ is not constant, then $F(-R_1) > 0$.

Proof. Since the sequence R_k is monotonic, it has a limit (finite or infinite). By d'Alembert's ratio test, the radius of convergence of the series $\sum\limits_{k=0}^{\infty} A_k z^k$ is equal to R. The positiveness of the function $F(x)$ on the interval $-R_1 < x < 0$ follows from the fact that the sequence $A_k|x|^k$ decreases since

$$\sum_{n=0}^{\infty}(-1)^n u_n > 0, \quad u_n \searrow 0, \quad n \to \infty.$$

The fact that the sequence $A_k|x|^k$ decreases is obvious from the inequalities

$$A_k|x^k| < A_{k-1}|x^{k-1}| \quad \Longleftrightarrow \quad |x| < R_k, \quad R_1 \le R_k \quad \forall k \in \mathbb{N}.$$

Finally, if $R > R_1$, then the series $F(-R_1) = \sum\limits_{k=2}^{\infty}(-1)^k A_k R_1^k$ converges, and the moduli of its terms decrease. The first term is positive and hence the sum is also positive. The lemma is proved. □

Corollary 3.3.1. *For any $\mu > 0$ and $\rho > 0$, the function $E_\rho(z;\mu)$ is positive on the ray $[-\Gamma(\mu + 1/\rho)/\Gamma(\mu), +\infty)$.*

To prove this corollary, is suffices to verify that the sequence of ratios

$$\left\{\Gamma\left(\mu + \frac{k-1}{\rho}\right) / \Gamma\left(\mu + \frac{k}{\rho}\right)\right\}_{k=1}^{\infty}$$

increases or, equivalently, that the second difference

$$l_k(\rho,\mu) = \ln\Gamma\left(\mu + \frac{k-1}{\rho}\right) - 2\ln\Gamma\left(\mu + \frac{k}{\rho}\right) + \ln\Gamma\left(\mu + \frac{k+1}{\rho}\right) \tag{3.3.1}$$

is positive for any $k \in \mathbb{N}$. This follows from the fact that $\psi(x) = (\ln\Gamma(x))'$ increases on the ray $0 < x < \infty$.

Thus, if the realness of all zeros of $E_\rho(z;\mu)$ has been proved, Corollary 3.3.1 (in the case $\mu > 0$) yields a nontrivial upper estimate of the first zero. Is it valid that the Mittag-Leffler function with arbitrary positive parameters (ρ, μ) has no zeros in the disk

$$|z| \le R_1(\rho,\mu) = \Gamma\left(\mu + \frac{1}{\rho}\right) / \Gamma(\mu)?$$

Probably, this is so, but we cannot prove this assertion for all values of the parameters $\rho > 0$ and $\mu > 0$. We present one of the results (of course, it can be strengthened).

In the sequel, we denote by ψ the logarithmic derivative of the Γ-function and use the following expansion:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right), \quad z \in \mathbb{C}, \quad z \ne 0, -1, -2, \dots, \tag{3.3.2}$$

where γ is the Euler constant. From (3.3.2) follows the identity

$$\psi'(z) = \sum_{k=0}^{\infty}(k+z)^{-2}. \tag{3.3.3}$$

Lemma 3.3.2. *For any $\rho \in (0,1]$ and $\mu \in (0, 2/\rho]$ and arbitrary z lying in the disk $|z| \le R_1(\rho,\mu)$, the real part of the function $E_\rho(z;\mu)$ is positive:*

$$\operatorname{Re} E_\rho(z;\mu) \ge \frac{\Gamma^2(\mu+1/\rho)}{2\Gamma^2(\mu)\Gamma(\mu+2/\rho)}. \tag{3.3.4}$$

Proof. In is known that for a function harmonic in a domain $\mathcal{D}$, its maximum and minimum in the closure of $\mathcal{D}$ is attained on the boundary $\partial\mathcal{D}$. Since the real part of an analytic function is harmonic, it suffices to prove inequality (3.3.4) for $z = e^{i\theta}R_1(\rho,\mu)$, $-\pi \le \theta \le \pi$. We have the equality

$$\operatorname{Re} E_\rho(R_1 e^{i\theta};\mu)\Gamma(\mu) = 1 + \cos\theta + \sum_{k=2}^{\infty} \frac{\Gamma^k\big(\mu+1/\rho\big)}{\Gamma^{k-1}(\mu)\Gamma\big(\mu+k/\rho\big)} \cos k\theta. \tag{3.3.5}$$

We use the following representation of the sum of cosine series (see [2, p. 100]):

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta = \sum_{k=1}^{\infty} \left[(a_{k-1} - 2a_k + a_{k+1}) \sum_{m=0}^{k-1} D_m(\theta) \right]; \tag{3.3.6}$$

here

$$D_m(\theta) = \frac{1}{2} + \sum_{\nu=0}^{m} \cos(\nu\theta), \quad m \in \mathbb{N},$$

is the Dirichlet kernel, $D_0(\theta) \equiv 1/2$. One can specify different conditions for the sequence of coefficients $\{a_k\}$ sufficient for the convergence of both series in (3.3.6) for certain values of θ and for the coincidence of their sums. For series (3.3.5), there is no problems; under the condition

$$\sum_{k=1}^{\infty} k^2 |a_k| < +\infty,$$

which holds in our case, both series in (3.3.6) absolutely converge (for all $\theta \in \mathbb{R}$) and their sums are identically equal. As is known,

$$\sum_{m=0}^{k-1} D_m(\theta) = \frac{1}{2}\left(\sin\frac{k\theta}{2} \operatorname{cosec}\frac{\theta}{2} \right)^2 \ge 0.$$

Therefore, for a convex sequence $\{a_k\}$, we have the relation

$$a_{k-1} - 2a_{k+1} + a_{k+2} \ge 0 \ (\forall k \in \mathbb{N}) \quad \Longrightarrow \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta \ge \frac{a_0}{2} - a_1 + \frac{a_2}{2}. \tag{3.3.7}$$

In series (3.3.5), we have

$$\frac{a_0}{2} = a_1 = 1, \quad a_2 = \frac{\Gamma^2\big(\mu+1/\rho\big)}{\Gamma(\mu)\Gamma\big(\mu+2/\rho\big)}. \tag{3.3.8}$$

Therefore, the inequality of the lemma immediately follows from (3.3.7) and (3.3.8) if for any $k \in \mathbb{N}$ the following inequality holds:

$$\frac{\Gamma^k\big(\mu+1/\rho\big)}{\Gamma^{k-1}(\mu)\Gamma\big(\mu+k/\rho\big)} - 2\frac{\Gamma^{k+1}\big(\mu+1/\rho\big)}{\Gamma^{k}(\mu)\Gamma\big(\mu+(k+1)/\rho\big)} + \frac{\Gamma^{k+2}\big(\mu+1/\rho\big)}{\Gamma^{k+1}(\mu)\Gamma\big(\mu+(k+2)/\rho\big)} \ge 0. \tag{3.3.9}$$

In the case $k \ge 2$, even a stronger inequality holds:

$$\frac{\Gamma^k\big(\mu+1/\rho\big)}{\Gamma^{k-1}(\mu)\Gamma\big(\mu+k/\rho\big)} \ge 2\frac{\Gamma^{k+1}\big(\mu+1/\rho\big)}{\Gamma^{k}(\mu)\Gamma\big(\mu+(k+1)/\rho\big)}, \tag{3.3.10}$$

or, equivalently,

$$\ln 2 \leq \left[\ln\Gamma\left(\mu+\frac{k+1}{\rho}\right)-\ln\Gamma\left(\mu+\frac{k}{\rho}\right)\right]-\left[\ln\Gamma\left(\mu+\frac{1}{\rho}\right)-\ln\Gamma(\mu)\right]. \tag{3.3.11}$$

Prove this inequality. The derivative of the right-hand side of (3.3.11) with respect to the parameter μ ia equal to

$$\left[\psi\left(\mu+\frac{k+1}{\rho}\right)-\psi\left(\mu+\frac{k}{\rho}\right)\right]-\left[\psi\left(\mu+\frac{1}{\rho}\right)-\psi(\mu)\right]=\frac{1}{\rho}\big(\psi'(\xi_k)-\psi'(\xi_0)\big),$$

where $\xi_k \in \left(\mu+\dfrac{k}{\rho},\mu+\dfrac{k+1}{\rho}\right)$. Since $\psi'(t)$ decreases on the ray $0<t<+\infty$, the last expression is negative. Therefore, the right-hand side of (3.3.11) decreases as the parameter μ increases, and it suffices to prove the inequality at the extreme right point $\mu=2/\rho$:

$$\ln 2 \leq \left[\ln\Gamma\left(\frac{k+3}{\rho}\right)-\ln\Gamma\left(\frac{k+2}{\rho}\right)\right]-\left[\ln\Gamma\left(\frac{3}{\rho}\right)-\ln\Gamma\left(\frac{2}{\rho}\right)\right]. \tag{3.3.12}$$

Since the function ψ increases, we see that the right-hand side of (3.3.12) increases with increasing k. Therefore, it suffices to prove (3.3.12) for $k=2$, i.e., deduce the inequality

$$\ln 2 \leq \ln\Gamma(5h)-\ln\Gamma(4h)-\ln\Gamma(3h)+\ln\Gamma(2h) \quad \forall h \geq 1. \tag{3.3.13}$$

For $h=1$, (3.3.13) becomes an equality. The derivative of the right-hand side of (3.3.13), owing to (3.3.2), is equal to

$$\begin{aligned}5\psi(5h)-4\psi(4h)-3\psi(3h)+2\psi(2h)&=\sum_{k=1}^{\infty}\left(\frac{4}{k+4h}+\frac{3}{k+3h}-\frac{5}{k+5h}-\frac{2}{k+2h}\right)\\&=\sum_{k=1}^{\infty}\frac{2kh(2k+7h)}{(k+2h)(k+3h)(k+4h)(k+5h)}>0.\end{aligned}$$

Thus, inequality (3.3.13) and hence inequality (3.3.11) are proved.

It remains to prove inequality (3.3.9) in the case $k=1$, namely,

$$1-\frac{2\Gamma^2\big(\mu+1/\rho\big)}{\Gamma(\mu)\Gamma\big(\mu+2/\rho\big)}+\frac{\Gamma^3\big(\mu+1/\rho\big)}{\Gamma^2(\mu)\Gamma\big(\mu+3/\rho\big)} \geq 0. \tag{3.3.14}$$

We introduce the following notation:

$$h=\frac{1}{\rho}, \quad u(\mu,h)=\frac{\Gamma^2(\mu+h)}{\Gamma(\mu)\Gamma(\mu+2h)}, \quad v(\mu,h)=\frac{\Gamma^3(\mu+h)}{\Gamma^2(\mu)\Gamma(\mu+3h)}.$$

The derivative of the left-hand side of (3.3.14) with respect to the parameter μ is equal to

$$\begin{aligned}&-2u(\mu,h)\frac{\partial \ln u(\mu,h)}{\partial\mu}+v(\mu,h)\frac{\partial \ln v(\mu,h)}{\partial\mu}\\&\quad=-2u(\mu,h)\big[-\psi(\mu)+2\psi(\mu+h)-\psi(\mu+2h)\big]+v(\mu,h)\big[3\psi(\mu+h)-2\psi(\mu)-\psi(\mu+3h)\big].\end{aligned}$$

We prove that the last expression is negative, i.e., we must deduce the inequality

$$v(\mu,h)\big[3\psi(\mu+h)-2\psi(\mu)-\psi(\mu+3h)\big]<2u(\mu,h)\big[-\psi(\mu)+2\psi(\mu+h)-\psi(\mu+2h)\big]. \tag{3.3.15}$$

By inequality (3.3.10) proved above ($k=2$), we have the inequality

$$0<v(\mu,h)\leq\frac{u(\mu,h)}{2}. \tag{3.3.16}$$

Both factors in the brackets in (3.3.15) are positive. The positiveness of $\psi(\mu+h)-\psi(\mu)-\psi(\mu+2h)$ is obvious from the concavity of ψ on $(0,+\infty)$; the positiveness of the function $3\psi(\mu+h)-2\psi(\mu)-\psi(\mu+3h)$ is proved as follows:

$$\begin{aligned} 0<3\psi(\mu+h)-2\psi(\mu)-\psi(\mu+3h) \quad &\Longleftrightarrow \quad \psi(\mu+3h)-\psi(\mu+h)<2\psi(\mu+h)-2\psi(\mu) \\ &\Longleftrightarrow \quad 2h\psi'(\eta_1)<2h\psi(\eta_2), \quad \eta_1\in(\mu+h,\mu+3h), \quad \eta_2\in(\mu,\mu+h). \end{aligned}$$

The last inequality is obvious since $\psi'(t)$ decreases on the ray $0<t<+\infty$.

Thus, we conclude that (3.3.15) follows from the inequality

$$\begin{aligned} &3\psi(\mu+h)-2\psi(\mu)-\psi(\mu+3h)<4\big(-\psi(\mu)+2\psi(\mu+h)-\psi(\mu+2h)\big) \\ &\qquad\Longleftrightarrow \quad 0<\psi(\mu+3h)-4\psi(\mu+2h)+5\psi(\mu+h)-2\psi(\mu), \quad h\ge 1, \quad 0<\mu\le 2h, \end{aligned}$$

which, in its turn, is implied by the expansion

$$\begin{aligned} &\psi(\mu+3h)-4\psi(\mu+2h)+5\psi(\mu+h)-2\psi(\mu) \\ &\qquad=\sum_{n=0}^{\infty}\left(\frac{4}{\mu+2h+n}+\frac{2}{\mu+n}-\frac{5}{\mu+h+n}-\frac{1}{\mu+3h+n}\right) \\ &\qquad=\sum_{n=0}^{\infty}\frac{2h^2(\mu+3h+n)}{(\mu+n)(\mu+n+h)(\mu+n+2h)(\mu+n+3h)}. \end{aligned}$$

Thus, the derivative of the left-hand side of (3.3.14) with respect to the parameter μ is negative. Therefore, it suffices to prove inequality (3.3.14) only for the maximal value $\mu=2/\rho=2h$.

So, we prove that

$$1-\frac{2\Gamma^2(3h)}{\Gamma(2h)\Gamma(4h)}+\frac{\Gamma^3(3h)}{\Gamma^2(2h)\Gamma(5h)}\ge 0 \quad \forall h\ge 1. \tag{3.3.17}$$

For $h=1$, we have an equality. Thus, it suffices to verify the positiveness of the derivative of the left-hand side of (3.3.17) on the ray $1<h<+\infty$. Introduce the notation

$$u(h)=\frac{\Gamma^2(3h)}{\Gamma(2h)\Gamma(4h)}, \quad v(h)=\frac{\Gamma^3(3h)}{\Gamma^2(2h)\Gamma(5h)}.$$

The required derivative is equal to

$$\begin{aligned} &-2u(h)\frac{d\ln u(h)}{dh}+v(h)\frac{d\ln v(h)}{dh} \\ &\qquad -2u(h)\big[6\psi(3h)-2\psi(2h)-4\psi(4h)\big]+v(h)\big[9\psi(3h)-4\psi(2h)-5\psi(5h)\big]. \end{aligned}$$

Therefore, we must prove the inequality

$$v(h)\big[9\psi(3h)-4\psi(2h)-5\psi(5h)\big]>2u(h)\big[6\psi(3h)-2\psi(2h)-4\psi(4h)\big]. \tag{3.3.18}$$

We have the equalities

$$\begin{aligned} 6\psi(3h)-2\psi(2h)-4\psi(4h)&=\sum_{n=1}^{\infty}\left(\frac{2}{2h+n}+\frac{4}{4h+n}-\frac{6}{3h+n}\right)=\sum_{n=1}^{\infty}\frac{-2nh}{(2h+n)(3h+n)(4h+n)}, \\ 9\psi(3h)-4\psi(2h)-5\psi(5h)&=\sum_{n=1}^{\infty}\left(\frac{4}{2h+n}+\frac{5}{5h+n}-\frac{9}{3h+n}\right)=\sum_{n=1}^{\infty}\frac{-6nh}{(2h+n)(3h+n)(5h+n)}. \end{aligned}$$

Therefore, inequality (3.3.18) takes the form

$$v(h)\sum_{n=1}^{\infty}\frac{-6nh}{(2h+n)(3h+n)(5h+n)} > u(h)\sum_{n=1}^{\infty}\frac{-4nh}{(2h+n)(3h+n)(4h+n)}$$

$$\iff \quad 3v(h)\sum_{n=1}^{\infty}\frac{n}{(2h+n)(3h+n)(5h+n)} < 2u(h)\sum_{n=1}^{\infty}\frac{n}{(2h+n)(3h+n)(4h+n)}.$$

It remains to prove that $3v(h) < 2u(h)$ or, equivalently,

$$\frac{3}{2} < \frac{\Gamma(2h)\Gamma(5h)}{\Gamma(3h)\Gamma(4h)} \quad \forall h \geq 1. \tag{3.3.19}$$

For $h = 1$, (3.3.19) is valid. Therefore, it suffices to verify that the right-hand side of (3.3.19) (or its logarithm) increases. We have the relations

$$\frac{d}{dh}\big[\ln\Gamma(5h) - \ln\Gamma(4h) - \ln\Gamma(3h) + \ln\Gamma(2h)\big] = 5\psi(5h) - 4\psi(4h) - 3\psi(3h) + 2\psi(2h)$$
$$= \sum_{n=1}^{\infty}\left(\frac{-5}{5h+n} + \frac{4}{4h+n} + \frac{3}{3h+n} - \frac{2}{2h+n}\right) = \sum_{n=1}^{\infty}\frac{2nh(2n+7h)}{(2h+n)(3h+n)(4h+n)(5h+n)} > 0,$$

which was required. The lemma is completely proved. □

It is interesting that for "small" values of ρ, Lemma 3.3.2 is almost the best possible in the sense that in a disk of radius slightly larger than $R_1(\rho,\mu)$, the function $E_\rho(z;\mu)$ already has a zero for any $\mu \in (0, 2/\rho]$.

Proposition 3.3.1. *For any $\rho \in (0, 1/6]$ and $\mu \in (0, 2/\rho]$, the function $E_\rho(z;\mu)$ has a zero lying in the interval*

$$\left(-\left(1 + 2\exp\left(-\frac{1}{3\rho}\right)\right)R_1(\rho,\mu), -R_1(\rho,\mu)\right).$$

Theorem 3.1.1, Lemma 3.1.1, and this assertion imply a highly exact asymptotics of the zero of the Mittag-Leffler function, which is closest to the origin:

$$z_1(\rho,\mu) = -\frac{\Gamma(\mu + 1/\rho)}{\Gamma(\mu)}\left[1 + O\left(\exp\left(-\frac{1}{3\rho}\right)\right)\right], \quad \rho \to 0+;$$

this asymptotics is uniform with respect to $\mu \in (0, 2/\rho]$ (the constant in O does not exceed 2).

Lemma 3.3.3. *Consider the function F from Lemma* 3.3.1. *If $\varepsilon > 0$ and*

$$\frac{R_1}{R_2} \leq \frac{\varepsilon}{(1+\varepsilon)^2},$$

then

$$F(-(1+\varepsilon)R_1) < 0.$$

Proof. We have

$$F\big(-(1+\varepsilon)R_1\big) = A_0 - (1+\varepsilon)A_1R_1 + (1+\varepsilon)^2A_2R_1^2 + \sum_{k=3}^{+\infty}(-1)^kA_k(1+\varepsilon)^kR_1^k.$$

In the sum $\sum\limits_{k=3}^{\infty}$, the moduli of terms decrease since the ratio of any subsequent term to the previous term is

$$\frac{(1+\varepsilon)R_1}{R_k} \leq \frac{(1+\varepsilon)R_1}{R_2} \leq \frac{\varepsilon}{1+\varepsilon} < 1.$$

Therefore, this sign-alternating sum is negative and

$$F\big(-(1+\varepsilon)R_1\big) < A_0 - (1+\varepsilon)A_1R_1 + (1+\varepsilon)^2A_2R_1^2$$
$$= A_0\left(1 - (1+\varepsilon)\frac{A_1R_1}{A_0} + (1+\varepsilon)^2\frac{A_2R_1^2}{A_0}\right). \tag{3.3.20}$$

By the definition of R_k we have

$$\frac{A_0}{A_1} = R_1, \quad \frac{A_0}{A_2} = R_1R_2.$$

This and (3.3.20) imply that

$$A_0^{-1}F\big(-(1+\varepsilon)R_1\big) < -\varepsilon + \frac{(1+\varepsilon)^2R_1}{R_2} \le 0,$$

which was required. The lemma is proved. ☐

Proof of Proposition 3.3.1. By Lemma 3.3.3, it suffices to verify that if one takes

$$\varepsilon = 2\exp\left(-\frac{1}{3\rho}\right),$$

then for $0 < \rho \le 1/6$ and $0 < \mu \le 2/\rho$, the inequality

$$\frac{R_1}{R_2} < \varepsilon(1+\varepsilon)^{-2}$$

holds. By Lemma 3.9.1 (see Sec. 3.8 below), we have the inequality

$$\frac{R_1}{R_2} < \frac{\varepsilon}{2}, \quad 0 < \mu \le \frac{2}{\rho}.$$

Since for $\rho \le 1/6$, the inequality $\varepsilon \le 2e^{-2} < 1/3$ is valid, we have $(1+\varepsilon)^{-2} > 1/2$ and the required inequality holds. The proposition is proved. ☐

3.4. Inequalities for the Gamma-Function and Its Derivatives

Lemma 3.4.1. *The following inequalities hold*:

$$\left(\frac{1}{\Gamma(x)}\right)' > \gamma, \quad 0 < x < 1, \qquad \left(\frac{1}{\Gamma(x)}\right)' < \gamma, \quad x > 1.$$

Remark. The equality

$$\left(\frac{1}{\Gamma(x)}\right)'_{|x=1} = \gamma$$

is valid, but the derivative $(1/\Gamma(x))'$ is not monotonic on $(0, +\infty)$; the graph of $y = 1/\Gamma(x)$ has two inflection points.

Proof of Lemma 3.4.1. By (3.3.2), we have the relations

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right) = \gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$+ \sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x}\right) = \gamma + 1 - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x-1}{(k+1)(k+x)} = -\gamma + \sum_{k=0}^{\infty}\frac{x-1}{(k+1)(k+x)}.$$

This implies

$$\left(\frac{1}{\Gamma(x)}\right)' = -\frac{\psi(x)}{\Gamma(x)} = \frac{1}{\Gamma(x)}\left(\gamma + \sum_{k=0}^{\infty}\frac{1-x}{(k+1)(k+x)}\right).$$

Therefore, we must prove the inequalities

$$\gamma < \frac{1}{\Gamma(x)}\left(\gamma + \sum_{k=0}^{\infty} \frac{1-x}{(k+1)(k+x)}\right), \qquad 0 < x < 1,$$

$$\frac{1}{\Gamma(x)}\left(\gamma + \sum_{k=0}^{\infty} \frac{1-x}{(k+1)(k+x)}\right) < \gamma, \qquad x > 1,$$

which after simple transformations become

$$\begin{aligned} \gamma\bigl(\Gamma(x)-1\bigr) &< \sum_{k=0}^{\infty} \frac{1-x}{(k+1)(k+x)}, \qquad 0 < x < 1, \\ \sum_{k=0}^{\infty} \frac{1-x}{(k+1)(k+x)} &< \gamma\bigl(\Gamma(x)-1\bigr), \qquad x > 1. \end{aligned} \tag{3.4.1}$$

Dividing both inequalities (3.4.1) by $1-x$ and changing the sign in the second of them, we obtain that it suffices to prove the inequality

$$\gamma\frac{\Gamma(x)-1}{1-x} < \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)}, \quad x \in (0,1) \cup (1,+\infty). \tag{3.4.2}$$

For $x > 2$, the left-hand side of (3.4.2) is negative and the inequality is obvious. For $1 < x \leq 2$, the left-hand side of (3.4.2) is equal to

$$-\gamma\Gamma'(\xi) = \gamma\psi(\xi)\Gamma(\xi),$$

where ξ is some point of the interval $(1,2)$. Since $0 < \Gamma(\xi) < 1$ for $1 < \xi < 2$ and since, owing to the increase of ψ on $\mathbb{R}_+$, we have the inequality

$$-\gamma\psi(\xi) \leq -\gamma\psi(1) = \gamma^2,$$

we conclude that for $x \in (1,2]$, the left-hand side of (3.4.2) does not exceed γ^2, whereas the right-hand side is not less than

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu} - \frac{1}{\nu+1}\right) = 1.$$

The inequality (3.4.2) for $x > 1$ is proved. For $x \in (0,1)$, we multiply both sides of (3.4.2) by x and obtain the equivalent inequality

$$\gamma\frac{\Gamma(x+1)-x}{1-x} < \sum_{k=0}^{\infty} \frac{x}{(k+1)(k+x)}. \tag{3.4.3}$$

Clearly, the right-hand side of (3.4.3) is greater than the term of the series with number $k = 0$, which is equal to 1. However, by the inequality $\Gamma(t) < 1$, $1 < t < 2$, the left-hand side of (3.4.3) in not greater than γ. Inequality (3.4.2) is now proved for $x \in (0,1)$. The proof of lemma is complete. □

Lemma 3.4.2. *The logarithmic derivative of the* Γ*-function satisfies the following estimates*:

$$-(2t-1)^{-1} + \ln t < \psi(t) < \ln t \quad \forall t > \frac{1}{2}, \tag{3.4.4}$$

$$\ln t < \psi(t+0.5) \quad \forall t > 0. \tag{3.4.5}$$

Proof. In addition to (3.3.2), the following expansion is valid in $\mathbb{C} \setminus (-\infty, 0]$ (see [3, Vol. 1, Chap. 1, Sec. 1.7]):

$$\psi(z) = \ln z + \sum_{k=0}^{\infty}\left[\ln\left(1 + \frac{1}{k+z}\right) - \frac{1}{k+z}\right], \tag{3.4.6}$$

in which ln means the principal branch of the logarithm. From (3.4.6) and the well-known relation

$$0 < u - \ln(1+u) < \frac{u^2}{2} \quad \forall u > 0$$

we obtain

$$\ln t - 0.5 \sum_{k=0}^{\infty} (k+t)^{-2} < \psi(t) < \ln t \quad \forall t > 0.$$

Since

$$a^{-2} < \int\limits_{a-0.5}^{a+0.5} u^{-2} du \quad \forall a > 0.5, \tag{3.4.7}$$

we have

$$\sum_{k=0}^{\infty} (k+t)^{-2} < \int\limits_{t-0.5}^{+\infty} u^{-2} du.$$

Therefore,

$$\psi(t) > \ln t - 0.5 \int\limits_{t-0.5}^{+\infty} u^{-2} du = \ln t - \frac{0.5}{t-0.5} = \ln t - (2t-1)^{-1}, \quad t > 0.5.$$

Inequality (3.4.4) is proved. It implies

$$\lim_{t\to+\infty} \psi(t) - \ln t = 0, \quad \lim_{t\to+\infty} \psi(t+0.5) - \ln t = 0.$$

Thus, to prove (3.4.5), it suffices to verify that the difference $g(t) = \psi(t+0.5) - \ln t$ decreases. By (3.3.3) we have the equality

$$g'(t) = \sum_{k=0}^{\infty} \frac{1}{(k+t+0.5)^2} - \frac{1}{t}.$$

Applying inequality (3.4.7) to each term of the series, we obtain the upper estimate

$$g'(t) < \int\limits_{-0.5}^{+\infty} \frac{du}{(u+t+0.5)^2} - \frac{1}{t} = \int\limits_{0}^{+\infty} \frac{dx}{(x+t)^2} - \frac{1}{t} = 0,$$

which was required. The lemma is proved. □

Lemma 3.4.3. *For any fixed $x > 0$, the function $\Gamma(t)x^{-t}$ decreases with respect to the variable t on the interval $0 < t < x$.*

Proof. We have the equality

$$\frac{d}{dt}\left[\ln\left(\Gamma(t)x^{-t}\right)\right] = \frac{d}{dt}\Big[\ln\Gamma(t) - t\ln x\Big] = \psi(t) - \ln x.$$

By Lemma 3.4.1,

$$\psi(t) - \ln x < \ln t - \ln x < 0.$$

The lemma is proved. □

Lemma 3.4.4. *For any $a, b > 0$, $a < b$, the following inequalities hold*:

$$\psi''(a)\frac{(b-a)^3}{24} + (b-a)\psi\left(\frac{a+b}{2}\right) < \ln\Gamma(b) - \ln\Gamma(a) < (b-a)\psi\left(\frac{a+b}{2}\right).$$

Proof. By Taylor's formula with the remainder in the Lagrange form, for any function $f \in C^3[x-h, x+h]$, $x, h \in \mathbb{R}$, $h > 0$, there exist points $\xi_1 \in (x-h, x)$ and $\xi_2 \in (x, x+h)$ such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(\xi_2),$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(\xi_1).$$

Subtracting the second equality from the first, we obtain the relation

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\big(f'''(\xi_1) + f'''(\xi_2)\big). \tag{3.4.8}$$

From this, taking into account the identities

$$\big(\ln\Gamma(z)\big)' = \psi(z), \quad \psi''(z) = -2\sum_{k=0}^{\infty}(k+z)^{-3},$$

which imply the increase and negativeness of $(\ln\Gamma(z))'''$ on the ray $(0, +\infty)$, for $f(z) = \ln\Gamma(z)$ we obtain the inequalities

$$\begin{aligned} &\ln\Gamma(x+h) - \ln\Gamma(x-h) < 2h\psi(x), \\ &\ln\Gamma(x+h) - \ln\Gamma(x-h) > 2h\psi(x) + \frac{h^3}{6}\psi''(x-h), \quad 0 < h < x. \end{aligned} \tag{3.4.9}$$

Applying (3.4.9) for $x = (a+b)/2$ and $h = (b-a)/2$, we deduce the assertion of the lemma. □

Lemma 3.4.5. *For any $a, b > 0$, $a < b$, we have the inequality*

$$\frac{\ln\Gamma(b) - \ln\Gamma(a)}{b-a} < \ln\left(\frac{a+b}{2}\right). \tag{3.4.10}$$

If, moreover, $2 \le a < b \le 2a$, then

$$-\frac{2(b-a)^2}{3(a+b)^2} + \psi\left(\frac{a+b}{2}\right) < \frac{\ln\Gamma(b) - \ln\Gamma(a)}{b-a}. \tag{3.4.11}$$

Proof. Inequality (3.4.10) immediately follows from the upper estimates for the difference $\ln\Gamma(b) - \ln\Gamma(a)$ and the function ψ, which were proved in Lemmas 3.4.4 and 3.4.2. It is easily seen from Lemma 3.4.4 that to prove (3.4.11), it suffices to verify the estimate

$$\psi''(a) = -2\sum_{k=1}^{\infty}\frac{1}{(k+a)^3} > -\frac{16}{(a+b)^2}, \quad 2 \le a < b \le 2a,$$

which can be obtained from the inequality

$$\sum_{k=0}^{\infty}(k+a)^{-3} < \frac{8}{9}a^{-2}.$$

Prove it. We have the inequality

$$\frac{1}{p^3} < \int\limits_{p-1/2}^{p+1/2}\frac{du}{u^3} \quad \forall p > \frac{1}{2}.$$

Applying these inequalities for $p = k + a$, $k \in \mathbb{N}$, and adding them, we obtain the inequality

$$\sum_{k=0}^{\infty}(k+a)^{-3} < \int\limits_{a-1/2}^{+\infty} u^{-3}du = 0.5(a-0.5)^{-2} \le \frac{8}{9}a^{-2}, \quad a \ge 2.$$

The lemma is proved. □

3.5. Proof of Lemma 3.1.1

Inequalities (3.1.7) ban be written in detail as follows:

$$\ln\left(\frac{\pi\rho}{\sin\pi\rho}\right)+\ln\left(\frac{n-1+\rho(\mu-1)}{\rho}\right) < \rho\left(\ln\Gamma\left(\mu+\frac{n}{\rho}\right)-\ln\Gamma\left(\mu+\frac{n-1}{\rho}\right)\right),\quad 2\le n\le\left[\frac{1}{3\rho}\right],$$

and

$$\delta\rho\ln 2+\rho\left(\ln\Gamma\left(\mu+\frac{n}{\rho}\right)-\ln\Gamma\left(\mu+\frac{n-1}{\rho}\right)\right) < \ln\left(\frac{\pi\rho}{\sin\pi\rho}\right)+\ln\left(\frac{n+\rho(\mu-1)}{\rho}\right),\quad 1\le n\le\left[\frac{1}{3\rho}\right],$$

where $\delta=1$ for $0<\mu\le 1/\rho$ and $\delta=0.5$ for $1/\rho<\mu\le 2/\rho$. We simplify them replacing by a stronger inequality using the estimate

$$0<\ln\left(\frac{\pi\rho}{\sin\pi\rho}\right)<2\rho^2,\quad 0<\rho\le\frac{1}{2},$$

and Lemma 3.4.5 taking

$$a=\mu+\frac{n-1}{\rho},\quad b=\mu+\frac{n}{\rho}.$$

The condition $2\le a\le b\le 2a$ of the lemma appearing when we estimate the difference $\ln\Gamma(b)-\ln\Gamma(a)$ from below holds owing to the restriction $n\ge 2$ in the first inequality.

We arrive at the proof of the inequalities

$$2\rho^2+\ln\left(\frac{n-1+\rho(\mu-1)}{\rho}\right)<\frac{-1}{6\rho^2(\mu+(n-0.5)/\rho)^2}+\psi\left(\mu+\frac{n-0.5}{\rho}\right),\quad 2\le n\le\left[\frac{1}{3\rho}\right],$$

and

$$\delta\rho\ln 2+\ln\left(\mu+\frac{n-0.5}{\rho}\right)<\ln\left(\frac{n+\rho(\mu-1)}{\rho}\right),\quad 1\le n\le\left[\frac{1}{3\rho}\right]. \tag{3.5.1}$$

Applying in (3.5.1) the lower estimate for the function from Lemma 3.4.2 and transferring the logarithms on the right-hand sides of the inequalities, we replace inequalities (3.5.1) by stronger inequalities

$$2\rho^2+\frac{1}{6(\rho\mu+n-0.5)^2}+\frac{1}{2(\mu+(n-0.5)/\rho)-1}<\ln\left(\frac{\rho\mu+n-0.5}{\rho\mu+n-1-\rho}\right),\quad 2\le n\le\left[\frac{1}{3\rho}\right],$$

and

$$0.7\delta\rho<\ln\left(\frac{\rho\mu+n-\rho}{\rho\mu+n-0.5}\right),\quad 1\le n\le\left[\frac{1}{3\rho}\right].$$

Prove these inequalities.

Rewrite these inequalities in a compact form using the notation $y=\rho\mu+n-0.5$. We obtain the inequalities

$$2\rho^2+\frac{1}{6y^2}+\frac{\rho}{2y-\rho}<\ln\left(\frac{y}{y-0.5-\rho}\right),\quad 2\le n\le\left[\frac{1}{3\rho}\right], \tag{3.5.2}$$

$$0.7\delta\rho<\ln\left(\frac{y+0.5-\rho}{y}\right),\quad 1\le n\le\left[\frac{1}{3\rho}\right]. \tag{3.5.3}$$

Since the function $(y + 0.5 - \rho)/y$ decreases on the ray $0 < y < +\infty$, it suffices to prove inequality (3.5.3) for the maximal value of y, i.e., for

$$0.35\rho < \ln\left(1 + \frac{0.5-\rho}{n+1.5}\right), \quad n = \left[\frac{1}{3\rho}\right],$$
$$0.7\rho < \ln\left(1 + \frac{0.5-\rho}{n+0.5}\right), \quad n = \left[\frac{1}{3\rho}\right].$$

We have the inequalities

$$0.35n\rho \le \frac{0.35}{3} < 0.12, \quad 0.7n\rho < \frac{0.7}{3} < 0.24.$$

On the other hand (we use the fact that $\ln(1+t) > 0.9t$ for $0 < t \le 0.2$),

$$n\ln\left(1 + \frac{0.5-\rho}{n+1.5}\right) > \frac{0.9n(0.5-\rho)}{n+1.5} \ge \frac{0.3n}{n+1.5} \ge \frac{0.6}{3.5} > 0.16 > 0.35n\rho,$$
$$n\ln\left(1 + \frac{0.5-\rho}{n+0.5}\right) > \frac{0.9n(0.5-\rho)}{n+0.5} \ge \frac{0.3n}{n+0.5} \ge \frac{0.6}{2.5} > 0.24 > 0.7n\rho.$$

The inequality is proved.

To prove (3.5.2), we use the lower estimate

$$\ln\left(\frac{y}{y-2h}\right) > \frac{2h}{y-h}, \quad 0 < h < \frac{y}{2},$$

which follows from (3.4.8) and the positiveness of the third derivative of the logarithm:

$$\ln\left(\frac{y}{y-0.5-\rho}\right) > \frac{0.5+\rho}{y-0.25-\rho/2} = \frac{1+2\rho}{2y-0.5-\rho} > \frac{1+2\rho}{2y-\rho}.$$

After this it remains to prove the inequality

$$2\rho^2 + \frac{1}{6y^2} < \frac{1+\rho}{2y-\rho} \quad \Longleftrightarrow \quad 2\rho(\rho y) + \frac{1}{6y} < \frac{1+\rho}{2-\rho/y}.$$

Since

$$\rho y \le \rho\left(\frac{1}{3\rho} + 1.5\right) \le \frac{1}{3} + \frac{1}{4} = \frac{7}{12}, \quad y \ge 1.5,$$

we must verify that

$$\frac{7\rho}{6} + \frac{1}{9} < \frac{1+\rho}{2}, \quad 0 < \rho \le \frac{1}{6}.$$

This inequality is valid, and the proof of inequality (3.5.2) is complete. Lemma 3.1.1 is proved.

3.6. Proof of Theorem 3.1.1 in the Case $0.4 \le \rho < 0.5$

First, we consider the case where $\mu \in (0, 1/\rho]$. The reasoning from the beginning of Sec. 3.2 shows that the problem is reduced to the proof of the relations

$$E_\rho(-\xi_n^{1/\rho}(\rho,\mu);\mu) = (-1)^n \quad \forall n \in \mathbb{N}. \tag{3.6.1}$$

Theorems 1.5.3 and (3.2.3) imply that these equalities follow from the inequalities (we occasionally omit the arguments ρ and μ of ξ_n)

$$1.5\xi_n^{-1/\rho} < 2\rho\xi_n^{1-\mu}\exp(\xi_n\cos\pi\rho) \quad \forall n \in \mathbb{N}.$$

Show that for $0.4 \leq \rho < 0.5$, $x \geq \xi_1$, the function $g(\rho) = 2\rho \exp(x \cos \pi\rho)$ is greater than 1. Indeed, $g(0.5) = 1$,

$$\big(\ln g(\rho)\big)' = \frac{1}{\rho} - \pi x \sin \pi\rho \leq \frac{1}{\rho} - \pi\xi_1 \sin \pi\rho = \frac{1}{\rho} - \pi^2(1 - \rho(\mu - 1))$$
$$< \frac{1}{\rho} - \pi^2(1-\rho) < 2.5 - \frac{\pi^2}{2} < 0.$$

Therefore, $2\rho \exp(\xi_n \cos \pi\rho) > 1$ for all $n \in \mathbb{N}$, and it suffices to prove the inequality

$$1.5\xi_n^{-1/\rho} < \xi_n^{1-\mu} \quad \Longleftrightarrow \quad 1.5 < \xi_n^{1+1/\rho-\mu}.$$

Since $\mu \leq 1/\rho$, we must verify that $\xi_1 > 1.5$. We have the inequalities

$$\xi_1 = \frac{\pi(1+\rho(\mu-1))}{\sin \pi\rho} > \frac{\pi(1-\rho)}{\sin \pi\rho} > \pi(1-\rho) > \frac{\pi}{2} > 1.5,$$

which was required.

In the case where $1/\rho < \mu \leq 2/\rho$, we use the identities ($\mu = \lambda + 1/\rho$)

$$E_\rho\left(z; \lambda + \frac{1}{\rho}\right) = \frac{1}{z}\left(E_\rho(z;\lambda) - \frac{1}{\Gamma(\lambda)}\right), \qquad \xi_n\left(\rho, \lambda + \frac{1}{\rho}\right) = \xi_{n+1}(\rho, \lambda). \tag{3.6.2}$$

Relations (3.6.2) and (3.6.1) shows that for $\mu \in \left(\frac{1}{\rho}, \frac{2}{\rho} - 1\right]$, the problem is reduced to the proof of the equalities

$$\operatorname{sgn}\left[E_\rho(-\xi_m)^{1/\rho}(\rho,\lambda);\lambda) - \frac{1}{\Gamma(\lambda)}\right] = (-1)^m \quad \forall m \geq 2, \quad \forall \lambda \in \left(0, \frac{1}{\rho} - 1\right], \quad \forall \rho \in [0.4, 0.5).$$

Appealing to Theorem 1.5.3, we see that it suffices to deduce the inequalities

$$|\omega_\rho(\xi_m, \lambda)| + \frac{1}{\Gamma(\lambda)} < 2\rho\xi_m^{1-\lambda} \exp(\xi_m \cos \pi\rho). \tag{3.6.3}$$

We start from the values $\lambda \in (0,1]$. Substituting in (3.6.3) the estimate of the modulus of the remainder from Theorem 1.5.3 (see Chap. 1), we reduce the problem to the proof of the inequality

$$\Gamma(3.5)\xi_m^{-1/\rho}\left(\frac{3}{2}\min(\lambda, 1-\lambda) + \frac{1}{\rho} - 2\right) + \frac{1}{\Gamma(\lambda)} < 2\rho\xi_m^{1-\lambda}\exp(\xi_m \cos \pi\rho). \tag{3.6.4}$$

We write a stronger inequality replacing the left-hand side of (3.6.4) by a greater expression and the right-hand side by a lesser expression using the estimates

$$\Gamma(3.5) < 4, \quad \xi_m^{-1/\rho} < \xi_2^{-2}, \quad \xi_m^{1-\lambda} \geq 1, \quad \xi_2 > \pi(2-\rho)\operatorname{cosec} \pi\rho.$$

Namely, we prove that

$$6\xi_2^{-2}(1-\lambda) + 4\xi_2^{-2}\left(\frac{1}{\rho} - 2\right) + \frac{1}{\Gamma(\lambda)} < 2\rho \exp\big(\pi(2-\rho)\cot \pi\rho\big).$$

Again, we replace the inequality by a stronger inequality using the estimates $\xi_2 \geq 1.5\pi$ (or, equivalently, $\xi_2^{-2} < 1/20$) and $e^t > 1 + t$, $t > 0$. We obtain that it suffices to prove the inequality

$$0.3(1-\lambda) + 0.2\left(\frac{1}{\rho} - 2\right) + \frac{1}{\Gamma(\lambda)} < 2\rho(1 + 1.5\pi \cot \pi\rho). \tag{3.6.5}$$

Lemma 3.3.3 implies that

$$\gamma(1-\lambda)+\frac{1}{\Gamma(\lambda)}\le 1,\quad 0<\lambda\le 1,$$

and, the more so,

$$0.3(1-\lambda)+\frac{1}{\Gamma(\lambda)}\le 1,\quad 0<\lambda\le 1. \tag{3.6.6}$$

After the change $\varepsilon=1/2-\rho$, $0<\varepsilon\le 0.1$, we obtain the estimate

$$0.2\left(\frac{1}{\rho}-2\right)=\frac{0.8\varepsilon}{1-2\varepsilon}\le\varepsilon, \tag{3.6.7}$$

$$\begin{aligned}2\rho(1+1.5\pi\cot\pi\rho)&=(1-2\varepsilon)(1+1.5\pi\tan\pi\varepsilon)>(1-2\varepsilon)(1+1.5\pi^2\varepsilon)\\&>(1-2\varepsilon)(1+14\varepsilon)=1+12\varepsilon-28\varepsilon^2>1+9\varepsilon.\end{aligned} \tag{3.6.8}$$

From (3.6.6) and (3.6.8) we deduce (3.6.5). Inequalities (3.6.3) in the case $0<\lambda\le 1$ are proved.

For $1<\lambda\le 1/\rho-1$, by Theorem 1.5.3 (see Chap. 1), the problem is reduced to the proof of the inequality

$$\left(\frac{1}{\rho}-2\right)\Gamma\left(\frac{1}{\rho}\right)\xi_m^{-1/\rho}+\frac{1}{\Gamma(\lambda)}<2\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho). \tag{3.6.9}$$

For the considered values of the parameters λ we have the inequality

$$\xi_m(\rho,\lambda)\ge\xi_2(\rho,\lambda)=\frac{\pi(2+\rho(\lambda-1))}{\sin\pi\rho}>\frac{2\pi}{\sin\pi\rho}>2\pi. \tag{3.6.10}$$

Therefore, using Lemmas 3.4.1 and 3.4.2, we obtain the inequalities ($\delta=1/\rho-2$)

$$\Gamma\left(\frac{1}{\rho}\right)\xi_m^{-1/\rho}\le\Gamma(2)\xi_m^{-2}\le\xi_2^{-2},\quad \frac{1}{\Gamma(\lambda)}<1+\gamma(\lambda-1)\le 1+\gamma\left(\frac{1}{\rho}-2\right)=1+\delta\gamma.$$

Therefore, inequality (3.6.9) can be replaced by the following stronger inequality:

$$\delta(2\pi)^{-2}+1+\gamma\delta<2\rho\xi_m^{-\delta}\exp(\xi_m\cos\pi\rho). \tag{3.6.11}$$

The function

$$x^{-\delta}\exp(x\cos\pi\rho)\equiv\exp(-\delta\ln x+x\sin\pi\varepsilon)$$

increases on the ray $x>2$ since (from (3.6.7) it is seen that $\delta\le 5\varepsilon$)

$$\frac{d}{dx}(-\delta\ln x+x\sin\pi\varepsilon)=-\frac{\delta}{x}+\sin\pi\varepsilon>-\frac{\delta}{2}+3\varepsilon>0.$$

Therefore (see (3.6.10)), we have the inequality

$$\begin{aligned}2\rho\xi_m^{-\delta}\exp(\xi_m\cos\pi\rho)&\ge 2\rho\exp(-\delta\ln 6+6\sin\pi\varepsilon)>2\rho\exp(-1.8\delta+18\varepsilon)\\&\ge 2\rho e^{9\varepsilon}=(1-2\varepsilon)e^{9\varepsilon}>(1-2\varepsilon)(1+9\varepsilon)=1+7\varepsilon-18\varepsilon^2\ge 1+5.2\varepsilon.\end{aligned}$$

However, the left-hand side of (3.6.11) does not exceed

$$1+0.62\delta\le 1+\frac{0.62\cdot 4\varepsilon}{1-2\varepsilon}\le 1+3.1\varepsilon.$$

Inequality (3.6.11) is proved, and the proof of Theorem 3.1.1 for the case $0.4\le\rho<0.5$ is complete.

3.7. Proof of Theorem 3.1.1 in the Case $0.25 < \rho < 0.4$

Arguing as in the previous section and applying Theorem 1.5.4, we see that for the proof of relations (3.6.1) in the case considered it suffices to verify the inequalities

$$0.74\xi_n^{-\mu} < 2\rho\xi_n^{1-\mu}\exp(\xi_n\cos\pi\rho), \quad 0 < \mu \le \frac{1}{\rho}, \quad n \ge 1, \tag{3.7.1}$$

$$0.74\xi_n^{-\lambda} + \frac{1}{\Gamma(\lambda)} < 2\rho\xi_n^{1-\lambda}\exp(\xi_n\cos\pi\rho), \quad 0 < \lambda \le \frac{1}{\rho} - 1, \quad n \ge 2. \tag{3.7.2}$$

We start from the proof of inequalities (3.7.1). Dividing them by $2\rho\xi_n^{1-\mu}$, we obtain the equivalent inequalities

$$\frac{0.74}{2\rho\xi_n} < \exp(\xi_n\cos\pi\rho), \tag{3.7.3}$$

which must be proved only for $n = 1$ owing to the fact that the right-hand side increases, and the left-hand side decreases, with respect to n. Since

$$\xi_1 = \xi_1(\rho,\mu) = \pi\big(1 + \rho(\mu - 1)\big)\operatorname{cosec}(\pi\rho) > \pi(1-\rho),$$

we have

$$2\rho\xi_1 > 2\pi\rho(1-\rho) > \frac{3\pi}{8}, \quad 0.25 < \rho < 0.4.$$

Therefore, the left-hand side of (3.7.3) is less than 1. Since the right-hand side of (3.7.3) is obviously greater than 1, the inequality is valid.

Now we prove inequality (3.7.2). In the case where $0 < \lambda \le 1$, the left-hand side of (3.7.2) is less than 1.74 and the right-hand side is greater than

$$0.5\exp(\xi_2\cos\pi\rho) = 0.5\exp\big(\pi(2 + \rho(\lambda-1))\cot\pi\rho\big)$$
$$> 0.5\exp\big(\pi(2-\rho)\cot\pi\rho\big) > 0.5\exp\big(1.6\pi\cot(0.4\pi)\big) > 2.$$

Consider the values $a = \lambda - 1 \in (0, 1/\rho - 2]$. Note that the function $x^{-a}e^{\tau x}$, $\tau > 0$, increases on the ray $a/\tau < x < +\infty$. We have $\tau = \cos\pi\rho$ and the inequality

$$\frac{a}{\cos\pi\rho} < \xi_2(\rho,\lambda) \quad \Longleftarrow \quad \frac{1}{\rho} - 2 < 2\pi\cot\pi\rho$$

holds since

$$\frac{1}{\rho} - 2 < \frac{1}{2\rho}, \quad \frac{1}{4} < \frac{\pi\rho}{\tan\pi\rho} \quad \text{for } 0.25 < \rho < 0.4.$$

This implies that inequality (3.7.2) must be proved only for $n = 2$. In this case it has the form

$$0.74\Gamma(\lambda)\xi_2^{-\lambda} + 1 < 2\rho\Gamma(\lambda)\xi_2^{1-\lambda}\exp(\xi_2\cos\pi\rho), \quad 1 < \lambda \le \frac{1}{\rho} - 1. \tag{3.7.4}$$

Since

$$\xi_2 = \xi_2(\rho,\lambda) = \pi(2 + \rho a)\operatorname{cosec}(\pi\rho), \tag{3.7.5}$$

we see that $\xi_2 > 2\pi$ and the left-hand side of (3.7.4) is less than $1 + 0.74\Gamma(\lambda)(2\pi)^{-\lambda}$, but this function, by Lemma 3.4.2, decreases with respect to λ (in our case $1 < \lambda \le 3$). Therefore, the left-hand side of (3.7.4) is less than $1 + 0.7/(2\pi) < 1.12$.

To obtain a lower estimate of the right-hand side of (3.7.4), we prove that it decreases with respect to the parameter λ or, equivalently, that the function

$$g(a) = \ln\Gamma(a+1) - a\ln\big(\pi(2+\rho a)\operatorname{cosec}\pi\rho\big) + \pi(2+\rho a)\cot\pi\rho$$

decreases with respect to the variable a on the segment $0 \le a \le \dfrac{1}{\rho} - 2 < 2$. We have the equality

$$g'(a) = \psi(a+1) - \ln\big(\pi(2+\rho a)\operatorname{cosec}\pi\rho\big) - \frac{\rho a}{2+\rho a} + \pi\rho\cot\pi\rho.$$

Since

$$\psi(a+1) \le \psi\left(\frac{1}{\rho} - 1\right) < \psi(3) = 1.5 - \gamma < 0.93, \quad \pi\rho\cot(\pi\rho) < \frac{\pi}{4} \quad \text{for } 0.25 < \rho < 0.4,$$

we have

$$g'(a) < 1.72 - \ln\big(\pi(2+\rho a)\operatorname{cosec}\pi\rho\big) < 1.72 - \ln(2\pi) < 0.$$

Thus, the right-hand side of (3.7.4) is not less than its value at the point $\lambda = 1/\rho - 1$. Substituting the expression (3.7.5) for ξ_2, in which $a = 1/\rho - 2$, in (3.7.4), we see that it remains to prove the inequality

$$1.12 < 2\rho\Gamma\left(\frac{1}{\rho} - 1\right)\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right)^{2-1/\rho} \exp\big(\pi(3-2\rho)\cot\pi\rho\big), \quad 0.25 < \rho < 0.4.$$

Consider the function

$$F(\rho) = \rho\Gamma\left(\frac{1}{\rho} - 1\right)\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right)^{2-1/\rho} \exp\big(\pi(3-2\rho)\cot\pi\rho\big).$$

We must prove that $F(\rho) > 0.56$ for $0.25 < \rho < 0.4$. For this, we verify that $F(\rho)$ decreases on this interval. Using the continuity of this function on the segment $0.25 \le \rho \le 0.4$, we see that the problem is reduced to the proof of the numerical inequality $F(0.4) > 0.56$. It is valid ($F(0.4) > 1.2$), and Theorem 3.1.1 for the case $0.25 < \rho < 0.4$ is proved in this case.

Denote $F_1(\rho) = \ln F(\rho)$. We have the relation

$$F_1'(\rho) = \frac{1}{\rho} - \frac{1}{\rho^2}\psi\left(\frac{1}{\rho} - 1\right) + \frac{1}{\rho^2}\ln\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right) + \left(\frac{1}{\rho} - 2\right)\left(\pi\cot\pi\rho + \frac{1}{1.5-\rho}\right) - \frac{\pi^2(3-2\rho)}{\sin^2\pi\rho} - 2\pi\cot\pi\rho.$$

From this we obtain the following representation for the function $\Phi(\rho) = \rho^2 F_1'(\rho)$:

$$\Phi(\rho) = \rho(1 - 2\pi\rho\cot\pi\rho) - \psi\left(\frac{1}{\rho} - 1\right) + \ln\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right) + (1-2\rho)\left(\pi\rho\cot\pi\rho + \frac{\rho}{1.5-\rho}\right) - (3-2\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2.$$

The function $\pi\rho\cot\pi\rho$ decreases on $(0, 1/2)$ and $\rho/(1.5-\rho)$ increases. Therefore, the following estimates hold:

$$\rho(1 - 2\pi\rho\cot\pi\rho) < \rho\big(1 - 0.8\pi\cot(0.4\pi)\big) < 0.19\rho,$$

$$(1-2\rho)\left(\pi\rho\cot\pi\rho + \frac{\rho}{1.5-\rho}\right) < \begin{cases} \dfrac{1}{2}\left(\dfrac{\pi}{4} + \dfrac{2}{7}\right) < 0.536, & \dfrac{1}{4} < \rho \le \dfrac{1}{3}, \\ \dfrac{1}{3}\left(\dfrac{\pi}{3\sqrt{3}} + \dfrac{4}{11}\right) < 0.324, & \dfrac{1}{3} < \rho < 0.4. \end{cases}$$

Therefore,

$$\Phi(\rho) < A + \ln\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right) - \psi\left(\frac{1}{\rho} - 1\right) - (3-2\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2, \tag{3.7.6}$$

where $A = 0.6$ for $1/4 < \rho \le 1/3$ and $A = 0.4$ for $1/3 < \rho < 0.4$. For $1/4 < \rho \le 1/3$, we have the inequalities

$$\ln\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right) < \ln(2.5\pi\sqrt{2}) < 2.41,$$
$$\psi\left(\frac{1}{\rho}-1\right) \ge \psi(2) = 1-\gamma > 0.42,$$
$$(3-2\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2 > \left(2+\frac{1}{3}\right)\frac{\pi^2}{8} > 2.76.$$

These numerical estimates and (3.7.6) imply the negativeness of $\Phi(\rho)$ on the semi-interval $(1/4, 1/3]$. For $1/3 < \rho < 0.4$, we have the inequalities

$$\ln\left(\frac{\pi(3-2\rho)}{\sin\pi\rho}\right) < \ln\left(2+\frac{1}{3}\right) + \ln\pi + \ln\left(\frac{2}{\sqrt{3}}\right) < 2.14,$$
$$(3-2\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2 > 2.2\cdot\frac{4\pi^2}{27} > 3,$$

and the inequality $\Phi(\rho) < 0$ holds. We have proved that the function F decreases.

3.8. Proof of Theorem 3.1.1 in the Case $1/6 < \rho \le 1/4$

If $\rho \in (1/6, 1/4]$, then by Theorem 1.5.4 the "principal part" of the function $E_\rho(-x^{1/\rho}, \mu)$ contains two terms:

$$2\rho x^{1-\mu}\Big[\exp(x\cos\pi\rho)\cos\big(x\sin\pi\rho - \pi\rho(\mu-1)\big) + \exp(x\cos3\pi\rho)\cos\big(x\sin3\pi\rho - 3\pi\rho(\mu-1)\big)\Big].$$

Therefore, to prove relations (3.2.1), it suffices to deduce the inequalities

$$\begin{aligned} &0.74\xi_n^{-\mu} < 2\rho\xi_n^{1-\mu}\Big[\exp(\xi_n\cos\pi\rho) - \exp(\xi_n\cos3\pi\rho)\Big], && 0 < \mu \le \frac{1}{\rho}, \quad n \ge 1, \\ &0.74\xi_n^{-\lambda} + \frac{1}{\Gamma(\lambda)} < 2\rho\xi_n^{1-\lambda}\Big[\exp(\xi_n\cos\pi\rho) - \exp(\xi_n\cos3\pi\rho)\Big], && 0 < \lambda \le \frac{1}{\rho}, \quad n \ge 2. \end{aligned} \tag{3.8.1}$$

Since

$$\begin{aligned} \frac{\exp(\xi_n\cos3\pi\rho)}{\exp(\xi_n\cos\pi\rho)} &= \exp\left(\pi\big(n+\rho(\mu-1)\big)\frac{\cos3\pi\rho - \cos\pi\rho}{\sin\pi\rho}\right) \\ &= \exp\Big(-2\pi\big(n+\rho(\mu-1)\big)\sin2\pi\rho\Big) \le \exp\big(-2\pi(1-\rho)\sin2\pi\rho\big) \le \exp\big(-\pi(1-\rho)\sqrt{3}\big) \\ &< \exp(-0.75\pi\sqrt{3}) < 0.02, \end{aligned}$$

inequalities (3.8.1) can be replaced by the following stronger inequalities:

$$\frac{0.74}{2\rho\xi_n} < 0.98\exp(\xi_n\cos\pi\rho), \tag{3.8.2}$$
$$0.74\xi_n^{-\lambda} + \frac{1}{\Gamma(\lambda)} < 1.96\rho\xi_n^{1-\lambda}\exp(\xi_n\cos\pi\rho). \tag{3.8.3}$$

From the estimates

$$2\rho\xi_n \ge 2\rho\xi_1 \ge 2\rho\pi(1-\rho)\operatorname{cosec}(\pi\rho) > 2(1-\rho) \ge 1.5$$

we obtain that

$$\frac{0.74}{2\rho\xi_n} < \frac{0.74}{1.5} < 0.5,$$

and the right-hand side of (3.8.2) is greater than 0.98. This shows that inequality (3.8.2) is valid.

We prove inequalities (3.8.3). For $0 < \lambda \le 1$, there is no difficulty, as in the previous section: the left-hand side is less than 1.74, while the right-hand side is greater than $1.9\rho \exp(\xi_2 \cos \pi\rho)$. But

$$\xi_2 \cos \pi\rho \ge \pi(2-\rho)\cot \pi\rho \ge \pi(2-\rho) \ge 1.75\pi,$$

and hence

$$\rho \exp(\xi_2 \cos \pi\rho) > (1/6)\exp(1.75\pi) > 30$$

and inequality (3.8.3) is valid.

Consider values $a = \lambda - 1 \in (0, 1/\rho - 1]$. We rewrite inequality (3.8.3), which must be proved, in the form

$$0.74\Gamma(\lambda)\xi_n^{-\lambda} + 1 < 1.96\rho\Gamma(a+1)\xi_n^{-a}\exp(\xi_n \cos \pi\rho). \tag{3.8.4}$$

Since the function $x^{-1}\exp(x \cos \pi\rho)$ increases on the ray $x > a\sec(\pi\rho)$ and the inequalities $a \le 5$ and

$$\xi_n \ge \xi_2 > 2\pi \operatorname{cosec}(\pi\rho) > 5\sec(\pi\rho),$$

hold in the case $1/6 < \rho \le 1/4$, we obtain that it suffices to prove inequalities (3.8.4) for $n = 2$. By Lemma 3.4.2, the function $\Gamma(\lambda)\xi_2^{-\lambda}$ decreases with respect to the variable λ on the segment $1 \le \lambda \le 6$, since for $\lambda \ge 1$ and $\rho \le 1/4$ we have the inequality

$$\xi_2 \ge 2\pi \operatorname{cosec}(\pi\rho) \ge 2\pi\sqrt{2}.$$

Therefore, the left-hand side of (3.8.4) is greater than $1 + 0.74/(2\pi\sqrt{2}) < 1.1$. For any fixed $\rho \in (1/6, 1/4]$, the right-hand side of (3.8.4) also decreases with respect to the variable a on the segment $0 \le a \le 5$. To prove this based on the explicit formula for ξ_2, it suffices to verify that the function

$$g(a) = \ln\Gamma(a+1) - a\ln\left(\frac{\pi(2+\rho a)}{\sin \pi\rho}\right) + \pi(2+\rho a)\cot(\pi\rho).$$

decreases. We have the equality

$$g'(a) = \psi(a+1) - \ln\left(\frac{\pi(2+\rho a)}{\sin \pi\rho}\right) - \frac{a\rho}{2+a\rho} + \pi\rho\cot(\pi\rho). \tag{3.8.5}$$

Prove that the derivative (3.8.5) is negative. First, we obtain a numerical upper estimate of the function

$$t\cot t + \ln \sin t, \quad t = \pi\rho \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right].$$

This function increases on $(0, \pi/4]$ and, therefore, does exceed its value $\pi/4 - 0.5\ln 2 < 0.44$ at the point $\pi/4$. Thus,

$$g'(a) < 0.44 + \psi(a+1) - \ln\big(\pi(2+\rho a)\big) - \rho a/(2+\rho a).$$

For $0 < a \le 3$ we have the inequality

$$g'(a) < 0.44 + \psi(4) - \ln(2\pi) = 0.44 + \frac{11}{6} - \gamma - \ln(2\pi) < 0.$$

For $3 < a \le 5$, since the function $\rho a/(2+\rho a)$ increases with respect to a and ρ and the estimate $\psi(t) < \ln t$ holds, we have the inequality

$$g'(a) < 0.44 + \psi(a+1) - \ln\big(\pi(2+3\rho)\big) - \frac{3\rho}{(2+3\rho)} < 0.44 + \psi(6) - \ln(2.5\pi) - 0.2 < 0.24 + \ln\left(\frac{6}{2.5\pi}\right) < 0.$$

From this we conclude that the right-hand side of (3.8.4) is not less than its value at the point $a = 1/\rho - 1$. It remains to prove the inequality

$$1.1 < 1.96\rho\Gamma\left(\frac{1}{\rho}\right)\left(\frac{\pi(3-\rho)}{\sin \pi\rho}\right)^{1-1/\rho}\exp\Big(\pi(3-\rho)\cot \pi\rho\Big), \quad \frac{1}{6} < \rho \le \frac{1}{4}. \tag{3.8.6}$$

Taking the logarithms, we obtain the inequality

$$\ln 1.1 < 0.1 < F(\rho),$$

where

$$F(\rho) = \ln 1.96 + \ln \rho + \ln \Gamma\left(\frac{1}{\rho}\right) - \left(\frac{1}{\rho} - 1\right) \ln \left(\frac{\pi(3-\rho)}{\sin \pi\rho}\right) + \pi(3-\rho)\cot \pi\rho.$$

We have the relation

$$F\left(\frac{1}{4}\right) = \ln(0.49) + \ln 6 - 3\ln(2.75\pi\sqrt{2}) + 2.75\pi > 1.$$

It remains to prove that $F(\rho)$ decreases on the semi-interval $(1/6, 1/4]$. We calculate the derivative:

$$F'(\rho) = \frac{1}{\rho} - \frac{1}{\rho^2}\psi\left(\frac{1}{\rho}\right) + \frac{1}{\rho^2}\ln\left(\frac{\pi(3-\rho)}{\sin\pi\rho}\right) + \left(\frac{1}{\rho} - 1\right)\left(\frac{1}{3-\rho} + \pi\cot\pi\rho\right) - \pi\cot\pi\rho - (3-\rho)\frac{\pi^2}{\sin^2\pi\rho}.$$

We set $\Phi(\rho) = \rho^2 f(\rho)$. Then

$$\Phi(\rho) = \rho - \psi\left(\frac{1}{\rho}\right) + \ln\left(\frac{\pi(3-\rho)}{\sin\pi\rho}\right) + \frac{\rho(1-\rho)}{3-\rho} + (1-2\rho)\pi\rho\cot\pi\rho - (3-\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2.$$

Replacing the functions $\pi\rho\cot\pi\rho$ and $(1-\rho)/(3-\rho)$ by their upper estimate (identity unit) in the formula for $\Phi(\rho)$, we obtain the inequality

$$\Phi(\rho) < 1 - \psi\left(\frac{1}{\rho}\right) + \ln\left(\frac{\pi(3-\rho)}{\sin\pi\rho}\right) - (3-\rho)\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2. \tag{3.8.7}$$

Since the function $t/\sin t$ increases on the interval $(0, \pi/2)$ for $1/6 < \rho < 1/2$, the estimate

$$\left(\frac{\pi\rho}{\sin\pi\rho}\right)^2 > \frac{\pi^2}{9}$$

holds. Therefore, we have the inequalities

$$\left(\frac{\pi\rho}{\sin\pi\rho}\right)(3-\rho) > \frac{2.75\pi^2}{9} > 30, \quad \ln\left(\frac{\pi(3-\rho)}{\sin\pi\rho}\right) < \ln\left(2\pi\left(3 - \frac{1}{6}\right)\right) < \ln 18 < 2.9. \tag{3.8.8}$$

From (3.8.7) and (3.8.8) we obtain that

$$\Phi(\rho) < 0.9 - \psi\left(\frac{1}{\rho}\right) \le 0.9 - \psi(4) < 0,$$

which was required. Inequality (3.8.6) is proved, and the proof of the realness of all zeros of the function $E_\rho(z, \mu)$ for $1/6 < \rho \le 1/4$ and $0 < \mu \le 2/\rho$ is complete.

3.9. Completion of Proof of Theorem 3.1.1 (Case $0 < \rho \le 1/6$). Proof of Theorem 3.1.2

As was said in Sec. 3.2, for the proof of Theorems 3.1.1 and 3.1.2 for $\rho \in (0, 1/6]$, we must deduce relations (3.2.1), (3.2.6), and (3.2.7).

First, we prove relations (3.2.1). Owing to Theorem 1.5.4, the problem is reduced to the proof of inequalities

$$0.74\xi_n^{-\mu} < 2\rho\xi_n^{1-\mu}\left[\exp(\xi_n\cos\pi\rho) - \sum_{k=2}^{\left[\frac{1}{2\rho}\right]} \exp\Big(\xi_n \cos\big(\pi(2k-1)\rho\big)\Big)\right]$$

for $0 < \mu \le 1/\rho$ and $n \ge N_1(\rho)$ and

$$0.74\xi_n^{-\lambda} + \frac{1}{\Gamma(\lambda)} < 2\rho\xi_n^{1-\lambda}\left[\exp(\xi_n\cos\pi\rho) - \sum_{k=2}^{\left[\frac{1}{2\rho}\right]}\exp\Big(\xi_n\cos\big(\pi(2k-1)\rho\big)\Big)\right]$$

for $0 < \lambda \le 1/\rho$ and $n \ge 1 + N_1(\rho)$.

We obtain an upper estimate of the sum of exponents

$$S_n = \sum_{k=2}^{\left[\frac{1}{2\rho}\right]}\exp\Big(\xi_n\cos\big(\pi(2k-1)\rho\big)\Big).$$

The ratio (3.2.5) of functions (3.2.4) with numbers $k+1$ and k is equal to

$$\begin{aligned}\Delta_{k,\rho}(x) &= \exp\big(-2x\sin\pi\rho\sin 2\pi k\rho\big) \le \exp\big(-2\xi_{N_1(\rho)}\sin\pi\rho\sin 2\pi k\rho\big)\\ &\le \exp\big(-2\pi(N_1(\rho)-\rho)\sin(2\pi k\rho)\big) \le \exp\big(-2\pi(N_1(\rho)-\rho)\sin 2\pi\rho\big)\\ &\le \exp\Big(-8\pi\rho\Big(N_1(\rho)-\rho\Big)\Big) \le \exp\Big[-8\pi\rho\Big(\frac{1}{3\rho}-1-\rho\Big)\Big] \le \exp\Big(-8\pi\Big(\frac{1}{3}-\rho-\rho^2\Big)\Big)\\ &\le \exp\left(-8\pi\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{36}\right)\right) = \exp\left(-\frac{10\pi}{9}\right) < \frac{1}{26},\end{aligned} \tag{3.9.1}$$

where $x \ge \xi_{N_1}$. Here we used the following relations:

$$\xi_n\sin\pi\rho = \pi\big(n+\rho(\mu-1)\big) > \pi(n-\rho),$$

$$\sin(2\pi k\rho) \ge \min\left[\sin 2\pi\rho,\ \sin\left(2\pi\left(\frac{1}{2\rho}-1\right)\rho\right)\right] = \sin 2\pi\rho, \quad 1 \le k \le \frac{1}{2\rho}-1,$$

$$\sin(\pi t) \ge 2t, \quad 0 \le t \le \frac{1}{2} \quad\Longrightarrow\quad \sin(2\pi\rho) \ge 4\rho, \quad 0 < \rho \le 1/4.$$

It is seen from (3.9.1) that for $x \ge \xi_{N_1}$, any subsequent function $\exp\Big[x\cos\big(\pi(2k-1)\rho\big)\Big]$ is less than the preceding at least 26 times. This leads to the estimate

$$S_n < \frac{1}{25}\exp(\xi_n\cos\pi\rho)$$

and allows one to proceed to the proof of simpler inequalities

$$0.74\xi_n^{-\mu} < 1.92\rho\xi_n^{1-\mu}\exp(\xi_n\cos\pi\rho), \quad 0 < \mu \le \frac{1}{\rho}, \quad n \ge N_1(\rho), \tag{3.9.2}$$

$$0.74\xi_n^{-\lambda} + \frac{1}{\Gamma(\lambda)} < 1.92\rho\xi_n^{1-\lambda}\exp(\xi_n\cos\pi\rho), \quad 0 < \lambda \le \frac{1}{\rho}, \quad n \ge 1 + N_1(\rho). \tag{3.9.3}$$

Inequality (3.9.2) is equivalent to the following:

$$\frac{0.74}{1.92\rho\xi_n} < \exp(\xi_n\cos\pi\rho). \tag{3.9.4}$$

We have the inequality

$$\rho\xi_n = \frac{\pi\rho\big(n+\rho(\mu-1)\big)}{\sin\pi\rho} > n+\rho(\mu-1) > n-\rho \ge 2-\rho > 1, \tag{3.9.5}$$

since $n \ge N_1(\rho) \ge 2$. Therefore, the left-hand side of (3.9.4) is less than 1 and the right-hand side is obviously greater than 1. Inequality (3.9.2) is proved.

We prove inequality (3.9.3). For $0 < \lambda \le 1$, there is no difficulty: the left-hand side of (3.9.3) is less than 1.74 and the right-hand side is greater than

$$1.92\rho \exp(\xi_n \cos \pi\rho) > 1.92\rho\xi_n \cos(\pi\rho) > 1.92(n-\rho)\cos\pi\rho \ge 1.92(2-\rho)\cos(\pi/6) > 2$$

and the required inequality is proved (we have used the estimates $e^t > t$ and (3.9.5)).

For $1 < \lambda \le 1/\rho$, we rewrite (3.9.3) in the equivalent form

$$0.74\Gamma(\lambda)\xi_n^{-\lambda} + 1 < 1.92\rho\Gamma(\lambda)\xi_n^{1-\lambda}\exp(\xi_n\cos\pi\rho).$$

By Lemma 3.4.2, the function $\Gamma(\lambda)\xi^{-\lambda}$ decreases on the segment $1 \le \lambda \le 1/\rho$ if $\xi > 1/\rho$. In our case, $\xi_n \ge 1/\rho$ (see (3.9.5)) and, therefore,

$$\Gamma(\lambda)\xi_n^{-\lambda} \le \Gamma(1)\xi_n^{-1} \le \frac{\rho}{2-\rho} \le \frac{1/6}{2-1/6} < 0.1.$$

Thus, we must prove that

$$0.1 < \ln 1.92 + \ln\rho + \ln\Gamma(\lambda) + (1-\lambda)\ln\xi_n + \xi_n\cos\pi\rho$$

or (in a detailed form with account of the inequality $\ln 1.92 > 0.6$)

$$\begin{gathered} 0 < 0.5 + \ln\rho + \ln\Gamma(a+1) - a\ln\left(\frac{\pi(n+a\rho)}{\sin\pi\rho}\right) + \pi(n+a\rho)\cot\pi\rho, \\ 0 < a \le \frac{1}{\rho} - 1, \quad n \ge 1 + \left[\frac{1}{3\rho}\right], \quad 0 < \rho \le \frac{1}{6}. \end{gathered} \tag{3.9.6}$$

We verify that the minimum of the right-hand side of (3.9.6) with respect to the variable a is attained at $a = 1/\rho - 1$. Fir this, it suffices to prove the negativeness of the derivative

$$\begin{aligned} \psi(a+1) - \ln\left(\frac{\pi(n+a\rho)}{\sin\pi\rho}\right) - \frac{a\rho}{n+a\rho} + \pi\rho\cot\pi\rho &< 1 + \psi(a+1) - \ln\left(\frac{\pi(n+a\rho)}{\sin\pi\rho}\right) \\ &< 1 + \ln\frac{1}{\rho} - \ln\frac{\pi n}{\sin\pi\rho} = 1 + \ln\left(\frac{\sin\pi\rho}{\pi\rho}\right) - \ln n < 1 - \ln n < 0; \end{aligned}$$

here we have used the inequalities $t\cot t < 1$ for $0 < t < \pi/2$, $\psi(t) < \ln t$ for $t > 0$, and

$$n \ge 1 + \left[\frac{1}{3\rho}\right] \ge 3.$$

Thus, it remain to prove (3.9.6) for $a = 1/\rho - 1$, i.e.,

$$0 < 0.5 + \ln\rho + \ln\Gamma\left(\frac{1}{\rho}\right) - \left(\frac{1}{\rho} - 1\right)\ln\left(\frac{\pi(n+1-\rho)}{\sin\pi\rho}\right) + \pi(n+1-\rho)\cot\pi\rho. \tag{3.9.7}$$

Clearly,

$$n + 1 - \rho > n > \frac{1}{3\rho}.$$

We show that the right-hand side of (3.9.7) will decrease if we replace $n+1-\rho$ by a smaller number $1/(3\rho)$. For this, we prove that the function

$$u(t) = \pi t\cot\pi\rho - \left(\frac{1}{\rho} - 1\right)\ln\left(\frac{\pi t}{\sin\pi\rho}\right)$$

increases on the ray $2 \le t < +\infty$ (containing the ray $[1/(3\rho), +\infty)$). We have the inequality

$$u'(t) = \pi\cot\pi\rho - \left(\frac{1}{\rho} - 1\right)\frac{1}{t} > \pi\cot\pi\rho - \frac{1}{\rho t} = \frac{1}{\rho t}(t\pi\rho\cot\pi\rho - 1)$$
$$> \frac{1}{\rho t}\left(\frac{\pi t}{4} - 1\right) \ge \frac{1}{\rho t}\left(\frac{\pi}{2} - 1\right) > 0.$$

Thus, (3.9.7) can be replaced by the following stronger inequality and just its proof:

$$0 < 0.5 + \ln\rho + \ln\Gamma\left(\frac{1}{\rho}\right) - \left(\frac{1}{\rho} - 1\right)\ln\left(\frac{\pi}{3\rho\sin\pi\rho}\right) + \frac{\pi}{3\rho}\cot\pi\rho, \quad 0 < \rho \le \frac{1}{6}. \tag{3.9.8}$$

For $\rho = 1/6$, inequality (3.9.8) holds; this can be verified by a direct calculation. Therefore, it suffices to prove that the function on the right-hand side of (3.9.8) decreases. The derivative of this function is equal to

$$\frac{1}{\rho} - \frac{1}{\rho^2}\psi\left(\frac{1}{\rho}\right) + \frac{1}{\rho^2}\ln\left(\frac{\pi}{3\rho\sin\pi\rho}\right) + \left(\frac{1}{\rho} - 1\right)\left(\frac{1}{\rho} + \frac{\pi}{\tan\pi\rho}\right) - \frac{\pi}{3}\left(\frac{\cot\pi\rho}{\rho^2} + \frac{\pi}{\rho\sin^2\pi\rho}\right).$$

Therefore, we must verify that the function

$$g(\rho) = \rho - \psi\left(\frac{1}{\rho}\right) + \ln\left(\frac{\pi}{3\rho\sin\pi\rho}\right) + (1-\rho)\Big(1 + \pi\rho\cot\pi\rho\Big) - \frac{\pi}{3}\left(\cot\pi\rho + \frac{\pi\rho}{\sin^2\pi\rho}\right)$$

is negative. To estimate this function, we use inequalities

$$\psi(t) > \ln t - (2t-1)^{-1}$$

(see Lemma 3.4.1) and $\pi\rho\cot\pi\rho < 1$. Introducing the notation $h(\rho) = \pi/(3\sin\pi\rho)$, we obtain

$$g(\rho) < 2 - \rho + \left(\frac{2}{\rho} - 1\right)^{-1} + \ln h(\rho) - \frac{\pi}{3}\cot\pi\rho - \frac{\pi\rho}{\sin\pi\rho}h(\rho). \tag{3.9.9}$$

Using the inequalities $\pi\rho/\sin\pi\rho > 1$, $h(\rho) > 1$, and $\ln h(\rho) < h(\rho) - 1$, from (3.9.9) we have

$$g(\rho) < 1 - \rho + \frac{\rho}{2-\rho} - \frac{\pi}{3}\cot\pi\rho < 1 - \frac{\pi}{3}\cot\pi\rho < 0, \quad 0 < \rho \le \frac{1}{6}.$$

The proof of relations (3.2.1) for $n \ge [1/(3\rho)\big]$ is complete.

Prove relations (3.2.6) and (3.2.7).

Lemma 3.9.1. *For any $\rho > 0$, $\mu \in \mathbb{R}$, and $n \in \mathbb{N}$ satisfying the condition $n + \rho\mu > 1$, the following inequality holds*:

$$\frac{R_n(\rho,\mu)}{R_{n+1}(\rho,\mu)} < \exp\left(-\frac{1}{\rho(n+\rho\mu)}\right).$$

Proof. By the definition of $R_n(\rho,\mu)$ we have the equality

$$\ln\left(\frac{R_{n+1}(\rho,\mu)}{R_n(\rho,\mu)}\right) = \ln\Gamma\left(\mu + \frac{n-1}{\rho}\right) - 2\ln\Gamma\left(\mu + \frac{n}{\rho}\right) + \ln\Gamma\left(\mu + \frac{n+1}{\rho}\right). \tag{3.9.10}$$

Note that if a real-valued function f is continuous on a segment $[x-h, x+h]$ ($x \in \mathbb{R}$, $h > 0$), has on the interval $(x-h, x+h)$ derivatives up to fourth order, and $f^{(4)}(t) > 0$ for all $t \in (x-h, x+h)$, then

$$f(x+h) - 2f(x) + f(x-h) > h^2 f''(x). \tag{3.9.11}$$

Indeed, applying Taylor's formula with the remainder in the Lagrange form, we have

$$f(x+h) = f(x) + hf(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(\xi_1), \quad \xi_1 \in (x, x+h),$$
$$f(x-h) = f(x) - hf(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(\xi_2), \quad \xi_2 \in (x-h, x).$$

Adding these equalities and subtracting $2f(x)$ from the result, we obtain

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(x) + \frac{1}{24}h^4\big(f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\big). \tag{3.9.12}$$

From (3.9.12) and the positiveness $f^{(4)}$ we immediately obtain (3.9.11). In our case $x = n/\rho$, $h = 1/\rho$, and

$$f(t) = f_\mu(t) = \ln\Gamma(\mu + t) \in C^\infty(0, +\infty).$$

We have the relations

$$f''_\mu(t) = \psi'(\mu + t) = \sum_{k=0}^{\infty}(k + \mu + t)^{-2}, \quad f^{(4)}_\mu(t) = 6\sum_{k=0}^{\infty}(k + \mu + t)^{-4} > 0 \quad \forall t > 0. \tag{3.9.13}$$

From (3.9.10), (3.9.11), and (3.9.13) we obtain

$$\ln\left(R_{n+1}(\rho,\mu)/R_n(\rho,\mu)\right) > \sum_{k=0}^{\infty}(k + \mu + n/\rho)^{-2}.$$

This relation together with the estimate

$$\sum_{k=0}^{\infty}(k+a)^{-2} > \int_0^{+\infty}(u+a)^{-2}du = a^{-1}$$

leads to the inequality

$$\ln\left(R_{n+1}(\rho,\mu)/R_n(\rho,\mu)\right) > \rho^{-2}\left(\mu + \frac{n}{\rho}\right)^{-1} = \rho^{-1}(\rho\mu + n)^{-1},$$

which after exponentiation yields the assertion of Lemma 3.9.1. □

For $n = 1$, relation (3.2.6) is proved in Lemma 3.3.1. We prove (3.2.7). Until the end of this section, we adopt the notation

$$A_k = \frac{1}{\Gamma(\mu + k/\rho)},$$

so that

$$R_n = \frac{A_{n-1}}{A_n}, \quad E_\rho(-x;\mu) = \sum_{k=0}^{\infty}(-1)^k A_k x^k. \tag{3.9.14}$$

Also, we omit the arguments ρ and μ in the notation of A_k and R_n.

We have the relation

$$E_\rho(-\sqrt{2}R_1;\mu) = A_0 - A_1\sqrt{2}R_1 + 2A_1R_1^2 + \sum_{k=3}^{\infty}(-1)^k A_k(\sqrt{2}R_1)^k. \tag{3.9.15}$$

The last term on the right-hand side of (3.9.15) is negative; this follows from the fact that the sequence $\{A_k(\sqrt{2}R_1)^k\}_{k=3}^{\infty}$ decreases, which is equivalent to the inequality

$$\sqrt{2}R_1 < R_k, \quad k \geq 3.$$

Prove it. From Lemma 3.9.1, taking into account the fact that $\{R_k\}$ increases, we obtain the inequality

$$\frac{R_1}{R_k} < \frac{R_1}{R_2} < \exp\big(-(\rho(1+\rho\mu))^{-1}\big)s.$$

But we have $\rho\mu \le 2$ and $\rho \le 1/6$; therefore,

$$\frac{R_1}{R_2} < \exp\big(-(3\rho)^{-1}\big) \le e^{-2} < 0.14, \tag{3.9.16}$$

and the required inequality is proved.

The negativeness of the sum

$$\sum_{k=3}^{\infty}(-1)^k A_k(\sqrt{2}R_1)^k$$

implies the estimate

$$E_\rho(-\sqrt{2}R_1,\mu) < A_0 - A_1\sqrt{2}R_1 + 2A_2R_1^2 = A_0 - \sqrt{2}A_0 + 2A_2(A_0/A_1)^2. \tag{3.9.17}$$

Multiplying both sides of (3.9.17) by $A_0^{-1} = \Gamma(\mu)$ and taking into account (3.9.16), we find

$$\Gamma(\mu)E_\rho(-\sqrt{2}R_1,\mu) < 1-\sqrt{2}+2A_0A_2A_1^{-2} = 1-\sqrt{2}+2(R_1/R_2) < 1-\sqrt{2}+0.28 < 0,$$

which was required.

Prove (3.2.6) for $n=2$. We have the relation

$$E_\rho(-R_2,\mu) = A_0 - A_1R_2 + A_2R_2^2 - A_3R_2^3 + A_4R_2^4 + \sum_{k=5}^{\infty}(-1)^k A_kR_2^k.$$

Since the sequence $\{A_kR_2^k\}_{k=5}^{\infty}$ decreases, the sum

$$\sum_{k=5}^{\infty}(-1)^k A_kR_2^k$$

is negative, and since $R_2 = A_1/A_2$, we have $A_2R_2^2 - A_1R_2 = 0$. Therefore,

$$E_\rho(-R_2,\mu) < A_0 - A_3R_2^3 + A_4R_2^4. \tag{3.9.18}$$

Dividing both sides of inequality (3.9.18) by $A_3R_2^3$, we rewrite it in the equivalent form

$$A_3^{-1}R_2^{-3}E_\rho(-R_2,\mu) < -1 + (A_0/A_3)R_2^{-3} + (A_4/A_3)R_2.$$

Introduce the notation

$$B = \left(\frac{A_0}{A_3}\right)R_2^{-3} = A_0A_1^{-3}A_2^3A_3^{-1}.$$

The last equality follows from the definition of A_k; moreover, we also see that $A_3/A_4 = R_4$. Therefore, we have the relations

$$\begin{aligned} &A_3^{-1}R_2^{-3}E_\rho(-R_2,\mu) < -1 + B + R_2/R_4,\\ &\ln B = \ln\Gamma\left(\mu+\frac{3}{\rho}\right) - 3\ln\Gamma\left(\mu+\frac{2}{\rho}\right) + 3\ln\Gamma\left(\mu+\frac{1}{\rho}\right) - \ln\Gamma(\mu). \end{aligned} \tag{3.9.19}$$

By the intermediate-value theorem applied to the third difference of the function $\ln\Gamma(z)$ at the point μ with step $1/\rho$, for some $\xi \in (\mu, \mu+3/\rho)$ we have the equality

$$\ln B = \rho^{-3}\psi''(\xi) = -2\rho^{-3}\sum_{k=0}^{\infty}(k+\xi)^{-3}.$$

This implies

$$\ln B < -2\rho^{-3}\sum_{k=0}^{\infty}\left(k+\mu+\frac{3}{\rho}\right)^{-3} < -2\rho^{-3}\int\limits_{0}^{+\infty}\left(t+\mu+\frac{3}{\rho}\right)^{-3}dt$$

$$= -\rho^{-3}\left(\mu+\frac{3}{\rho}\right)^{-2} = -\rho^{-1}(\rho\mu+3)^{-2}. \quad (3.9.20)$$

By the restrictions $\rho\mu \le 2$ and $\rho \le 1/6$ from (3.9.20) we obtain the estimate

$$B < \exp\left(-\frac{1}{25\rho}\right) \le \exp\left(-\frac{6}{25}\right) < 0.8. \tag{3.9.21}$$

By Lemma 3.9.1 we have the inequalities

$$\begin{aligned}\frac{R_2}{R_3} &< \exp\big(-(\rho(2+\rho\mu))^{-1}\big) \le \exp\left(-\frac{1}{4\rho}\right) \le \exp(-1.5),\\ \frac{R_3}{R_4} &< \exp\big(-(\rho(3+\rho\mu))^{-1}\big) \le \exp\left(-\frac{1}{5\rho}\right) \le \exp(-1.2).\end{aligned} \tag{3.9.22}$$

Therefore,

$$\frac{R_2}{R_4} = \frac{R_2}{R_3}\frac{R_3}{R_4} < \exp(-1.5-1.2) = \exp(-2.7) < 0.1.$$

From this, (3.9.19), and (3.9.21) we obtain the inequality

$$A_3^{-1}R_2^{-3}E_\rho(-R_2,\mu) < -1+0.8+0.1 < 0,$$

i.e., $E_\rho(-R_2;\mu) < 0$, which was required.

Further, we have the equality

$$E_\rho\big(-\sqrt{2}R_2;\mu\big) = A_0 - \sqrt{2}A_1R_2 + 2A_2R_2^2 - 2\sqrt{2}A_3R_2^3 + \sum_{k=4}^{\infty}(-1)^k A_k(\sqrt{2}R_2)^k.$$

The sequence $\{A_k(\sqrt{2}R_2)^k\}_{k=4}^{\infty}$ decreases since the ratio of its elements with numbers k and $k+1$ is equal to $\sqrt{2}R_2/R_k \le \sqrt{2}R_2/R_4$ and by (3.9.22) is less than 1. Therefore, the sum

$$\sum_{k=4}^{\infty}(-1)^k A_k(\sqrt{2}R_2)^k$$

is positive and we arrive at the inequality

$$E_\rho(-\sqrt{2}R_2;\mu) > -\sqrt{2}A_1R_2 + 2A_2R_2^2 - 2\sqrt{2}A_3R_2^3. \tag{3.9.23}$$

Dividing both sides of (3.9.23) by $2A_2R_2^2$ and using the fact that $A_1/A_2 = R_2$, $A_2/A_3 = R_3$, we obtain the inequality

$$0.5A_2^{-1}R_2^{-2}E_\rho(-\sqrt{2}R_2;\mu) > 1 - \frac{1}{\sqrt{2}} - \sqrt{2}\frac{R_2}{R_3}.$$

In this case, by the restriction $n \le [1/(3\rho)]-1$, the inequality $\rho \le 1/9$ holds. Therefore, by Lemma 3.9.1 with account of (3.9.22) we have the inequality

$$\frac{R_2}{R_3} \le \exp\left(-\frac{1}{4\rho}\right) \le \exp\left(-\frac{9}{4}\right) < 0.11.$$

Therefore,

$$0.5A_2^{-1}R_2^{-2}E_\rho(-\sqrt{2}R_2;\mu) > 1 - 0.61\sqrt{2} > 0.$$

In the case $n=2$, relations (3.2.6) and (3.2.7) are proved.

Prove (3.2.7) for $n \geq 3$. In this case, $\rho \leq 1/9$ since only for these values of ρ is the inequality $n \leq N_1(\rho) = [1/(3\rho)]$ valid. Note (this will be used below) that for $0 < \rho \leq 1/9$, $0 < \mu \leq 2/\rho$, and $3 \leq n \leq N_1(\rho)$, the following inequality holds:

$$\frac{R_n}{R_{n+1}} < \frac{1}{6}. \tag{3.9.24}$$

Indeed, by Lemma 3.9.1 we have the inequality

$$\frac{R_n}{R_{n+1}} < \exp\big(-(\rho(n+\rho\mu))^{-1}\big) \leq \exp\big(-(\rho(n+2))^{-1}\big) = \exp\big(-(2\rho+n\rho)^{-1}\big).$$

Since $n\rho \leq 1/3$ and $2\rho \leq 2/9$, we have

$$\frac{R_n}{R_{n+1}} < \exp\left(-\left(\frac{1}{3}+\frac{2}{9}\right)^{-1}\right) = \exp(-1.8) < 1/6.$$

Relation (3.9.24) is proved.

From the power expansion of the Mittag-Leffler function, we obtain the equality

$$(-1)^n E_\rho(-\sqrt{2}R_n,\mu) = S_{n,0} - A_{n-1}\big(\sqrt{2}R_n\big)^{n-1} + A_n\big(\sqrt{2}R_n\big)^n - A_{n+1}\big(\sqrt{2}R_n\big)^{n+1} + S_{n,1},$$

where

$$S_{n,0} = \sum_{k=0}^{n-2}(-1)^{k-n}A_k\big(\sqrt{2}R_n\big)^k, \quad S_{n,1} = \sum_{k=n+2}^{\infty}(-1)^{k-n}A_k\big(\sqrt{2}R_n\big)^k.$$

We prove that the sums $S_{n,0}$ and $S_{n,1}$ are positive. Since they are alternating and the terms with numbers $k = n \pm 2$ are positive, it suffices to prove that the sequence $A_k\big(\sqrt{2}R_n\big)^k$ increases for $0 \leq k \leq n-2$ and decreases for $k \geq n+2$. The ratio of the elements of this sequence with numbers $k+1$ and k is equal to

$$d_k = \frac{A_{k+1}\big(\sqrt{2}R_n\big)^{k+1}}{A_k\big(\sqrt{2}R_n\big)^k} = \sqrt{2}R_n\frac{A_{k+1}}{A_k} = \sqrt{2}\frac{R_n}{R_{k+1}}. \tag{3.9.25}$$

Equality (3.9.25) together with (3.9.24) and the fact proved above that R_k increases imply the estimates

$$d_k > \sqrt{2} > 1, \quad 0 \leq k \leq n-2, \qquad d_k \leq \sqrt{2}\frac{R_n}{R_{n+1}} < \frac{\sqrt{2}}{6} < 1, \quad k \geq n+2,$$

which prove the required assertion. By the positiveness of the sums $S_{n,0}$ and $S_{n,1}$ we have the inequality

$$(-1)^n E_\rho\big(-\sqrt{2}R_n,\mu\big) > -A_{n-1}\big(\sqrt{2}R_n\big)^{n-1} + A_n\big(\sqrt{2}R_n\big)^n - A_{n+1}\big(\sqrt{2}R_n\big)^{n+1}. \tag{3.9.26}$$

Dividing both sides of (3.9.26) by $A_n\big(\sqrt{2}R_n\big)^n$ and taking into account the relations

$$\frac{A_{n-1}}{A_n} = R_n, \quad \frac{A_n}{A_{n+1}} = R_{n+1}, \quad \frac{R_n}{R_{n+1}} < \frac{1}{6},$$

we obtain

$$(-1)^n A_n^{-1}\big(\sqrt{2}R_n\big)^{-n}E_\rho\big(-\sqrt{2}R_n,\mu\big) > 1 - \frac{1}{\sqrt{2}} - \sqrt{2}\frac{R_n}{R_{n+1}} > 1 - \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{6} > 0.$$

Relations (3.2.7) are completely proved.

We introduce the following notation:

$$u_{n_{ij}} = R_{n+j}R_n^{-2}R_{n-j}, \quad 1 \leq j \leq n-1, \quad u_n = u_{n,1}. \tag{3.9.27}$$

Note that by the concavity of the sequence

$$\ln R_n = \ln \Gamma\left(\mu + \frac{n}{\rho}\right) - \ln \Gamma\left(\mu + \frac{n-1}{\rho}\right)$$

(this follows from the negativeness of ψ'') we have the inequalities

$$u_{n,j} < 1. \tag{3.9.28}$$

In what follows, we need the following lemma.

Lemma 3.9.2. *For any $\rho \le 1/9$, $0 < \mu \le 2/\rho$, and $m \ge 5$, the following inequality holds*:

$$u_{m+1}^3 < u_m.$$

Proof. By the definition of u_n and R_n, we have the representation

$$\frac{u_{m+1}^3}{u_m} = \left(\frac{R_{m+2}R_m}{R_{m+1}^2}\right)^3 \frac{R_m^2}{R_{m-1}R_{m+1}} = \left(\frac{A_{m+1}}{A_{m+2}}\right)^3 \left(\frac{A_m}{A_{m+1}}\right)^{-7} \left(\frac{A_{m-1}}{A_m}\right)^5 \left(\frac{A_{m-2}}{A_{m-1}}\right)^{-1}, \tag{3.9.29}$$

where $A_k = 1/\Gamma(\mu + k/\rho)$. Taking the logarithm of both sides of (3.9.29) we obtain

$$\ln\left(\frac{u_{m+1}^3}{u_m}\right) = 3\ln\Gamma\left(\mu + \frac{m+2}{\rho}\right) - 10\ln\Gamma\left(\mu + \frac{m+1}{\rho}\right)$$
$$+ 12\ln\Gamma\left(\mu + \frac{m}{\rho}\right) - 6\ln\Gamma\left(\mu + \frac{m-1}{\rho}\right) + \ln\Gamma\left(\mu + \frac{m-2}{\rho}\right). \tag{3.9.30}$$

Prove the following assertion. Let $h, x \in \mathbb{R}$, $h > 0$, g be a real-valued, four times continuously differentiable function on the segment $[x - 2h, x + 2h]$, and $g^{(4)}$ be positive and decrease on this segment. Then the following inequality holds:

$$3g(x+2h) - 10g(x+h) + 12g(x) - 6g(x-h) + g(x-2h)$$
$$< 2h^3 g'''(x) + h^4\left(\frac{4}{3}g^{(4)}(x) + \frac{2}{3}g^{(4)}(x-2h)\right). \tag{3.9.31}$$

Indeed, by Taylor's formula with the remainder in the Lagrange form, there exist numbers $\xi_1, \eta_1 \in (x, x+2h)$ and $\xi_2, \eta_2 \in (x-2h, x)$ such that

$$\begin{aligned}
g(x+2h) &= g(x) + 2hg'(x) + 2h^2 g''(x) + \frac{4}{3}h^3 g'''(x) + \frac{2}{3}h^4 g^{(4)}(\eta_1),\\
g(x+h) &= g(x) + hg'(x) + \frac{1}{2}h^2 g''(x) + \frac{1}{6}h^3 g'''(x) + \frac{1}{24}h^4 g^{(4)}(\xi_1),\\
g(x-h) &= g(x) - hg'(x) + \frac{1}{2}h^2 g''(x) - \frac{1}{6}h^3 g'''(x) + \frac{1}{24}h^4 g^{(4)}(\xi_2),\\
g(x-2h) &= g(x) - 2hg'(x) + 2h^2 g''(x) - \frac{4}{3}h^3 g'''(x) + \frac{2}{3}h^4 g^{(4)}(\eta_2).
\end{aligned} \tag{3.9.32}$$

From (3.9.32) we have

$$3g(x+2h) - 10g(x+h) + 12g(x) - 6g(x-h) + g(x-2h)$$
$$= 2h^3 g'''(x) + h^4\left(2g^{(4)}(\eta_1) - \frac{5}{12}g^{(4)}(\xi_1) - \frac{1}{4}g^{(4)}(\xi_2) + \frac{2}{3}g^{(4)}(\eta_2)\right). \tag{3.9.33}$$

From the upper estimate of the coefficient of h^4 in (3.9.33) obtained by using the inequalities

$$g^{(4)}(\eta_1), g^{(4)}(\xi_1) < g^{(4)}(x), \quad g^{(4)}(\eta_2), g^{(4)}(\xi_2) < g^{(4)}(x+2h),$$

which follow from the fact that $g^{(4)}$ decreases, we arrive at (3.9.31).

We take $g(t) = \ln\Gamma(t)$, $x = \mu + m/\rho$, and $h = 1/\rho$. Since $g'(t) = \psi(t)$, the fourth derivative

$$g^{(4)}(t) = \psi'''(t) = 6\sum_{k=0}^{\infty}(k+t)^{-4}$$

is positive and decreases. Therefore, from (3.9.30) and (3.9.31) we obtain

$$\ln\left(\frac{u_{m+1}^3}{u_m}\right) < \frac{2\psi''(x)}{\rho^3} + \frac{1}{\rho^4}\left(\frac{4}{3}\psi'''(x) + \frac{2}{3}\psi'''\left(x - \frac{2}{\rho}\right)\right).$$

Using the standard method of estimating the sum of a series by the corresponding integral, we obtain the inequalities

$$\psi''(x) = \sum_{k=0}^{\infty}\frac{-2}{(k+x)^3} < -\frac{1}{x^2} \quad \forall x > 0, \quad \psi'''(x) = \sum_{k=0}^{\infty}\frac{6}{(k+x)^4} < \frac{2}{(x-0.5)^3} \quad \forall x > 0.5,$$

$$\ln\left(\frac{u_{m+1}^3}{u_m}\right) < -\frac{2}{\rho^3 x^2} + \frac{4}{3\rho^4}\left(\frac{2}{(x-0.5)^3} + \frac{1}{(x-2/\rho-0.5)^3}\right).$$

To complete the proof of the lemma, it remains to verify the negativeness of the last expression, i.e., to check the inequality

$$\frac{2}{(x-0.5)^3} + \frac{1}{(x-2/\rho-0.5)^3} < \frac{3\rho}{2x^2}$$

$$\iff \quad \frac{2}{\rho(x-0.5)}\left(\frac{x}{x-0.5}\right)^2 + \frac{1}{\rho(x-2/\rho-0.5)}\left(\frac{x}{x-2/\rho-0.5}\right)^2 < \frac{3}{2}. \quad (3.9.34)$$

The function $t/(t-a)$, $a > 0$, decreases on the ray $t > a$, and by the condition $m \geq 5$, the inequality $x > 5/\rho \geq 45$ holds. Therefore, the following estimates hold:

$$\rho(x-0.5) \geq 5 - \frac{1}{18}, \quad \rho(x-2/\rho-0.5) \geq 3 - \frac{1}{18},$$

$$\frac{x}{x-0.5} < \frac{90}{89}, \quad \frac{x}{x-2/\rho-0.5} < \frac{5/\rho}{3/\rho-0.5} = \frac{5}{3-0.5\rho} \leq \frac{5}{3-1/18} = \frac{90}{53}.$$

This implies that the left-hand side of (3.9.34) does not exceed

$$\frac{2}{5-1/18}\left(\frac{90}{89}\right)^2 + \frac{1}{3-1/18}\left(\frac{90}{53}\right)^2 = \frac{146197013400}{104953669813} < \frac{3}{2}.$$

The lemma is proved. □

Complete the proof of Theorems 3.1.1 and 3.1.2, i.e., deduce the equalities (3.2.6) for $3 \leq n \leq [1/(3\rho)]$.

From the power expansion of the Mittag-Leffler function we obtain

$$E_\rho(-R_n,\mu) = \sum_{k=0}^{n-2}(-1)^k A_k R_n^k + (-1)^{n-1}\bigl(A_{n-1}R_n^{n-1} - A_n R_n^n\bigr) + \sum_{k=n+1}^{\infty}(-1)^k A_k R_n^k. \quad (3.9.35)$$

Grouping in (3.9.35) terms with numbers $k = n - \nu - 1$ and $k = n + \nu$, $1 \leq \nu \leq n-1$, and taking into account the equality

$$A_{n-1}R_n^{n-1} - A_n R_n^n = A_n R_n^{n-1}\left(\frac{A_{n-1}}{A_n} - R_n\right) = 0,$$

which is valid owing to (3.9.14), we obtain the representation

$$E_\rho(-R_n,\mu) = (-1)^{n+1}\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) + \sum_{k=2n}^{\infty}(-1)^k A_k R_n^k, \quad (3.9.36)$$

where

$$v_\nu(x) = A_{n+\nu}x^{n+\nu} - A_{n-\nu-1}x^{n-\nu-1}.$$

It follows from (3.9.36) that relation (3.2.6) means the positiveness of the sums

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) + (-1)^{n+1}\sum_{k=2n}^{\infty}(-1)^k A_k R_n^k. \tag{3.9.37}$$

Now we prove that the positiveness of sums (3.9.37) follows from the inequalities

$$0 < v_{\nu+1}(R_n) < v_\nu(R_n), \qquad 1 \le \nu \le n-2, \quad 3 \le n \le N_1(\rho), \tag{3.9.38}$$

$$A_{2n}R_n^{2n} < v_{n-1}(R_n), \quad 3 \le n \le N_1(\rho). \tag{3.9.39}$$

Indeed, by (3.9.38), the moduli of terms in the alternating sum

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n)$$

decrease and the first term is positive. Therefore,

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) > 0$$

for all $n \in [3, N_1(\rho)]$. This immediately implies the required assertion for odd n since

$$\sum_{k=2n}^{\infty}(-1)^k A_k R_n^k$$

is also an alternating series which has terms with decreasing moduli and the first term is positive. If n is even, then

$$\sum_{\nu=1}^{n-2}(-1)^{\nu-1}v_\nu(R_n) > 0, \quad (-1)^{n+1}\sum_{k=2n+1}^{\infty}(-1)^k A_k R_n^k > 0$$

by the above reasoning. Further, by (3.9.39)

$$(-1)^{n-2}v_{n-1}(R_n) + (-1)^{3n+1}A_{2n}R_n^{2n} = v_{n-1}(R_n) - A_{2n}R_n^{2n} > 0,$$

and we also obtain the positiveness of sum (3.9.37).

Thus, to complete the proof, it remains to deduce inequalities (3.9.38) and (3.9.39). It is seen from the definition of A_k and R_n and the functions $v_\nu(x)$ that the positiveness of $v_\nu(R_n)$ for any $n \in \mathbb{R}$ and $1 \le \nu \le n-1$ is equivalent to the concavity of the sequence $\{\ln R_n\}_{n=1}^{\infty}$. Indeed,

$$0 < v_\nu(R_n) \quad \Longleftrightarrow \quad A_{n-\nu-1}R_n^{n-\nu-1} < A_{n+\nu}R_n^{n+\nu} \quad \Longleftrightarrow \quad \frac{A_{n-\nu-1}}{A_{n+\nu}} < R_n^{2\nu+1}. \tag{3.9.40}$$

From (3.9.14) we have the representation

$$\frac{A_{n-\nu-1}}{A_{n+\nu}} = \prod_{p=n-\nu}^{n+\nu} R_p = R_n \prod_{j=1}^{\nu}\big(R_{n-j}R_{n+j}\big). \tag{3.9.41}$$

By (3.9.40) and (3.9.41) we have the inequalities

$$0 < v_\nu(R_n) \quad \Longleftrightarrow \quad \prod_{j=1}^{\nu}\big(R_{n-j}R_{n+j}\big) < R_n^{2\nu} \quad \Longleftrightarrow \quad \sum_{j=1}^{\nu}\big(\ln R_{n+j} - 2\ln R_n + \ln R_{n-j}\big) < 0; \tag{3.9.42}$$

the last inequality follows fromt the concavity of the sequence $\{\ln R_n\}$ (this concavity was proved above).

We prove the inequalities

$$v_{\nu+1}(R_n) < v_\nu(R_n),$$

which, by the definition of the function v_ν, are equivalent to the following:

$$A_{n-\nu-1}R_N^{n-\nu-1} - A_{n-\nu-2}R_n^{n-\nu-2} < A_{n+\nu}R_n^{n+\nu} - A_{n+\nu+1}R_n^{n+\nu+1}. \tag{3.9.43}$$

Divide both sides of (3.9.43) by $A_{n+\nu}R_n^{n+\nu}$. Applying formula (3.9.41), we can rewrite the inequalities obtained in the form

$$\left(1 - \frac{R_{n-\nu-1}}{R_n}\right)\prod_{j=1}^{\nu}\left(R_{n+j}R_{n-j}R_n^{-2}\right) < 1 - \frac{R_n}{R_{n+\nu+1}}. \tag{3.9.44}$$

Introduce the notation

$$a_{n,\nu} = \frac{R_{n-\nu-1}}{R_n}, \quad b_{n,\nu} = \frac{R_n}{R_{n+\nu+1}}, \quad u_{n,p} = R_{n+p}R_n^{-2}R_{n-p}. \tag{3.9.45}$$

By (3.9.45), inequalities (3.9.44) take the form

$$(1 - a_{n,\nu})\prod_{j=1}^{\nu} u_{n,\nu} < 1 - b_{n,\nu}. \tag{3.9.46}$$

Dividing both sides of (3.9.46) by $1 - b_{n,\nu}$ and using the identity

$$(1-a)(1-b)^{-1} = 1 + b(1-b)^{-1}(1-a/b)$$

(note that $a_{n,\nu}/b_{n,\nu} = u_{n,\nu+1}$), we arrive at the inequalities

$$\left(1 + b_{n,\nu}(1-b_{n,\nu})^{-1}(1-u_{n,\nu+1})\right)\prod_{j=1}^{\nu} u_{n,j} < 1, \tag{3.9.47}$$

which are equivalent to (3.9.44). We will prove (3.9.47).

First, we prove (3.9.47) for $3 \le n \le 12$ (for all $\nu \in [1, n-2]$) and for $\nu \ge 0.5n - 1$ (for all $n \ge 12$). We show that in both these cases, the following, even stronger than (3.9.47), inequalities hold[6]:

$$\left(1 + b_{n,\nu}(1-b_{n,\nu})^{-1}\right)u_{n,1} < 1. \tag{3.9.48}$$

We obtain upper estimates of

$$u_{n,1} = R_{n+1}R_n^{-2}R_{n-1} = A_{n+1}^{-1}A_n^3A_{n-1}^{-3}A_{n-2}$$

for any $n \in \mathbb{N}$, where, as above, $A_n = 1/\Gamma(\mu + n/\rho)$. We have

$$\ln u_{n,1} = \ln\Gamma\left(\mu + \frac{n+1}{\rho}\right) - 3\ln\Gamma\left(\mu + \frac{n}{\rho}\right) + 3\ln\Gamma\left(\mu + \frac{n-1}{\rho}\right) - \ln\Gamma\left(\mu + \frac{n-2}{\rho}\right). \tag{3.9.49}$$

Applying the formula of finite increments to the third difference (3.9.49), we obtain that for some $\xi \in \big((n-2)/\rho, (n+1)/\rho\big)$, the following relation holds:

$$\ln u_{n,1} = \rho^{-3}\left(\ln\Gamma(\mu+z)\right)'''_{|z=\xi} = \rho^{-3}\psi''(\mu+\xi) = -2\rho^{-3}\sum_{k=0}^{\infty}(k+\mu+\xi)^{-3}.$$

[6] By the above (see (3.9.28)), $u_{n,j} < 1$, $1 \le j \le n-1$.

This implies the estimate

$$\ln u_{n,1} < -2\rho^{-3}\sum_{k=0}^{\infty}\left(k+\mu+\frac{n+1}{\rho}\right)^{-3} < -2\rho^{-3}\int_{0}^{+\infty}\left(t+\mu+\frac{n+1}{\rho}\right)^{-3}dt$$
$$= -\rho^{-3}\left(\mu+\frac{n+1}{\rho}\right)^{-2} = -\rho^{-1}(\rho\mu+n+1)^{-2} \quad \forall n\in\mathbb{N}. \quad (3.9.50)$$

From (3.9.50) and the restrictions $\rho\mu \le 2$ and $\rho \le 1/9$, for $n \le 12$ we obtain the inequality

$$u_{n,1} < \exp\left(-\frac{\rho^{-1}}{225}\right) \le \exp\left(-\frac{9}{225}\right) = \exp\left(-\frac{1}{25}\right). \quad (3.9.51)$$

By Lemma 3.9.1, for any $n, \nu \in \mathbb{N}$ we have the inequality

$$b_{n,\nu} = \frac{R_n}{R_{n+\nu+1}} \le \frac{R_n}{R_{n+2}} = \frac{R_n}{R_{n+1}}\frac{R_{n+1}}{R_{n+2}}$$
$$< \exp\left(-\frac{1}{\rho(n+\rho\mu)} - \frac{1}{\rho(n+1+\rho\mu)}\right) \le \exp\left(-\frac{1}{\rho(n+2)} - \frac{1}{\rho(n+3)}\right).$$

Since $\rho n \le \rho N_1(\rho) \le 1/3$, $\rho \le 1/9$, we have

$$b_{n,\nu} < \exp\left(-\frac{1}{1/3+2/9} - \frac{1}{1/3+1/3}\right) = \exp(-1.8-1.5) = \exp(-3.3) < \frac{1}{26}.$$

This implies

$$b_{n,\nu}(1-b_{n,\nu})^{-1} < \frac{1}{25}, \quad n \le N_1(\rho), \quad 0 < \rho \le \frac{1}{9}, \quad 1 \le \nu \le n-2. \quad (3.9.52)$$

From (3.9.52) and (3.9.51) we conclude that for $3 \le n \le 12$, the left-hand side of (3.9.48) does not exceed

$$\frac{26}{25}\exp\left(-\frac{1}{25}\right) < 1,$$

which was required.

Now let $\nu \ge 0.5n-1$. Then by Lemma 3.9.1 and the restriction $\nu \le n-2$ we obtain the inequalities

$$b_{n,\nu} = \prod_{j=0}^{\nu}\left(\frac{R_{n+j}}{R_{n+j+1}}\right) < \exp\left(-\frac{1}{\rho}\sum_{j=0}^{\nu}\frac{1}{n+j+\rho\mu}\right) \le \exp\left(-\frac{1}{\rho}\cdot\frac{\nu+1}{n+\nu+\rho\mu}\right)$$
$$\le \exp\left(-\frac{1}{\rho}\cdot\frac{\nu+1}{n+\nu+2}\right) \le \exp\left(-\frac{1}{\rho}\cdot\frac{0.5n}{2n}\right) = \exp\left(-\frac{1}{4\rho}\right).$$

Thus, for $0.5n-1 \le \nu \le n-2$, the following estimate holds:

$$b_{n,\nu}(1-b_{n,\nu})^{-1} < 2\exp\left(-\frac{1}{4\rho}\right) < 2\rho \quad (3.9.53)$$

(for any $t \ge 9$, we have the inequality $t < \exp(t/4)$, which implies $\exp(-1/(4\rho)) < \rho$).

Now we obtain another upper estimate for $u_{n,1}$, which differs from (3.9.51). From (3.9.50) we obtain

$$u_{n,1} < \exp\left(-\frac{1}{\rho(n+3)^2}\right) = \exp\left(-\frac{\rho}{(n\rho+3\rho)^2}\right).$$

Recall that $n\rho \le 1/3$ and $\rho \le 1/9$; then $(n\rho+3\rho)^2 \le 4/9$ and hence $u_{n,1} < \exp(-2\rho)$. This and (3.9.53) imply the inequality

$$\left(1+b_{n,\nu}(1-b_{n,\nu})^{-1}\right)u_{n,1} < (1+2\rho)\exp(-2\rho) < 1,$$

which proves (3.9.48) for the case considered.

Finally, it remains to prove (3.9.47) for the case where $n \ge 13$ and $1 \le \nu < 0.5n - 1$. According to (3.9.52), we replace inequalities (3.9.47) by stronger inequalities

$$\Bigl(1 + \frac{1}{25}\bigl(1 - u_{n,\nu+1}\bigr)\Bigr) \prod_{j=1}^{\nu} u_{n,j} < 1. \tag{3.9.54}$$

Prove (3.9.54). We start from the proof of the estimate

$$u_{n,\nu}^6 < u_{n,\nu+1}, \quad 1 \le \nu < 0.5n - 1, \quad n \ge 13. \tag{3.9.55}$$

If $\nu = 1$, then

$$u_{n,2} = u_{n-1,1} u_{n,1}^2 u_{n+1,1}, \quad n - 1 \ge 12,$$

and by Lemma 3.9.2 (where $u_{n,1}$ was denoted by u_n) and the fact that the sequence $\{u_{p,1}\}_{p=2}^{\infty}$ increases, we obtain the inequality

$$u_{n,1}^6 < u_{n-1,1} u_{n,1}^3 < u_{n-1,1} u_{n,1}^2 \cdot u_{n+1,1} = u_{n,2}.$$

For $\nu = 1$, inequality (3.9.55) is proved.

If $\nu \ge 2$, then, applying the representation

$$u_{n,\nu} = u_{n,1}^{\nu} \prod_{k=1}^{\nu-1} \bigl(u_{n-k,1} u_{n+k,1}\bigr)^{\nu-k} \tag{3.9.56}$$

and Lemma 3.9.2, we obtain the estimate[7]

$$\begin{aligned} u_{n,\nu}^6 &= u_{n,1}^{6\nu} \prod_{k=1}^{\nu-1} \bigl(u_{n-k,1} u_{n+k,1}\bigr)^{6(\nu-k)} < u_{n,1}^{2\nu} \prod_{k=1}^{\nu-1} \bigl(u_{n-k,1} u_{n+k,1}\bigr)^{2(\nu-k)} \cdot u_{n-\nu+1,1}^3 \cdot u_{n+\nu-1,1}^3 \\ &< u_{n,1}^{2\nu} \prod_{k=1}^{\nu-1} \bigl(u_{n-k,1} u_{n+k,1}\bigr)^{2(\nu-k)} \cdot \bigl(u_{n-\nu,1} u_{n+\nu,1}\bigr) \le u_{n,1}^{\nu+1} \prod_{k=1}^{\nu} \bigl(u_{n-k,1} u_{n+k,1}\bigr)^{\nu+1-k} = u_{n,\nu+1}. \end{aligned}$$

Here we have used the obvious inequality $\nu + 1 - k \le 2(\nu - k)$, $0 \le k \le \nu - 1$, $\nu \in \mathbb{N}$. The possibility of applying Lemma 3.9.2 follows from the inequality $n - \nu \ge 8$ (for $n = 13$ it is obvious since $\nu \le 0.5n - 1 \le 6.5 - 1$ and hence $\nu \le 5$, and for $n \ge 14$, by the restriction $\nu < 0.5n - 1$, we have the inequality $n - \nu \ge 0.5n + 1 \ge 8$). Thus, inequality (3.9.55) is proved. It allows one to replace inequalities (3.9.54) by stronger inequalities

$$\Bigl(1 + \frac{1}{25}\bigl(1 - u_{n,\nu}^6\bigr)\Bigr) u_{n,\nu} < 1; \tag{3.9.57}$$

we will prove (3.9.57).

Consider the function

$$f(t) = \Bigl(1 + \frac{1}{25}(1 - t^6)\Bigr) t = \frac{26}{25} t - \frac{t^7}{25}$$

on the segment $0 \le t \le 1$. Its derivative

$$f'(t) = \frac{26}{25} - \frac{7t^6}{25}$$

is positive for for $t \in [0, 1]$. Therefore, $f(t)$ increases on $[0, 1]$, and since $f(1) = 1$, we see that $f(t) < 1$ for all $t \in (0, 1)$. From this and the inclusion $u_{n,\nu} \in (0, 1)$ proved above, we obtain (3.9.57). Relations (3.9.38) are proved.

[7]Here and in the sequel, we use the inequalities $u_{n,k} < 1$, $n, k \in \mathbb{N}$, $n \ge 2$, $1 \le k \le n - 1$.

Now we prove inequalities (3.9.39), which have the form

$$A_{2n}R_n^{2n} < A_{2n-1}R_n^{2n-1} - A_0. \tag{3.9.58}$$

Transferring A_0 on the left-hand side and dividing (3.9.58) by $A_{2n-1}R_n^{2n-1}$, we obtain the equivalent inequalities

$$\left(\frac{A_0}{A_{2n-1}}\right) R_n^{1-2n} + \left(\frac{A_{2n}}{A_{2n-1}}\right) R_n < 1,$$

which, with account of (3.9.14), (3.9.41), and (3.9.45), can be rewritten in the following form:

$$\prod_{j=1}^{n-1} u_{n,j} + \frac{R_n}{R_{2n}} < 1. \tag{3.9.59}$$

Representing $u_{n,j}$ by formula (3.9.56), omitting factors less than 1, and taking into account the fact that $\{R_k\}$ increases, we strengthen inequality (3.9.59):

$$u_{n,1}^{\frac{n(n-1)}{2}} + \frac{R_n}{R_{n+1}} < 1. \tag{3.9.60}$$

To obtain an upper estimate of the left-hand side of (3.9.60) (we denote it by U_n), we use inequalities (3.9.24) and (3.9.50). We have

$$U_n < \frac{1}{6} + \exp\left(-\frac{n(n-1)}{2\rho(n+1+\rho\mu)^2}\right) < \frac{1}{6} + \exp\left(-\frac{n(n-1)}{2\rho(n+3)^2}\right).$$

Since

$$n(n-1)(n+3)^{-2} \geq \frac{1}{6}, \quad \rho \leq \frac{1}{9}$$

for $n \geq 3$, we have

$$U_n < \frac{1}{6} + e^{-\frac{9}{12}} < \frac{1}{6} + \frac{1}{2} < 1,$$

which was required. Equalities (3.2.6) are proved and the proof of Theorems 3.1.1 and 3.1.2 is complete.

3.10. Proof of Theorem 3.1.3

The realness, negativeness, and simpleness of all zeros of the function $E_{1/N}(z, N+1)$ follow from Theorem 3.1.1. The required upper estimates of the first $[N/3]-1$ zeros for $N \geq 6$ are proved in Theorem 3.1.2, and the upper estimate of the first zero (for any N) is a consequence of Lemma 3.3.1. It remains to obtain

(1) more exact than in Theorem 3.1.2, lower estimates of the first $[N/3]-1$ zeros in the case $N \geq 6$ and
(2) more exact than in Theorem 3.1.1, two-sided estimates of zeros with numbers $n \geq [N/3]$.

We start from the simpler problem 2. We have the identity

$$\frac{N}{2} E_{1/N}(-w^N; 1) = \sum_{k=1}^{[N/2]} \exp\left[w \cos\left(\frac{\pi(2k-1)}{N}\right)\right] \cos\left[w \sin\left(\frac{\pi(2k-1)}{N}\right)\right] + \delta_N e^{-w}, \tag{3.10.1}$$

where $\delta_N = 1/2$ if N odd and $\delta_N = 0$ if N is even. Identity (3.10.1) can be proved as follows. If

$$f(x) = \sum_{m=0}^{\infty} a_m z^m$$

is an arbitrary entire function, $N \in \mathbb{N}$, then

$$N \sum_{\nu=0}^{\infty} a_{\nu N} w^{\nu N} = \sum_{k=1}^{N} f\left(w \exp\left(\frac{2\pi i k}{N}\right)\right). \tag{3.10.2}$$

Applying (3.10.2) to the equality

$$f(z) = \exp\left(ze^{-\pi i/N}\right) \equiv \sum_{m=0}^{\infty} \frac{z^m}{m!} \exp\left(-\frac{\pi i m}{N}\right),$$

we obtain the following representation of the classical Mittag-Leffler function of order $1/N$:

$$NE_{1/N}\left(-w^N; 1\right) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu w^{\nu N}}{(\nu N)!} = \sum_{k=1}^{N} \exp\left[w \exp\left(\frac{\pi i(2k-1)}{N}\right)\right].$$

Combining in the last sum the conjugate pairs $\exp\left(\dfrac{\pi i(2k-1)}{N}\right)$ and applying the identity

$$\exp(we^{i\varphi}) + \exp(we^{-i\varphi}) = 2\exp(w\cos\varphi)\cos(w\sin\varphi),$$

we arrive at (3.10.1).

Theorem 3.1.1 implies that on each of the intervals

$$I_{N,n} = \left(-\left(\frac{\pi(n+1)}{\sin(\pi/N)}\right)^N, -\left(\frac{\pi n}{\sin(\pi/N)}\right)^N\right), \quad n \in \mathbb{N},$$

the function $E_{1/N}(z; N+1)$ has exactly one root. We must prove that for $n \geq [N/3]$, a root is contained on a narrower interval

$$I'_{N,n} = \left(-\left(\frac{\pi n + \pi/2 + x_n(N)}{\sin(\pi/N)}\right)^N, -\left(\frac{\pi n + \pi/2 - x_n(N)}{\sin(\pi/N)}\right)^N\right)$$

(the values $x_n(N)$ were defined in the statement of Theorem 3.1.3). Note that for any $N \in \mathbb{N}$, $N \geq 3$, the sequence $x_n(N)$ decreases and tends to zero. It maximal value is equal to $x_{[N/3]}(N)$; we denote it by y_N. We have the estimate

$$y_N \leq y_3 = \exp(-\pi/\sqrt{3}) < 0.164, \quad 3 \leq N \leq 11. \tag{3.10.3}$$

For $N \geq 12$, owing to the inequality $\sin(2\pi/N) \geq 6/N$, we have the estimate

$$\begin{aligned} y_N &\leq 1.01 \exp\left(-2\pi\left[\frac{N}{3}\right]\frac{6}{N}\right) \leq 1.01 \exp\left(\frac{-12\pi(N-2)}{3N}\right) \\ &= 1.01 \exp\left(\frac{-4\pi(N-2)}{N}\right) \leq 1.01 \exp\left(\frac{-40\pi}{12}\right) < 1.01 e^{-10.47} < e^{-10.46}. \end{aligned} \tag{3.10.4}$$

By (3.10.3) and (3.10.4), we see that the largest value of $x_n(N)$ for $N \geq 3$ and $n \geq [N/3]$ is equal to $e^{-\pi/\sqrt{3}} < 0.164$. This implies the inclusion

$$I'_{N,n} \subset I_{N,n} \quad \forall N \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad N \geq 3, \quad n \geq \left[\frac{N}{3}\right].$$

Compare the lengths of the intervals $I_{N,n}$ and $I'_{N,n}$. It is easy to verify the asymptotics

$$\begin{aligned} |I_{N,n}| &\sim (Nn)^N \left(\exp\left(\frac{N}{n}\right) - 1\right), \qquad N \to \infty, \quad n \geq N/3, \\ |I'_{N,n}| &\sim N^{N+1} n^{N-1} \exp\left(-\frac{4\pi^2 n}{N}\right), \quad N \to \infty, \quad \frac{n}{N} \to \infty. \end{aligned}$$

These asymptotics show that the length of $I_{N,n}$ tends to ∞, whereas the length of $I'_{N,n}$ tends to zero as $n \geq (N \ln N)^2$, $N \to \infty$.

Continue the proof of the theorem. Introduce the notation

$$w_n^{\pm}(N) = \Big(\pi n + \frac{\pi}{2} \pm x_n(N)\Big) \operatorname{cosec}\Big(\frac{\pi}{N}\Big).$$

To prove the existence of a zero of the function $E_{1/N}(z; N+1) \equiv \big(E_{1/N}(z,1) - 1\big)/z$ on each of the intervals $I'_{N,n}$, it suffices to verify that the function

$$F_N(w) = \frac{N}{2}\left(E_{1/N}(-w^N; 1) - 1\right)$$

has opposite signs at the points $w_n^-(N)$ and $w_n^+(N)$ for $n \geq [N/3]$. We have the relations

$$\cos\Big(w_n^{\pm}(N)\sin\Big(\frac{\pi}{N}\Big)\Big) = \mp(-1)^n \sin\left(x_n(N)\right). \tag{3.10.5}$$

First, we consider values $N \in [3, 6]$. Since the moduli of products of exponents with negative powers by cosines are not greater than 1, we obtain that

$$\Delta_N(w) \equiv \Big|F_N(w) - \exp\Big(w\cos\Big(\frac{\pi}{N}\Big)\Big)\cos\Big(w\sin\Big(\frac{\pi}{N}\Big)\Big)\Big| < N - 1. \tag{3.10.6}$$

Relations (3.10.5) and (3.10.6) and representation (3.10.1) show that the difference of signs of the numbers $F_N(w_n^{\pm}(N))$ follows from the inequality

$$N - 1 \leq \sin\left(x_n(N)\right)\exp\Big(w_n^{\pm}(N)\cos\Big(\frac{\pi}{N}\Big)\Big).$$

Since $w_n^-(N) < w_n^+(N)$, it suffices to prove that

$$(N-1)\exp\Big(-w_n^-(N)\cos\Big(\frac{\pi}{N}\Big)\Big) < \sin x_n(N).$$

Using the estimate $x_n(N) < 0.17$, we obtain the inequalities

$$\sin\left(x_n(N)\right) > (3/\pi)x_n(N),$$
$$w_n^-(N)\cos\Big(\frac{\pi}{N}\Big) > \Big(\pi n + \frac{\pi}{2} - 0.17\Big)\cot\Big(\frac{\pi}{N}\Big) > (\pi n + 1.4)\cot\Big(\frac{\pi}{N}\Big).$$

Therefore, the problem is reduced to the proof of the inequality

$$\frac{\pi}{3}(N-1)\exp\Big(-\pi n\cot\Big(\frac{\pi}{N}\Big)\Big)\exp\Big(-1.4\cot\Big(\frac{\pi}{N}\Big)\Big) < x_n(N)$$
$$\Longleftrightarrow \quad \frac{\pi(N-1)}{3} < \exp\Big(1.4\cot\Big(\frac{\pi}{N}\Big)\Big). \tag{3.10.7}$$

The validity of (3.10.7) for $N = 3, 4, 5, 6$ can be proved by a straightforward calculation. The two-sided estimates of zeros for $3 \leq N \leq 6$ are proved.

For $N > 6$, the estimate of $\Delta_N(w)$ is proved in a different way. We have the inequalities

$$\Big|\delta_N e^{-w} - \frac{N}{2}\Big| \leq \frac{N}{2}, \tag{3.10.8}$$

$$\Big|\sum_{k=2}^{[N/2]} \exp\Big(w\cos\Big(\frac{\pi(2k-1)}{N}\Big)\Big)\cos\Big(w\sin\Big(\frac{\pi(2k-1)}{N}\Big)\Big)\Big|$$
$$\leq \sum_{k=2}^{[N/2]} \exp\Big(w\cos\Big(\frac{\pi(2k-1)}{N}\Big)\Big). \tag{3.10.9}$$

The ratio of terms with numbers $k+1$ and k in the last sum of exponents is equal to

$$\omega_{N,\,k}(w) = \exp\left[w\left(\cos\left(\frac{\pi(2k+1)}{N}\right) - \cos\left(\frac{\pi(2k-1)}{N}\right)\right)\right]$$
$$= \exp\left(-2w\sin\left(\frac{\pi}{N}\right)\sin\left(\frac{2\pi k}{N}\right)\right). \quad (3.10.10)$$

We have the inequalities

$$w_n^{\pm}\sin\left(\frac{\pi}{N}\right) > \pi n + 1.4. \quad (3.10.11)$$

Therefore, for $w \geq w^{-}_{[N/3]}(N)$, $1 \leq k \leq [N/2]-1$, the following inequality holds:

$$\omega_{N,\,k}(w) < \exp\left(-2\pi\left[\frac{N}{3}\right]\sin\left(\frac{2\pi k}{N}\right)\right)$$
$$\leq \exp\left(-2\pi\left[\frac{N}{3}\right]\sin\left(\frac{2\pi}{N}\right)\right) \leq \exp\left(-\frac{8\pi}{N}\left[\frac{N}{3}\right]\right). \quad (3.10.12)$$

Here we have used the inequalities

$$\sin\left(\frac{2\pi k}{N}\right) \geq \sin\left(\frac{2\pi}{N}\right), \quad 1 \leq k \leq \left[\frac{N}{3}\right] - 1,$$
$$\sin t \geq \frac{2t}{\pi}, \qquad 0 \leq t \leq \frac{\pi}{2}.$$

Since $n \geq [N/3]$ and for $N \geq 7$ we have

$$\frac{8}{N}\left[\frac{N}{3}\right] \geq 2,$$

(3.10.12) implies that

$$\omega_{N,k}(w) < \exp(-2\pi) < \frac{1}{535}, \quad w \geq w^{-}_{[N/3]}(N). \quad (3.10.13)$$

From (3.10.13) and (3.10.10) we obtain the following estimate of sum (3.10.9):

$$\sum_{k=2}^{[N/2]} \exp\left(w\cos\left(\frac{\pi(2k-1)}{N}\right)\right) < \left(1 + \frac{1}{534}\right)\exp\left(w\cos\left(\frac{3\pi}{N}\right)\right), \quad w \geq w^{-}_{[N/3]}(N). \quad (3.10.14)$$

From (3.10.1), (3.10.8), and (3.10.14) we obtain the estimate

$$\Delta_N(w) \leq \frac{N}{2} + \left(1 + \frac{1}{534}\right)\exp\left(w\cos\left(\frac{3\pi}{N}\right)\right), \quad N \geq 7, \quad w \geq w^{-}_{[N/3]}(N). \quad (3.10.15)$$

Now we must obtain an estimate of the form

$$\frac{N}{2} < c_N \exp\left(w\cos(3\pi/N)\right),$$

in which the constant c_N "is not large." The case $N = 7$ will be considered separately. Clearly, we can take

$$c_7 = 3.5\exp\left(-w_2^{-}(7)\cos\left(\frac{3\pi}{7}\right)\right).$$

Owing to (3.10.11), we have the relation

$$c_7 < 3.5\exp\left[-\frac{(2\pi + 1.4)\cos\left(\frac{3\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\right] = 3.5\exp\left(-\frac{\pi + 0.7}{\cos\left(\frac{\pi}{14}\right)}\right) < 0.098.$$

This and (3.10.15) imply that

$$\Delta_7(w) < 1.1 \exp\left(w \cos\left(\frac{3\pi}{7}\right)\right), \quad w \geq w_2^-(7). \tag{3.10.16}$$

From (3.10.16), (3.10.5), and (3.10.6) we conclude that the difference of signs of the numbers $F_7\left(w_n^{\pm}(7)\right)$ for $n \geq 2$ follows from the inequality

$$1.1 \exp\left(w_n^{\pm}(7) \cos\left(\frac{3\pi}{7}\right)\right) \leq \sin\left(x_n(7)\right) \exp\left(w_n^{\pm}(7)\right) \cos\left(\frac{\pi}{7}\right), \quad n \geq 2$$
$$\iff \quad 1.1 \exp\left(-2w_n \pm (7) \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{2\pi}{7}\right)\right) \leq \sin\left(x_n(7)\right), \quad n \geq 2.$$

Applying the lower estimate (3.10.11), we see that now we must prove the inequality

$$1.1 \exp\left(-(2\pi n + 2.8) \sin\left(\frac{2\pi}{7}\right)\right) \leq \sin\left(x_n(7)\right)$$
$$\iff \quad 1.1 x_n(7) \exp\left(-2.8 \sin\left(\frac{2\pi}{7}\right)\right) \leq \sin\left(x_n(7)\right), \quad n \geq 2.$$

Since $\sin(2\pi/7) > 4/7$, it remains to verify that

$$1.1 e^{-1.6} < \frac{\sin x_n(7)}{x_n(7)}. \tag{3.10.17}$$

For $N \geq 7$, $n \geq [N/3]$, we have the estimate

$$x_n(N) \leq 1.01 \exp\left(-2\pi \left[\frac{N}{3}\right] \sin\left(\frac{2\pi}{N}\right)\right) \leq 1.01 \exp\left(-\frac{8\pi}{N}\left[\frac{N}{3}\right]\right) \leq 1.01 e^{-2\pi} < \frac{1}{500}.$$

This implies that

$$\frac{\sin x_n(N)}{x_n(N)} \geq 1 - \frac{x_n^2(N)}{6} > 1 - \frac{4 \cdot 10^{-6}}{6} > 1 - 10^{-6}, \quad N \geq 7, \quad n \geq [N/3]. \tag{3.10.18}$$

From (3.10.18) we see that inequality (3.10.17) is valid. The two-sided estimates of zeros in the case $N = 7$, $n \geq 2$ are proved.

Further, consider the case $N \geq 8$. As c_N we can take

$$c_N = \frac{N}{2} \exp\left(-w_{[N/3]}^-(N) \cos\left(\frac{3\pi}{N}\right)\right) < \frac{N}{2} \exp\left(-\frac{\pi[N/3] \cos\left(3\pi/N\right)}{\sin\left(\pi/N\right)}\right).$$

Since

$$\cos\left(\frac{3\pi}{N}\right) \Big/ \sin\left(\frac{\pi}{N}\right) \geq 1, \quad N \geq 8,$$

we have the inequality

$$c_N \leq \frac{N}{2} \exp\left(-\pi \left[\frac{N}{3}\right]\right) \leq 4 e^{-2\pi} < \frac{4}{535}.$$

This and (3.10.15) imply the inequality

$$\Delta_N(w) < \left(1 + \frac{1}{106}\right) \exp\left(w \cos\left(\frac{3\pi}{N}\right)\right), \quad N \geq 8, \quad w \geq w_{[N/3]}^-(N). \tag{3.10.19}$$

Thus, to prove that the numbers $F_N\left(w_n^{\pm}(N)\right)$ have opposite signs, it suffices to verify the inequality

$$1.01\exp\left(w_n^{\pm}(N)\cos\left(\frac{3\pi}{N}\right)\right) < \sin\left(x_n(N)\right)\exp\left(w_n^{\pm}(N)\right)\cos\left(\frac{\pi}{N}\right)$$
$$\iff \quad 1.01\exp\left(-2w_n^{\pm}(N)\sin\left(\frac{\pi}{N}\right)\sin\left(\frac{2\pi}{N}\right)\right) < \sin\left(x_n(N)\right).$$

Applying the lower estimate (3.10.11), we see that now we must prove the inequality

$$1.01\exp\left(-(2\pi n+2.8)\sin\left(\frac{2\pi}{N}\right)\right) < \sin\left(x_n(N)\right).$$

Using the definition of $x_n(N)$, for $8 \le N \le 1400$ we replace this inequality by the following:

$$1.01\exp\left(-2.8\sin\left(\frac{2\pi}{N}\right)\right) < \frac{\sin x_n(N)}{x_n(N)}. \tag{3.10.20}$$

Since $\sin(2\pi/N) > 4/N$, $N > 4$, the left-hand side of (3.10.20) for $8 \le N \le 560$ is less than $1.01\exp(-0.02) < \exp(-0.01) < 0.991$, whereas the right-hand side of (3.10.20), by (3.10.18), is greater than 0.999. The required inequality is proved. For $560 < N \le 1400$, applying the stronger estimate $\sin(2\pi/N) > \dfrac{6}{N}$, $N > 12$, we obtain that the left-hand side of (3.10.20) is less than $1.01\exp(-0.012) < \exp(-0.02)$, which is again less than the right-hand side. Thus, for $8 \le N \le 1400$, the two-sided estimates of zeros of the function $E_{1/N}(z; N+1)$ with numbers $n \ge [N/3]$ are proved.

In the case $N > 1400$, we do not roughen estimate (3.10.19) and see that we must prove the inequality

$$\left(1+\frac{1}{106}\right)\exp\left(-(2\pi n+2.8)\sin\left(\frac{2\pi}{N}\right)\right) \le \sin\left(x_n(N)\right).$$

Using the definition of $x_n(N)$, we transfer the equivalent inequality

$$\frac{107}{106\cdot 1.01}\exp\left(-2.8\sin\left(\frac{2\pi}{N}\right)\right) < \frac{\sin x_n(N)}{x_n(N)}. \tag{3.10.21}$$

The left-hand side of (3.10.21) is less than $\dfrac{107}{107.06} < 1 - 10^{-4}$; this value is less than the right-hand side (see (3.10.18)). We have proved the two-sided estimates of zeros of the function $E_{1/N}(z; N+1)$ with numbers $\ge [N/3]$.

Now we prove lower estimates of the first $[N/3]-1$ zeros for $N \ge 6$. The required lower estimate of the first zero immediately follows from Lemma 3.3.3 if we take $\varepsilon = 1.5R_1/R_2$ and verify the inequality

$$\frac{R_1}{R_2} < \frac{\varepsilon}{(1+\varepsilon)^2} \quad\iff\quad (1+\varepsilon)^2 < 1.5 \quad\iff\quad 2\varepsilon+\varepsilon^2 < 0.5. \tag{3.10.22}$$

We have the estimate

$$\frac{R_1(N)}{R_2(N)} = \prod_{k=0}^{N-1}\left(\frac{2N-k}{3N-k}\right) < \left(\frac{2}{3}\right)^N \le \left(\frac{2}{3}\right)^6 < 0.1, \quad N \ge 6.$$

Therefore, $\varepsilon < 0.15$, $2\varepsilon+\varepsilon^2 < 0.4$, and inequality (3.10.22) holds. The lower estimate of the first zero is proved.

Obtain a lower estimate of the second zero (then $N \ge 9$). If F is the function from Lemma 3.3.1 and

$$\varepsilon > 0, \quad R_2(1+\varepsilon) < R_5, \tag{3.10.23}$$

then

$$F(-R_2(1+\varepsilon)) = \sum_{k=0}^{\infty}(-1)^k A_k R_2^k (1+\varepsilon)^k > -A_1R_2(1+\varepsilon) + A_2R_2^2(1+\varepsilon)^2 - A_3R_2^3(1+\varepsilon)^3,$$

since the moduli of terms of the alternating sum

$$\sum_{k=4}^{\infty}(-1)^k A_k R_2^k (1+\varepsilon)^k$$

decrease and hence the sum is positive (the ratio of the terms with numbers $k+1$ and k is equal to

$$\frac{R_2(1+\varepsilon)}{R_{k+1}} \le \frac{R_2(1+\varepsilon)}{R_5} < 1$$

by (3.10.23)). If we take $\varepsilon = 1.5\dfrac{R_2}{R_3}$, then we obtain the inequality

$$\frac{F(-R_2(1+\varepsilon))}{A_2R_2^2} > -\frac{A_1(1+\varepsilon)}{A_2R_2} + (1+\varepsilon)^2 - \frac{A_3}{A_2}R_2(1+\varepsilon)^3 = -(1+\varepsilon) + (1+\varepsilon)^2$$
$$-\frac{R_2}{R_3}(1+\varepsilon)^3 = \varepsilon + \varepsilon^2 - \frac{2}{3}\varepsilon(1+\varepsilon)^3 = \varepsilon\left(\frac{1}{3} - \varepsilon - 2\varepsilon^2 - \frac{2}{3}\varepsilon^3\right). \quad (3.10.24)$$

Impose on ε the restriction

$$\varepsilon \le 0.2 \quad \Longleftrightarrow \quad 7.5R_2 \le R_3. \quad (3.10.25)$$

By the monotonicity of R_n, this condition is stronger than in (3.10.23). The positiveness of $F(-R_2(1+\varepsilon))$ immediately follows from (3.10.24) and (3.10.25).

We verify that for the values $R_n(N)$ defined before the statement of Theorem 3.1.3, the inequality $7.5R_2(N) \le R_3(N)$ holds for $N \ge 9$, and hence

$$E_{1/N}\left(-R_2\left(1 + \frac{3R_2}{2R_3}\right); N+1\right) > 0.$$

We have the inequality

$$\frac{R_2(N)}{R_3(N)} = \prod_{k=0}^{N-1}\left(\frac{3N-k}{4N-k}\right) < \left(\frac{3}{4}\right)^N \le \left(\frac{3}{4}\right)^9 < 0.1.$$

The lower estimate of the second zero is proved since by (3.2.6)

$$E_{1/N}(-R_2; N+1) < 0.$$

Further, $3 \le n \le [N/3] - 1$ (here n is the ordinal number of a zero) and, naturally, $N \ge 12$. Until the end of this section, we assume that the number N is fixed and omit the dependence of different quantities on it. We set

$$E_{1/N}(z; N+1) = \sum_{k=0}^{\infty}\frac{z^k}{(kN+N)!} \equiv \sum_{k=0}^{\infty}A_k z^k.$$

Clearly, $R_k = A_{k-1}/A_k$, $k \in \mathbb{N}$. Introduce the notation

$$v_{n,k}(x) = A_{n+k}x^{n+k} - A_{n-k-1}x^{n-k-1}, \quad 1 \le k \le n-1.$$

Lemma 3.10.1. *For any $x \in [R_n, R_n+1)$, $n \ge 3$, the sequence $v_{n,k}(x)$ decreases when the index k varies from 1 to $n-1$. Moreover, the following inequality is valid*:

$$A_{2n}x^{2n} < v_{n,n-1}(x). \quad (3.10.26)$$

Proof. First, we prove the inequality

$$v_{n,\,k+1}(x) < v_{n,\,k}(x), \quad 1 \le k \le n-2, \quad R_n \le x < R_{n+1}$$
$$\iff \quad A_{n-k-1}x^{n-k-1} - A_{n-k-2}x^{n-k-2} < A_{n+k}x^{n+k} - A_{n+k+1}x^{n+k+1}. \quad (3.10.27)$$

Introduce the notation

$$u_{n,k} = R_{n-k}R_n^{-2}R_{n+k} = \prod_{j=1}^{N} \frac{(N(n-k)+j)\,(N(n+k)+j)}{(Nn+j)^2} = \prod_{j=1}^{N}\left[1-\left(\frac{kN}{Nn+j}\right)^2\right]. \quad (3.10.28)$$

Then

$$\frac{A_{n-k-1}}{A_{n+k}R_n^{2k+1}} = R_n^{-2k-1}\prod_{m=n-k}^{n+k} R_m = \prod_{j=1}^{k}\left(R_{n-j}R_n^{-2}R_{n+j}\right) = \prod_{j=1}^{k} u_{n,\,j}. \quad (3.10.29)$$

From (3.10.29) we see that inequality (3.10.27) after dividing both sides by $A_{n+k}x^{n+k}$ can be written in the following equivalent form:

$$\frac{A_{n-k-1}}{A_{n+k}x^{2k+1}} - \frac{A_{n-k-2}}{A_{n+k}x^{2k+2}} < 1 - \frac{xA_{n+k+1}}{A_{n+k}}$$
$$\iff \quad \left(\frac{R_n}{x}\right)^{2k+1} \frac{A_{n-k-1}}{A_{n+k}R_n^{2k+1}}\left(1-\frac{A_{n-k-2}}{A_{n-k-1}x}\right) < 1 - \frac{x}{R_{n+k+1}}$$
$$\iff \quad \left(1-\frac{R_{n-k-1}}{x}\right)\prod_{j=1}^{k} u_{n,\,j} < \left(1-\frac{x}{R_{n+k+1}}\right) < \left(1-\frac{x}{R_{n+k+1}}\right)\left(\frac{x}{R_n}\right)^{2k+1}.$$

Setting

$$\delta_{n,k}(x) = \left(1-\frac{R_{n-k-1}}{x}\right)\left(1-\frac{x}{R_{n+k+1}}\right)^{-1},$$

we see that we must prove the inequality

$$\delta_{n,\,k}(x)\prod_{j=1}^{k} u_{n,\,j} < \left(\frac{x}{R_n}\right)^{2k+1}, \quad 1 \le k \le n-2, \quad R_n \le x < R_{n+1}. \quad (3.10.30)$$

We obtain an upper estimate of the function $\delta_{n,\,k}(x)$. We have the relations

$$\delta_{n,\,k}(x) - 1 = \frac{1-R_{n-k-1}/x}{1-x/R_{n+k+1}} - 1 = \frac{x/R_{n+k+1} - R_{n-k-1}/x}{1-x/R_{n+k+1}}$$
$$= \frac{1-R_{n-k-1}x^{-2}R_{n+k+1}}{R_{n+k+1}/x-1} = \frac{1-(R_{n/x})^2u_{n,\,k+1}}{R_{n+k+1}/x-1} < \frac{1-(R_{n/x})^2u_{n,\,k+1}}{R_{n+2}/R_{n+1}-1}. \quad (3.10.31)$$

Further, we have for $n \le [N/3]-1$ and $N \ge 12$

$$\frac{R_{n+2}}{R_{n+1}} = \prod_{j=0}^{N-1}\left(\frac{(n+3)N-j}{(n+2)N-j}\right) > \left(\frac{n+3}{n+2}\right)^N > \left(\frac{N/3+2}{N/3+1}\right)^N$$
$$= \left(\frac{N+6}{N+3}\right)^N \ge \left(\frac{18}{15}\right)^{12} > 8 \quad (3.10.32)$$

(we have used the fact that the functions $(t+1)/t$ and $((t+6)/(t+3))^t$ decrease on the ray $0 < t < +\infty$). Introduce the variable $y = x/R_n \ge 1$ and, using (3.10.31) and (3.10.32), replace inequality

(3.10.30) by the following stronger inequality:

$$\left(1+\frac{1-y^{-2}u_{n,\,k+1}}{7}\right)\prod_{j=1}^{k}u_{n,\,j}\le y^{2k+1}. \tag{3.10.33}$$

Note that it suffices to prove inequality (3.10.33) only for $y=1$ since the derivative $(2k+1)y^2k$ of its right-hand side is greater than the derivative of its left-hand side, which is equal to

$$\frac{2y^{-3}u_{n,\,k+1}}{7}\prod_{j=1}^{k}u_{n,\,j}<\frac{2y^{-3}}{7}\le\frac{2}{7}.$$

From 3.10.33 we obtain the estimate

$$u_{n,k}<\left(1-\left(\frac{k}{n+1}\right)^2\right)^N<\exp\left(-\frac{k^2N}{(n+1)^2}\right); \tag{3.10.34}$$

this immediately implies that $u_{n,\,k}<1$. Thus, we must prove the inequality

$$\left(1+\frac{1-u_{n,\,k+1}}{7}\right)\prod_{j=1}^{k}u_{n,\,j}\le 1. \tag{3.10.35}$$

If $k\ge n/4$ and $n\ge 3$, then from (3.10.34) we see that

$$\begin{aligned}\prod_{j=1}^{k}u_{n,\,j}<\exp\left(-\frac{N}{(n+1)^2}\sum_{j=1}^{k}j^2\right)&=\exp\left(-\frac{N(2k+1)k(k+1)}{6(n+1)^2}\right)\\ \le\exp\left(-\frac{(n/2+1)(n/2)(n/4+1)}{(n+1)^2}\right)&=\exp\left(-\frac{(n+2)(n+4)n}{16(n+1)^2}\right)\\ &\le\exp\left(-\frac{n+2}{16}\right)\le\exp\left(-\frac{5}{16}\right)\end{aligned} \tag{3.10.36}$$

(here $n\ge 12$ and hence $N/6\ge 2$). Therefore, the left-hand side of (3.10.35) is less than

$$\left(1+\frac{1}{7}\right)\exp\left(-\frac{5}{16}\right)<\exp\left(\frac{1}{7}-\frac{5}{16}\right)<1$$

and we have proved what was required.

If $1\le k<n/4$, then we replace inequality (3.10.35) by the following stronger but simpler inequality:

$$\left(1+\frac{1-u_{n,\,k+1}}{7}\right)u_{n,\,k}\le 1, \tag{3.10.37}$$

and obtain the estimate

$$u_{n,\,k}^5<u_{n,k+1},\quad 1\le k<n/4. \tag{3.10.38}$$

From (3.10.38) we immediately obtain (3.10.37) since the function $u^{1/5}\left(1+\dfrac{1-u}{7}\right)$ does not exceed 1 on the segment $[0,1]$.

Thus, to complete the proof of the first part of the lemma, it remains to verify estimate (3.10.38). From (3.10.28) we obtain the equality

$$\frac{u_{n,\,k}^5}{u_{n,\,k+1}}=\prod_{j=1}^{N}\frac{(N(n-k)+j)^5\,(N(n+k)+j)^5}{(N(n-k-1)+j)\,(Nn+j)^8\,(N(n+k+1)+j)}. \tag{3.10.39}$$

We prove that any factor in (3.10.39) is less than 1, i.e.,

$$\frac{(N(n-k)+j)^5\,(N(n+k)+j)^5}{(N(n-k-1)+j)\,(Nn+j)^8\,(N(n+k+1)+j)} < 1, \quad 1 \le j \le N, \quad 1 \le k < n/4.$$

Then the product is also less than 1, which means that inequality (3.10.38) is valid. Denoting $Nn+j$ by t and Nk by τ, we see that it suffices to prove the inequality

$$(t^2-\tau^2)^5 < t^8(t-\tau-N)(t+\tau+N), \quad N \le \tau \le \frac{t}{4}. \tag{3.10.40}$$

The right-hand side of (3.10.40) is equal to

$$t^8\left(t^2-(\tau+N)^2\right) \ge t^8(t^2-4\tau^2).$$

Therefore, it suffices to prove the following stronger inequality

$$t^{10}-5t^8\tau^2+10t^6\tau^4-10t^4\tau^6+5t^2\tau^8-\tau^{10} < t^{10}-4t^8\tau^2$$
$$\iff \quad 10t^6\tau^4-10t^4\tau^6+5t^2\tau^8-\tau^{10} < t^8\tau^2. \tag{3.10.41}$$

Rejecting on the left-hand side of (3.10.41) the negative term $-\tau^{10}$ and dividing both sides of (3.10.41) by $t^2\tau^2$, we obtain the following, stronger but simpler, inequality

$$10t^4\tau^2-10t^2\tau^4+5\tau^6 < t^6. \tag{3.10.42}$$

Since $\tau \le t/4$, the left-hand side of (3.10.42) is less than

$$10t^4\tau^2 = t^6\cdot 10\left(\frac{\tau}{t}\right)^4 \le \frac{10}{16}t^6 < t^6,$$

which was required. The first part of the lemma is proved.

Prove inequality (3.10.26), which, due to the definition of $v_{n,n-1}(x)$, can be written in the form

$$A_0+A_{2n}x^{2n} < A_{2n-1}x^{2n-1} \quad \iff \quad \frac{A_0}{A_{2n-1}}x^{1-2n}+\frac{A_{2n}x}{A_{2n-1}} < 1.$$

We strengthen the last inequality taking into account the inclusion $x \in [R_n, R_{n+1})$:

$$\frac{A_0}{A_{2n-1}}R_n^{1-2n}+\frac{A_2n}{A_{2n-1}}R_{n+1} < 1 \quad \iff \quad \prod_{k=1}^{n-1}u_{n,k}+\frac{R_{n+1}}{R_{2n}} < 1. \tag{3.10.43}$$

From estimates (3.10.32) and (3.10.36) we see that inequality (3.10.43) holds. The lemma is completely proved. □

Since relations (3.2.6) have been proved, it suffices to deduce the equalities

$$\operatorname{sgn} E_{1/N}(-R_n(1+q_n); N+1) = (-1)^n, \quad 3 \le n \le \left[\frac{N}{3}\right]-1. \tag{3.10.44}$$

Introduce the following notation for the mth remainder of the power series for the function $E_{1/N}(-x, N+1)$:

$$\sigma_m(x) = \sum_{\nu=m}^{\infty}(-1)^\nu A_\nu x^\nu.$$

We have the relation

$$E_{1/N}(-x, N+1) = \sum_{\nu=0}^{2n-1} (-1)^{\nu} A_{\nu} x^{\nu} + \sigma_{2n}(x)$$
$$= \sum_{k=0}^{n-1} (-1)^{n+k} (A_{n+k} x^{n+k} - A_{n-k-1} x^{n-k-1}) + \sigma_{2n}(x)$$
$$= (-1)^n \sum_{k=0}^{n-1} (-1)^k \nu_{n,k}(x) + \sigma_{2n}(x). \quad (3.10.45)$$

By (3.10.44) and (3.10.45), we must prove that

$$\begin{aligned} &\sum_{k=0}^{n-1} (-1)^k \nu_{n,k}(x) + \sigma_{2n}(x) > 0 \qquad \text{for even } n, \\ &\sigma_{2n}(x) < \sum_{k=0}^{n-1} (-1)^k \nu_{n,k}(x) \qquad \text{for odd } n, \end{aligned} \qquad x = (1+q_n)R_n. \quad (3.10.46)$$

First, we prove the inequality

$$v_{n,1}((1+q_n)R_n) < v_{n,0}((1+q_n)R_n), \quad 3 \le n \le \left[\frac{N}{3}\right] - 1 \quad (3.10.47)$$

independently of the parity of n. We have the relations

$$v_{n,0}(x) = A_n x^n - A_{n-1} x^{n-1} = A_{n-1} x^{n-1} \left(\frac{x A_n}{A_{n-1}} - 1 \right) = A_{n-1} x^{n-1} \left(\frac{x}{R_n} - 1 \right),$$
$$v_{n,1}(x) = A_{n+1} x^{n+1} - A_{n-2} x^{n-2} = A_{n-2} x^{n-2} \left(\frac{x^3 A_{n+1}}{A_{n-2}} - 1 \right) = A_{n-2} x^{n-2} \left(\frac{x^3}{R_{n-1} R_n R_{n+1}} - 1 \right).$$

Therefore, inequality $v_{n,1}(x) < v_{n,0}(x)$ can be rewritting in the following form:

$$A_{n-2} x^{n-2} \left(\frac{x^3}{R_{n-1} R_n R_{n+1}} - 1 \right) < A_{n-1} x^{n-1} \left(\frac{x}{R_n} - 1 \right)$$
$$\iff \quad \frac{x^3}{R_{n-1} R_n R_{n+1}} - 1 < \frac{x}{R_{n-1}} \left(\frac{x}{R_n} - 1 \right).$$

From this we conclude that inequality (3.10.47) is equivalent to the following inequality:

$$(1+q_n)^3 \frac{R_n^2}{R_{n-1} R_{n+1}} - 1 < q_n (1+q_n) \frac{R_n}{R_{n-1}}. \quad (3.10.48)$$

Substituting $q_n = 1.5 R_n / R_{n+1}$ in (3.10.48) and setting $t_n = (1+q_n) R_n^2 R_{n-1}^{-1} R_{n+1}^{-1}$, we rewrite (3.10.48) in the following form:

$$(1+q_n)^2 t_n < 1 + 1.5 t_n \quad \iff \quad (2q_n + q_n^2) t_n < 1 + 0.5 t_n.$$

It is easy to see that the last inequality holds if $q_n < 0.2$, i.e.,

$$\frac{R_n}{R_{n+1}} < \frac{2}{15}, \quad 3 \le n \le \left[\frac{N}{3}\right] - 1.$$

From (3.1.8) we obtain

$$\frac{R_{n+1}}{R_n} = \prod_{k=1}^{N} \left(\frac{nN+N+k}{nN+k}\right) = \prod_{k=1}^{N} \left(1+\frac{N}{nN+k}\right) = \prod_{k=1}^{N} \left(1+\frac{1}{n+k/N}\right)$$

$$> \left(1+\frac{1}{n+1}\right)^N \geq \left(1+\frac{1}{[N/3]}\right)^N \geq \left(1+\frac{1}{p}\right)^{3p}, \quad p = \left[\frac{N}{3}\right] \geq 4.$$

Therefore, taking into account the fact that the sequence $\left(1+\dfrac{1}{p}\right)^p$ increases, we obtain the estimate

$$\frac{R_n}{R_{n+1}} \leq \left(\frac{4}{5}\right)^{12} < \frac{1}{12},$$

which was required.

Owing to (3.10.47), inequalities (3.10.46) can be replaced by stronger inequalities

$$\begin{aligned} &\sum_{k=2}^{n-1} (-1)^k v_{n,\,k}(x) + \sigma_{2n}(x) > 0 \qquad \text{for even } n, \\ &\sigma_{2n}(x) < \sum_{k=2}^{n-1} (-1)^k v_{n,k}(x) \qquad \text{for odd } n, \end{aligned} \qquad x = (1+q_n)R_n.$$

Prove these inequalities. The case where n is even is simple: by Lemma 3.10.1, the sequence $v_{n,k}(x)$ decreases and all its elements are positive. Therefore,

$$\sum_{k=2}^{n-1} (-1)^k v_{n,k}(x) > 0.$$

The fact that the alternating sum $\sigma_{2n}(x)$ is positive follows from a similar reasoning: the first term is positive and the moduli of terms decrease when $0 < x < R_{2n}$. Since $x < R_{n+1} < R_{2n}$, we obtain the required inequality. For odd n, we reject the sum

$$\sum_{k=2}^{n-1} (-1)^k v_{n,k}(x)$$

on the right-hand side (as was proved above, it is positive for for $n \geq 5$ and empty for $n = 3$) and keep $A_{2n}x^{2n} > \sigma_{2n}(x)$ on the left-hand side. Thus, we must prove the inequality

$$A_{2n}x^{2n} < (-1)^{n-1} v_{n,\,n-1}(x) \equiv v_{n,\,n-1}(x),$$

which has been proved in Lemma 3.10.1. Therefore, relations (3.10.44) are verified and Theorem 3.1.3 is completely proved.

3.11. Proof of Theorem 3.1.4

If an entire function

$$F(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$$

of order less than 1 has only real negative roots[8] (not necessarily simple), then it can be represented in the form

$$F(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{r_n}\right),$$

[8]Here we use the term "root" instead of "zero" to avoid the inconsistent expression "negative zero."

where

$$\sum_{n=1}^{\infty} {r_n}^{-1} < +\infty,$$

where r_n is the sequence of all zeros of $F(z)$ taken with opposite sign and arranged in increasing order of their moduli with account of their multiplicity. Then

$$B_1 = \sum_{n=1}^{\infty} r_n^{-1}, \quad B_2 = \sum_{1 \le p < q < +\infty} (r_p r_q)^{-1}.$$

Obviously, $2B_2 < B_1^2$. Therefore, we conclude that if the function $E_\rho(z, \mu)$ for some positive value of the parameter μ has only real negative roots (in this case, the order ρ of the function $E_\rho(z, \mu)$ does not exceed 1/2), then, owing to the representation

$$\Gamma(\mu) E_\rho(z, \mu) = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\mu + k/\rho)} z^k,$$

we have the inequality

$$2\frac{\Gamma(\mu)}{\Gamma(\mu + 2/\rho)} < \left(\frac{\Gamma(\mu)}{\Gamma(\mu + 1/\rho)}\right)^2.$$

We rewrite it in the following equivalent form:

$$\ln 2 < \ln\Gamma(\mu) - 2\ln\Gamma\left(\mu + \frac{1}{\rho}\right) + \ln\Gamma\left(\mu + \frac{2}{\rho}\right). \tag{3.11.1}$$

We prove that for

$$\mu \ge 0.9 + (\rho^2 \ln 2)^{-1} - \rho^{-1}$$

inequality (3.11.1) does not hold and hence not all zeros of $E_\rho(z, \mu)$ are real. For this, we obtain an upper estimate of the second difference on the right-hand side of inequality (3.11.1). The power expansion

$$F(x \pm h) = \sum_{k=0}^{\infty} F^{(k)}(x) \frac{h^k (\pm 1)^k}{k!},$$

which holds for any F analytic in a neighborhood of the disk $\{z \in \mathbb{C} \mid |z - x| \le h\}$, implies the identity

$$F(x + h) - 2F(x) + F(x - h) = 2\sum_{\ell=1}^{\infty} F^{(2\ell)}(x) \frac{h^{2\ell}}{(2\ell)!}. \tag{3.11.2}$$

From (3.11.2) and the expansions

$$(\ln\Gamma(z))^{(k)} = \psi^{(k-1)}(z) = (-1)^k (k-1)! \sum_{m=0}^{\infty} (m + z)^{-k}$$

we obtain

$$\ln\Gamma(x + h) - 2\ln\Gamma(x) + \ln\Gamma(x - h) = \sum_{\ell=1}^{\infty} \ell^{-1} \sum_{m=0}^{\infty} (m + x)^{-2\ell} h^{2\ell}, \quad 0 < h < x. \tag{3.11.3}$$

Since

$$a^{-p} < \int_{a-0.5}^{a+0.5} t^{-p} dt$$

for all $a > 0.5$ and $p > 1$, we have the estimate

$$\sum_{m=0}^{\infty}(m+x)^{-2\ell} < \int_{x-0.5}^{+\infty} t^{-2\ell}dt = (2\ell-1)^{-1}(x-0.5)^{1-2\ell}, \quad x > 0.5. \tag{3.11.4}$$

From (3.11.3) and (3.11.4) we obtain the inequality

$$\ln\Gamma(x+h)-2\ln\Gamma(x)+\ln\Gamma(x-h) < \sum_{\ell=1}^{\infty}\ell^{-1}(2\ell-1)^{-1}h^{2\ell}(x-0.5)^{1-2\ell}, \quad h>0, \quad x>h+0.5. \tag{3.11.5}$$

We find an upper estimate of the sum on the right-hand side of (3.11.5) using the relation

$$\sum_{\ell=2}^{\infty}\frac{1}{2\ell(2\ell-1)} = \sum_{\ell=2}^{\infty}\left(\frac{1}{2\ell-1}-\frac{1}{2\ell}\right) = \ln 2 - \frac{1}{2} < 0.2.$$

For $x > h + 0.5$, we have the relation

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{h^{2\ell}(x-0.5)^{1-2\ell}}{\ell(2\ell-1)} &= \frac{h^2}{x-0.5} + h\sum_{\ell=2}^{\infty}\left(\frac{h}{x-0.5}\right)^{2\ell-1}\cdot\frac{1}{\ell(2\ell-1)} \\ &< \frac{h^2}{x-0.5} + \frac{h^4}{(x-0.5)^3}\sum_{\ell=2}^{\infty}\frac{1}{\ell(2\ell-1)} < \frac{h^2}{x-0.5} + \frac{0.4h^4}{(x-0.5)^3}.\end{aligned} \tag{3.11.6}$$

We denote $x-0.5$ by y. Then under the condition $h^2 < y$, $h > 1$, from (3.11.6) and the easily verifiable inequality

$$\frac{h^2}{y} + 0.4\frac{h^4}{y^3} < \frac{h^2}{y(y-0.4)}, \quad h>0, \quad y>0.4, \quad h^2<y,$$

we obtain

$$\sum_{\ell=1}^{\infty}\frac{h^{2\ell}}{\ell(2\ell-1)(x-0.5)^{2\ell-1}} < \frac{h^2}{y-0.4} = \frac{h^2}{x-0.9}. \tag{3.11.7}$$

From (3.11.7) and (3.11.5) for $h > 1$ and $h^2 < x - 0.5$ we obtain the inequality

$$\ln\Gamma(x+h) - 2\ln\Gamma(x) + \ln\Gamma(x-h) < \frac{h^2}{x-0.9}. \tag{3.11.8}$$

We set $h = 1/\rho$ and $x = \mu + 1/\rho$. By the condition of the theorem,

$$\mu + \frac{1}{\rho} \geq 0.9 + (\rho^2\ln 2)^{-1},$$

i.e.,

$$y = x - 0.5 > (\rho^2\ln 2)^{-1} > h^2.$$

Therefore, inequality (3.11.8) can be applied to the upper estimate of the second difference from (3.11.1), and we obtain the inequality

$$\ln\Gamma(\mu) - 2\ln\Gamma\left(\mu+\frac{1}{\rho}\right) + \ln\Gamma\left(\mu+\frac{2}{\rho}\right) < \frac{h^2}{\mu+1/\rho-0.9} \leq h^2\rho^2\ln 2 = \ln 2,$$

which contradicts inequality (3.11.1). Therefore, the function $E_\rho(z,\mu)$ has zeros outside $\mathbb{R}$ for $\mu \geq 0.9 + (\rho^2\ln 2)^{-1} - \rho^{-1}$. Theorem 3.1.4 is proved.

3.12. Proof of Theorem 3.1.5

Lemma 3.12.1. *If* $\rho \in (1/3, 1/2)$ *and* $1 - 1/\rho \le \mu < -1$ *or* $0 < \rho \le 1/3$ *and* $-2 < \mu < -1$, *then the following inequality holds*:

$$\frac{4\Gamma^2(\mu + 1/\rho)}{\Gamma(\mu)\Gamma(\mu + 2/\rho)} < 1. \tag{3.12.1}$$

Proof. Inequality (3.12.1) is equivalent to the following:

$$4\mu(\mu + 1)(\mu + 2) < \frac{\Gamma(\mu + 2/\rho)\Gamma(\mu + 3)}{\Gamma(\mu + 1/r)\Gamma(\mu + 1/\rho)}. \tag{3.12.2}$$

Note that

$$\max\Big\{4\mu(\mu + 1)(\mu + 2) \mid -2 \le \mu \le -1\Big\} = \frac{2}{\sqrt{27}}.$$

Therefore, the left-hand side of (3.12.2) is less than 1.6. We obtain a lower estimate for the right-hand side.

If $1/3 \le \rho < 1/2$, then $1 \le \mu + 1/\rho < 2¡1 < \mu + 2 < 2$. Therefore, owing to the inequalities $0.8 < \Gamma(t) \le 1$, $1 \le t \le 2$, the lower estimate

$$\frac{\Gamma(\mu + 3)}{\Gamma(\mu + 1/\rho)} > 0.8$$

is valid. In addition, setting $\mu + 1/\rho = \tau$, we have

$$\frac{\Gamma(\mu + 2/\rho)}{\Gamma(\mu + 1/r)} = \frac{\Gamma(\tau + 1/\rho)}{\Gamma(\tau)} > \frac{\Gamma(\tau + 2)}{\Gamma(\tau)} = \tau(\tau + 1) \ge 2.$$

(Note that since $\Gamma(x)$ increases on the ray $x \ge 2$ and the inequality $3 \le t + 2 < t + 1/\rho$ holds, we have the inequality $\Gamma(t + 1/\rho) > \Gamma(t + 2)$.) This implies that the right-hand side of (3.12.2) is greater than 1.6, and inequality (3.12.2) is proved.

If $1/4 \le \rho < 1/3$, then $3 - 1/\rho < 0$, and from the Lagrange intermediate-value theorem for derivatives, we obtain the estimate ($\xi \in (\mu + 3, m + 1/\rho)$)

$$\ln\Gamma(\mu + 3) - \ln\Gamma\left(\mu + \frac{1}{\rho}\right) = \left(3 - \frac{1}{\rho}\right)\psi'(\xi) \ge \left(3 - \frac{1}{\rho}\right)\max\left\{\psi^+(t) \;\middle|\; \mu + 3 \le t \le m + \frac{1}{\rho}\right\}$$
$$\ge \left(3 - \frac{1}{\rho}\right)\psi(3) \ge -\psi(3) > -1.$$

Therefore, the right-hand side of (3.12.2) is greater than

$$\frac{\Gamma(\mu + 2/\rho)}{e\Gamma(\mu + 1/\rho)} = \frac{\Gamma(\tau + 1/\rho)}{e\Gamma(\tau)} \ge \frac{\tau + 3}{e\Gamma(\tau)} = \frac{\tau(\tau + 1)(\tau + 2)}{e} \ge \frac{6}{e} > 2,$$

which proves the required inequality in the case considered.

If $\rho < 1/4$, then we have the relation ($\tau = \mu + 1/\rho$)

$$\ln\Gamma(\mu + 3) - \ln\Gamma\left(1 + \frac{1}{\rho}\right) \ge \left(3 - \frac{1}{\rho}\right)\psi(\tau), \quad \ln\Gamma\left(\tau + \frac{1}{\rho}\right) - \ln\Gamma(\tau) \ge \frac{1}{\rho}\psi(\tau).$$

since ψ increases; therefore, the right-hand side of (3.12.2) is greater than

$$\exp\big(3\psi(\tau)\big) > \exp\big(3\psi(2)\big) = \exp\big(3(1 - \gamma)\big) > e.$$

The lemma is completely proved. ☐

The proof of Theorem 3.1.5 is based on the same ideas as the proof of Theorems 3.1.1 and 3.1.2. We start from the proof of the inequalities

$$E_\rho(-R_1;\mu) > 0, \quad E_\rho(-2R_1;\mu) < 0, \tag{3.12.3}$$

where

$$R_1 = \frac{\Gamma(\mu+1/\rho)}{\Gamma(\mu)}.$$

Recall (this will be extensively used) that $\Gamma(\mu) > 0$ for $-2 < \mu < -1$. We have the relations

$$E_\rho(-R_1;\mu) = \sum_{k=2}^{\infty} \frac{(-1)^k R_1^k}{\Gamma(\mu+k/\rho)}, \tag{3.12.4}$$

$$\Gamma(\mu)E_\rho(-2R_1;\mu) = -1 + \frac{4\Gamma^2(\mu+1/\rho)}{\Gamma(\mu)\Gamma(\mu+2/\rho)} + \sum_{k=3}^{\infty} \frac{(-1)^k(2R_1)^k\Gamma(\mu)}{\Gamma(\mu+k/\rho)}. \tag{3.12.5}$$

The sum of the alternating series (3.12.4) is positive since the moduli of its terms decrease:

$$\frac{R_1^{k+1}}{\Gamma\big(\mu+(k+1)/\rho\big)} < \frac{R_1^k}{\Gamma(\mu+k/\rho)} \quad\Longleftrightarrow\quad \frac{\Gamma(\mu+1/\rho)}{\Gamma(\mu)} < \frac{\Gamma\big(\mu+(k+1)/\rho\big)}{\Gamma(\mu+k/\rho)}. \tag{3.12.6}$$

Since the sequence of ratios $\left\{\dfrac{\Gamma(\mu+(k+1)/\rho)}{\Gamma(\mu+k/\rho)}\right\}_{k\in\mathbb{N}}$ increases, it suffices to prove inequality (3.12.6) only for $k=1$. This immediately follows from Lemma 3.12.1. The first inequality (3.12.3) is proved.

The sum of the alternating series in (3.12.5) is negative since the sequence of the moduli of its terms is monotonic; we must verify the inequality

$$\frac{2\Gamma(\mu+1/\rho)}{\Gamma(\mu)} < \frac{\Gamma\big(\mu+(k+1)/\rho\big)}{\Gamma(\mu+k/\rho)},$$

which also follows from Lemma 3.12.1. Therefore,

$$\Gamma(\mu)E_\rho(-2R_1;\mu) < -1 + \frac{4\Gamma^2(\mu+1/\rho)}{\Gamma(\mu)\Gamma(\mu+2/\rho)}.$$

By Lemma 3.12.1, the last expression is negative. Inequalities (3.12.3) are proved.

As in the proof of Theorem 3.1.1, the cases $0 < \rho \le 1/6$ and $1/6 < \rho < 1/2$ are examined separately. We start from the values $\rho \in (1/6, 1/2)$. Prove the relations

$$\operatorname{sgn} E_\rho\big(-\xi_n^{1/\rho}(\rho,\mu);\mu\big) = (-1)^n \quad \forall n \in \mathbb{N}, \quad n \ge 2, \tag{3.12.7}$$

$$2R_1 < \xi_2^{1/\rho} \tag{3.12.8}$$

(as a rule, for brevity we omit the arguments ρ and μ of ξ_n and R_n). From (3.12.7) we obtain the existence of at least one zero of the function $E_\rho(z;\mu)$ on each of the intervals $\big(-\xi_{n+1}^{1/\rho}; -\xi_n^{1/\rho}\big)$ for all $n \ge 2$, and from (3.12.3), (3.12.7), and (3.12.8) we obtain the existence of zeros on the intervals $\big(-\xi_2^{1/\rho}; -2R_1\big)$, $(-2R_1, -R_1)$. Therefore, for any $n \in \mathbb{N}$, $n \ge 2$, the interval $\big(-\xi_n^{1/\rho}, 0\big)$ of the real axis contains no less than n zeros of the function $E_\rho(z;\mu)$. On the other hand, by Theorems 2.1.4 and 2.2.2, for any sufficiently large n, the disk $|z| \le \xi_n^{1/\rho}$ contains exactly n zeros of the Mittag-Leffler function. This means that all zeros are real, negative, and simple.

Now we prove relations (3.12.8) and (3.12.7). Write inequality (3.12.8) in detailed form:

$$\frac{2\Gamma(\mu+1/\rho)}{\Gamma(\mu)} < \left(\frac{\pi\big(2+\rho(\mu-1)\big)}{\sin\pi\rho}\right)^{1/\rho}. \tag{3.12.9}$$

The restrictions of Theorem 3.1.5 on the parameter μ imply the estimate $2+\rho(\mu-1)\ge 1$. We also have the inequality

$$\frac{1}{\Gamma(\mu)}=\frac{\mu(\mu+1)(\mu+3)}{\Gamma(\mu+3)}\le\frac{2}{\sqrt{27}\Gamma(\mu+3)}<\frac{1}{2},\quad \mu\in(-2,-1).$$

Here we have applied the well-known estimate $1/\Gamma(x)<1.2$, $1<x<2$. Therefore, (3.12.9) can be replaced by the stronger but simpler inequality

$$\Gamma(\mu+1/\rho)<\left(\frac{\pi}{\sin\pi\rho}\right)^{1/\rho},$$

which can be easily proved. If $1/4<\rho<1/2$, then $1\le\mu+1/\rho<3$ and

$$\Gamma\left(\mu+\frac{1}{\rho}\right)<2,$$

but

$$(\pi\operatorname{cosec}\pi\rho)^{1/\rho}>\pi^{1/\rho}>\pi^2.$$

If $1/6<\rho\le 1/4$, then

$$\Gamma\left(\mu+\frac{1}{\rho}\right)\le\Gamma\left(\frac{1}{\mu}-1\right)<\Gamma(5)=24,$$

but

$$(\pi\operatorname{cosec}\pi\rho)^{1/\rho}>\pi^4.$$

We prove equalities (3.12.7). Denote λ by $\mu+1/\rho$. We have the inequalities

$$\begin{aligned}
&1\le\lambda<\frac{1}{\rho}-1, && \frac{1}{3}<\rho<\frac{1}{2},\\
&1\le\frac{1}{\rho}-2<\lambda<\frac{1}{\rho}-1, && 0<\rho\le\frac{1}{3}.
\end{aligned}\tag{3.12.10}$$

By the identity

$$E_\rho(z;\mu)=\frac{1}{\Gamma(\mu)}+zE_\rho(z;\lambda),\quad \xi_n(\rho,\mu)=\xi_{n-1}(\rho,\lambda)$$

relations (3.12.7) are equivalent to the following:

$$\operatorname{sgn}\left[E_\rho\big(-\xi_m^{1/\rho}(\rho,\lambda);\lambda\big)-\frac{1}{\Gamma(\mu)\xi_m^{1/\rho}(\rho,\lambda)}\right]=(-1)^m\quad\forall m\in\mathbb{N}.\tag{3.12.11}$$

In the proof of Theorem 3.1.1, we obtained the equalities

$$\operatorname{sgn}E_\rho\big(-\xi_m^{1/\rho}(\rho,\lambda);\lambda\big)=(-1)^m\quad\forall m\in\mathbb{N},\quad 1/6<\rho<1/2.\tag{3.12.12}$$

Clearly, for odd m, equalities (3.12.12) are stronger than (3.12.11), and conversely for even m. Therefore, in the case $1/6<\rho<1/2$ it remains to deduce the inequalities

$$E_\rho\big(-\xi_m^{1/\rho}(\rho,\lambda);\lambda\big)>\frac{1}{\Gamma(\mu)\xi_m^{1/\rho}(\rho,\lambda)}\quad\text{for any even } m.\tag{3.12.13}$$

The estimates of the remainder in the representation of the function $E_\rho(z;\lambda)$ (Theorems 1.5.3 and 1.5.4), the expression $2\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho)$ for the principal term at the point $z=-\xi_m^{1/\rho}$ for even m, and the estimate of the second term (see Sec. 3.8) imply the following inequalities, which are valid for arbitrary $\lambda \in (0, 1/\rho]$:

$$E_\rho(-\xi_m^{1/\rho};\lambda) > \begin{cases} 2\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho) - 1.5\xi_m^{-1/\rho}, & 0.4 \le \rho < 0.5, \\ 2\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho) - 0.74\xi_m^{-\lambda}, & 0.25 < \rho < 0.4, \\ 1.96\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho) - 0.74\xi_m^{-\lambda}, & \dfrac{1}{6} < \rho \le \dfrac{1}{4}. \end{cases} \tag{3.12.14}$$

From (3.12.14), (3.12.10), and the estimate $0 < 1/\Gamma(\mu) < 1/2$, $-2 < \mu < -1$, obtained above, we see that inequality (3.12.13) follows from the inequalities

$$\begin{aligned} &\xi_m^{-1/\rho} < \rho\xi_m^{1-\lambda} exp(\xi_m\cos\pi\rho), && 0.4 \le \rho < 0.5, \quad 1 \le \lambda < \frac{1}{\rho} - 1, \\ &0.5\xi_m^{-1/\rho} + 0.74\xi_m^{-\lambda} < 1.96\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho), && \frac{1}{6} < \rho < 0.4, \quad 1 \le \lambda < \frac{1}{\rho} - 1. \end{aligned} \tag{3.12.15}$$

Multiplying inequalities (3.12.15) by $\xi_m^{\lambda-1}$ and applying the estimates $\xi_m > 1$ (see Secs. 3.5–3.7) and $\lambda - 1 - 1/\rho \le -2$, we reduce the problem to the proof of the following inequalities:

$$\begin{aligned} &\xi_m^{-2} < \rho exp(\xi_m\cos\pi\rho), \quad 0.4 \le \rho < 0.5, \quad 1 \le \lambda < \frac{1}{\rho} - 1, \\ &0.5\xi_m^{-2} + 0.74\xi_m^{-1} < 1.96\rho\xi_m^{1-\lambda} exp(\xi_m\cos\pi\rho), \quad \frac{1}{6} < \rho < 0.4, \quad 1 \le \lambda < \frac{1}{\rho} - 1. \end{aligned} \tag{3.12.16}$$

We obtain a more exact lower estimate for $\xi_m = \xi_m(\rho,\lambda)$. Since $\lambda \ge 1$ and $m \ge 2$, we have

$$\xi_m(\rho,\lambda) = \frac{\pi\big(m + \rho(\lambda - 1)\big)}{\sin\pi\rho} \ge \frac{\pi m}{\sin\pi\rho} > 2\pi. \tag{3.12.17}$$

From (3.12.17) and the trivial estimate $\exp(\xi_m\cos\pi\rho) > 1$ we immediately obtain (3.12.16). The proof of the theorem for values $\rho \in (1/6, 1/2)$ is complete.

In the case $\rho \in (0, 1/6]$, inequality (3.12.13) must be proved for any even $m \ge [1/(3\rho)]$. Arguing as above, we obtain the existence of at least one zero of the function $E_\rho(z;\mu)$ on each of the intervals

$$\left(-\xi_{n+1}^{1/\rho}(\rho,\mu),\ -\xi_n^{1/\rho}(\rho,\mu)\right), \quad n \ge \left[\frac{1}{3\rho}\right] + 1 = M.$$

We must also prove the equalities (for $n = 1$ this has been done in (3.12.3) and the inequality $E_\rho(-R_1(\rho,\mu);\mu) > 0$ was proved)

$$\operatorname{sgn} E_\rho\left(-\frac{R_n(\rho,\mu)}{2};\mu\right) = (-1)^{n-1}, \quad 2 \le n \le M, \tag{3.12.18}$$

$$\operatorname{sgn} E_\rho\left(-2R_n(\rho,\mu);\mu\right) = (-1)^n, \qquad 1 \le n \le M - 1. \tag{3.12.19}$$

This, together with the inequalities

$$0 < R_1 < 2R_1 < \frac{1}{2}R_2 < 2R_2 < \frac{1}{2}R_3 < \ldots < 2R_{M-1} < \frac{1}{2}R_M < \xi_M^{1/\rho} \tag{3.12.20}$$

yields the existence (the signs of the function E_ρ at the points $-R_M/2$ and $-\xi_M^{1/\rho}$ are different) of no less than M zeros on the interval $\left(-\xi_M^{1/\rho}, 0\right)$. Then we obtain that on the interval $\left(-\xi_n^{1/\rho}(\rho,\mu), 0\right)$ for any $n \ge M$, the function $E_\rho(z;\mu)$ has no less than n distinct zeros. Applying the theorem from Chap. 2 that asserts that in the disk $|z| \le \xi_n^{1/\rho}(\rho,\mu), 0 < \rho < 1/2$ (where $n \in \mathbb{N}$ is an arbitrary sufficiently large number), the function $E_\rho(z;\mu)$ has exactly n zeros, we conclude that it has no other zeros except for real and negative ones. If we prove inequalities (3.12.20), then the proof of Theorem 3.1.5 will be complete.

The inequality $R_M < \xi_M^{1/\rho}$ is proved in Lemma 3.1.1 since

$$R_M(\rho,\mu) = R_{M-1}(\rho,\lambda), \quad \xi_M(\rho,\mu) = \xi_{M-1}(\rho,\lambda), \quad \text{where} \quad \lambda = \mu + \frac{1}{\rho}.$$

The inequality $4R_N < R_{N+1}$ for $n = 1$ was proved in the lemma, and for $2 \le n \le M-1$ ($m = n-1$, $(m+1)\rho \le 1/3$), by Lemma 3.9.1, we have the inequality

$$\frac{R_n(\rho,\mu)}{R_{n+1}(\rho,\mu)} = \frac{R_m(\rho,\lambda)}{R_{m+1}(\rho,\lambda)} < \exp\left(-\frac{1}{\rho(m+\rho\lambda)}\right) < \exp\left(-\frac{1}{\rho(m+1)}\right) < e^{-3} < \frac{1}{20}.$$

We prove Eqs. (3.12.18). From the power expansion of the Mittag-Leffler function we obtain

$$E_\rho\left(-\frac{R_n}{2};\mu\right) = S_{n,0} + \left(\frac{(-1)^{n-1}}{2}A_{n-1}\left(\frac{R_n}{2}\right)^{n-1} + (-1)^n A_n\left(\frac{R_n}{2}\right)^n\right) + S_{n,1}, \tag{3.12.21}$$

where

$$S_{n,0} = \sum_{k=0}^{n-2}(-1)^k A_k\left(\frac{R_n}{2}\right)^k + \frac{(-1)^{n-1}}{2}A_{n-1}\left(\frac{R_n}{2}\right)^{n-1}, \quad S_{n,1} = \sum_{k=n+1}^{\infty}(-1)^k A_k\left(\frac{R_n}{2}\right)^k.$$

From the definition of $R_n = A_{n-1}/A_n$ we conclude that the parenthesized expression on the right-hand side of (3.12.21) vanishes. Therefore,

$$E_\rho\left(-\frac{R_n}{2};\mu\right) = S_{n,0} + S_{n,1}. \tag{3.12.22}$$

The moduli of terms of the alternating sum $S_{n,1}$ decrease; indeed, the ratio of any subsequent term to the preceding is equal to

$$\frac{A_{k+1}(R_n/2)^{k+1}}{A_k(R_n/2)^k} = \frac{R_n}{2R_{k+1}} < \frac{R_n}{R_{n+1}} < 1$$

since the sequence R_n increases. Therefore, the sign of the sum $S_{n,1}$ coincides with the sign of its maximal (in modulus) term:

$$\operatorname{sgn} S_{n,1} = (-1)^{n+1} = (-1)^{n-1}. \tag{3.12.23}$$

The moduli of terms of the alternating sum $S_{n,0}$ increase; we can similarly verify that the maximal ratio of a subsequent term to the previous is equal to $4R_{n-1}/R_n < 1$. Therefore, the sign of the sum $S_{n,0}$ coincides with the sign of its maximal (in modulus) term:

$$\operatorname{sgn} S_{n,0} = (-1)^{n-1}. \tag{3.12.24}$$

From (3.12.22)–(3.12.24) we obtain (3.12.18).

Prove equalities (3.12.19). Rejecting in the expansion of $E_\rho(-2R_n;\mu)$ the part

$$(-1)^{n-1}A_{n-1}\,(2R_n)^{n-1} + \frac{(-1)^n}{2}A_n\,(2R_n)^n,$$

which is equal to zero, we obtain the representation

$$E_\rho(-2R_n;\mu) = \sigma_{n,0} + \sigma_{n,1}, \tag{3.12.25}$$

where

$$\sigma_{n,0} = \sum_{k=0}^{n-2}(-1)^k A_k\,(2R_n)^k, \quad \sigma_{n,1} = \frac{(-1)^n}{2}A_n\,(2R_n)^n + \sum_{k=n+1}^{\infty}(-1)^k A_k\,(2R_n)^k.$$

We can easily verify that the moduli of terms of the sum $\sigma_{n,0}$ increase and hence

$$\operatorname{sgn}\sigma_{n,0} = (-1)^{n-2} = (-1)^n. \tag{3.12.26}$$

The maximal ratio of moduli of terms of the sum $\sigma_{n,1}$ is equal to

$$\frac{4R_n}{R_{n+1}} < 4\exp\left(-\frac{1}{\rho(n+\rho\mu)}\right) < 4\exp\left(-\frac{1}{n\rho}\right) \le 4e^{-2};$$

here we have use Lemma 3.9.1 and the inequality

$$-\frac{1}{n\rho} \le -2 \quad \Longleftrightarrow \quad 2n\rho \le 1,$$

which follows from the estimate

$$2\rho\left(1+\left[\frac{1}{3\rho}\right]\right) \le 2\rho\left(1+\frac{1}{3\rho}\right) = 2\rho + \frac{2}{3} \le 1, \quad 0<\rho\le\frac{1}{6}.$$

This means that $4R_n/R_{n+1} < 1$ and hence the moduli of terms of the sum $\sigma_{n,1}$ decrease and $\operatorname{sgn}\sigma_{n,1} = (-1)^n$. From this, (3.12.25), and (3.12.26) we obtain (3.12.19).

It remains to prove inequalities (3.12.13) for any even $m \ge [1/(3\rho)]$. By the method applied to the case $1/6 < \rho \le 1/4$ and reasoning similarly to Sec. 3.8, we reduce the problem to the proof of the inequality

$$0.5\xi_m^{-1/\rho} + 0.74\xi_m^{-\lambda} < 1.92\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho), \quad m \ge \left[\frac{1}{3\rho}\right].$$

Replace this inequality by the following stronger inequality:

$$1.24 < 1.92\rho\xi_m^{1-\lambda}\exp(\xi_m\cos\pi\rho). \tag{3.12.27}$$

Since

$$\lambda = \mu + \frac{1}{\rho} \ge -2 + \frac{1}{\rho} \ge 4 > 1,$$

we have

$$\rho\xi_m = \frac{\pi\rho\big(m+\rho(\lambda-1)\big)}{\sin\pi\rho} > \frac{\pi\rho m}{\sin\pi\rho} > m.$$

Since $\exp(\xi_m\cos\pi\rho) > 1$, inequality (3.12.27) is valid. Theorem 3.1.5 is proved.

Chapter 4

NONASYMPTOTIC PROPERTIES OF ZEROS

4.1. Real zeros

In this section, for real μ we solve the problem on the number of positive (for $\rho \ge 1$) and negative (for $\rho > 1$) roots of the function $E_\rho(z;\mu)$. Here and in the sequel, we use the term "root" instead of "zero" to avoid the inconsistent expressions "positive zero" and "negative zero."

4.1.1. Negative roots.

Theorem 4.1.1. *Let $\rho > 1$. Then the following assertions hold*:

(1) *if*

$$\mu \in \left(\bigcup_{n=0}^{\infty}\left[-n+\frac{1}{\rho}, -n+1\right]\right) \cup [1,+\infty)$$

then the function $E_\rho(z;\mu)$ has no negative roots;

(2) *if*

$$\mu \in \bigcup_{n=0}^{\infty} \left(-n, -n + \frac{1}{\rho}\right),$$

then the function $E_\rho(z;\mu)$ has a unique negative root, and this root is simple.

The case $\rho = 1$ will be considered in Sec. 4.5.

Proof. The proof is based on formulas (1.1.14), (1.1.12), and (1.1.10), which holds for $\rho > 1$ and $\mu < 1 + 1/\rho$. For $z = -r$, we rewrite these formulas in the form

$$E(r) := \frac{1}{\rho} r^{\rho(\mu-1)} E_\rho(-r;\mu) = \int\limits_0^\infty e^{-(tr)^\rho} t^{\rho(1-\mu)} g(t)dt, \quad r > 0, \tag{4.1.1}$$

$$g(t) = \frac{t \sin \pi\mu + \sin \pi(\mu - 1/\rho)}{\pi(t^2 + 2t\cos \pi/\rho + 1)}, \quad t > 0.$$

We see that the number $-r_0$ (where $r_0 > 0$) is a negative root of the function $E_\rho(z;\mu)$ if and only if the number r_0 is a positive root of the function $E(r)$ of the same multiplicity.

(1) If $-n + 1/\rho \le \mu \le -n + 1$, $n \in \mathbb{Z}_+$, then either the numbers

$$\sin \pi\mu, \quad \sin \pi \left(\mu - \frac{1}{/}\rho\right) \tag{4.1.2}$$

have the same signs or one of them is equal to zero. Therefore, for such μ the function $g(t)$ preserves its sign on $\mathbb{R}_+$, namely, $\operatorname{sign} g(t) = (-1)^n$, $t > 0$. By (4.1.1),

$$\operatorname{sign} E_\rho(x;\mu) = (-1)^n, \quad x < 0, \quad -n + \frac{1}{\rho} \le \mu \le -n + 1, \quad n \in \mathbb{Z}_+, \tag{4.1.3}$$

and the function $E_\rho(z;\mu)$ does not vanish on the negative real half-line. In assertion (1), it remains to consider values $\mu > 1$.

Let $1 < \mu < 1 + 1/\rho$. Then $E_\rho(0;\mu) > 0$. Theorem 1.2.1 implies that $E_\rho(x;\mu) > 0$ also in an appropriate neighborhood of the point $-\infty$. Therefore, either $E(r)$ has no positive roots or the number of them (with account of their multiplicities) is even.

Assuming that r_0 is a root of $E(r)$, we consider the function

$$G(t) = \int\limits_t^\infty e^{-(r_0 t)^\rho} t^{\rho(1-\mu)} g(t)dt, \quad t \ge 0. \tag{4.1.4}$$

Then $G(0) = 0$. Since $1 < \mu < 1 + 1/\rho$, the first number of (4.1.2) is negative and the second is positive. Therefore, $g(t)$ has a unique positive root t_0 and, moreover,

$$g(t) > 0 \;\text{ for }\; 0 < t < t_0 \quad , g(t) < 0 \;\text{ for }\; t > t_0. \tag{4.1.5}$$

From (4.1.4) we see that

$$\operatorname{sign} G'(t) = -\operatorname{sign} g(t)$$

and hence $G(t)$ decreases on $[0, t_0]$ and increases on $[t_0, \infty)$. But $G(0) = 0$ and $G(t) < 0$ for $t > t_0$. Therefore, $G(t) < 0$ for $t > 0$.

From (4.1.1) we obtain that

$$E'(r_0) = \rho r_0^{\rho-1} \int\limits_0^\infty t^\rho dG(t).$$

Integrating by parts, we obtain the formula

$$E'(r_0) = -\rho^2 r_0^{\rho-1} \int_0^\infty t^{\rho-1} G(t) dt. \tag{4.1.6}$$

This and the negativeness of $G(t)$ imply that the derivative of the function $E(r)$ at its possible positive root is necessarily positive. However, this is impossible since the number of positive roots of $E(r)$ is even. Therefore, for $1 < \mu < 1 + 1/\rho$ the function $E_\rho(z;\mu)$ has no negative roots; moreover, $E_\rho(x;\mu) > 0$ for $x < 0$.

The last inequality and formula (4.1.3) with $n = 0$ show that

$$E_\rho(x;\mu) > 0 \quad \text{for} \quad x < 0 \tag{4.1.7}$$

if $1/\rho \le \mu < 1 + 1/\rho$. This and the formula

$$E_\rho(z;\mu) = \mu E_\rho(z;\mu+1) + \frac{z}{\rho} E'_\rho(z;\mu+1) \tag{4.1.8}$$

imply that if $1/\rho \le \mu < 1+1/\rho$, then the derivative of the function $E_\rho(z;\mu+1)$ at its possible negative root is negative. Again, the function $E_\rho(z;\mu+1)$ is positive at the origin and (by Theorem 1.2.1) in a neighborhood of the point $-\infty$. Therefore, the function $E_\rho(z;\mu+1)$ has no negative roots for $1/\rho \le \mu < 1 + 1/\rho$, i.e., the function $E_\rho(z;\mu)$ has no negative roots for $1 + 1/\rho \le \mu < 2 + 1/\rho$; moreover, such μ satisfy property (4.1.7). Repeating this reasoning, we obtain property (4.1.7) for $2 + 1/\rho \le \mu < 3 + 1/\rho$, etc. Finally, we obtain the absence of negative roots of the function $E_\rho(z;\mu)$ for $\mu > 1$. Assertion (1) is proved.

(2) Let $\mu \in (-n, -n+1)$ and $n \in \mathbb{Z}_+$. Then

$$\operatorname{sign} E_\rho(0;\mu) = \operatorname{sign} \Gamma(\mu) = (-1)^n,$$

and by Theorem 1.2.1

$$\operatorname{sign} E_\rho(x;\mu) = \operatorname{sign} \Gamma\left(\mu - \frac{1}{\rho}\right) = (-1)^{n-1}, \quad x < -R,$$

for sufficiently large $R > 0$. Therefore, at the origin and in a neighborhood of the point $-\infty$ the function $E_\rho(x;\mu)$ has values of opposite signs; therefore, it has at least one negative root. We must prove that it is unique and simple.

Now numbers (4.1.2) have opposite signs and

$$\operatorname{sign} g(t) = \operatorname{sign} \sin \pi\mu = (-1)^n, \quad t > t_0,$$

where t_0 is a unique root of the function $g(t)$. Reasoning as in the case $1 < \mu < 1 + 1/\rho$, we conclude that

$$\operatorname{sign} G(t) = (-1)^n, \quad t > 0.$$

By (4.1.6), the derivative of the function $E(r)$ at each of its positive roots has the same sign, which is possible only if the root is unique and simple. Assertion (2) is proved. Theorem 4.1.1 is now completely proved. □

Remark 4.1.1. In the sequel, it will be important that property (4.1.7) holds for $\rho > 1$ and $1 < \mu < 1 + 1/\rho$ (this was established in the proof of Theorem 4.1.1).

4.1.2. Positive roots. First, we recall some facts that immediately follow from the definition: for $\mu \geq 0$, the function $E_\rho(x;\mu)$ is positive on the half-line $x > 0$ and, therefore, has no positive roots, and for $\mu = -m$, $m \in \mathbb{Z}_+$, it has a root at the point $x = 0$: simple if $\rho > 1$ and $(m+1)$-multiple if $\rho = 1$.

Theorem 4.1.2.

(I). *Let $\rho > 1$ and $\mu < 0$. Then the following assertions hold*:

(1) *if $\mu \in [-2n-1, -2n)$, $n \in \mathbb{Z}_+$, then the function $E_\rho(z;\mu)$ has a unique positive simple root*;

(2) *if $\mu \in [-2n, -2n+1)$, $n \in \mathbb{N}$, then one of the following possibilities is realized: the set of positive roots of the function $E_\rho(z;\mu)$ is empty, or consists of two simple roots, or consists of one double root; more precisely, there exist sequences ν_n and μ_n, $n \in \mathbb{N}$, such that $-2n + 1/\rho < \nu_n \leq \mu_n < -2n+1$ and*

(a) *for $-2n \leq \mu < \nu_n$, the function $E_\rho(z;\mu)$ has no positive roots*;

(b) *for $\mu_n < \mu < -2n+1$, the function $E_\rho(z;\mu)$ has two simple positive roots*;

(c) *for $\mu = \nu_n, \mu_n$, the function $E_\rho(z;\mu)$ has a double positive root.*

(II). *The function $E_1(z;\mu)$ has a unique positive simple root if $\mu \in (-2n-1, -2n)$, $n \in \mathbb{Z}_+$, and has no positive roots if $\mu \in [-2n, -2n+1]$, $n \in N$.*

Proof. (I). From representations (1.1.16), (1.1.12), and (1.1.10) we obtain the formula

$$E(r) = E(r;\mu) := \frac{1}{\rho} r^{\rho(\mu-1)} E_\rho(r;\mu) = e^{r^\rho} + \int_0^\infty e^{-(rt)^\rho} t^{\rho(1-\mu)} g(t)\,dt, \quad r > 0, \tag{4.1.9}$$

which is valid for $\rho > 1$ and $\mu < 1 + 1/\rho$, where

$$g(t) = \frac{t \sin \pi\mu - \sin \pi(\mu - 1/\rho)}{\pi(t^2 - 2t\cos \pi/\rho + 1)}, \quad t > 0. \tag{4.1.10}$$

(1) Let $\mu \in (-2n-1, -2n)$, $n \in \mathbb{Z}_+$. Then $\Gamma(\mu) < 0$ and $E_\rho(0,\mu) < 0$; since $E_\rho(+\infty;\mu) = +\infty$ (this follows from the definition), the function $E(r)$ $(r > 0)$ has at least one root.

If $\mu = -2n-1$, then $1/\Gamma(\mu) = E_\rho(0;\mu) = 0$. But $\Gamma(\mu + 1/\rho) < 0$ (here the condition $\rho > 1$ is important) and hence $E'_\rho(0;\mu) < 0$. Therefore, $E_\rho(r;\mu) < 0$ for sufficiently small $r > 0$, and again $E(r)$ has at least one root.

We prove that for $\mu \in [-2n-1, -2n)$, the root of the function $E(r)$, $r > 0$, is unique and simple.

We keep notation (4.1.4) for $G(t)$, where r_0 is some positive root of $E(r)$. We claim that $G(t) < 0$ for $t > 0$.

Indeed, if $-2n-1 \leq \mu \leq -2n-1+1/\rho$, then the numbers

$$\sin \pi\mu, \quad -\sin \pi(\mu - 1/\rho) \tag{4.1.11}$$

are not positive and do not vanish simultaneously. Therefore, $g(t) < 0$ for $t > 0$, and hence $G(t) < 0$ for $t > 0$.

If $-2n-1+1/\rho < \mu < -2n$, then we set $r = r_0$ in (4.1.9). Since $E(r_0) = 0$, we obtain that $G(0) = -\exp(r_0^\rho) < 0$. Now the first of the numbers (4.1.11) is negative and the second is positive. Therefore, similarly to the case $1 < \mu < 1 + 1/\rho$ in Theorem 4.1.1, we see that inequalities (4.1.5) hold. This implies (see the reasonings after (4.1.5)) that $G(t) < 0$ for $t > 0$. The assertion on the sign of $G(t)$ is proved.

By (4.1.9),

$$E'(r) = E'(r;\mu) = \rho r^{\rho-1} \left(e^{r^\rho} - \int_0^\infty e^{-(rt)^\rho} t^{\rho(2-\mu)} g(t)\,dt \right). \tag{4.1.12}$$

Therefore,

$$E'(r_0) = \rho r_0^{\rho-1} \left(e^{r_0^\rho} + \int_0^\infty t^\rho dG(t) \right),$$

and after integration by parts we have the expression

$$E'(r_0) = \rho r_0^{\rho-1} \left(e^{r_0^\rho} - \rho \int_0^\infty t^{\rho-1} G(t) dt \right).$$

Since $G(t) < 0$, we obtain that the derivative of the function $E(r)$ at its positive root is positive, which is possible only in the case of a unique simple root. Assertion (1) is proved.

(2) Let $\mu \in [-2n, -2n+1)$, $n \in \mathbb{N}$. Then $\mu - 1 \in [-2n-1, -2n)$, and by assertion (1) the function $E_\rho(z; \mu - 1)$ has a unique positive simple root. By the formula

$$E'(r; \mu) = \rho r^{\rho-1} E(r; \mu - 1), \tag{4.1.13}$$

which follows from (4.1.12), (4.1.9), and (4.1.10), the derivative $E'(r; \mu)$ for $r > 0$ has a unique simple root. This and the property

$$E(+0; \mu) = E(+\infty; \mu) = +\infty$$

imply that either $E(r)$ has no roots or has two simple roots or has one double root. We have proved the first claim of assertion (2).

If $\mu \in [-2n, -2n + 1/\rho]$, then numbers (4.1.11) are nonnegative and do not vanish simultaneously. Therefore, $g(t) > 0$ for $t > 0$, and (4.1.9) implies that for such μ the function $E_\rho(z; \mu)$ has no positive roots.

Let $\mu \to -2n + 1 - 0$, where n is fixed. Then

$$E_\rho(z; \mu) \to E_\rho(z; -2n + 1)$$

uniformly in any disk. But the function $E_\rho(z; -2n+1)$ has a simple root at the point $z = 0$ and has no other roots in some neighborhood $U(0)$ of the origin. By the Hurwitz theorem, for all μ sufficiently close to $-2n+1$ and such that $\mu < -2n+1$, the function $E_\rho(z; \mu)$ has exactly one root in $U(0)$. By Theorem 4.1.1, for such values of μ the function $E_\rho(z; \mu)$ has no negative roots. If $U(0)$ contains a nonreal root z_1, then the conjugate root $\bar{z}_1$ would also lie in $U(0)$ (since $E_\rho(x; \mu)$ is real). Therefore, the neighborhood $U(0)$ must contain a simple positive root of the function $E(r)$. In this case, by the conclusion after formula (4.1.13), the function $E(r)$ has two simple positive roots if $\mu < -2n+1$ and μ is sufficiently close to $-2n+1$. Denote ny μ_n the infimum of such μ.

The following auxiliary assertion holds. Let, for some $\mu_0 \in (-2n, -2n+1)$, the function $E_\rho(x; \mu_0)$ possess one of the following properties:

(1) it has no positive roots;
(2) it has two simple positive roots.

Then for all μ from some (real) neighborhood of the point μ_0, the function $E_\rho(x; \mu)$ possesses the same property.

Indeed, we introduce the notation

$$m(\mu) = \min(E_\rho(x; \mu) : x \in [0, +\infty)).$$

Since

$$E_\rho(0; \mu_0) > 0, \quad E_\rho(+\infty; \mu_0) = +\infty,$$

we have

$$m(\mu_0) > 0 \quad \text{in the case (1)}, \quad m(\mu_0) < 0 \quad \text{in the case (2)}.$$

By the continuity of the function $m(\mu)$, the last inequalities are valid in some neighborhood of the point μ_0, and the assertion stated holds.

We prove that for $\mu = \mu_n$, the function $E_\rho(x;\mu)$ has a double positive root. Indeed, in the opposite case, by the conclusion stated after formula (4.1.13), the function $E_\rho(x;\mu_n)$ either has two positive simple roots or has no roots at all. By the auxiliary assertion, this is also valid for the function $E_\rho(x;\mu)$ from some neighborhood of the point μ_n. However, this contradict the sense of the value μ_n.

Similarly, if we denote by ν_n the supremum of numbers $\mu > -2n$ for which $E_\rho(z;\mu)$ has no positive roots, then $E_\rho(z;\nu_n)$ has a double positive root and $-2n + 1/\rho < \nu_n$. Obviously, $\nu_n \le \mu_n$, and Part (I) of Theorem 4.1.2 is proved.

(II). Let $\mu < 2$. Passing to the limit in formulas (4.1.9) and (4.1.10) as $\rho \to 1 + 0$, we obtain that formula (4.1.9) remains valid for $\rho = 1$ and formula (4.1.10) becomes

$$g(t) = \frac{\sin \pi\mu}{\pi(t+1)}.$$

If $\mu \in [-2n, -2n+1]$, $n \in \mathbb{N}$, then $g(t) \ge 0$ for $t > 0$, and by (4.1.9) we see that $E(r) > 0$ for $r > 0$. Therefore, for such μ, the function $E_\rho(z;\mu)$ has no positive roots.

If $\mu \in (-2n-1, -2n)$, $n \in \mathbb{Z}_+$, then

$$E_1(0;\mu) < 0, \quad E_1(+\infty;\mu) = +\infty.$$

Therefore, $E_1(x;\mu)$ has at least one positive root. The fact that this root is unique and simple can be proved as in assertion (1) of Part (I). The proof is based on the negativeness of the function $G(t)$ for $t > 0$, but this property holds in the present case since $g(t) < 0$ for $t > 0$ with the specified values of μ. We have verified Part (II). Theorem 4.1.2 is completely proved. □

Remark 4.1.2. Assertion (2) of Part (I) characterizes positive roots for $\mu \in [-2n, -2n+1)$, $n \in \mathbb{N}$ incompletely. We venture to suggest that $\nu_n = \mu_n$ as a matter of fact.

4.1.3. Behavior of real roots. Let $\rho > 1$. For $\mu \in (-n, -n+1/\rho)$, $n \in \mathbb{Z}_+$, by Theorem 4.1.1, the function $E_\rho(z;\mu)$ has a unique negative root; we denote it by $x_-(\mu)$. For $\mu \in [-2n-1, -2n)$, $n \in \mathbb{Z}_+$, by Theorem 4.1.2, the function $E_\rho(z;\mu)$ has a unique positive simple root; we denote it by $x_+(\mu)$. The two positive simple roots $E_\rho(z;\mu)$ in the case $\mu \in (\mu_n, -2n+1)$ are denoted by $x_+^1(\mu)$ and $x_+^2(\mu)$; assume that $x_+^1(\mu) < x_+^2(\mu)$. The double positive root of the function $E_\rho(z;\mu)$ in the case $\mu = \mu_n$ is denoted by $x_+^*(\mu_n)$. We keep the notation $x_+(\mu)$ in the case $\rho = 1$.

Theorem 4.1.3. (1) *Let $\rho > 1$. Then the following assertions hold*:

$$\begin{aligned}
&x_+(\mu) \to 0 && \text{as } \mu \to -2n-0, && n \in \mathbb{Z}_+,\\
&x_+(\mu) \to x_+(-2n-1) && \text{as } \mu \to -2n-1+0, && n \in \mathbb{Z}_+,\\
&x_-(\mu) \to 0 && \text{as } \mu \to -n+0, && n \in \mathbb{Z}_+,\\
&x_-(\mu) \to -\infty && \text{as } \mu \to -n+\frac{1}{\rho}-0, && n \in \mathbb{Z}_+,\\
&x_+^1(\mu),\ x_+^2(\mu) \to x_+^*(\mu_n) && \text{as } \mu \to \mu_n+0, && n \in \mathbb{N},\\
&x_+^1(\mu) \to 0,\ x_+^2(\mu) \to x_+(-2n+1) && \text{as } \mu \to -2n+1-0, && n \in \mathbb{N}.
\end{aligned}$$

(2) *If $\rho = 1$, then*

$$x_+(\mu) \to 0 \quad \text{as } \mu \to -2n-1+0 \quad \text{and as } \mu \to -2n-0, \quad n \in \mathbb{Z}_+.$$

Proof. The point $x = 0$ is a simple root of the function $E_\rho(x;-2n)$. Therefore, for any sufficiently small $\varepsilon > 0$, the values $E_\rho(\pm\varepsilon;-2n)$ have opposite signs. By the continuity of the function $E_\rho(\pm\varepsilon;\mu)$ by the variable μ for all μ sufficiently close to $-2n$ and less than $-2n$, values of $E_\rho(\pm\varepsilon;\mu)$ have opposite signs and hence in an ε-neighborhood of the point $x = 0$, there exists a root of the function $E_\rho(x;\mu)$, which is obviously nonzero. By Theorem 4.1.1, for such μ, the function $E_\rho(x;\mu)$ has no negative roots. Therefore, the root in question is $x_+(\mu)$ and $x_+(\mu) \to 0$ as $\mu \to -2n - 0$.

In the case $\mu \to -2n - 1 + 0$, we replace a neighborhood of the point $x = 0$ by a neighborhood of the point $x_+(-2n - 1)$, and all arguments are similar (and even simpler since we do not need Theorem 4.1.1). We have proved the first two limit relations.

For similar reasons, for μ sufficiently close to $-n$ and such that $\mu > -n$, a punctured neighborhood of the point $x = 0$ contains a root of the function $E_\rho(x;\mu)$. For even n, it cannot be positive by Theorem 4.1.2, and for odd n, a unique positive root of $E_\rho(x;\mu)$ lies in a neighborhood of the point $x_+(-2n - 1)$ by the second limit relation of Theorem 4.1.3. Therefore, this root is $x_-(\mu)$, and $x_-(\mu) \to 0$ as $\mu \to -n + 0$. We have proved the third limit relation.

Let $\mu \to -n + 1/\rho - 0$. Then

$$E_\rho(x;\mu) \to E_\rho(x; -n + 1/\rho)$$

uniformly on each segment. By Theorem 4.1.1, the function $E_\rho(x; -n + 1/\rho)$ has no negative roots; obviously, the point $x = 0$ is not a root. Assume that there exist $R > 0$ and a sequence $\mu_j \to -n + 1/\rho - 0$ such that

$$-R \le x_-(\mu_j) < 0.$$

Then, extracting from the sequence $x_-(\mu_j)$ a converging subsequence and using the uniform convergence, we conclude that its limit x_- is a root of the function $E_\rho(x; -n + 1/\rho)$, which is impossible since $x_- \in [-R, 0]$. This means that the fourth limit relation of Theorem 4.1.3 is valid.

The fifth limit relation follows from the Hurwitz theorem. The proof of the sixth relation is similar to that of the first two. Part (1) of Theorem 4.1.3 is proved.

Part (2) can be proved similarly to the first limit relation of part (1). Theorem 4.1.3 is completely proved. □

Theorems 4.1.1 and 4.1.2 imply the following assertion.

Corollary 4.1.1. *For $\rho > 1$ and $\mu \in \mathbb{R}$, the function $E_\rho(z;\mu)$ has no real roots of multiplicity higher than* 2.

4.2. Distribution of Roots in Angles

4.2.1. Absence of roots on the right-hand angle. Let $\rho > 1/2$, μ be real, and $\mu \le 1 + 1/\rho$. Then by Theorem 2.1.1, all sufficiently large (in modulus) roots of the function $E_\rho(z;\mu)$ lie outside the angle

$$|\arg z| \le \frac{\pi}{2\rho}. \tag{4.2.1}$$

In this section, we describe a possibly wide set of pairs of the parameters $\rho > 1/2$ and $\mu \in \mathbb{R}$ such that *all* roots of the function $E_\rho(z;\mu)$ lie outside the angle (4.2.1). Theorem 2.1.1 shows that the condition $\mu \le 1 + 1/\rho$ is necessary.

Theorem 4.2.1.

(I). *Let one of the following conditions hold*:

(1) $\rho > 1$, $\mu \in [1, 1 + 1/\rho]$;

(2) $1/2 < \rho < 1$, $\mu \in [1/\rho - 1, 1] \cup [1/\rho, 2]$.

Then all roots of the function $E_\rho(z;\mu)$ lie outside the angle (4.2.1).

In particular, all roots of the classical Mittag-Leffler function $E_\rho(z;1)$, $\rho > 1/2$, $\rho \ne 1$, lie outside the angle (4.2.1).

(II). *If $1/2 < \rho < 1$ and $\mu = 0$, then all nonzero roots of the function $E_\rho(z;\mu)$ lie outside the angle* (4.2.1).

The case $\rho = 1$ will be considered in detail below.

Lemma 4.2.1. *Let $f(x)$, $x > 0$, be a nontrivial real-valued function of class $L^1(\mathbb{R}_+)$, which does not change sign on $\mathbb{R}_+$. Then its Laplace transform*

$$F(z) = \int_{\mathbb{R}_+} e^{-zv} f(v)dv, \quad \operatorname{Re} z \geq 0, \tag{4.2.2}$$

is continuous in the closed half-plane $\operatorname{Re} z \geq 0$ *and possesses the following property*:

$$|F(z)| < |F(0)|, \quad \operatorname{Re} z \geq 0, \quad z \neq 0.$$

Proof of Lemma 4.2.1. The assertion on the continuity is a direct consequence of the condition $f \in L^1$. Since the function $f(x)$ does not change sign, the inequality

$$|F(z)| < |F(0)|, \quad \operatorname{Re} z > 0,$$

is obvious, and it remains to prove that

$$|F(iy)| < |F(0)|, \quad y \in \mathbb{R}, \quad y \neq 0.$$

Assume the contrary, i.e., $|F(iy)| = |F(0)|$ for some $y \neq 0$. Then

$$F(iy) = F(0)e^{i\beta}, \quad \beta \in \mathbb{R},$$

and (4.2.2) and the fact that f is real-valued imply that

$$0 = \operatorname{Re}(F(0) - e^{-i\beta}F(iy)) = \operatorname{Re}\int_{\mathbb{R}_+}(1 - e^{-i(yv+\beta)})f(v)dv = \int_{\mathbb{R}_+}(1 - \cos(yv+\beta))f(v)dv.$$

Since the integrand in the last integral does not change sign and is nontrivial, we arrive at a contradiction. The lemma is proved. □

Proof of Theorem 4.2.1. We use the following representation: if $\rho > 1/2$, $\mu < 1 + 1/\rho$, and $\alpha = 1/\rho$, then

$$z^{\mu-1}E_\rho(z^\alpha;\mu) = \rho e^z + F_{\alpha\mu}(z), \quad \operatorname{Re} z > 0, \tag{4.2.3}$$

where

$$F_{\alpha\mu}(z) = \frac{1}{\pi}\int_{\mathbb{R}_+} e^{-zv} v^{\alpha-\mu} \frac{\sin\pi(\alpha-\mu) + v^\alpha \sin\pi\mu}{v^{2\alpha} + 1 - 2v^\alpha\cos\pi\alpha} dv.$$

Since in both sides of formula (4.2.3) we have functions that are analytic in the half-plane $\operatorname{Re} z > 0$, it suffices to prove the representation for $z = x > 0$.

Let an arbitrary $x > 0$ be fixed. Then applying formula (1.1.1) with $\alpha_2 = -\alpha_1 = \pi$, $\sigma > x$, $z = x^\alpha$, and $\alpha = 1/\rho$, and then the residue theorem, we have the equality

$$E_\rho(x^\alpha;\mu) = \rho e^x x^{1-\mu} + \frac{1}{2\pi i}\int_\gamma \frac{e^t t^{\alpha-\mu} dt}{t^\alpha - x^\alpha}, \quad x > 0,$$

where γ is the negative real half-line bypassed in mutually opposite directions. Further, setting $t = -u$, we first represent the integral over γ as two integrals over the positive real half-line and then, combining them by a simple calculation (similar to that used in the proof of Theorem 1.1.2) and the change of variables $u = vx$, we obtain the required representation.

Let

$$\rho > \frac{1}{2}, \rho \neq 1, \quad \mu \in [1,2] \cap \left[\frac{1}{\rho}, 1 + \frac{1}{\rho}\right)$$

or

$$\frac{1}{2} < \rho < 1, \quad \mu = 0.$$

Then the function

$$f(v) = \frac{1}{\pi} v^{\alpha-\mu} \frac{\sin \pi(\alpha - \mu) + v^{\alpha} \sin \pi\mu}{v^{2\alpha} + 1 - 2v^{\alpha} \cos \pi\alpha}, \quad v > 0,$$

satisfies all the conditions of Lemma 4.2.1. By this lemma,

$$|F_{\alpha\mu}(z)| < |F_{\alpha\mu}(0)|, \quad \operatorname{Re} z \geq 0, \quad z \neq 0.$$

This and (4.2.3) imply that

$$\left| z^{\mu-1} E_\rho(z^\alpha; \mu) \right| \geq \left| \rho e^z \right| - \left| F_{\alpha\mu}(z) \right| > \rho - \left| F_{\alpha\mu}(0) \right|, \quad \operatorname{Re} z \geq 0, \quad z \neq 0. \tag{4.2.4}$$

To find $F_{\alpha\mu}(0)$, we use formula (4.2.3). Namely, we substitute $z = 0$ in (4.2.3) if $\mu \neq 0$ or pass to the limit as $z \to 0$ if $\mu = 0$. Since $E_\rho(0; \mu) = 1/\Gamma(\mu)$ and

$$E_\rho(z^\alpha; 0) \sim \frac{z^\alpha}{\Gamma(1/\rho)}, \quad z \to 0, \quad \alpha = \frac{1}{\rho} > 1,$$

we obtain that

$$F_{\alpha\mu}(0) = -\rho \quad \text{for } \mu > 1 \text{ and } \mu = 0, \qquad F_{\alpha\mu}(0) = 1 - \rho \quad \text{for } \mu = 1. \tag{4.2.5}$$

Since the resulting inequality (4.2.4) is strong, we see that the function $E_\rho(z^\alpha; \mu)$ has no zeros on the set $\operatorname{Re} z \geq 0$, $z \neq 0$, i.e., the function $E_\rho(z; \mu)$ has no zeros in the angle (4.2.1) with punctured point $z = 0$. We have proved Part (II) of the theorem. Since $E_\rho(0; \mu) \neq 0$ for $\mu \neq 0$, we have also proved the absence of zeros in the angle (4.2.1) for pairs of the parameters ρ and μ for which $\mu \neq 0$.

To prove part (I) of the theorem relating to condition (1), it remains to consider the value $\mu = 1 + 1/\rho$ for $\rho > 1$. For this, we note that (4.2.4) and (4.2.5) for $\mu = 1$ imply the inequality

$$|E_\rho(z, 1)| > \rho - |1 - \rho| = 1, \quad |\arg z| \leq \frac{\pi}{2\rho}, \quad z \neq 0.$$

Applying it to the right-hand side of the formula

$$z E_\rho\left(z; 1 + \frac{1}{\rho}\right) = E_\rho(z, 1) - 1,$$

we see that the function $E_\rho(z; \mu)$ does not vanish in the angle (4.2.1). Case (1) is completely considered.

We have also considered the part of case (2) relating to the values $\mu \in [1/\rho, 2]$. To extend it to the missing values $\mu \in [1/\rho - 1, 1]$, we need the following "conditional" lemma.

Lemma 4.2.2. *Let $0 < \rho < 1$ and $\mu > 1$. If all zeros a_k of the function $E_\rho(z; \mu)$ lie outside the angle*

$$|\arg z| \leq \beta, \quad \frac{\pi}{2} \leq \beta < \pi,$$

then all zeros of the function $E_\rho(z; \mu - 1)$ also lie outside this angle.

The proof of Lemma 4.2.2 is based on the following auxiliary assertion.

Let $G(z)$ be a nontrivial meromorphic function of the form

$$G(z) = \frac{\gamma_0}{z} + \sum_{k=1}^{\infty} \frac{\gamma_k}{z - a_k},$$

where

$$\gamma_k \geq 0, \quad k \in \mathbb{Z}_+, \quad \sum_{k=1}^{\infty} \frac{\gamma_k}{|a_k|} < \infty.$$

If the sequence $(a_k)_{k=1}^{\infty}$ is contained in the angle

$$A_{\eta,\delta} := (z : \eta < \arg z < \eta + \delta), \quad \eta \in [0, 2\pi), \quad 0 < \delta \leq \pi,$$

then the function $G(z)$ does not vanish outside $A_{\eta,\delta}$.

First, we verify this assertion in the case $\eta = 0$ and $\delta = \pi$, i.e., when $A_{\eta,\delta}$ is the open upper half-plane. We have the equality

$$\operatorname{Im} G(z) = -\gamma_0 \frac{\operatorname{Im} z}{|z|^2} - \sum_{k=1}^{\infty} \gamma_k \frac{\operatorname{Im} z - \operatorname{Im} a_k}{|z - a_k|^2}.$$

Since $\operatorname{Im} a_k > 0$ for $k \geq 1$, we see that $\operatorname{Im} G(z) > 0$ for $\operatorname{Im} z \leq 0$, and the assertion is valid in this case.

Rotating the complex plane about the origin, we can show that the assertion is also valid for $\delta = \pi$ and any $\eta \in [0, 2\pi)$. This and the representation

$$A_{\eta,\,\delta} = A_{\eta,\,\pi} \cap A_{\eta+\delta+\pi,\,\pi}$$

imply that it is valid in the general case.

Now we prove Lemma 4.2.2. We use the well-known formula

$$E_\rho(z; \mu - 1) = (\mu - 1) E_\rho(z; \mu) + \frac{z}{\rho} E'_\rho(z; \mu)$$

(see [6]). Setting $\gamma = \rho(\mu - 1)$, we rewrite it in the form

$$\rho E_\rho(z; \mu - 1) = z E_\rho(z; \mu) \left(\frac{\gamma}{z} + \frac{E'_\rho(z; \mu)}{E_\rho(z; \mu)} \right). \tag{4.2.6}$$

Denote by $(a_k)_{k=1}^{\infty}$ the set of all zeros of the function $E_\rho(z; \mu)$. Since $E_\rho(z; \mu)$ is an entire function of order $\rho < 1$, we have

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|} < \infty$$

and by Hadamard's theorem

$$E_\rho(z; \mu) = \frac{1}{\Gamma(\mu)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k} \right).$$

Therefore,

$$\frac{E'_\rho(z; \mu)}{E_\rho(z; \mu)} = \sum_{k=1}^{\infty} \frac{1}{z - a_k}.$$

By the condition of the lemma

$$(a_k)_{k=1}^{\infty} \subset A_{\beta,\, 2(\pi - \beta)}.$$

Since $\beta \geq \pi/2$, we have $0 < 2(\pi - \beta) \leq \pi$. Moreover, $\gamma > 0$. Therefore, we can apply to the function

$$G(z) = \frac{\gamma}{z} + \frac{E'_\rho(z; \mu)}{E_\rho(z; \mu)}$$

the auxiliary assertion, which implies that $G(z)$ does not vanish in the angle $|\arg z| \leq \beta$. By (4.2.6), the function $E_\rho(z; \mu - 1)$ does not vanish in this angle, and Lemma 4.2.2 is proved.

Now the missing part of case (2) follows from the fact that it is valid for $\mu \in [1/\rho, 2]$ and Lemma 4.2.2. Theorem 4.2.1 is proved. □

4.2.2. Absence of zeros on the left-hand angle. In this section, we examine the condition on pairs of the parameters $\rho > 1$ and $\mu \in \mathbb{R}$ for which the function $E_\rho(z;\mu)$ has no zeros in the angle

$$\frac{\pi}{\rho} \le |\arg z| \le \pi. \tag{4.2.7}$$

Denote by A_ρ ($\rho > 1$) the set of all $\mu \in \mathbb{R}$ such that all nonzero roots of the function $E_\rho(z;\mu)$ lie outside the angle (4.2.7).

Theorem 4.2.2. *The following inclusions hold*:

$$(1)\quad A_\rho \subset \left(\bigcup_{n=0}^{\infty}\left[-n+\frac{1}{\rho}, -n+1\right]\right) \cup [1,+\infty)$$

$$(2)\quad A_\rho \supset \left[1, 1+\frac{1}{\rho}\right] \cup (-\mathbb{Z}_+) \cup \left(-\mathbb{Z}_+ + \frac{1}{\rho}\right);$$

$$(3)\quad A_2 \supset \left(\bigcup_{n=0}^{\infty}\left[-n+\frac{1}{2}, -n+1\right]\right) \cup \left[1, \frac{3}{2}\right].$$

For fixed $R, \rho > 0$, we introduce the notation

$$I(x) = \int_0^\infty e^{-Rt^\rho}\frac{t-x}{t^2-2tx+1}dt, \quad x \le 1, \tag{4.2.8}$$

where for $x = 1$ the integral is understood as singular.

Lemma 4.2.3. *The function $I(x)$ decreases for $x \le 1$.*

Proof. If $x < 1$, then $t^2 - 2tx + 1 > 0$ and, integrating by parts, we obtain

$$I(x) = \frac{1}{2}\int_0^\infty e^{-Rt^\rho} d\ln(t^2-2tx+1) = \frac{R\rho}{2}\int_0^\infty e^{-Rt^\rho}t^{\rho-1}\ln(t^2-2tx+1)dt. \tag{4.2.9}$$

Obviously, the right-hand side decreases on the half-line $x \le 1$. However, now we know only that it coincides with $I(x)$ for $x < 1$. Hence it remains to verify that the formal substitution of $x = 1$ on the right-hand side of (4.2.9) yields the singular integral $I(1)$. Denoting by $B(\varepsilon)$ the set $\mathbb{R}_+ \setminus (1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, by the definition of a singular integral we have the equality

$$I(1) = \lim_{\varepsilon\to 0}\int_{B(\varepsilon)} e^{-Rt^\rho}\frac{dt}{t-1} = \lim_{\varepsilon\to 0}\int_{B(\varepsilon)} e^{-Rt^\rho} d\ln|t-1|.$$

Integrating by parts, we obtain

$$I(1) = \lim_{\varepsilon\to 0}\left(\left(e^{-R(1-\varepsilon)^\rho} - e^{-R(1+\varepsilon)^\rho}\right)\ln\varepsilon + \frac{R\rho}{2}\int_{B(\varepsilon)} e^{-Rt^\rho}t^{\rho-1}\ln(t-1)^2dt\right)$$
$$= \frac{R\rho}{2}\int_0^\infty e^{-Rt^\rho}t^{\rho-1}\ln(t-1)^2dt;$$

moreover, the last integral exists in the ordinary sense. Comparing this with (4.2.9), we see that the right-hand side in (4.2.9) is $I(1)$. The lemma is proved. ☐

Lemma 4.2.4. *Let a function $H(z)$ be harmonic and bounded in the angle $\theta_0 < \arg z = \theta < \theta_1$ and continuous for $\theta_0 \le \theta \le \theta_1$. Moreover, let $H = 0$ on one side of the angle and $H > 0$ on the other. Then $H(z) > 0$ in this angle.*

Proof. If this angle is the half-plane $0 < \theta < \pi$, then the lemma follows from the representation of the function $H(z)$ by the Poisson integral of $H(x)$ and the positiveness of the Poisson kernel.

The general case is reduced to this by a conformal mapping of the angle to the half-plane. The lemma is proved. □

Proof of Theorem 4.2.2. (1) This assertion follows from Theorem 4.1.1.

Theorem 4.1.1 and the fact that zeros are complex conjugate imply that in the proof of assertions (2) and (3), it suffices to consider the angle $\pi/\rho \le \theta < \pi$ instead of angle (4.2.7).

We apply Theorem 1.1.2, which for $\rho > 1$ and $\mu < 1 + 1/\rho$ yields the following representations:

$$E_\rho(z;\mu) = F_\rho(z;\mu), \quad \frac{\pi}{\rho} < |\arg z| \le \pi, \tag{4.2.10}$$

$$E_\rho(z,\mu) = F_\rho(z;\mu) + \frac{\rho}{2} z^{\rho(1-\mu)} e^{z^\rho}, \quad \arg z = \pm\frac{\pi}{\rho}, \tag{4.2.11}$$

$$E_\rho(z;\mu) = F_\rho(z;\mu) + \rho z^{\rho(1-\mu)} e^{z^\rho}, \quad |\arg z| < \frac{\pi}{\rho}, \tag{4.2.12}$$

where

$$F_\rho(z;\mu) = \frac{\rho}{2\pi i} \int_0^\infty e^{-t^\rho} t^{\rho(1-\mu)} f_\rho(t;z,\mu)dt, \tag{4.2.13}$$

$$f_\rho(t;z,\mu) = \frac{e^{-i\pi\mu}}{ze^{-i\pi/\rho} - t} - \frac{e^{i\pi\mu}}{ze^{i\pi/\rho} - t}. \tag{4.2.14}$$

In the sequel, we write $f(t)$ instead of $f_\rho(t;z,\mu)$.

(2) Let $\mu = m = 1, 0, -1, \ldots$. Then by formula (4.2.14) we have

$$(-1)^{m-1} f(t) = \frac{1}{ze^{i\pi/\rho} - t} - \frac{1}{ze^{-i\pi/\rho} - t},$$

$$(-1)^{m-1} \operatorname{Re} f(t) = \frac{r\cos(\theta + \pi/\rho) - t}{r^2 - 2tr\cos(\theta + \pi/\rho) + t^2} - \frac{r\cos(\theta - \pi/\rho) - t}{r^2 - 2tr\cos(\theta - \pi/\rho) + t^2}.$$

Denoting

$$a = \cos\left(\theta - \frac{\pi}{\rho}\right), \quad b = \cos\left(\theta + \frac{\pi}{\rho}\right), \quad R = r^\rho,$$

by formula (4.2.13) we have the equality

$$\begin{aligned}(-1)^{m-1} \operatorname{Im} F_\rho(z;\mu) &= -\frac{\rho}{2\pi} \int_0^\infty e^{-Rt^\rho} \operatorname{Re} f(t)dt \\ &= \frac{\rho}{2\pi} \int_0^\infty e^{-Rt^\rho} \left(\frac{t-b}{t^2 - 2bt + 1} - \frac{t-a}{t^2 - 2at + 1} \right) dt = \frac{\rho}{2\pi}(I(b) - I(a))\end{aligned}$$

in notation (4.2.8); moreover, $a, b \le 1$. We have the formula

$$a - b = 2\sin\theta \sin\frac{\pi}{\rho} > 0 \quad \text{for} \quad 0 < \theta < \pi.$$

By Lemma 4.2.3,

$$(-1)^{m-1} \operatorname{Im} F_\rho(z;\mu) > 0, \quad 0 < \theta < \pi, \quad r > 0. \tag{4.2.15}$$

Using this and formula (4.2.10), we see that

$$(-1)^{m-1}\operatorname{Im} E_\rho(z;\mu)>0,\quad \frac{\pi}{\rho}\le\theta<\pi,\quad r>0,\quad \mu\in(-\mathbb{Z}_+)\cup\{1\}, \tag{4.2.16}$$

and hence $(-\mathbb{Z}_+)\cup\{1\}\subset A_\rho$.

From (4.2.16) and the formula

$$E_\rho(z;\mu)=\frac{1}{\Gamma(\mu)}+zE_\rho\left(z;\mu+\frac{1}{\rho}\right) \tag{4.2.17}$$

(this follows from the definition) we immediately obtain that

$$\operatorname{Im}\left(zE_\rho\left(z;m+\frac{1}{\rho}\right)\right)\neq 0,\quad \frac{\pi}{\rho}\le\theta<\pi,\quad r>0,$$

i.e., $m+1/\rho\in A_\rho$ for $m=1,0,-1,\ldots$.

To examine the rest of the set of parameters $\mu\in(1,1+1/\rho)$, for fixed $\rho>1$, $\mu\in\mathbb{R}$, and $s\in(0,1)$ we consider the auxiliary function

$$G(z)=G(z;\rho,\mu,s):=E_\rho(z;\mu)-z^sE_\rho\left(z;\mu+\frac{s}{\rho}\right),$$

which is analytic for $y=\operatorname{Im}z>0$ and continuous for $y\ge 0$. Theorem 1.2.1 implies the boundedness of the function $G(z)$ for $y>0$. Therefore, the function $H(z)=\operatorname{Im}G(z)$ is harmonic in the half-plane $y>0$, bounded in it, and continuous for $y\ge 0$.

Since the values

$$E_\rho(x;\mu),\quad E_\rho\left(x;\mu+\frac{s}{\rho}\right)$$

are real for $x\in\mathbb{R}$, we have

$$H(x)=0\quad\text{for}\quad x>0. \tag{4.2.18}$$

Further,

$$\operatorname{sign}H(x)=-\operatorname{sign}\operatorname{Im}\left(|x|^se^{i\pi s}E_\rho\left(x;\mu+\frac{s}{\rho}\right)\right)=-\operatorname{sign}E_\rho\left(x;\mu+\frac{s}{\rho}\right),\quad x<0. \tag{4.2.19}$$

Fix $\mu=m=1$. By Remark 4.1.1,

$$E_\rho\left(x;1+\frac{s}{\rho}\right)>0,\quad x<0.$$

Therefore,

$$H(x)<0\quad\text{for}\quad x<0. \tag{4.2.20}$$

From this and (4.2.18), by Lemma 4.2.4, we have the inequality

$$H(z)<0,\quad y>0. \tag{4.2.21}$$

In the sense of the function $G(z)$ this means that

$$\operatorname{Im}\left(z^sE_\rho\left(z;1+\frac{s}{\rho}\right)\right)>\operatorname{Im}E_\rho(z;1),\quad y>0, \tag{4.2.22}$$

if $0<s<1$. By (4.2.16), the right-hand side in (4.2.22) is positive in the angle $\pi/\rho\le\theta<\pi$. Therefore, the left-hand side is also positive, i.e., the function $E_\rho(z;1+s/\rho)$ does not vanish in this angle if $0<s<1$. Assertion (2) is proved.

(3) Owing to assertions (2) and (1), we must prove that the function $E_2(x;\mu)$ has no zeros in the angle $\pi/2\le\theta<\pi$ if

$$\mu\in\left(-m+\frac{1}{2},\,-m+1\right),\quad m\in\mathbb{Z}_+. \tag{4.2.23}$$

First, let $\mu = -m + 1/2$, $m \in \mathbb{Z}_+$. Then by formula (4.2.14)

$$(-1)^m f(t) = \frac{i}{iz+t} - \frac{i}{iz-t}.$$

Therefore, if $z = iy$, $y > 0$, then $\operatorname{Re} f(t) = 0$, and by formula (4.2.13)

$$\operatorname{Im} F_\rho(z, \mu) = 0$$

on the positive imaginary half-line. Then by (4.2.11)

$$\operatorname{sign} \operatorname{Im} E_2(z;\mu) = \operatorname{sign} \operatorname{Im}(e^{-z^2} z^{2(1-\mu)}) = \operatorname{sign} \sin \pi \left(\frac{1}{2} + m\right) = (-1)^m$$

on this half-line. Obviously, $\operatorname{Im} E_2(x;\mu) = 0$ for $x \in \mathbb{R}$. By Theorem 1.2.1, the function $E_2(z;\mu)$ is bounded in the angle $\pi/2 < \theta < \pi$. Thus, the function $\operatorname{Im} E_2(z;\mu)$ is harmonic and bounded in the angle $\pi/2 < \theta < \pi$ and is such that on one side of this angle it identically vanishes and preserves its sign $(-1)^m$ on the other side. By Lemma 4.2.4, everywhere in the angle $\pi/2 < \theta < \pi$ it preserves its sign $(-1)^m$. Therefore,

$$\operatorname{sign} \operatorname{Im} E_2\left(z; -m + \frac{1}{2}\right) = (-1)^m, \quad \frac{\pi}{2} \leq \theta < \pi, \quad m \in \mathbb{Z}_+. \tag{4.2.24}$$

We fix $m \in \mathbb{Z}_+$ and consider the auxiliary function

$$G(z) = G\left(z; 2, -m + \frac{1}{2}, s\right), \quad 0 < s < 1.$$

Recall that property (4.2.18) holds.

First, let m be even. Then by (4.1.3)

$$E_2\left(x; -m + \frac{1+s}{2}\right) > 0, \quad x < 0,$$

and hence by formula (4.2.19) property (4.2.20) holds. This and (4.2.18) by Lemma 4.2.4 imply property (4.2.21). Therefore,

$$\operatorname{Im}\left(z^s E_2\left(z; -m + \frac{1+s}{2}\right)\right) > \operatorname{Im} E_2\left(z; -m + \frac{1}{2}\right), \quad y > 0.$$

By (4.2.24), the left-hand side in the angle $\pi/2 \leq \theta < \pi$ is positive. We have proved that all values of μ from the set (4.2.23) with even m belong to the set A_2.

If m is odd, then by property (4.3.4)

$$E_2\left(x; -m + \frac{1+s}{2}\right) < 0, \quad x < 0,$$

and therefore, formula (4.2.19) yields the inequality

$$H(x) > 0 \quad \text{for } x < 0, \quad 0 < s < 1.$$

This and (4.2.18), by Lemma 4.2.4, imply the inequality

$$H(z) > 0, \quad y > 0.$$

But now

$$\operatorname{Im} E_2\left(z; -m + \frac{1}{2}\right) < 0, \quad \frac{\pi}{2} \leq \theta < \pi,$$

by (4.2.24). Therefore,

$$\operatorname{Im}\left(z^s E_2\left(z; -m + \frac{1+s}{2}\right)\right) < \operatorname{Im} E_2\left(z; -m + \frac{1}{2}\right) < 0, \quad \frac{\pi}{2} \leq \theta < \pi,$$

and assertion (3) is verified. Theorem 4.2.2 is completely proved. □

Theorem 4.2.3. *For $\rho > 1$ and $\mu \in [1, 1 + 1/\rho]$, all zeros of the function $E_\rho(z;\mu)$ lie on the set*

$$\frac{\pi}{2\rho} < |\arg z| < \frac{\pi}{\rho}, \quad 2\pi n < |\operatorname{Im} z^\rho| < (2n+1)\pi, \quad n \in \mathbb{Z}_+. \tag{4.2.25}$$

In particular, for $\rho > 1$, all zeros of the classical Mittag-Leffler function $E_\rho(z;1)$ lie on the set (4.2.25)*.*

Proof. For $\rho > 1$ and $1 \le \mu \le 1 + 1/\rho$, all zeros of the function $E_\rho(z;\mu)$ lie outside the angle (4.2.1) (by Theorem 4.2.1) and outside the angle (4.2.7) (by Theorem 4.2.2). It remains to verify the second double inequality in (4.2.25). Owing to the complex conjugacy of zeros it suffices to consider the angle

$$\frac{\pi}{2\rho} < \theta < \frac{\pi}{\rho}.$$

Let $\mu = m = 1$. Then formula (4.2.12) and inequality (4.2.15), respectively, imply that

$$\operatorname{Im} E_\rho(z;1) = \operatorname{Im} F_\rho(z;1) + \rho \operatorname{Im} e^{z^\rho} > \rho e^{\operatorname{Re} z^\rho} \sin(\operatorname{Im} z^\rho), \quad 0 < \theta < \frac{\pi}{\rho},$$

$$\operatorname{Im} E_\rho(z;1) > 0, \quad 2\pi n < \operatorname{Im} z^\rho < (2n+1)\pi, \quad n \in \mathbb{Z}_+, \quad 0 < \theta < \frac{\pi}{\rho}, \tag{4.2.26}$$

and for $\mu = 1$ Theorem 4.2.3 is valid. Now, gathering (4.2.26) with formula (4.2.17), and then with inequality (4.2.22), we obtain the assertion of the theorem for $\mu = 1 + 1/\rho$ and $\mu \in (1, 1 + 1/\rho)$. Theorem 4.2.3 is proved. □

Corollary 4.2.1. *For $\rho > 1$ and $1 \le \mu \le 1 + 1/\rho$, the function $E_\rho(z;\mu)$ has no zeros in the disk $|z| \le \pi^{1/\rho}$.*

4.3. Case $\rho = 1/2$

In this section, we prove that for $\mu \in (1,2) \cup (2,3)$, all zeros of the function $E_{1/2}(z;\mu)$ are negative and simple and indicate the interval containing these zeros. If values of the parameter μ are close to 2 or 3, then we indicate more exact bounds of intervals containing zeros with small numbers. We start from the case $\mu > 3$: we show that the function $E_{1/2}(z;\mu)$ has no real zeros and then clarify this fact by proving that there are no zeros inside some parabola containing the negative real half-line.

4.3.1. Case $\mu > 3$. Our reasoning is based on the following formulas, which were proved in [6]:

$$E_{1/2}(-z^2;\mu) = \frac{1}{\Gamma(\mu-1)} \int_0^1 (1-t)^{\mu-2} \cos zt\, dt, \quad \mu > 1, \tag{4.3.1}$$

$$zE_{1/2}(-z^2;\mu) = \frac{1}{\Gamma(\mu-2)} \int_0^1 (1-t)^{\mu-3} \sin zt\, dt, \quad \mu > 2. \tag{4.3.2}$$

Lemma 4.3.1 (see [19, 20, 34])**.** *Let a function $f(t)$, $t > 0$, be nonnegative and nonincreasing and let*

$$\frac{tf(t)}{t+1} \in L^1(\mathbb{R}_+).$$

Moreover, let $f(t)$ decrease on some interval. Then the function

$$F(y) = \int_{\mathbb{R}_+} f(t) \sin yt\, dt, \quad y \in \mathbb{R},$$

is positive on the half-line $y > 0$ and, in particular, has no nonzero roots.

Introduce the notation

$$x_\mu = \sqrt{(\mu-3)(\mu-2)} + \operatorname{artanh}\sqrt{\frac{\mu-3}{\mu-2}}, \quad \mu > 3. \tag{4.3.3}$$

Theorem 4.3.1. *Let $\mu > 3$. Then the following assertions hold*:

(1) *the function $E_{1/2}(z;\mu)$ has no real zeros*;

(2) *the function $E_{1/2}(z;\mu)$ has no zeros on the set*

$$0 \le \operatorname{Re}\sqrt{z} \le x_\mu, \tag{4.3.4}$$

i.e., in the left-hand curvilinear half-plane bounded by the parabola

$$x = x_\mu^2 - \frac{y^2}{4x_\mu^2}.$$

Proof. (1) The fact that $E_{1/2}(z;\mu)$ has no nonnegative roots follows from the definition of the Mittag-Leffler function. We prove that $E_{1/2}(z;\mu)$ has no negative roots.

Introduce the notation

$$f(t) = \begin{cases} (1-t)^{\mu-3}, & 0 < t < 1, \\ 0, & t \ge 1; \end{cases}$$

we see that for $\mu > 3$ the function $f(t)$ satisfies all conditions of Lemma 4.3.1. By this lemma and formula (4.3.2), the function $E_{1/2}(z;\mu)$ has no negative roots. Assertion (1) is proved.

(2) The proof of this assertion is also based on formula (4.3.2), which we rewrite in the form

$$\Gamma(\mu-2) z E_{1/2}(z^2;\mu) = \int_0^1 (1-t)^{\mu-3} \sinh zt\, dt, \quad \mu > 2.$$

This implies that

$$\Gamma(\mu-2)\operatorname{Im}(zE_{1/2}(z^2;\mu)) = \int_0^1 (1-t)^{\mu-3}\cosh xt \sin yt\, dt, \quad z = x+iy, \quad \mu > 2. \tag{4.3.5}$$

For fixed $x \ge 0$, we set $f_x(t) = 0$ for $t \ge 0$ and

$$f_x(t) = (1-t)^{\mu-3}\cosh xt, \quad 0 \le t \le 1.$$

Then the right-hand side of (4.3.5) has the form of the function $F(y)$ from Lemma 4.3.1. The function $f_x(t)$ satisfies all condition of this lemma, except for, perhaps, the monotonicity. Therefore, if for some x the function $f_x(t)$ becomes decreasing on the interval $(0,1)$, by Lemma 4.3.1, the function $E_{1/2}(z^2;\mu)$ has no zeros on the line $\operatorname{Re} z = x$ with punctured point x. By assertion (1), for $\mu > 3$, a real point cannot be a zero of this function. Therefore, in the case where $f_x(t)$ decreases, the function $E_{1/2}(z;\mu)$ has no zeros on the line $\operatorname{Re} z = x$.

Since

$$(f_x(t))' = (1-t)^{\mu-3}\left(-\frac{\mu-3}{1-t}\cosh xt + x\sinh xt\right), \quad 0 < t < 1,$$

the condition $(f_x(t))' \le 0$ (expressing the fact that the function $f_x(t)$ does not increase) is equivalent to the condition

$$x \tanh xt \le \frac{\mu-3}{1-t}, \quad 0 < t < 1. \tag{4.3.6}$$

Obviously, condition (4.3.6) holds for all small $x > 0$ and fails for all sufficiently large x. The right-hand side of (4.3.6) is a function strongly convex on the interval $(0,1)$, and the left-hand side, for fixed $x > 0$, is a function strongly concave on it. This and the continuous dependence of the left-hand side on x imply the existence of a unique value $x = x_\mu > 0$ such that the graphs of these function touch each other for some $t = t_\mu \in (0,1)$, and for $t \neq t_\mu$, $0 < t < 1$, the strong inequality in (4.3.6) holds. This means that for $0 \leq x \leq x_\mu$, the function $f_x(t)$ decreases on the interval $(0,1)$ and hence the function $E_{1/2}(z^2;\mu)$ has no zeros on the strip $0 \leq \operatorname{Re} z \leq x_\mu$. Therefore, the function $E_{1/2}(z;\mu)$ has no zeros on the set (4.3.4). It remains to calculate x_μ.

The touching of the graphs mentioned is equivalent to the equality (4.3.6) for some $t \in (0,1)$ and the equality of the derivatives at the same value of t, i.e., is equivalent to the system of equations

$$x \tanh xt = \frac{\mu - 3}{1-t}, \qquad \frac{x^2}{\cosh^2 xt} = \frac{\mu-3}{(1-t)^2}. \tag{4.3.7}$$

To find $x = x_\mu$, we must eliminate t from this system.

Raising both sides of the first equation to the second power, adding the second equation, and applying the formual $1 - \tanh^2 t = 1/\cosh^2 t$, we obtain

$$x^2 = \frac{(\mu-3)(\mu-2)}{(1-t)^2}.$$

From this we have

$$xt = x - \sqrt{(\mu-3)(\mu-2)}, \qquad \frac{\mu-3}{1-t} = \sqrt{\frac{\mu-3}{\mu-2}}\, x,$$

and substituting these relations in the first equation (4.3.7), we obtain

$$\tanh\left(x - \sqrt{(\mu-3)(\mu-2)}\right) = \sqrt{\frac{\mu-3}{\mu-2}}.$$

Since a solution of system (4.3.7) exists and is unique, it can be found by formula (4.3.3). The theorem is proved. □

Note that assertion (1) of Theorem 4.3.1 is contained in assertion (2). We have presented its proof since it is much simpler than the proof of assertion (2).

4.3.2. Zeros of finite Fourier cosine and sine transforms. Let $f \in L^1(0,1)$. Consider the corresponding Fourier cosine and sine transforms

$$U(z) = \int_0^1 f(t) \cos zt \, dt, \tag{4.3.8}$$

$$V(z) = \int_0^1 f(t) \sin zt \, dt. \tag{4.3.9}$$

The integrals in formulas (4.3.1) and (4.3.2) have the form $U(z)$ and $V(z)$, respectively. Therefore, it is natural to use known results on the distribution of zeros of the functions $U(z)$ and $V(z)$ (these results were essentially obtained by Pólya). We formulate them in Theorems A and B; by an *exceptional function* we mean an echelon function with discontinuities at rational points.

Introduce the sequences of intervals

$$\left(\pi\left(n-\frac{1}{2}\right),\ \pi\left(n+\frac{1}{2}\right)\right), \quad n \in \mathbb{N}, \tag{4.3.10}$$

$$\big(\pi n,\ \pi(n+1)\big), \quad n \in \mathbb{N}, \tag{4.3.11}$$

$$\left(\pi\left(n-\frac{1}{2}\right),\ \pi n\right), \quad n \in \mathbb{N}, \tag{4.3.12}$$

$$\big(\pi(2n-1),\ 2\pi n\big),\ \left(2\pi n, \pi\left(2n+\frac{1}{2}\right)\right), \quad n \in \mathbb{N}. \tag{4.3.13}$$

The following results are well known.

Theorem A. *Assume that a function $f(t)$ is positive and nondecreasing on the interval $(0,1)$ (and is not an exceptional function). Then all roots of the function $U(z)$ (respectively, $V(z)$) are real and simple; moreover, all positive roots of the function $U(z)$ (respectively, $V(z)$) lie singly in the intervals* (4.3.10) (*respectively, in the intervals* (4.3.11)).

Theorem B. *Assume that a function $f(t)$ is positive, increasing, and convex on the interval $(0,1)$ (and its right-sided derivative is not an exceptional function). Then all roots of the function $V(z)$ (respectively, $U(z)$) are real and simple; moreover, all positive roots lie singly in the intervals* (4.3.13) (*respectively, in the intervals* (4.3.12)).

Theorem A and the part of Theorem B relating to roots of the function $U(z)$ belong to Pólya (see [20]) and the part of Theorem B relating to roots of the function $V(z)$ belong to Sedletskii [30].

We consider formula (4.3.1) for $\mu \in (1,2)$ and formula (4.3.2) for $\mu \in (2,3)$. These formulas show that the functions $E_{1/2}(-z^2;\mu)$ and $zE_{1/2}(-z^2;\mu)$ have the form $U(z)$ and $V(z)$, respectively, where, up to a coefficient,

$$f(t) = (1-t)^{\mu-2} \quad \text{for} \ \ 1 < \mu < 2, \tag{4.3.14}$$

$$f(t) = (1-t)^{\mu-3} \quad \text{for} \ \ 2 < \mu < 3. \tag{4.3.15}$$

In both cases, $f(t)$ satisfies the conditions of Theorem B. By this theorem, all roots of the function

$$E_{1/2}(-z^2;\mu), \quad \operatorname{Re} z \geq 0,$$

are positive and simple and lie singly in the intervals (4.3.12) (respectively, in the intervals (4.3.13)). Since the function is even, it suffices to consider only the right-hand side half-plane. Therefore, the following theorem holds.

Theorem 4.3.2. *For $1 < \mu < 2$ and $2 < \mu < 3$, all roots z_n of the function $E_{1/2}(z;\mu)$ are negative and simple and the points $(-z_n)^{1/2}$ lie singly in the intervals* (4.3.12) *and* (4.3.13), *respectively.*

In Chap. 3 we proved stronger assertions, which were obtained by painstaking and cumbersome calculations, whereas Theorem 4.3.2 is an immediate consequence of well-known results. Moreover, this theorem motivates an additional analysis of formulas (4.3.1) and (4.3.2), which allows one to localize zeros with small numbers more exactly. We perform this in the general case, i.e., for the functions $U(z)$ and $V(z)$, and then, using formulas (4.3.1) and (4.3.2), apply the obtained assertions to the function $E_{1/2}(-z^2;\mu)$.

Assume that a function $f(t)$ is positive and nondecreasing in the interval $(0,1)$, i.e., the conditions of Theorem A holds (nota that this theorem, unlike Theorem B, has not yet been used). Denote by u_n (respectively, v_n), $n \in \mathbb{N}$, positive zeros of the function $U(z)$ (respectively, $V(z)$), numbered in ascending order, i.e.,

$$\pi\left(n-\frac{1}{2}\right)<u_n<\pi\left(n+\frac{1}{2}\right),\qquad \pi n<v_n<\pi(n+1),\qquad n\in\mathbb{N}.$$

Introduce the notation

$$I=I_f:=\int\limits_{1/2}^{1}(f(t)-f(1-t))dt.$$

The method used in the proof is effective (compared with Theorem 4.3.2) under the condition that I is not too large and $f(+0)>0$.

Theorem 4.3.3. *Assume that a function $f(t)$ does not decrease in the interval $(0,1)$ and $f(+0)=1$. Then the following assertions hold:*

(1) *if, for some $N\in\mathbb{N}$,*

$$I<\frac{1}{\pi(N+1/2)},\tag{4.3.16}$$

then

$$\pi n-\arcsin(\pi nI)<u_n<\pi n+\arcsin\left(\pi\left(n+\frac{1}{2}\right)I\right),\qquad n=\overline{1,N};\tag{4.3.17}$$

(2) *if, moreover, the function f increases and is convex in $(0,1)$ and, for some $N\in\mathbb{N}$,*

$$I<\frac{1}{\pi N},\tag{4.3.18}$$

then

$$\pi n-\arcsin(\pi nI)<u_n<\pi n,\qquad n=\overline{1,N}.$$

Theorem 4.3.4. *Assume that a function $f(t)$ does not decrease in the interval $(0,1)$ and is not an exceptional function. Let $f(+0)=1$. Then the following assertions hold:*

(1) *if, for some $N\in\mathbb{N}$,*

$$2I<\frac{1}{\pi N},\tag{4.3.19}$$

then

$$2\pi n-\arccos\big(1-2\pi nI\big)<v_{2n-1}<2\pi n,\qquad n=\overline{1,N};\tag{4.3.20}$$

(2) *if, for some $n\in\mathbb{N}$,*

$$2I<\frac{1}{\pi(N+1/2)},$$

then

$$2\pi n<v_{2n}<2\pi n+\arccos\big(1-\pi(2n+1)I\big),\qquad n=\overline{1,N};\tag{4.3.21}$$

(3) *if, moreover, the function f increases and is convex in $(0,1)$ and, for some $N\in\mathbb{N}$,*

$$2I<\frac{1}{\pi(N+1/4)},\tag{4.3.22}$$

then

$$2\pi n<v_{2n}<2\pi n+\arccos\left(1-\pi\left(2n+\frac{1}{2}\right)I\right),\qquad n=\overline{1,N}.$$

Note that by the conditions imposed on I, all values in Theorem 4.3.4 represented as arccosines lie in the interval $(0,\pi/2)$.

In the proof we use the following Steffensen inequality.

Theorem C (see [4])**.** *Let a function $f(t)$ be integrable, positive, and nondecreasing in the interval $(0,1)$ and*

$$0 \le g(t) \le 1, \quad 0 < t < 1, \quad g \in L^1(0,1).$$

Then

$$\int_0^1 f(t)g(t)dt \le \int_{1-c}^1 f(t)dt,$$

where

$$c = \int_0^1 g(t)dt.$$

Assume that the function f is the same as in Theorem C and

$$-a \le G(t) \le a \quad (a > 0), \quad 0 < t < 1, \quad G \in L^1(0,1). \tag{4.3.23}$$

Then for the functions f and g, where

$$g(t) = \frac{G(t)+a}{2a},$$

the conditions of Theorem C hold. By this theorem,

$$\int_0^1 f(t)\frac{G(t)+a}{2a}dt \le \int_{1-c}^1 f(t)dt,$$

where

$$c = \frac{1}{2} + \frac{1}{2a}\int_0^1 G(t)dt. \tag{4.3.24}$$

We see that the following lemma is valid.

Lemma 4.3.2. *Let a function $f(t)$ be integrable, positive, and nondecreasing in the interval $(0,1)$, and a function $G(t)$ satisfy conditions (4.3.23). Then*

$$\int_0^1 f(t)G(t)dt \le a\left(2\int_{1-c}^1 f(t)dt - \int_0^1 f(t)dt\right),$$

where c is calculated by formula (4.3.24).

Proof of Theorem 4.3.3. In formula (4.3.8), we set $z = x$ and then subtract the formula

$$\frac{\sin x}{x} = \int_0^1 \cos xt\, dt.$$

We obtain that at zeros of the function U, which are real by Theorem A, the following inequality holds:

$$-\frac{\sin x}{x} = \int_0^1 (f(t)-1)\cos xt\, dt.$$

Let n be odd. If $x = u_n = \pi n$, then there is nothing to prove. Let $x = u_n \ne \pi n$. Then by Theorem A

$$\sin x \gtrless 0 \quad \text{for} \quad x \lessgtr \pi n, \tag{4.3.25}$$

and, therefore, in the formula

$$\frac{\sin(\pm(\pi n - x))}{x} = \int_0^1 (f(t)-1)(\mp\cos xt)dt, \quad x = u_n \lesseqgtr \pi n, \tag{4.3.26}$$

both sides are positive. The functions

$$f(t) - 1 \quad \text{and} \quad \mp\cos xt$$

satisfy the conditions of Lemma 4.3.2 with $a = 1$. By this lemma,

$$\frac{\sin(\pm(\pi n - x))}{x} \le 2\int_{1-c}^{1} (f(t)-1)dt - \int_0^1 (f(t)-1)dt, \tag{4.3.27}$$

where

$$c = \frac{1}{2}\left(1 + \int_0^1 \mp\cos xt\,dt\right) = \frac{1}{2}\left(1 \mp \frac{\sin x}{x}\right). \tag{4.3.28}$$

From (4.3.25) it follows that $c \le 1/2$ and hence

$$\frac{\sin(\pm(\pi n - x))}{x} \le \left(2\int_{1/2}^{1} - \int_0^1\right)(f(t)-1)dt = I. \tag{4.3.29}$$

We have proved estimate (4.3.29) for odd n.

If n is even, then in (4.3.25) inequalities for $\sin x$ must be replaced by the opposite. Therefore, on the right-hand side of (4.3.26) the signs $\mp$ interchange; both sides in (4.3.26) are positive. Applying Lemma 4.3.2, we obtain inequality (4.3.27), where c is calculated by formula (4.3.28), in which the signs $\mp$ also interchange. Taking into account the sign of the sine, we see that again $c \le 1/2$. Therefore, estimate (4.3.29) is also valid for even n.

From (4.3.29) it follows that

$$\begin{aligned} &0 < \sin(\pi n - x) < \pi n I, && \text{if} \quad \pi\left(n - \frac{1}{2}\right) < x < \pi n, \\ &0 < \sin(x - \pi n) < \pi\left(n + \frac{1}{2}\right) I, && \text{if} \quad \pi n < x < \pi\left(n + \frac{1}{2}\right). \end{aligned} \tag{4.3.30}$$

Taking into account condition (4.3.16), we see that these inequalities yield the required inequalities (4.3.17). Assertion (1) is proved.

If f is convex, then by Theorem B $x = u_n \in (\pi(n - 1/2), \pi n)$, and by (4.3.18) assertion (2) follows from (4.3.30). Theorem 4.3.3 is proved. □

Proof of Theorem 4.3.4. In formula (4.3.9), we set $z = x$ and then subtract the formula

$$\frac{1 - \cos x}{x} = \int_0^1 \sin xt\,dt.$$

We obtain that at zeros of the function V, which are real by Theorem A,

$$\frac{1 - \cos x}{x} = \int_0^1 (f(t)-1)(-\sin xt)dt.$$

By Theorem A, both sides are positive. By Lemma 4.3.2, for $x = v_n$ we have

$$\frac{1-\cos x}{x} \leq 2\int\limits_{1-c}^{1}(f(t)-1)dt - \int\limits_{0}^{1}(f(t)-1)dt,$$

where

$$c = \frac{1}{2}\left(1 + \int\limits_{0}^{1}(-\sin xt)dt\right) = \frac{1}{2}\left(1 - \frac{1-\cos x}{x}\right).$$

Therefore, $c \leq 1/2$ and hence

$$\frac{1-\cos x}{x} \leq I, \quad x = v_n. \tag{4.3.31}$$

Since by Theorem A

$$\pi(2n-1) < v_{2n-1} < 2\pi n < v_{2n} < \pi(2n+1), \tag{4.3.32}$$

from (4.3.31) we obtain

$$\begin{aligned} &\cos(2\pi n - x) > 1 - 2\pi n I, &&\text{if} \quad x = v_{2n-1}, \\ &\cos(x - 2\pi n) > 1 - \pi(2n+1)I, &&\text{if} \quad x = v_{2n}. \end{aligned}$$

Therefore, the following inequalities hold:

$$\begin{aligned} 2\pi n - x &< \arccos(1 - 2\pi n I), \quad x = v_{2n-1}, \\ x - 2\pi n &< \arccos(1 - \pi(2n+1)I), \quad x = v_{2n}. \end{aligned}$$

Together with (4.3.32), they yield the required inequalities (4.3.20) and (4.3.21). Assertions (1) and (2) are proved.

If f increases and is convex, then, by Theorem B, instead of the right-hand inequality (4.3.32) we have the inequality

$$v_{2n} < 2\pi n + \frac{\pi}{2}.$$

Making this change, we obtain assertion (3). Theorem 4.3.4 is proved. □

Now we turn to consequences we are interested. They are based on formulas (4.3.1) and (4.3.2), which show that the functions

$$\Gamma(\mu-1)E_{1/2}(-z^2;\mu), \quad \Gamma(\mu-2)zE_{1/2}(-z^2;\mu)$$

have the form $U(z)$ and $V(z)$ (see (4.3.8) and (4.3.9)), where the corresponding functions $f(t)$ are defined by formulas (4.3.14) and (4.3.15). In these formula, we restricted the values of μ not by accident: under these restrictions, the function $f(t)$ satisfies the conditions of Theorems 4.3.3 and 4.3.4, respectively, and is convex in $(0,1)$. Therefore, to apply these theorems, we must express I and the conditions for I in Theorems 4.3.3 and 4.3.4 through the parameter μ.

We have the following relation:

$$\begin{aligned} I &= \frac{2^{2-\mu}-1}{\mu-1} \qquad \text{for} \quad 1 < \mu < 2, \\ I &= \frac{2^{3-\mu}-1}{\mu-2} \qquad \text{for} \quad 2 < \mu < 3. \end{aligned}$$

Therefore, if $1 < \mu < 2$, then condition (4.3.18) takes the form

$$\frac{2^{2-\mu}-1}{\mu-1} < \frac{1}{\pi N}; \tag{4.3.33}$$

if $2<\mu<3$, then conditions (4.3.19) and (4.3.22) become, respectively,

$$\frac{2^{3-\mu}-1}{\mu-2}<\frac{1}{2\pi N},\quad \frac{2^{3-\mu}-1}{\mu-2}<\frac{1}{\pi(2N+1/2)}. \tag{4.3.34}$$

Denote by $x_n=x_n(\mu)$, $n\in\mathbb{N}$, zeros of the function $E_{1/2}(z;\mu)$. Since

$$(-x_n)^{1/2}=u_n \quad \text{for} \quad 1<\mu<2, \quad (-x_n)^{1/2}=v_n \quad \text{for} \quad 2<\mu<3,$$

by Theorems 4.3.3 and 4.3.4, we obtain the following assertion on zeros x_n with small numbers.

Corollary 4.3.1. *Let $1<\mu<2$. If, for some $N\in\mathbb{N}$, condition* (4.3.33) *holds, then*

$$\pi n-\arcsin\left(\pi n\frac{2^{2-\mu}-1}{\mu-1}\right)<(-x_n)^{1/2}<\pi n, \quad n=\overline{1,N}.$$

Corollary 4.3.2. *Let $2<\mu<3$. Then the following assertions hold*:

(1) *if, for some $N\in\mathbb{N}$, the first condition* (4.3.34) *holds, then*

$$2\pi n-\arccos\left(1-2\pi n\frac{2^{3-\mu}-1}{\mu-2}\right)<(-x_{2n-1})^{1/2}<2\pi n, \quad n=\overline{1,N};$$

(2) *if, for some $N\in\mathbb{N}$, the second condition* (4.3.34) *holds, then*

$$2\pi n<(-x_n)^{1/2}<2\pi n+\arccos\left(1-\pi\left(2n+\frac{1}{2}\right)\frac{2^{3-\mu}-1}{\mu-2}\right), \quad n=\overline{1,N}.$$

Corollaries 4.3.1 and 4.3.2 allow one to draw conclusions on the rate of approximation of the points $(-x_n)^{1/2}$ to positive roots of the functions $\sin z$ and $1-\cos z$, respectively, as $\mu\to 2-0$ and $\mu\to 3-0$.

Introduce the notation

$$\rho_n(\mu)=\pi n-(-x_n(\mu))^{1/2} \quad \text{for} \quad 1<\mu<2$$

and

$$\rho_{2n-1}(\mu)=2\pi n-(-x_{2n-1}(\mu))^{1/2}, \quad \rho_{2n}(\mu)=(-x_{2n}(\mu))^{1/2}-2\pi n \quad \text{for} \quad 2<\mu<3.$$

Since

$$\arcsin x\sim x, \quad \arccos(1-x)\sim\sqrt{2x}, \quad x\to+0,$$

from Corollaries 4.3.1 and 4.3.2 we obtain the following assertion.

Corollary 4.3.3. *For any fixed $n\in\mathbb{N}$,*

$$\varlimsup_{\mu\to 2-0}\frac{\rho_n(\mu)}{2-\mu}\le \pi n\ln 2,$$

$$\varlimsup_{\mu\to 3-0}\frac{\rho_n(\mu)}{\sqrt{3-\mu}}\le\sqrt{2\pi\left(n+\frac{3-(-1)^n}{4}\right)\ln 2}.$$

Conditions (4.3.33) and (4.3.34) contain transcendental functions. If we apply the inequality

$$2^x-1<x, \quad 0<x<1,$$

then we can replace these condition by rougher but simpler conditions of applicability of Corollaries 4.3.1 and 4.3.2:

$$2-\frac{1}{\pi N+1}<\mu<2,$$

$$3-\frac{1}{2\pi N+1}<\mu<3, \quad 3-\frac{1}{\pi(2N+1/2)+1}<\mu<3.$$

4.4. Absence of Multiple Zeros

The results of Chap. 2 show that, except for the unique case $\rho = 1/2$, $\mu = 3$, the number of multiple zeros of the function $E_\rho(z;\mu)$ is no more than finite.

We still have few information on multiple zeros.

First, all zeros of the function $E_{1/2}(z;3)$ in the specified exceptional case are double.

Second, the case of the zero multiple root admits a complete description. Indeed, the definition implies that the point $z = 0$ is a s-multiple root of the function $E_\rho(z;\mu)$ if and only if

$$\frac{1}{\Gamma(\mu)} = \frac{1}{\Gamma(\mu + 1/\rho)} = \ldots = \frac{1}{\Gamma(\mu + s/\rho)} = 0, \quad \frac{1}{\Gamma(\mu + (s+1)/\rho)} \neq 0.$$

These conditions are equivalent to the following:

$$\mu,\ \mu + \frac{1}{\rho},\ \ldots,\ \mu + \frac{s}{\rho} \in -\mathbb{Z}_+,\ \mu + \frac{s+1}{\rho} \notin -\mathbb{Z}_+.$$

Clearly, this is possible only in the case where $\rho = 1/n$, $n \in \mathbb{N}$, and choosing μ appropriately, we can obtain a multiple root of arbitrarily high multiplicity. Thus, we have the following assertion.

Proposition 4.4.1. *The following assertions hold.*

(1) *A multiple root of the function $E_\rho(z;\mu)$ is possible only in the case where $\rho = 1/n$, $n \in \mathbb{N}$.*
(2) *If $\rho = 1/n$, $n \in \mathbb{N}$, then the function $E_\rho(z;\mu)$ has at the point $z = 0$ an s-multiple root if and only if $\mu = -s/\rho$.*

Third, by Theorem 4.1.2, for any $\rho > 1$, there exists a sequence $\mu_n < 0$, $\mu_n \to -\infty$, such that the function $E_\rho(z;\mu_n)$ has a double positive root, and by Corollary 4.1.1, for $\rho > 1$ and $\mu \in \mathbb{R}$, the function $E_\rho(z;\mu)$ has no real roots of multiplicity higher than 2.

The following lemma gives an answer to the question on multiple roots of the Mittag-Leffler functions in the case of natural ρ.

Lemma 4.4.1. *Let z_0 be a root of the function $E_n(z;\mu)$, $n \in \mathbb{N}$, $\mu \in \mathbb{C}$. It is multiple if and only if the following condition holds*:

$$\frac{1}{\Gamma(\mu - 1)} + \frac{z_0}{\Gamma\mu - 1 + 1/n)} + \ldots + \frac{z_0^{n-1}}{\Gamma(\mu - 1 + (n-1)/n)} = 0. \tag{4.4.1}$$

Proof. The proof is based on the formula

$$E_\rho(z;\mu) = \mu E_\rho(z;\mu+1) + \frac{z}{\rho} E'_\rho(z;\mu+1) \tag{4.4.2}$$

and formula (4.2.17), which we rewrite in the following form:

$$E_\rho(z;\mu-1) = \frac{1}{\Gamma(\mu-1)} + zE_\rho\left(z;\mu - 1 + \frac{1}{\rho}\right). \tag{4.4.3}$$

In our case $\rho = n$. We replace μ by $\mu + 1/n$ in formula (4.4.3):

$$E_n\left(z;\mu - 1 + \frac{1}{n}\right) = \frac{1}{\Gamma(\mu - 1 + 1/n)} + zE_n\left(z;\mu - 1 + \frac{2}{n}\right).$$

Substituting the left-hand side of this formula on the right-hand side of formula (4.4.3), we obtain

$$E_n(z;\mu-1) = \frac{1}{\Gamma(\mu-1)} + \frac{z}{\Gamma(\mu - 1 + 1/n)} + z^2 E_n\left(z;\mu - 1 + \frac{2}{n}\right). \tag{4.4.4}$$

Now, replacing μ by $\mu + 2/n$ in (4.4.3), we obtain an expression for $E_n(z; \mu - 1 + 2/n)$ and then substitute it on the right-hand side of (4.4.4). Repeating this process, after n steps we obtain

$$E_n(z;\mu-1) = \frac{1}{\Gamma(\mu-1)} + \frac{z}{\Gamma(\mu-1+1/n)} + \ldots + \frac{z^{n-1}}{\Gamma(\mu-1+(n-1)/n)} + z^n E_n(z;\mu). \qquad (4.4.5)$$

Let z_0 be a root of the function $E_n(z; \mu)$.

Then the last term in (4.4.5) vanishes, and formula (4.4.2) shows that a root of the function $E_n(z; \mu)$ is multiple if and only if it is a root of the function $E_n(z; \mu - 1)$. Thus, the fact that a root z_0 of the function $E_n(z; \mu)$ is multiple is equivalent to vanishing of the left-hand side of (4.4.1). The lemma is proved. □

Theorem 4.4.1. *The following assertions hold.*

(1) *The function $E_1(z; \mu)$, $\mu \in \mathbb{C} \setminus (-\mathbb{N})$, has no multiple roots.*
(2) *The function $E_n(z; \mu)$, $1 < n \in \mathbb{N}$, $1 \le \mu < 1 + 1/n$, has no multiple roots. In particular, the classical Mittag-Leffler function $E_n(z; 1)$ of integer order has no multiple roots.*
(3) *The number of multiple roots of the function $E_n(z; \mu)$, $1 < n \in \mathbb{N}$, $\mu \in \mathbb{C}$, does not exceed $n - 1$.*
(4) *The function $E_2(z; \mu)$*
   (a) *has no roots of multiplicity higher than 2 for $\mu \in \mathbb{C}$,*
   (b) *has no multiple nonpositive roots for $\mu \in \mathbb{R}$,*
   (c) *has no multiple roots for $\mu \ge 0$.*
(5) *The function $E_3(z; \mu)$ has no roots of multiplicity higher than 2 for $\mu \in \mathbb{R}$.*
(6) *The function $E_{1/2}(z; \mu)$, $\mu \in \mathbb{C}$, has no nonzero roots of multiplicity higher than 2.*

Proof. (1) For $n = 1$, condition (4.4.1) means that $1/\Gamma(\mu - 1) = 0$. This is possible only if $\mu = 1, 0, -1, \ldots$. In this case, $E_1(z; \mu)$ has an explicit form (see the beginning of Chap. 2), which implies that for $\mu = 1, 0$, the function $E_1(z; \mu)$ has no multiple roots. The values $\mu \in -\mathbb{N}$ (when $z = 0$ is a root of multiplicity $1 - \mu$) have been excluded. Assertion (1) is proved.

(2) Since $1 \le \mu \le 1 + 1/n$ and the function $\Gamma(x)$ decreases on $(0, 1]$, a necessary condition (4.4.1) of a multiple root is the equation

$$a_0 + a_1 z + \ldots + a_{n-1} z_0^{n-1} = 0$$

with monotonic coefficients $0 < a_0 < \ldots < a_{n-1}$. It is well known (see [13, Chap. 4, Sec. 3] or [22, Part 3, Chap. 1, Sec. 2]) that all roots of this equation lie in the disk $|z| < 1$. By Corollary 4.2.1, the function $E_n(z; \mu)$ has no roots in this disk. Therefore, the function $E_n(z; \mu)$ has no multiple roots.

(3) Assertion (3) follows from the fact that the order of Eq. (4.4.1) is not higher than $n - 1$.

(4) If $\mu \in \mathbb{R}$, then by Theorem 4.1.1, a possible negative root of the function $E_2(z; \mu)$ is simple. If there was a nonreal multiple root, then the conjugate number was also a multiple root. This means that the function $E_2(z; \mu)$ would have no less than two multiple roots, which contradicts assertion (3). The case $\mu \in \mathbb{R}$ has been examined. It also implies the case $\mu \ge 0$ since $E_2(x; \mu) > 0$ for $x > 0$ and $\mu \ge 0$.

Let z_0 be a root of the function $E_2(\ \mu)$ of multiplicity no higher than 2. By (4.4.2), z_0 is a multiple root of the function $E_2(z; \mu - 1)$. By Lemma 4.4.1 and Proposition 4.4.1, z_0 is a nontrivial solution of the following system:

$$\frac{1}{\Gamma(\mu-1)} + \frac{z_0}{\Gamma(\mu-1/2)} = 0, \quad \frac{1}{\Gamma(\mu-2)} + \frac{z_0}{\Gamma(\mu-3/2)} = 0.$$

We have proved that $\mu \ne 2, 3/2$. The first of these equations can be rewritten in the form

$$\frac{1}{(\mu-2)\Gamma(\mu-2)} + \frac{z_0}{(\mu-3/2)\Gamma(\mu-3/2)} = 0.$$

Comparing this with the second equation, we see that the system cannot have nontrivial solutions. The case $\mu \in \mathbb{C}$ has also been examined.

(5) Let z_0 be a root of the function $E_\rho(z;\mu)$ of multiplicity higher than 2. Then by formula (4.3.33), z_0 is a multiple root of the function $E_\rho(z;\mu-1)$. Therefore, by Lemma 4.4.1, in addition to condition (4.4.1), the following condition must hold:

$$\frac{1}{\Gamma(\mu-2)} + \frac{z_0}{\Gamma(\mu-2+1/n)} + \ldots + \frac{z_0^{n-1}}{\Gamma(\mu-2+(n-1)/n)} = 0.$$

For $n = 3$, both these conditions mean that z_0 is a common root of two quadratic equations. The relation $\Gamma(s+1) = s\Gamma(s)$ implies that the coefficient of these equations are not proportional and hence these equations can have no more than one common root z_0. By Theorems 4.1.1 and 4.1.2, $z_0 \notin \mathbb{R}$, but then $\bar{z}_0$ (since μ is real) must be a common root, which is impossible. Assertion (5) is proved.

(6) Let z_0 be a root of the function $E_{1/2}(z;\mu)$ of multiplicity not higher than 2. Twice applying formula (4.4.2), we conclude that z_0 is a root of the function $E_{1/2}(z;\mu-2)$. Then formula (4.4.3) with $\rho = 1/2$ shows that the following condition holds:

$$\frac{1}{\Gamma(\mu-2)} = 0;$$

this is possible only for $\mu = 2, 1, 0, -1, \ldots$. But for these μ, the function $E_{1/2}(z;\mu)$ has an explicit form (see the begining of Chap. 2), which shows that a multiple root can exist only at the point $z = 0$. Assertion (6) is verified, and the theorem is completely proved. $\square$

4.5. Zeros of the Function $E_1(z;\mu)$, Incomplete Gamma-Function, and the Error Function

Owing to the formula

$$E_1(z;\mu) = \frac{1}{\Gamma(\mu-1)} \int_0^1 e^{zt}(1-t)^{\mu-2}dt, \quad \mu > 1, \tag{4.5.1}$$

the case $\rho = 1$ admits a more detailed analysis of zeros, which is presented in the present section. As consequences we obtain assertions on the distribution of roots of two functions that can be expressed through $E_1(z;\mu)$: the incomplete gamma-function and the error function (Gaussian error function related to the density of normal distribution in probability theory).

4.5.1. Zeros of the function $E_1(z;\mu)$. The following lemma, by some modification of formula (4.5.1), allows one to broaden the set of values of the parameter μ.

Lemma 4.5.1. *The following representation holds*:

$$\Gamma(\mu)e^{-z}E_1(z;\mu) = 1 + (1-\mu)\int_0^1 (1-e^{-zt})t^{\mu-2}dt, \quad \mu > 0.$$

Proof. Since both sides of the equality are entire functions, it suffices to prove it for $z = x > 0$. By formula (4.5.1), we have the expression

$$E_1(x;\mu+1) = \frac{e^x}{\Gamma(\mu)} \int_0^1 e^{-tx}t^{\mu-1}dt. \tag{4.5.2}$$

Integration by parts yields

$$x\int_0^1 e^{-tx}t^{\mu-1}dt = -(e^{-x}-1) + (\mu-1)\int_0^1 (e^{-tx}-1)t^{\mu-2}dt.$$

Combining this relation, the well-known formula

$$E_1(x;\mu) = \frac{1}{\Gamma(\mu)} + xE_1(x;\mu+1),$$

and formula (4.5.2), we obtain the equality

$$E_1(x;\mu) = \frac{1}{\Gamma(\mu)} + \frac{e^x}{\Gamma(\mu)}\left(-\big(e^{-x}-1\big) + (\mu-1)\int_0^1 \big(e^{-tx}-1\big)t^{\mu-2}dt\right)$$
$$= \frac{e^x}{\Gamma(\mu)}\left(1 + (\mu-1)\int_0^1 \big(e^{-tx}-1\big)t^{\mu-2}dt\right).$$

This is equivalent to the required formula for $z = x > 0$. The lemma is proved. □

Lemma 4.5.2. *For $\mu > 0$, the following formula holds*:

$$\operatorname{Im}\big(e^{-z}E_1(z;\mu)\big) = \frac{1-\mu}{\Gamma(\mu)}\int_0^1 e^{-xt}t^{\mu-2}\sin yt\,dt, \quad z = x+iy. \tag{4.5.3}$$

Lemma 4.5.2 immediately follows from Lemma 4.5.1.

The main result of this section is the following theorem.

Theorem 4.5.1. *The following assertions hold.*

(1) *For $\mu > 2$, all roots of the function $E_1(z;\mu)$ belong to the set*

$$\operatorname{Re} z > \mu - 2, \quad |\operatorname{Im} z| > \pi.$$

(2) *For $1 < \mu < 2$, all roots of the function $E_1(z;\mu)$ belong to the set*

$$\operatorname{Re} z < \mu - 2, \quad |\operatorname{Im} z| > \pi. \tag{4.5.4}$$

(3) *For $0 < \mu < 1$, all roots of the function $E_1(z;\mu)$, except for a unique negative root, belong to the set* (4.5.4).

(4) *For $\mu \in (-n, -n+1)$, $n \in \mathbb{Z}_+$, the function $E_1(z;\mu)$ has a unique negative root.*

Note that positive roots of the function $E_1(z;\mu)$ were characterized in Theorem 4.1.2, and recall that for $\mu = 1, 0, -1, \ldots$, this function has an explicit representation (see the beginning of Chap. 2).

Proof. (1)–(3) Let $\mu > 0$. Then for $0 < y \le \pi$ (respectively, $-\pi \le y < 0$), the integrand in formula (4.5.3) is positive (respectively, negative) in the interval $(0,1)$. Therefore, by formula (4.5.3), the function $E_1(z;\mu)$ has no roots on the set $0 < |\operatorname{Im} z| \le \pi$.

Formula (4.5.1) implies the absence of real zeros of the function $E_1(z;\mu)$, $\mu > 1$.

Let $0 < \mu < 1$. The definition of the Mittag-Leffler function implies that there are no nonnegative roots of the function $E_1(z;\mu)$. The assertion on the uniqueness of a negative root will be proved in part (4) (we have used assertion (4), which has not been proved, but this does not lead to a vicious circle in the proof).

Thus, to prove assertions (1)–(3), it remains to examine the distribution of nonreal roots in the corresponding half-planes. The analysis is based on representations (4.5.1) and (4.5.3) and Lemma 4.3.1.

Let $\mu \in (0,1) \cup (1,2)$. For fixed $x \in \mathbb{R}$, we set

$$f(t) = \begin{cases} e^{-xt} t^{\mu-2}, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

We show that for $x \geq \mu - 2$, the function $f(t)$ satisfies the conditions of Lemma 4.3.1.

The condition

$$\frac{tf(t)}{t+1} \in L^1(\mathbb{R}_+)$$

is obvious. Further,

$$f'(t) = e^{-xt} t^{\mu-3}(-xt + (\mu - 2)), \quad 0 < t < 1.$$

If $x \geq 0$, then $f'(t) < 0$ on $(0,1)$ since $\mu - 2 < 0$. If $x < 0$, then $-xt + \mu - 2 < -x + \mu - 2$ for $t \in (0,1)$. Therefore, $f'(t) < 0$ on $(0,1)$ also for $\mu - 2 \leq x < 0$. Thus, $f(t)$ decreases on $(0,1)$. Finally, for $x \geq \mu - 2$, all conditions of Lemma 4.3.1 hold. Applying this lemma to integral (4.5.3), we obtain that

$$E_1(z; \mu) \neq 0 \quad \text{for} \quad x = \operatorname{Re} z \geq \mu - 2, \quad y \neq 0.$$

Thus, assertions (2) and (3) are proved.

Let $\mu > 2$. From (4.5.1) it follows that

$$\Gamma(\mu - 1) \operatorname{Im} E_1(z; \mu) = \int\limits_{\mathbb{R}_+} f(t) \sin yt \, dt,$$

where

$$f(t) = \begin{cases} e^{xt}(1-t)^{\mu-2}, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

If we prove that the function $f(t)$ decreases on $(0,1)$ for $x \leq \mu - 2$, then, by Lemma 4.3.1, the function $E_1(z; \mu)$ has no roots in the half-plane $\operatorname{Re} z = x \leq \mu - 2$, and assertion (1) will be proved.

We have

$$f'(t) = e^{xt}(1-t)^{\mu-3}(x(1-t) - (\mu - 2)),$$

which implies that for $x \leq \mu - 2$, the derivative of the function $f(t)$ is negative on $(0,1)$. The function $f(t)$ decreases on $(0,1)$. We have proved assertion (3).

(4) Let $\mu \in (-n, -n+1)$, $n \in \mathbb{Z}_+$. By the definition of the Mittag-Leffler function and Theorem 1.2.1, we have the equality

$$E_1(0, \mu) = \frac{1}{\Gamma(\mu)}, \quad \operatorname{sign} E_1(-\infty; \mu) = \operatorname{sign} \Gamma(\mu - 1).$$

Since $\Gamma(\mu)$ and $\Gamma(\mu - 1)$ have opposite signs, this implies that the function $E_1(x; \mu)$ has at least one negative root. By Theorem 4.4.1, it is simple. Prove the uniqueness.

Assume the contrary, i.e., let the function $E_1(x; \mu)$ have at least two negative roots x_1 and x_2. By Theorem 4.4.1, they are simple. Since

$$E_\rho(z; \mu) \to E_1(z; \mu), \quad \rho \to 1 + 0,$$

uniformly in any disk, by the Hurwitz theorem, for ρ sufficiently close to 1, $\rho > 1$, small neighborhoods of the points x_1 and x_2 contain exactly one root of the function $E_\rho(z; \mu)$. Since conjugate roots form pairs, we conclude that both these roots are real, which contradicts Theorem 4.1.1. We have proved the uniqueness of a negative root, and hence assertion (4) holds. The theorem is completely proved. □

In Chap. 5, assertion (2) of Theorem 4.5.1 will be substantially strengthened by another approach.

4.5.2. Zeros of the incomplete gamma-function and the error function. Consider the *incomplete gamma-function* of the variable z:

$$\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt, \quad \operatorname{Re}\alpha > 0. \tag{4.5.5}$$

This function is analytic everywhere, except for a possible branch point $z = 0$; the principal branch is defined by the cut along the negative real half-line. Expanding e^{-t} in a power series and integrating, we obtain the representation

$$\gamma(\alpha, z) = z^{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!(n+\alpha)},$$

which allows one to continue $\gamma(\alpha, z)$ to all values $\alpha \neq 0, -1, -2, \dots$. This implies that the modified incomplete gamma-function

$$\gamma^*(\alpha, z) = \frac{\gamma(\alpha, z)}{z^{\alpha}\Gamma(\alpha)}$$

is an entire functions (also with respect to α) and the following formula holds:

$$\gamma^*(\alpha, z) = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+n+1)}$$

(see [18, Chap. 2, Sec. 5]). Therefore, for all $\alpha \in \mathbb{C}$,

$$\gamma^*(\alpha, z) = e^{-z} E_1(z; 1+\alpha). \tag{4.5.6}$$

This representation allows one to apply theorems on zeros of the function $E_{\rho}(z; \mu)$ for $\rho = 1$ and $\mu = 1 + \alpha$ to the function $\gamma^*(\alpha, z)$. We exclude from consideration the degenerate case $\alpha \in -\mathbb{Z}_+$, where $\gamma^*(\alpha, z) = z^{-\alpha}$.

Corollary 4.5.1. *Let $\alpha \notin -\mathbb{Z}_+$. Then all zeros w_n of the function $\gamma^*(\alpha, w)$ are simple. They can be numbered such that $n \in \mathbb{Z} \setminus 0$ and*

$$w_n = 2\pi n i + (\alpha - 1)\ln 2\pi i n + \ln\frac{1}{\Gamma(\alpha)} + (1-\alpha)^2 \frac{\ln 2\pi i n}{2\pi i n} + O\left(\frac{1}{n}\right), \quad n \to \pm\infty. \tag{4.5.7}$$

Corollary 4.5.1 follows from formula (4.5.6) and Theorems 2.2.1, 4.2.2, and 4.1.1.

Corollary 4.5.2. *The following assertions hold.*

(1) *For $\alpha > 1$, all roots of the function $\gamma^*(\alpha, w)$ belong to the set*

$$\operatorname{Re} w > \alpha - 1, \quad |\operatorname{Im} w| > \pi.$$

(2) *For $0 < \alpha < 1$, all roots of the function $\gamma^*(\alpha, w)$ belong to the set*

$$\operatorname{Re} w < \alpha - 1, \quad |\operatorname{Im} w| > \pi. \tag{4.5.8}$$

(3) *For $-1 < \alpha < 0$, all roots of the function $\gamma^*(\alpha, w)$, except for a unique negative root, belong to the set* (4.5.8)*.*

(4) *For $\alpha \in (-n-1, -n)$, $n \in \mathbb{Z}_+$ the function $\gamma^*(\alpha, w)$ has a unique negative root.*

Corollary 4.5.2 follows from formula (4.5.6) and Theorem 4.5.1.

In concluding this section, we consider the *error function*

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{4.5.9}$$

It is an entire function having a simple root $z = 0$ and having no other real roots. From (4.5.5) and (4.5.9) we obtain the formula

$$\operatorname{erf} z = \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, z^2\right)$$

(see also [18, Chap. 2, Sec. 2]), which shows, together with the definition of $\gamma^*(\alpha, z)$, that all nonzero roots of the function $\operatorname{erf} z$ coincide with roots of the function $\gamma^*(1/2, z^2)$. By Corollary 4.5.2, roots of the function $\gamma^*(1/2, z^2)$ belong to the set

$$\operatorname{Re} z^2 < -1/2, \quad |\operatorname{Im} z^2| > \pi.$$

This set lies in the angles

$$\left|\pm\frac{\pi}{2} - \arg z\right| < \frac{\pi}{4}$$

and consists of the intersections of two curvilinear sectors, which are symmetric with respect to the real axis and are bounded by branches of the hyperbola

$$y^2 - x^2 = \frac{1}{2} \quad (z = x + iy),$$

with the outer region of the hyperbolic cross $|xy| \leq \pi$.

Consider the mapping $w = z^2$ of the angle

$$\left|\frac{\pi}{2} - \arg z\right| < \frac{\pi}{4} \tag{4.5.10}$$

on the half-plane $\operatorname{Re} w < 0$, which, by Corollary 4.5.2, contains roots w_n of the function $\gamma^*(1/2, w)$; recall that these roots are simple.

Denote by z_n^+ roots of the function $\operatorname{erf} z$ in the angle (4.5.10). Then $n \in \mathbb{Z}\setminus 0$, and by formula (4.5.7) with $\alpha = 1/2$ we have

$$(z_n^+)^2 = 2\pi ni - \frac{1}{2}\ln 2\pi in - \ln\Gamma\left(\frac{1}{2}\right) + \frac{\ln 2\pi in}{8\pi in} + O\left(\frac{1}{n}\right), \quad n \to \pm\infty.$$

Since $\operatorname{erf} z$ is odd, the symmetric angle $\left|\pi/2 + \arg z\right| < \pi/4$ contains a symmetric chain of roots z_n^- of the function $\operatorname{erf} z$. The function $\operatorname{erf} z$ has no roots other than the point $z = 0$. Since $\Gamma(1/2) = \sqrt{\pi}$, we have proved the following assertion.

Corollary 4.5.3. *All zeros of the function* $\operatorname{erf} z$ *are simple. Nonzero roots form two chains* z_n^+ *and* z_n^-, $n \in \mathbb{Z}\setminus 0$, *which are symmetric with respect to the real axis and lie in the intersection of the hyperbolic sectors* $y^2 - x^2 > 1/2$ *with the outer region of the hyperbolic cross* $|xy| \leq \pi/2$. *Moreover,*

$$(z_n^\pm)^2 = 2\pi ni - \frac{1}{2}\ln 2\pi in - \frac{1}{2}\ln\pi + \frac{\ln 2\pi in}{8\pi in} + O\left(\frac{1}{n}\right), \quad n \to \pm\infty.$$

Theorem 4.2.1 belongs to Ostrovskii and Peresyolkova (see [19]). Other results of this chapter were proved by Sedletskii (see [31, 32, 36, 37]).

CHAPTER 5

ZEROS OF LAPLACE TRANSFORMS AND DEGENERATE HYPERGEOMETRIC FUNCTION

5.1. Statement of the Problem

In this chapter, we examine the distribution of zeros of functions that, in some sense, are related to the Mittag-Leffler function for $\rho = 1$. Namely, we consider the *Laplace transforms* of functions concentrated in the interval $(0,1)$, i.e., entire functions of the form

$$F(z) = \int_0^1 e^{zt} f(t)dt, \quad f \in L^1(0,1), \tag{5.1.1}$$

and the *confluent hypergeometric function* (the *Kummer function*)

$$\Phi(a,c;z) := 1 + \sum_{s=1}^{\infty} \frac{a(a+1)\dots(a+s-1)}{c(c+1)\dots(c+s-1)} \frac{z^s}{s!}, \quad c \notin -\mathbb{Z}_+ \tag{5.1.2}$$

(see, e.g., [10, 18, 40]), as a function of the variable z for fixed values of the parameters $a, c \in \mathbb{C}$.

These function are often used in various branches of analysis, for example, in spectral theory, in the theory of differential-difference equations, in the study of nonharmonic Fourier series, etc. Functional classes (5.1.1) and (5.1.2) are extremely vast. In particular, class (5.1.1), contains the functions $E_1(z;\mu)$, $\operatorname{Re}\mu > 1$, owing to the formula

$$E_1(z;\mu) = \frac{1}{\Gamma(\mu-1)} \int_0^1 e^{zt}(1-t)^{\mu-2}dt, \quad \operatorname{Re}\mu > 1 \tag{5.1.3}$$

(see (1.4.21)). The well-known integral representation

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{c-a-1}dt, \quad 0 < \operatorname{Re} a < \operatorname{Re} c \tag{5.1.4}$$

(see [10, 18, 40]) shows that

$$\Phi(1,c;z) = \Gamma(c)E_1(z;c), \quad \operatorname{Re} c > 1.$$

Note that the modified Bessel function can be represented by the Kummer function for $c = 2a$ (see [40]).

5.2. Zeros of Finite Laplace Transforms

The starting point is the following Pólya theorem from [20] (see also [34]).

Theorem A. *Let a function $f(t)$ be integrable and positive and not decrease in the interval $(0,1)$. Then all zeros of the function (5.1.1) lie in the left-hand half-plane $\operatorname{Re} z \le 0$. Moreover, if $f(t)$ is not an echelon function of the form*

$$f(t) = c_i, \quad t_i < t < t_{i+1}, \quad i = 0,\dots,n; \quad t_0 = 0, \quad t_{n+1} = 1, \quad 0 < c_i < c_{i+1},$$

where numbers t_i are rational, then all zeros of the function (5.1.1) lie in the open left-hand half-plane $\operatorname{Re} z < 0$.

In the class of function f satisfying the conditions of this theorem, we consider a sufficiently wide subclass and prove for it more thorough assertions on the distribution of zeros of function (5.1.1), namely, assertions on the localization of all zeros of function (5.1.1) in some proper subsets of the set $\operatorname{Re} z < 0$. The additional condition defining this subclass consists of the requirement that the function f is logarithmically convex in some left-hand neighborhood of the point $t = 1$. For this subclass, we find the shape of a (left-hand) curvilinear half-plane, and under the condition $f(+0) > 0$ the shape of a curvilinear strip that contains all zeros of function (5.1.1). In both cases, the description of the set containing all zeros is exact, in some sense. Moreover, we prove that if f is logarithmically convex on the whole interval $(0, 1)$, then, independently of the monotonicity of f, all zeros of function (5.1.1) lie in the union of horizontal strips $(2n-1)\pi < |\operatorname{Im} z| < 2\pi n$, $n \in \mathbb{N}$.

5.2.1. Distribution of zeros in curvilinear half-planes and strips.

Definition. A function f that is positive in an interval (a, b) is said to be *logarithmically convex* in this interval if the function $\ln f$ is convex in (a, b).

For a twice differentiable positive function, its logarithmic convexity in (a, b) is equivalent to the condition

$$(f'(t))^2 \le f(t)f''(t), \quad t \in (a, b)$$

(see [22]). In the general case, the criterion of the logarithmic convexity is as follows.

Lemma 5.2.1. *Let a function f be positive in an interval (a, b). It is logarithmically convex in this interval if and only if the function*

$$f_x(t) := e^{xt} f(t)$$

is convex in (a, b) for all $x \in \mathbb{R}$.

Proof. In the proof of both necessity and sufficiency, we deal with the continuous functions f and $\ln f$. We use the fact that for a function $\varphi \in C(a, b)$ the convexity of φ in (a, b) is equivalent to the condition

$$\varphi\left(\frac{t_1 + t_2}{2}\right) \le \frac{\varphi(t_1) + \varphi(t_2)}{2} \tag{5.2.1}$$

for all points $t_1, t_2 \in (a, b)$ (see [8]). We obtain that the logarithmic convexity of the function f in (a, b) is equivalent to the condition

$$f\left(\frac{t_1 + t_2}{2}\right) \le \sqrt{f(t_1)f(t_2)} \tag{5.2.2}$$

for all $t_1, t_2 \in (a, b)$.

Let a function f be logarithmically convex in (a, b). Then property (5.2.2) holds. Multiplying (5.2.2) by $\exp(x(t_1 + t_2)/2)$, we obtain the inequality

$$f_x\left(\frac{t_1 + t_2}{2}\right) \le \sqrt{f_x(t_1)f_x(t_2)}.$$

Applying to the right-hand side the Cauchy inequality, we obtain for the function $\varphi = f_x$ property (5.2.1), which means the convexity of the function f_x.

Conversely, let the function f_x be convex in (a, b). Then for $\varphi = f_x$ property (5.2.1) is valid. We can rewrite it in the form

$$f\left(\frac{t_1 + t_2}{2}\right) \le \frac{1}{2}(e^{x(t_1 - t_2)/2} f(t_1) + e^{x(t_2 - t_1)/2} f(t_2)). \tag{5.2.3}$$

By the condition, for fixed t_1 and t_2 this inequality holds for all $x \in \mathbb{R}$, in particular, for a value of x for which the minimum of the right-hand side is attained. Differentiating, we see that for this value of x, terms on the right-hand side of (5.2.3) coincide. Then their half-sum is equal to their geometric mean and we obtain property (5.2.2), which is equivalent to the logarithmic convexity of the function f. The lemma is proved. □

Lemma 5.2.2 (see [9]). *If a function* φ *is convex in an interval* $(0, 2\pi)$ (*respectively, in the interval* $(\pi/2, 5\pi/2)$), *then*

$$\int_0^{2\pi} \varphi(t)\cos t\, dt \geq 0$$

(respectively,

$$\int_{\pi/2}^{5\pi/2} \varphi(t)\sin t\, dt \geq 0).$$

Recalling that $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $r = |z|$, $\theta = \arg z$, we introduce the notation

$$\Phi_u(z) = \int_u^1 e^{-zt} g(t)dt. \tag{5.2.4}$$

Lemma 5.2.3. *Let a function* g *have bounded variation of a segment* $[a_0, 1]$, $0 \leq a_0 < 1$, *and* $g(1-0) \neq 0$. *Then for any fixed* $b_0 \in (a_0, 1)$, *we have the estimate*

$$|\Phi_u(z)| \geq C\frac{e^{-x}}{r}(1 + o(1)), \quad x \to -\infty, \tag{5.2.5}$$

where $C > 0$ *is independent of* $u \in [a_0, b_0]$.

Proof. Assume that $g(1) = g(1-0)$; by the condition, $g(1) \neq 0$. Using the continuity of the function $g(t)$ at the point 1 from the left, we fix $\delta \in (0, 1-b_0)$ so small that

$$\operatorname{var}(g(t) : 1-\delta \leq t \leq 1) \leq \frac{|g(1)|}{2}. \tag{5.2.6}$$

Integrate by parts:

$$\Phi_u(z) = -\frac{1}{z}\left(e^{-z}g(1) - e^{-zu}g(u) - \int_u^1 e^{-zt}dg(t)\right). \tag{5.2.7}$$

We divide the last integral into two terms K_1 and K_2 corresponding to the segments $[u, 1-\delta]$ and $[1-\delta, 1]$, respectively. For $x \leq 0$, taking into account (5.2.6), we obtain the estimates

$$|K_1| \leq Ve^{-(1-\delta)x}, \quad |K_2| \leq \frac{|g(1)|}{2}e^{-x}.$$

These estimates together with the triangle inequality applied to (5.2.7) yield the inequality

$$|\Phi_u(z)| \geq \frac{|g(1)|}{2}\frac{e^{-x}}{r}(1 - M_1e^{(1-\delta)x} - M_2e^{\delta x}), \quad x \leq 0,$$

which proves Lemma 5.2.3. □

Theorem 5.2.1. *Let the conditions of Theorem* A *hold and, moreover, let a function* $f(t)$ *be logarithmically convex on an interval* $(1-a, 1)$, $0 < a \leq 1$. *Then all zeros of function* (5.1.1) *with sufficiently large moduli lie in the set*

$$x \leq -\ln\left(y^2 \int_0^{\pi/(2|y|)} tf(1-t)dt\right) + C, \quad |y| > y_0,$$

where C is some constant.

Note that the meaningfulness of Theorem 5.2.1 (in comparison with Theorem A) is manifested in the case $f(1-0)=+\infty$, where

$$y^2 \int\limits_0^{\pi/(2|y|)} tf(t)dt > \frac{\pi^2}{8} f\left(1-\frac{\pi}{2|y|}\right) \to +\infty, \quad y \to \pm\infty. \tag{5.2.8}$$

Proof. Since $F(\bar{z}) = \bar{F}(z)$, it suffices to consider only $y>0$. We have the expressions

$$e^{-z}F(z) = \int\limits_0^1 e^{-zt} f(1-t)dt =: G(z) \tag{5.2.9}$$

and

$$V(z) := -\operatorname{Im} G(z) = \int\limits_0^1 e^{-xt} f(1-t)\sin ytdt.$$

We obtain a lower estimate of $|V(z)|$ for sufficiently large y and for $x<0$ (recall that, by Theorem A, the function $F(z)$ has no zeros for $x \geq 0$). We denote

$$N = N(y) := \max\left(n \in \mathbb{N} : \frac{\pi}{2y} + \frac{2\pi n}{y} \leq a\right),$$

$$a_N := \frac{\pi}{2y} + \frac{2\pi N}{y}$$

and note that $a_N \leq a$. Moreover, since

$$\frac{\pi}{2y} + \frac{2\pi(N+1)}{y} > a,$$

we have $a_N > a - 2\pi/y$ and therefore

$$\frac{a}{2} < a_N \leq a \tag{5.2.10}$$

for sufficiently large y. Let

$$V(z) = \left(\int\limits_0^{\pi/(2y)} + \int\limits_{\pi/(2y)}^{a_N} + \int\limits_{a_N}^1 \right) e^{-xt} f(1-t)\sin yt\, dt =: V_1 + V_2 + V_3.$$

Since $x<0$, we have

$$V_1 > \frac{2}{\pi} y \int\limits_0^{\pi/(2y)} tf(1-t)dt.$$

Further, by the condition, the function $f(1-t)$ is logarithmically convex on $(0,a)$. By Lemma 5.2.1, the function $e^{-xt}f(1-t)$ is convex on $(0,a)$ and hence by Lemma 5.2.2, $V_2 \geq 0$. Since

$$V_3 = -\operatorname{Im} \int\limits_{a_N}^1 e^{-zt} f(1-t)dt = \operatorname{Im}\left(\frac{1}{z}\int\limits_{a_N}^1 f(1-t)de^{-zt}\right)$$

$$= \operatorname{Im}\left(\frac{1}{z}\left(f(+0) - f(1-a_N)e^{-za_N} - \int\limits_{a_N}^1 e^{-zt}df(1-t)\right)\right),$$

taking into account (5.2.10), for $x < 0$ we have the relations

$$|V_3| \le A\frac{e^{-x}}{r} \le A\frac{e^{-x}}{y}, \quad A = 3f\left(1 - \frac{a}{2}\right).$$

The function $G(z)$ (and hence the function $F(z)$) has no zeros in the region where $V_1 + V_2 > |V_3|$. Therefore, the estimates obtained for V_i $i = 1, 2, 3$, imply that $F(z)$ has no zeros in the region where

$$\frac{2}{\pi} y \int\limits_0^{\pi/(2y)} t f(1-t)dt > \frac{A}{y} e^{-x}, \quad y > y_0 > 0.$$

Taking the logarithm, we obtain the assertion of the theorem. Theorem 5.2.1 is proved. □

Theorem 5.2.1, the inequality in (5.2.8), and Theorem A imply the following assertion.

Corollary 5.2.1. *Let the conditions of Theorem* 5.2.1 *hold. Then all zeros of function* (5.1.1) *lie in the union of the set*

$$x \le \min\left(0, \; -\ln f\left(1 - \frac{\pi}{2|y|}\right) + C\right), \quad |y| > \frac{\pi}{2},$$

where C is a constant, and the semi-strip $x < 0$, $|y| \le \pi/2$.

Denote by $(z_n)_{n=1}^{\infty}$, $|z_{n+1}| \ge |z_n|$, the sequence of all zeros of function (5.1.1)[9]. We are interested in condition for the function f under which

$$\operatorname{Re} z_n \to -\infty, \quad n \to \infty. \tag{5.2.11}$$

It was proved in [28] that, under the conditions of Theorem A, all zeros of function (5.1.1) lie in some vertical strip $-\infty < h \le \operatorname{Re} z \le 0$ if and only if

$$0 < f(+0) \le f(1-0) < +\infty.$$

Therefore, if condition (5.2.11) holds, then at least one of the following conditions is necessary:

$$f(+0) = 0, \quad f(1-0) = +\infty.$$

Corollary 5.2.1 shows that, under the conditions of Theorem 5.2.1, the second of them is also sufficient.

Corollary 5.2.2. *Let a function f satisfy the conditions of Theorem* 5.2.1. *If*

$$f(1-0) = +\infty,$$

then zeros of function (5.1.1) *satisfy the relation* (5.2.11).

The right bound for zeros of function (5.1.1) guaranteed by Theorem 5.2.1 depends on the behavior of the function f in a left-hand neighborhood of the point 1. Now we find a left bound for zeros. It also depends on the behavior of the function f in a right-hand neighborhood of the point 0. We impose the condition

$$f(+0) > 0. \tag{5.2.12}$$

Theorem 5.2.2. . *Let the conditions of Theorem* 5.2.1 *hold and, moreover, let condition* (5.2.12) *hold. Then there exist constants $C \in \mathbb{R}$ and $H, y_0 > 0$ such that all zeros of function* (5.1.1) *lie in the union of the set*

$$x \ge -\ln\left(|y| \int\limits_0^{\pi/(2|y|)} f(1-t)dt\right) + C, \quad |y| > y_0,$$

with the set $-H \le x < 0$, $|y| \le y_0$.

[9]It is well known (see, e.g., [12]) that function (5.1.1) has an infinite set of zeros.

Proof. Let $G(z)$ and a_N be the same as in the proof of Theorem 5.2.1. It suffices to consider $y \geq 0$. First, we show that any sector of the form $\cos\theta \leq -\delta < 0$ contains no more than a finite number of zeros of the function $G(z)$. For this, we denote by J_1 and J_2 the parts of the integral in (5.2.9) corresponding to the intervals $(0, 1/2)$ and $(1/2, 1)$, respectively. Then

$$|J_1| \leq e^{-x/2}\|f\|_1, \quad x \leq 0.$$

Since $J_2 = \Phi_{1/2}$ in notation (5.2.4), by Lemma 5.2.3 (with $g(t) = f(1-t)$), for J_2 we have estimate (5.2.5). Therefore, if H is sufficiently large, then by the triangle inequality for $x \leq H < 0$ we obtain the estimate

$$|G(z)| \geq C_1\frac{e^{-x}}{r} - C_2 e^{-x/2} \geq C_1\frac{e^{-x}}{r}(1 - C_3 r e^{x/2}).$$

The expressions in the last parentheses is positive if $\cos\theta \leq -\delta < 0$ and r is sufficiently large, and the intermediate assertion for sectors is proved.

Thus, to find a left bound for zeros, we can consider instead of the left-hand half-plane the set

$$x < H, \quad y > \frac{r}{2}, \tag{5.2.13}$$

where H is sufficiently large. We can assume that $a < 1$. We set

$$G(z) = \left(\int_0^{a_N} + \int_{a_N}^1\right) e^{-zt} f(1-t)dt =: G_1(z) + G_2(z).$$

By (5.2.10) and Lemma 5.2.3 we have estimate (5.2.5) for $\Phi_{a_N}(z) = G(z)$. Therefore, on the set (5.2.13) for sufficiently large H the following estimate is valid:

$$|G_2(z)| \geq B\frac{e^{-x}}{y}, \quad B > 0. \tag{5.2.14}$$

Now let $G_1 = U - iV$, where

$$U = \int_0^{a_N} e^{-xt} f(1-t)\cos yt\,dt, \quad V = \int_0^{a_N} e^{-xt} f(1-t)\sin yt\,dt.$$

We write

$$V = \int_0^{\pi/y} + \int_{\pi/y}^{3\pi/(2y)} + \int_{3\pi/(2y)}^{a_N - \pi/y} + \int_{a_N-\pi/y}^{a_N} =: V_1 + V_2 + V_3 + V_4.$$

Then $V_2 < 0$ by the negativeness of $\sin yt$. Applying Lemmas 5.2.1 and 5.2.2, we obtain inequality $V_3 \leq 0$. Since the function $f(1-t)$ does not increase and $-x/y \leq M_1 < +\infty$ on the set (5.2.13), we have the double inequality

$$0 < V_1 < 2e^{-\pi x/y}\int_0^{\pi/(2y)} f(1-t)dt < C\int_0^{\pi/(2y)} f(1-t)dt.$$

Further, taking into account (5.2.10) and the monotonicity of the function f and denoting I_N by $(a_N - \pi/y, a_N)$ and C_1 by $3f(1-a/2)$, for $y > y_0$ we obtain that

$$|V_4| = \left| \mathrm{Im} \left(\frac{1}{z} \int\limits_{I_N} f(1-t) d e^{-zt} \right) \right| \le \frac{1}{r} |f(1-a_N) e^{-z a_N}$$
$$- f\left(1 - a_N + \frac{\pi}{y}\right) e^{-z(a_N - \pi/y)} - \int\limits_{I_N} e^{-zt} df(1-t)| < C_1 \frac{e^{-ax}}{y}. \quad (5.2.15)$$

But

$$V = \int\limits_0^{\pi/(2y)} + \int\limits_{\pi/(2y)}^{a_N} > 0, \quad (5.2.16)$$

since in the first term of the integrand the function is positive and to the second term, owing to Lemma 5.2.1, we can apply Lemma 5.2.2. Since $V_2, V_3 \le 0$, we have

$$0 < V = (V_1 + V_4) + (V_2 + V_3) \le V_1 + V_4 \le V_1 + |V_4|,$$

i.e.,

$$|V| \le C \int\limits_0^{\pi/(2y)} f(1-t) dt + C_1 \frac{e^{-ax}}{y}. \quad (5.2.17)$$

Similarly, for the function U,

$$U = \int\limits_0^{\pi/(2y)} + \int\limits_{\pi/(2y)}^{\pi/y} + \int\limits_{\pi/y}^{a_N - 3\pi/(2y)} + \int\limits_{a_N - 3\pi/(2y)}^{a_N} =: U_1 + U_2 + U_3 + U_4.$$

Note that $U_2 < 0$ by the negativeness of $\cos yt$ and $U_3 \le 0$ by Lemmas 5.2.1 and 5.2.2. Further, on the set (5.2.13) the inequality

$$0 < U_1 < e^{-\pi x/(2y)} \int\limits_0^{\pi/(2y)} f(1-t) dt < C \int\limits_0^{\pi/(2y)} f(1-t) dt$$

is valid and $|U_4|$ satisfies estimate (5.2.15), i.e.,

$$|U_4| < C \frac{e^{-ax}}{y}.$$

Similarly to (5.2.16), we have the inequality

$$U = \int\limits_0^{a_N - \pi/(2y)} + \int\limits_{a_N - \pi/(2y)}^{a_N} > 0$$

and hence for $|U|$ we obtain estimate (5.2.17). Finally, estimate (5.2.17) is valid for $|G_1|$.

Recalling that $G = G_1 + G_2$ and applying estimate (5.2.17) for $|G_1|$ and estimate (5.2.14), we arrive at the inequality

$$|G(z)| \ge B \frac{e^{-x}}{y} - C_2 \int\limits_0^{\pi/(2y)} f(1-t) dt - C_3 \frac{e^{-ax}}{y}.$$

This implies that if H is sufficiently large, then, for $x < -H$, the function $G(z)$ (and hence the function $F(z)$) has no zeros on the set

$$\left(z = x + iy : \ \frac{B}{2} e^{-x} > C_2 y \int\limits_0^{\pi/(2y)} f(1-t)dt, \quad y > y_0 \right).$$

This proves Theorem 5.2.2. □

From Theorems 5.2.1, 5.2.2, and A we obtain the following assertion.

Corollary 5.2.3. *Let the conditions of Theorem* 5.2.2 *hold. Then there exist constants* $C_1, C_2 \in \mathbb{R}$ *and* $H, y_0 > 0$ *such that all zeros of function* (5.1.1) *lie in the curvilinear strip, which is the union of the set*

$$-\ln\left(|y| \int\limits_0^{\pi/(2|y|)} f(1-t)dt \right) + C_1 \le x \le -\ln\left(y^2 \int\limits_0^{\pi/(2|y|)} tf(1-t)dt \right) + C_2 < 0, \tag{5.2.18}$$

where $|y| > y_0$*, with the set* $-H \le x < 0$*,* $|y| \le y_0$*.*

Remark 5.2.1. Due to Corollary 5.2.1, the set (5.2.18) in Corollary 5.2.3 can be replaced by the set

$$-\ln\left(|y| \int\limits_0^{\pi/(2|y|)} f(1-t)dt \right) + C_1 \le x \le -\ln f\left(1 - \frac{\pi}{2|y|}\right) + C_2 < 0.$$

For a function f regularly varying in a neighborhood of the point 1, asymptotics of integrals in (5.2.18) can be easily calculated, which allows one to simplify the form of the set (5.2.18).

A function g positive and measurable in a right-hand neighborhood of the point 0 is called a *regularly varying function* of order $\alpha \in \mathbb{R}$ if for all $\lambda > 0$,

$$\lim_{t \to +0} \frac{g(\lambda t)}{g(t)} = \lambda^\alpha.$$

A regularly varying function of order $\alpha = 0$ is called a *slowly varying* function. A regularly varying function g of order α has the form

$$g(t) = t^\alpha l(t), \tag{5.2.19}$$

where $l(t)$ is a slowly varying function. For any slowly varying function $l(t)$, there exists an equivalent, as $t \to +0$, continuously differentiable function $l_0(t)$ satisfying the condition

$$l_0'(t) = o\left(\frac{l_0(t)}{t}\right), \quad t \to +0. \tag{5.2.20}$$

The definitions and facts presented above are taken from [38].

Introduce the notation

$$g_1(u) = \frac{1}{u} \int\limits_0^u g(t)dt, \quad g_2(u) = \frac{1}{u^2} \int\limits_0^u tg(t)dt, \quad 0 < u < 1.$$

Lemma 5.2.4. *Let a function* g *be integrable on* $(0, 1)$ *and be a regularly varying function of order* $\alpha > -1$*. Then*

$$g_1(u) \sim \frac{1}{1+\alpha} g(u), \quad g_2(u) \sim \frac{1}{2+\alpha} g(u), \quad u \to +0.$$

Proof. By the condition, representation (5.2.19) holds, where $l(t)$ is a slowly varying function. Since the replacement of the function g by an equivalent function does not change the asymptotics of the functions g_1 and g_1, we can assume that $l(t)$ satisfies property (5.2.20). By the L'Hôpital rule,

$$\lim_{u\to+0}\frac{g_1(u)}{g(u)}=\lim_{u\to+0}\frac{1}{u^{1+\alpha}l(u)}\int_0^u t^\alpha l(t)dt$$
$$=\lim_{u\to+0}\frac{u^\alpha l(u)}{(1+\alpha)u^\alpha l(u)+u^{1+\alpha}l'(u)}=\lim_{u\to+0}\frac{1}{1+\alpha+(ul'(u)/l(u))},\quad (5.2.21)$$

which, together with property (5.2.20), implies the required asymptotics for g_1. If in (5.2.21) we replace g_1 by g_2, then α is replaced by $1+\alpha$, and everything repeats. The lemma is proved. □

Corollary 5.2.4. *Let the condition of Theorem* 5.2.2 *hold and. moreover, let a function $f(1-t)$ be a regularly varying function of order $\alpha\in(-1,0]$. Then all zeros of function* (5.1.1) *lie in the curvilinear strip, which is the union of the set $\{-H\le x<0,\ |y|\le y_0\}$ with the set*

$$\left|x+\ln f\left(1-\frac{1}{|y|}\right)\right|\le C,\quad |y|>y_0,$$

where H, y_0, and C are positive numbers.

Indeed, introducing the notation

$$g(t)=f(1-t),\quad u=\frac{\pi}{2|y|}$$

and applying Lemma 5.2.4, we see that the operands of the logarithms in (5.2.18) are asymptotically proportional to the expressions $g_1(u)$ and $g_2(u)$, respectively. Therefore, Corollary 5.2.4 follows from Corollary 5.2.3 and the definition of regularly varying functions, which allows one to replace $f(1-\pi/(2|y|))$ by $f(1-1/|y|)$ in the resulting set.

The example

$$f(1-t)=g(t)=\frac{1}{t\ln^2 t},$$

where the functions

$$g_1(u)=\frac{1}{u|\ln u|},\quad g_2(u)\sim\frac{1}{u\ln^2 u},\quad u\to+0,$$

are not asymptotically proportional, shows that in the case $\alpha=-1$ we must be satisfied with Remark 5.2.1.

5.2.2. Distribution of zeros in horizontal strips.

Theorem 5.2.3. *Let a function f be integrable and logarithmically convex in an interval $(0,1)$. Then all zeros of function* (5.1.1) *lie in the union of the horizontal strips*

$$(2n-1)\pi<|\operatorname{Im}z|<2\pi n,\quad n\in\mathbb{N}.$$

Proof. Let

$$G(z)=e^{-z}F(z)=\int_0^1 e^{-zt}f(1-t)dt=:U(z)-iV(z).$$

It suffices to prove that the function $G(z)$ has no zeros in the strips

$$2\pi n\le y\le(2n+1)\pi,\quad n\in\mathbb{Z}_+.$$

First, let

$$\left(2n+\frac{1}{2}\right)\pi \le y \le (2n+1)\pi, \quad n\in\mathbb{Z}_+. \tag{5.2.22}$$

If $n=0$, then $\sin yt > 0$ for $0<t<1$, and since $f>0$, we have $V>0$.

If $n\ge 1$, then we write

$$V(z) = \left(\int\limits_0^{\pi/(2y)} + \sum_{k=0}^{n-1}\int\limits_{\pi(1/2+2k)/y}^{\pi(5/2+2k)/y} + \int\limits_{\pi(1/2+2n)/y}^{1}\right) e^{-xt} f(1-t)\sin yt\,dt =: I_1 + \sum_{k=1}^{n-1} V_k + I_2.$$

For I_1 and I_2, we have the inequality $\sin yt > 0$. For I_1 this is obvious. For I_2, we have $\pi(1/2+2n)/y < t < 1$ and hence

$$\frac{\pi}{2} + 2\pi n < yt < \pi + 2\pi n$$

(we have taken into account the right-hand side inequality (5.2.22)). Since $f>0$, we have $I_1+I_2>0$.

Applying Lemmas 5.2.1 and 5.2.2, we obtain the inequality $V_k\ge 0$ on the set (5.2.22). Therefore, this set does not contain zeros of the function $G(z)$.

It remains to consider the strips

$$2\pi n \le y < \frac{\pi}{2} + 2\pi n, \quad n\in\mathbb{Z}_+. \tag{5.2.23}$$

Here we examine the function $U(z)$.

If $n=0$, then $\cos yt > 0$ for $0<t<1$ and hence $U>0$.

For $n\ge 1$ we have

$$U(z) = \sum_{k=1}^{n}\left(\int\limits_{2\pi(k-1)/y}^{2\pi k/y} + \int\limits_{2\pi n/y}^{1}\right) e^{-xt} f(1-t)\cos yt\,dt =: \sum_{k=1}^{n} U_k + J.$$

By Lemmas 5.2.1 and 5.2.2, again $U_k\ge 0$, and $J>0$ due to the positiveness of the integrand. Indeed, if $2\pi n/y < t < 1$, then by the right-hand inequality (5.2.23), we have the relation

$$2\pi n < yt < y \le \frac{\pi}{2} + 2\pi n.$$

Therefore, $U>0$ on the set (5.2.23), and the function $G(z)$ also has no zeros on this set. The theorem is proved. □

Theorems A and 5.2.3 imply the following assertion.

Corollary 5.2.5. *Let a function f be integrable, nondecreasing, and logarithically convex in the interval $(0,1)$. Then all zeros of function (5.1.1) lie in the union of the semi-strips*

$$\operatorname{Re} x < 0, \quad (2n-1)\pi < |\operatorname{Im} z| < 2\pi n, \quad n\in\mathbb{N}.$$

Formula (5.1.3) shows that for $\mu>1$, the function $E_1(z;\mu)$ has the form (5.1.1), where

$$f(t) = \frac{(1-t)^{\mu-2}}{\Gamma(\mu-1)}.$$

If $1<\mu<2$, then this function satisfies all conditions of Theorem 5.2.3. Moreover, by Theorem 4.3.4, in this case all zeros of the function $E_1(z;\mu)$ lie in the half-plane $\operatorname{Re} z < \mu-2$. Thus, Theorems 4.3.4 and 5.2.3 imply the following assertion.

Corollary 5.2.6. *For $1<\mu<2$, all zeros of the function $E_1(z;\mu)$ lie in the union of semi-strips*

$$\operatorname{Re} z < \mu-2, \quad (2n-1)\pi < |\operatorname{Im} z| < 2\pi n, \quad n\in\mathbb{N}.$$

The results of this section are taken from the paper [33].

5.3. Zeros of the Confluent Hypergeometric Function

In this section, we find the asymptotics of zeros of the confluent hypergeometric function (see formula (5.1.2)) for all values of the parameters a and c for which the set of zeros is infinite, and indicate a modus of numeration of all zeros matched with asymptotics. As a particular case, we consider a subclass of sine-type functions important for applications. We also obtain nonasymptotic properties of zeros, namely, their distributions in the left- and right-hand half-planes and in horizontal strips.

5.3.1. Asymptotics of zeros and its matching with numeration. If $a \in -\mathbb{Z}_+$, then the series in (5.1.2) terminates, i.e., $\Phi(a, c; z)$ is a polynomial. Further, if $c - a \in -\mathbb{Z}_+$, then $\Phi(c - a, c, z)$ is a polynomial and the Kummer formula (see [18])

$$e^{-z}\Phi(a, c; z) = \Phi(c - a, c; -z) \tag{5.3.1}$$

shows that the function $\Phi(a, c; z)$ has no more than a finite number of zeros. Therefore, in the study of zeros of function (5.1.2) we impose the condition $a, c, c - a \notin -\mathbb{Z}_+$.

Theorem 5.3.1. *Let $a, c \in \mathbb{C}$, $a, c, c - a \notin -\mathbb{Z}_+$. Then the following assertions hold:*

(1) *all zeros z_n of the function $\Phi(a, c; z)$ are simple and the following asymptotic formula holds:*

$$z_n = 2\pi i n + \left((c - 2a)\ln 2\pi|n| + \ln\frac{\Gamma(a)}{\Gamma(c - a)} \pm i\frac{\pi}{2}(c - 2)\right)\left(1 + \frac{c - 2a}{2\pi i n}\right) + \frac{2a(a - c) - c}{2\pi i n} + O\left(\frac{\ln|n|}{n^2}\right), \quad n \to \pm\infty; \tag{5.3.2}$$

(2) *the numeration of all zeros of the function $\Phi(a, c; , z)$ is matched with asymptotics* (5.3.2) *by the index set $T = \mathbb{Z} \setminus \{0\}$.*

Proof. (1) Since for all $z \in \mathbb{C}$ the function $w = \Phi(a, c; z)$ is a solution of the equation

$$zw'' + (c - z)w' - aw = 0 \tag{5.3.3}$$

(see [18, Chap. 7, Sec. 9]), the absence of multiple zeros is proved as for the Bessel function (see [18, Chap. 7, Sec. 6]), i.e., by using the uniqueness of solution of the Cauchy problem. Namely, let z_0 be a multiple zero of the function $w = \Phi$. Then

$$w(z_0) = w'(z_0) = 0. \tag{5.3.4}$$

But the function $w \equiv 0$ is also a solution of Eq. (5.3.3) with initial condition (5.3.4). By the theorem on the uniqueness of a solution of the Cauchy problem we have $\Phi \equiv 0$. The contradiction obtained proves the simpleness of all zeros of the function $\Phi(a, c; z)$.

To deduce formula (5.3.2), we use the well-known asymptotics (see [10, 40])

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c - a)}(-z)^{-a}\left(1 - \frac{a(1 + a - c)}{z} + O\left(\frac{1}{z^2}\right)\right) + \frac{\Gamma(c)}{\Gamma(a)}z^{a-c}e^z\left(1 + \frac{(1 - a)(c - a)}{z} + O\left(\frac{1}{z^2}\right)\right), \quad z \to \infty, \tag{5.3.5}$$

where

$$-\pi < \arg z \le \pi, \quad -\pi < \arg(-z) \le \pi. \tag{5.3.6}$$

For brevity, we introduce the notation

$$\gamma = \frac{\Gamma(a)}{\Gamma(c - a)}, \quad A = 2a(a - c) - c.$$

If $a, c-a \notin -\mathbb{Z}_+$, then γ is defined and $\gamma \neq 0$. Since

$$\left(1-\frac{a(a-c+1)}{z}+O\left(\frac{1}{z^2}\right)\right)\Big/\left(1+\frac{(1-a)(c-a)}{z}+O\left(\frac{1}{z^2}\right)\right)=1+\frac{A}{z}+O\left(\frac{1}{z^2}\right),$$

the formula (5.3.5) yields the following equation for sufficiently large (in modulus) zeros z_n of the function $\Phi(z)=\Phi(a,c;z)$:

$$z^{a-c}(-z)^a e^z=-\gamma\left(1+\frac{A}{z}+O\left(\frac{1}{z^2}\right)\right). \tag{5.3.7}$$

This implies that these zeros lie in the set

$$\arg z\mp\frac{\pi}{2}=O\left(\frac{\ln|z|}{|z|}\right), \quad |z|>r_0.$$

If $0<\arg z<\pi$ (respectively, $-\pi<\arg z<0$), then by condition (5.3.6), $-\pi<\arg(-z)<0$ (respectively, $0<\arg(-z)<\pi$). Therefore,

$$(-z)^a=z^a e^{\mp i\pi a}, \quad \operatorname{Im} z \gtrless 0. \tag{5.3.8}$$

Taking this into account and setting $-1=e^{\mp i\pi}$ respectively in the cases $\operatorname{Im} z\gtrless 0$, we rewrite Eq. (5.3.7) in the form

$$e^{z+(2a-c)\ln z}=\gamma e^{\mp i\pi(1-a)}\left(1+\frac{A}{z}+O\left(\frac{1}{z^2}\right)\right). \tag{5.3.9}$$

This implies that for sufficiently large $|n|$, the following relation holds:

$$z_n+(2a-c)\ln z_n=2\pi i n\mp i\pi(1-a)+\ln\gamma+\frac{A}{z_n}+O\left(\frac{1}{n^2}\right).$$

Twice iterating this formula, we obtain the required formula (5.3.2). Assertion (1) is proved.

(2) Step 1. Assuming that r_0 is sufficiently large and introducing the notation $K(r_0)=(z:|z|>r_0)$, we consider the sets

$$P_+=(z:|z^{a-c}(-z)^a e^z|>2|\gamma|)\cap K(r_0),$$
$$P_-=\left(z:|z^{a-c}(-z)^a e^z|<\frac{1}{2}|\gamma|\right)\cap K(r_0),$$
$$P=\left(z:\frac{1}{2}|\gamma|\le|z^{a-c}(-z)^a e^z|\le 2|\gamma|\right)\cap K(r_0),$$

which are the left- and right-hand curvilinear half-planes and the union of two curvilinear semi-strips lying in the upper and lower half-planes, respectively. Clearly,

$$\mathbb{C}\backslash(z:|z|\le r_0)=P_+\cup P_-\cup P.$$

We need convenient estimates $|\Phi(z)|$ on these sets. To obtain them, we consider the function

$$\Phi_1(z)=\frac{\Gamma(a)}{\Gamma(c)}(-z)^a\Phi(a,c;z).$$

From (5.3.5) follows that

$$\Phi_1(z)=z^{a-c}(-z)^a e^z\left(1+O\left(\frac{1}{z}\right)\right)+\gamma\left(1+O\left(\frac{1}{z}\right)\right), \quad z\to\infty. \tag{5.3.10}$$

On the sets P_+ and P_-, estimates for $|\Phi_1(z)|$ can be obtained from (5.3.10), the triangle inequality, and the definitions of these sets. Indeed, for sufficiently large r_0, from (5.3.10) we obtain the inequalities

$$|\Phi_1(z)| \le \frac{3}{2}|z^{a-c}(-z)^a e^z| + 2|\gamma|, \tag{5.3.11}$$

$$|\Phi_1(z)| \ge \frac{3}{4}|z^{a-c}(-z)^a e^z| - \frac{5}{4}|\gamma|, \tag{5.3.12}$$

$$|\Phi_1(z)| \ge -\frac{5}{4}|z^{a-c}(-z)^a e^z| + \frac{3}{4}|\gamma|. \tag{5.3.13}$$

Therefore, if $z \in P_+$, then (5.3.11) and (5.3.12) imply the estimates

$$\frac{1}{8}|z^{a-c}(-z)^a e^z| \le |\Phi_1(z)| \le \frac{5}{2}|z^{a-c}(-z)^a e^z|, \quad z \in P_+, \tag{5.3.14}$$

and if $z \in P_-$, then (5.3.11) and (5.3.13) imply the estimates

$$\frac{1}{8}|\gamma| \le |\Phi_1(z)| \le \frac{11}{4}|\gamma|, \quad z \in P_-. \tag{5.3.15}$$

If $z \in P$, then (5.3.11) yields the upper estimate

$$|\Phi_1(z)| \le 5|\gamma|, \quad z \in P. \tag{5.3.16}$$

However, a lower estimate cannot be obtained as easily as for the sets P_+ and P_-; the reason is the fact that the set P contains zeros of the function $\Phi_1(z)$. Therefore, we use the mapping

$$w = z + (2a - c)\ln z. \tag{5.3.17}$$

From (5.3.8) we see that

$$\operatorname{Re} w = \operatorname{Re}(z + (2a-c)\ln z) = \ln|z^{a-c}(-z)^a e^z| \mp \pi \operatorname{Im} a$$

respectively for $\operatorname{Im} z \gtrless 0$; therefore, the images of the components of the set P for sufficiently large $|\operatorname{Im} z|$ coincide with the semi-strips

$$\ln\frac{|\gamma|}{2} \mp \pi \operatorname{Im} a \le |\operatorname{Re} w| \le \ln(2|\gamma|) \mp \pi \operatorname{Im} a, \quad \operatorname{Im} w \gtrless 0, \quad |\operatorname{Im} w| > v_0. \tag{5.3.18}$$

By (5.3.10), the image $\Psi(w)$ of the function $\Phi_1(z)$ under mapping (5.3.17) has the form

$$\Psi(w) = e^w e^{\mp i\pi a}(1 + o(1)) + \gamma(1 + o(1)), \quad \operatorname{Im} w \to \infty, \quad \operatorname{Im} w \gtrless 0.$$

Obviously, in semi-strips (5.3.18), but outside small disks of a fixed radius δ centered at zeros of the function $\Psi(w)$ (i.e., at the points $w_n = 2\pi i n + \ln\gamma \mp i\pi(1-a) + o(1)$, $n \to \pm\infty$), the following estimate holds:

$$|\Psi(w)| \ge C(\delta) > 0.$$

Since δ can be taken arbitrarily small, this implies the existence of a sequence $r_k \uparrow +\infty$ such that

$$|\Phi_1(z)| \ge C_0 > 0, \quad z \in P, \quad |z| = r_k. \tag{5.3.19}$$

Now, by estimates (5.3.14)–(5.3.16) and (5.3.19), we obtain the resulting estimates for the function $\Phi(z)$:

$$|\Phi(z)| \asymp e^x r^{\operatorname{Re}(a-c)}, \quad z = x + iy = re^{i\theta} \in P_+, \tag{5.3.20}$$

$$|\Phi(z)| \asymp r^{-\operatorname{Re} a}, \quad z \in P_-, \tag{5.3.21}$$

$$|\Phi(z)| \asymp r^{-\operatorname{Re} a}, \quad z \in P, \quad r = r_k \uparrow +\infty. \tag{5.3.22}$$

Step 2. The definition of the set P_+ implies that on its boundary

$$x \asymp \ln r \quad \text{for} \quad \operatorname{Re}(2a - c) \ne 0$$

and

$$x = O(1) \quad \text{for} \quad \operatorname{Re}(2a - c) = 0$$

(both relations for $r > r_0$). Therefore, denoting by $r\exp(i\theta_r^\pm)$ the intersection point of the boundary of the set P_+ with the circle $|z| = r > r_0$, $\theta_r^\pm \gtrless 0$, we have the relations

$$\cos\theta_r^\pm = \frac{x}{r} = O\left(\frac{\ln r}{r}\right), \quad r \to \infty.$$

Applying the formulas $\cos\theta = \sin(\pi/2 \mp \theta)$, we obtain that

$$\theta_r^\pm = \pm\frac{\pi}{2} + O\left(\frac{\ln r}{r}\right), \quad r \to \infty. \tag{5.3.23}$$

Step 3. The final step of the proof is based on the Jensen formula

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\Phi(re^{i\theta})|d\theta, \tag{5.3.24}$$

where $n(t)$ is the number of zeros of the function $\Phi(z) = \Phi(a, c; z)$ in the disk $|z| < t$. We write

$$\int_{-\pi}^{\pi} \ln|\Phi(re^{i\theta})|d\theta = \int_{\theta_r^-}^{\theta_r^+} + \int_{\theta_r^+}^{\pi} + \int_{-\pi}^{\theta_r^-} =: J_1 + J_2 + J_3. \tag{5.3.25}$$

Using estimate (5.3.20) and relations (5.3.23) we have the relations

$$\begin{aligned} J_1 &= \int_{\theta_r^-}^{\theta_r^+} \Big(r\cos\theta + \operatorname{Re}(a - c)\ln r\Big)d\theta + O(1) \\ &= r\Big(\sin\theta_r^+ - \sin\theta_r^-\Big) + \Big(\theta_r^+ - \theta_r^-\Big)\operatorname{Re}(a - c)\ln r + O(1) \\ &= 2r + \pi\operatorname{Re}(a - c)\ln r + O(1), \quad r \to \infty. \end{aligned}$$

To estimate J_2 and J_3, we apply relations (5.3.21) and (5.3.22) and formulas (5.3.23). We obtain

$$\begin{aligned} J_2 + J_3 &= \left(\int_{\theta_r^+}^{\pi} + \int_{-\pi}^{\theta_r^-}\right)(-\operatorname{Re} a\ln r + O(1))d\theta \\ &= \left(\pi + O\left(\frac{\ln r}{r}\right)\right)(-\operatorname{Re} a\ln r + O(1)) = -\pi\operatorname{Re} a\ln r + O(1), \end{aligned}$$

where $r = r_k$ is a sequence from estimate (5.3.22). Substituting the estimate for J_k in formula (5.3.25) and then the obtained result in (5.3.24), we arrive at the following asymptotics:

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{r}{\pi} - \frac{\operatorname{Re} c}{2}\ln r + O(1), \quad r = r_k \to \infty. \tag{5.3.26}$$

On the other hand, we estimate the left-hand side in (5.3.26) by using formula (5.3.2).

By (5.3.2), there exists an integer m such that if Z is a sequence of all zeros of the function $\Phi(z)$, then

$$Z = Z_+ \cup Z_-, \quad Z_+ = (z_n^+)_{n=m}^{+\infty}, \quad Z_- = (z_n^-)_{n=-1}^{-\infty}, \quad Z_+ \cap Z_- = \emptyset, \tag{5.3.27}$$

and relation (5.3.2) is valid for $z_n^\pm$ as $n \to \pm\infty$. Thus, the numeration of (5.3.27) is matched with asymptotics (5.3.2), and it remains to prove that $m = 1$.

From (5.3.2) it follows that

$$|z_n| = \big((\operatorname{Im} z_n)^2 + (\operatorname{Re} z_n)^2\big)^{1/2} = |\operatorname{Im} z_n| \left(1 + O\left(\left(\frac{\operatorname{Re} z_n}{\operatorname{Im} z_n}\right)^2\right)\right) = |\operatorname{Im} z_n| + o(1), \quad n \to \pm\infty.$$

Taking this into account and using formula (5.3.2), we obtain the relation

$$|z_n^{\pm}| = 2\pi|n| \pm \operatorname{Im}(c-2a)\ln|n| + \pi\left(\frac{\operatorname{Re} c}{2} - 1\right) \\ \pm (\operatorname{Im}(c-2a)\ln 2\pi + \arg\gamma) + o(1), \quad n \to \pm\infty. \quad (5.3.28)$$

We use Lemma 2.2.2. If a positive sequence $\Lambda = (\lambda_n)_{n=m}^{+\infty}$ has the form

$$\lambda_n = a_1 n + a_2 \ln n + a_3 + o(1), \quad n \to +\infty, \quad a_1 > 0, \quad a_2, a_3 \in \mathbb{R},$$

and $\Lambda(t)$ is the number of points λ_n in the interval $(0, t)$, then

$$\int_0^r \frac{\Lambda(t)}{t} dt = \frac{r}{a_1} - \frac{a_2}{2a_1}\ln^2 r + \left(\frac{1}{2} - m - \frac{a_3}{a_1} + \frac{a_2}{a_1}\ln a_1\right)\ln r + o(\ln r), \quad r \to \infty. \quad (5.3.29)$$

Let us apply this lemma to the sequences $|Z_+| = (|z_n^+|)_{n=m}^{+\infty}$ and $|Z_-| = (|z_{-n}^-|)_{n=1}^{\infty}$ with the parameters

$$a_1 = 2\pi, \quad a_2 = \pm\operatorname{Im}(c-2a), \quad a_3 = \pi\left(\frac{\operatorname{Re} c}{2} - 1\right) \pm (\operatorname{Im}(c-2a)\ln 2\pi + \arg\gamma) \quad (5.3.30)$$

contained in (5.3.28).

We write relation (5.3.29) first for $\Lambda = |Z_+|$ and then for $\Lambda = |Z_-|$, and in the latter case, we must set $m = 1$ in relation with numeration (5.3.27). After this, we add the obtained relations. On the right-hand side, the terms corresponding to terms in (5.3.30) with opposite signs vanish, and we obtain

$$\int_0^r \frac{n(t)}{t} dt = \frac{r}{\pi} + \left(1 - m - \frac{\operatorname{Re} c}{2}\right)\ln r + o(\ln r), \quad r \to \infty.$$

Comparing this with formula (5.3.26), we see that $m = 1$. The theorem is proved. □

In connection with formula (5.3.2), we note that the choice of the value of $\ln(\Gamma(a)/\Gamma(c-a))$ is not important. Indeed, replacement of one value by another leads to the renumbering of the sequence of zeros by the same index set, after which asymptotics (5.3.2) does not change.

5.3.2. Sine-type functions and their generalizations. Denote by S_α, $\alpha \in \mathbb{R}$, the class of entire function of exponential type satisfying the estimate

$$|F(z)| \asymp |z|^{-\alpha} e^{\pi|\operatorname{Im} z|}, \quad \text{when} \quad |\operatorname{Im} z| \geq h = h(F) > 0. \quad (5.3.31)$$

The S_0 consists of so-called *sine-type functions* introduced by Levin (see [11]). The classes S_α and, in particular, the class S_0 play an important role in nonharmonic analysis (see [34]).

The Fourier–Stieltjes transform of a finite measure concentrated on the segment $[-\pi, \pi]$, i.e., the function

$$F(z) = \int_{-\pi}^{\pi} e^{izt} d\sigma(t), \quad \operatorname{var}\sigma(t) < +\infty, \quad (5.3.32)$$

is a sine-type function if and only if $\sigma(t)$ has jumps at both points $\pm\pi$; in this case, zeros z_n of the function $F(z)$ satisfy the condition

$$z_n = n + O(1), \quad n \to \pm\infty. \quad (5.3.33)$$

In a number of works, sine-type functions that are not Fourier–Stieltjes transforms, i.e., that cannot be represented in the form (5.3.32), were constructed; in all cases, zeros of the functions constructed also satisfy condition (5.3.33). All mentioned results can be found in [34].

It is known (see [34]) that an entire function of the form

$$F(z) = \int_{\pi}^{\pi} e^{izt} \frac{k(t)dt}{(\pi^2 - t^2)^{1-\beta}}, \quad 0 < \operatorname{Re}\beta < 1, \quad \operatorname{var} k(t) < +\infty, \quad k(\pm\pi \mp 0) \neq 0, \tag{5.3.34}$$

belongs to the class $S_{\operatorname{Re}\beta}$, and its zeros also have the form (5.3.33).

It turns out that under certain conditions on a and c, the function

$$F(z) = e^{-i\pi z}\Phi(a, c; 2\pi iz) \tag{5.3.35}$$

belongs to the class $S_{\operatorname{Re}a}$ and possesses some additional properties. The following theorem holds.

Theorem 5.3.2. *Let* $a, c \in \mathbb{C}$, $a, c, c - a \notin -\mathbb{Z}_+$, *and*

$$\operatorname{Re} c = 2 \operatorname{Re} a. \tag{5.3.36}$$

Then the following assertions hold:

(1) *function* (5.3.35) *belongs to the class* $S_{\operatorname{Re}\alpha}$;

(2) *zeros of function* (5.3.35) *have the asymptotics*

$$z_n = n + \left(\frac{\operatorname{Im}(c-2a)}{2\pi}\ln 2\pi|n| - \frac{i}{2\pi}\ln\frac{\Gamma(a)}{\Gamma(c-a)} \pm \frac{c-2}{4}\right)\left(1 + \frac{\operatorname{Im}(c-2a)}{2\pi n}\right) - \frac{2a(c-a)-c}{4\pi^2 n} + O\left(\frac{\ln|n|}{n^2}\right), \quad n \to \pm\infty; \tag{5.3.37}$$

(3) *the numeration of all zeros of function* (5.3.35) *is matched with asymptotics* (5.3.37) *by the index set* $T = \mathbb{Z} \setminus \{0\}$;

(4) *if, moreover,*

$$c \neq 2a, \tag{5.3.38}$$

then function (5.3.35) *for* $\operatorname{Re} a = 0$ *is not a Fourier–Stieltjes transform, and for* $0 < \operatorname{Re} a < 1$ *it cannot be represented in the form* (5.3.34).

Proof. First, formula (5.3.5) implies that the entire function (5.3.35) is of exponential type. Second, this formula together with condition (5.3.36) shows that if positive h is sufficiently large, then the following estimates hold:

$$|\Phi(z)| \asymp |z|^{-\operatorname{Re}a} \qquad \text{when} \quad \operatorname{Re} z \leq -h,$$
$$|\Phi(z)| \asymp |z|^{-\operatorname{Re}a}e^{\operatorname{Re}z} \qquad \text{when} \quad \operatorname{Re} z > h > 0.$$

Therefore, function (5.3.35) satisfies the estimate (5.3.31) with $\alpha = \operatorname{Re} a$, i.e., function (5.3.35) belongs to the class $S_{\operatorname{Re}a}$, and assertion (1) is proved.

Assertions (2) and (3) immediately follow from Theorem 5.3.1 if we replace z_n by $2\pi iz_n$ on the left-hand side of formula (5.3.2).

Finally, if condition (5.3.38) holds, then formula (5.3.37), by which

$$z_n - n \sim C\ln|n|, \quad n \to \pm\infty, \quad C \neq 0,$$

is incompatible with the necessary condition (5.3.33) of the representability of the function $F(z)$ in the form (5.3.32) for $\operatorname{Re} a = 0$ and in the form (5.3.34) for $0 < \operatorname{Re} a < 1$. Assertion (4) is also valid. The theorem is proved. □

Thus, Theorem 5.3.2 gives a whole class of sine-type functions whose asymptotics of zeros cannot be represented by formula (5.3.33). Further studies of these functions have been carried out in [47].

5.3.3. Nonasymptotic properties of zeros of the confluent hypergeometric function.

Theorem 5.3.3. *The following assertions hold.*

(1) *Let* $1 \le a < c \le a+1$ *and, moreover,* $c \ne 2$ *if* $a = 1$. *Then all zeros of the function* $\Phi(a, c; z)$ *lie in the half-plane*
$$\operatorname{Re} z < -\left(\sqrt{a-1} + \sqrt{1-(c-a)}\right)^2. \tag{5.3.39}$$

(2) *Let* $0 < a \le 1$ *and* $c \ge 1 + a$ *and, moreover,* $c \ne 2$ *if* $a = 1$. *Then all zeros of the function* $\Phi(a, c; z)$ *lie in the half-plane*
$$\operatorname{Re} z > \left(\sqrt{c-a-1} + \sqrt{1-a}\right)^2.$$

(3) *Let* $0 < a \le 1$ *and* $a < c \le 1 + a$ *and, moreover,* $c \ne 2$ *if* $a = 1$. *Then all zeros of the function* $\Phi(a, c; z)$ *lie in the horizontal strips*
$$(2n-1)\pi < |\operatorname{Im} z| < 2\pi n, \quad n \in \mathbb{N}. \tag{5.3.40}$$

Proof. We have the formula
$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt, \quad \operatorname{Re} a,\ \operatorname{Re}(c-a) > 0 \tag{5.3.41}$$
(see [10, 18, 40]), which allows one to apply the Pólya theorem [20] (below we recall its statement) and results of the first part of this chapter on zeros of the Laplace transform
$$F(z) = \int_0^1 e^{zt} f(t) dt, \quad f \in L^1(0, 1). \tag{5.3.42}$$

Theorem A (see [20, 34]). *Let a function* $f(t)$ *be positive, differentiable, nonconstant in the interval* $(0, 1)$, *and, moreover,*
$$\frac{f'(t)}{f(t)} \ge -h, \quad 0 < t < 1,$$
for some $h \in \mathbb{R}$. *Then all zeros of function* (5.3.42) *lie in the half-plane* $\operatorname{Re} z < h$.

Thus, having in mind formula (5.3.41), we must find the minimum of the function $f'(t)/f(t)$ on $(0, 1)$, where
$$f(t) = t^\alpha (1-t)^\beta, \quad \alpha = a - 1 > -1, \quad \beta = c - a - 1 > -1; \tag{5.3.43}$$
the case $\alpha = \beta = 0$ is excluded by the condition of the theorem ($c \ne 2$ if $a = 1$). In this case, all zeros of the function $\Phi(a, c; z)$ lie on the boundary of the half-plane (5.3.39).

We have the formula
$$\frac{f'(t)}{f(t)} = \frac{\alpha - (\alpha + \beta)t}{t(1-t)}. \tag{5.3.44}$$
For the boundedness from below of function (5.3.44) on the interval $(0, 1)$, the nonnegativeness of the numerator for $t = 0, 1$ is necessary; this requirement implies the condition $\beta \le 0 \le \alpha$. Thus (see (5.3.43)), we consider values
$$-1 < \beta \le 0 \le \alpha. \tag{5.3.45}$$

If $\alpha + \beta = 0$ (i.e., $c = 2$), then $\alpha > 0$, and the minimum of function (5.3.44) on $(0, 1)$ is equal to $4\alpha = 4(a - 1)$. By Theorem A, all zeros of the function $\Phi(a, c; z)$ lie in the half-plane $\operatorname{Re} z < 4(1 - a)$, which coincides with (5.3.39) for $c = 2$.

If $\alpha = 0$ (i.e., $a = 1$), then $\beta < 0$, and the minimum of function (5.3.44) on $[0, 1)$ is equal to $-\beta$. By Theorem A, all zeros of the function $\Phi(a, c; z)$ lie in the half-plane $\operatorname{Re} z < \beta$, which coincides with (5.3.39) for $a = 1$.

Let $\alpha + \beta \neq 0$ and $\alpha > 0$. Then

$$\left(\frac{f'(t)}{f(t)}\right)' = \frac{-(\alpha+\beta)t^2 + 2\alpha t - \alpha}{t^2(1-t)^2} = -\frac{\alpha}{t^2} - \frac{\beta}{(1-t)^2}. \tag{5.3.46}$$

The middle part of this formula vanishes at the points

$$t_1 = \frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{-\beta}}, \quad t_2 = \frac{\sqrt{\alpha}}{\sqrt{\alpha} - \sqrt{-\beta}}.$$

Clearly, $t_1 \in (0, 1]$, $t_2 \geq 1$, and the minimum of function (5.3.44) on $(0, 1]$ is attained at the point t_1. Substituting this value on the right-hand side of (5.3.44), we obtain the inequality

$$\frac{f'(t)}{f(t)} \geq (\sqrt{\alpha} + \sqrt{-\beta})^2, \quad 0 < t < 1.$$

This and Theorem A imply assertion (1). It should be taken into account that condition (5.3.45) turns into the condition $1 \leq a < c \leq a + 1$.

Assertion (2) follows from assertion (1) and formula (5.3.1).

To prove assertion (3), we use Theorem 5.2.3, which asserts that if a function $f(t) \neq C$ is positive and logarithmically convex in $(0, 1)$, then all zeros of function (5.3.42) lie in strips (5.3.40). For this, we must find the set of the parameters $\alpha, \beta > -1$ for which function (5.3.43) is logarithmically convex in $(0, 1)$, i.e., the function $\ln f(t)$ is convex in $(0, 1)$. Since $(\ln f(t))''$ is the right-hand side of formula (5.3.46), the function $\ln f(t)$ is convex only for values $\alpha, \beta > -1$ satisfying the condition

$$\alpha(1-t)^2 + \beta t^2 \leq 0 \tag{5.3.47}$$

for $t \in (0, 1)$, and, by the continuity, also for $t \in [0, 1]$. Substituting here the values $t = 0, 1$, we obtain a necessary condition $\alpha, \beta \leq 0$. Clearly, it is also sufficient for the validity of the condition (5.3.47). Recalling the relation (5.3.43) between the pairs of parameters α, β and a, b, we complete the proof of assertion (3). Theorem 5.3.3 is proved.

In connection with Theorem 5.3.3, we draw the attention of the reader to papers [42, 43] of Tsvetkov on zeros of the Whittaker function $M_{k,m}(z)$. This function is related with the Kummer function by the formula

$$M_{k,m}(z) = e^{-z/2} z^{m+1/2} \Phi\left(\frac{1}{2} + m - k, 1 + 2m; z\right).$$

This allows one to reformulate results of [42, 43] for the function $\Phi(a, c; z)$ as follows.

If $0 < c < 2a$ (respectively, $c > \max(0, 2a)$), then the function $\Phi(a, c; z)$ has no complex roots in the half-plane $\operatorname{Re} z \geq c - 2a$ (respectively, $\operatorname{Re} z \leq c - 2a$; see [42]).

If $1 < c < 2a$ (respectively $c > \max(1, 2a)$), then the function $\Phi(a, c; z)$ has complex roots only in the half-plane $\operatorname{Re} z < c - 2a$ (respectively, $\operatorname{Re} z > c - 2a$; see [43]).

We see that the result of [42] covers a wider set of the parameters a and c compared with Theorem 5.3.3. On the other hand, Theorem 5.3.3 indicates the half-plane of roots more exactly. Indeed, if $1 < a < c < a + 1$, then

$$-\left(\sqrt{a-1} + \sqrt{1-(c-a)}\right)^2 < c - 2a.$$

Therefore, the half-plane (5.3.39) from Theorem 5.3.3 is a proper subset of the half-plane $\operatorname{Re} z < c - 2a$ appearing in [42, 43]. □

Presentation of the material of this section follows [35].

Chapter 6

REAL ROOTS OF THE MITTAG-LEFFLER FUNCTION OF ORDER $\rho \in (1/2, 1)$

6.1. Statement of the Problem and the Main Results

Theorem 2.1.1 implies that for any $\rho > 1/2$ and $\mu \in \mathbb{C}$, the set of real roots of the Mittag-Leffler function is either empty or finite. In tis connection, we state the following problem.

Problem. For any $\rho \in (1/2, 1)$ and $\mu \in \mathbb{R}$, find or estimate, as exactly as possible, the number $\mathcal{N}(\rho, \mu)$, of real roots of the function $E_\rho(z; \mu)$.

For $\rho \geq 1$, this problem was almost completely solved in Chap. 4. For $\mu = 2$, the problem was stated by Nakhushev [17] in 1977, but a solution (not complete) was obtained only in 2006 by Popov [24]. The first publication known to the authors containing results on this problem appeared in 2005.

Theorem 6.1.1 (see [26])**.** *The following assertions hold.*

(1) *There exists a function f decreasing on the segment $[1/2, 1]$ and such that for any $\rho \in [1/2, 1]$ and $\mu > f(\rho)$, the function $E_\rho(z; \mu)$ has no real roots. On the other hand, for any $\mu \in (0, f(\rho))$, the function $E_\rho(z; \mu)$ attains negative values on some interval of the real axis and, therefore, has at least one real root. For any $\rho \in [1/2, 1)$, the function $E_\rho(z; f(\rho))$ is nonnegative on $\mathbb{R}$, but has at least one real root of even multiplicity.*

(2) *The following double inequality holds*:

$$\frac{1}{\rho} < f(\rho) < \frac{3}{2\rho}, \quad \frac{1}{2} < \rho < 1. \tag{6.1.1}$$

The author of [26] did not consider values $\mu \leq 0$. However, for

$$\mu \in \bigcup_{n=0}^{\infty} \left(-2n - 3 + \frac{1}{\rho}, -2n\right], \tag{6.1.2}$$

the existence of a root of $E_\rho(z; \mu)$ on $\mathbb{R}$ is obvious: for such μ, either

$$\frac{1}{\Gamma(\mu)} = E_\rho(0; \mu) \leq 0,$$

or

$$\frac{1}{\Gamma(\mu - 1/\rho)} = \lim_{x \to -\infty} \big(-xE_\rho(x; \mu)\big) < 0,$$

and since

$$\lim_{x \to +\infty} E_\rho(x; \mu) = +\infty$$

for any $\mu \in \mathbb{R}$, we obtain the existence of a real root of the Mittag-Leffler function; here we have used the identity

$$\operatorname{sgn}\left(\frac{1}{\Gamma(t)}\right) = (-1)^n, \quad -n < t < 1 - n, \quad n \in \mathbb{N}.$$

It is much more difficult to prove the existence of a real root $E_\rho(z; \mu)$ for negative μ that does not belong to the set (6.1.2), i.e., for

$$\mu \in \bigcup_{m \in \mathbb{N}} \left(-2m, -2m - 1 + \frac{1}{\rho}\right]. \tag{6.1.3}$$

We have reason to believe that $E_\rho(z;\mu)$ has at least one root on $\mathbb{R}$ for any pair $(\rho,\mu) \in (1/2,1) \times (-\infty,0]$, but we have not been able to prove this assertion completely.

Theorem 6.1.2. *If* $\rho \in (1/2, 3/4]$, *then* $E_\rho(z;\mu)$ *has at least one real root for any* $\mu < 0$, *and if* $\rho \in (3/4,1]$, *then the existence of a real root is guaranteed for any* $\mu \le -(1-\rho)^{-2}$.

Further results are related to the decreasing of the gap in the two-sided estimate (6.1.1). Following Nakhushev, we introduce the parameter $\alpha = 2 - 1/\rho$. Then inequality (6.1.1) takes the form

$$2-\alpha < f(\rho) < 3 - 1.5\alpha, \quad 0<\alpha<1.$$

We strengthen this result.

Theorem 6.1.3. *The following double inequalities hold*:

$$3-3\alpha+0.7\alpha^2 < f(\rho) < 3-2\alpha, \quad 0<\alpha\le\frac{1}{2} \quad \Longleftrightarrow \quad \frac{1}{2}<\rho\le\frac{2}{3}, \tag{6.1.4}$$

$$\frac{1}{\rho}+h(\rho) < f(\rho) < \frac{4}{3\rho}, \quad \frac{2}{3}<\rho<1, \tag{6.1.5}$$

where

$$h(\rho) = \exp\Big[-\pi\cot\big(\pi(1-\rho)\big)\Big].$$

From (6.1.4) we immediately obtain that

$$f(\rho) = 3 + O\left(\rho-\frac{1}{2}\right), \quad \rho\to\frac{1}{2}+,$$

which implies the continuity of the function f at the point 1/2 (the results of Chap. 5 imply the equality $f(1/2)=3$). However, the continuity of f at the point 1 has not yet been proved (below at the end of Sec. 6.5 we prove that $f(1)=1$).

We first observe the following phenomenon. For any $\mu<3$, the value $\mathcal{N}(\rho,\mu)$ tends to $+\infty$ as $\rho\to 1/2+$. If $\mu\ge 3$, then, owing to (6.1.1), we have the equality $\mathcal{N}(\rho,\mu)=0$ for any $\rho>1/2$.

Theorem 6.1.4. *For any* $\mu<3$ *and* $\rho\to 1/2+$ (*then* $\varepsilon=\rho-1/2\to 0+$), *the following asymptotics holds*:

$$\mathcal{N}(\rho,\mu) = \frac{3-\mu}{\pi^2\varepsilon}\left(\ln\left(\frac{1}{\varepsilon}\right)+O\left(\ln\ln\left(\frac{1}{\varepsilon}\right)\right)\right), \quad \mu\notin\mathbb{Z},$$

$$\mathcal{N}(\rho,\mu) = \frac{4-\mu}{\pi^2\varepsilon}\left(\ln\left(\frac{1}{\varepsilon}\right)+O\left(\ln\ln\left(\frac{1}{\varepsilon}\right)\right)\right), \quad \mu\in\mathbb{Z}.$$

In connection with [17], the case $\mu=2$ is studied in more detail. Since the number $\lambda\in\mathbb{C}$ is an eigenvalue of the boundary-value problem[10]

$$y''(x)+\lambda y^{(\alpha)}(x)=0, \quad 0<x<1, \quad y(0)=y(1)=0, \quad y\in C[0,1]\cap C^2(0,1), \tag{6.1.6}$$

if and only if $E_\rho(-\lambda,2)=0$, we consider roots of the function $E_\rho(-z,2)$ and arrange them in the sequence $\{\lambda_n\}_{n\in\mathbb{N}}=\{\lambda_n(\alpha)\}_{n\in\mathbb{N}}$, which is nondescending by moduli. Each element of this sequence is found in it as many times as its multiplicity. In what follows, "eigenvalue" means an eigenvalue of problem (6.1.6).

Theorem 6.1.5. *The following assertions hold.*

(1) *For* $0.45\le\alpha\le 1$, *there is no real eigenvalues.*

---

[10] Here $y^{(\alpha)}(x) \equiv \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\left(\int\limits_0^x (x-t)^{-\alpha}\,y(t)dt\right)$, $0<\alpha<1$, is the Riemann–Liouville derivative of a function $y(x)$ of order α.

(2) *For $0 < \alpha \le 0.364$, there are at least two real eigenvalues.*
(3) *If, for $\alpha \in (0.346, 0.45)$, there are real eigenvalues, then they lie on the interval $(-32, -3)$.*
(4) *If $0 < \alpha \le 1/3$, then all real eigenvalues lie on the interval $\big(\Gamma(4-\alpha), R^{2-\alpha}(\alpha)\big)$, where*

$$R(\alpha) = 2\left[\ln\left(\frac{2}{\alpha}\right) + \ln\ln\left(\frac{2}{\alpha}\right)\right]\operatorname{cosec}(\pi\varepsilon), \quad \varepsilon = \rho - \frac{1}{2} = \frac{\alpha}{4-2\alpha}.$$

(5) *Let*

$$R_1(\alpha) = 2\ln\left(\frac{1}{\alpha}\right)\operatorname{cosec}(\pi\varepsilon), \quad x_n(\alpha) = \pi(n+\rho)\sec(\pi\varepsilon).$$

Then, in the case $0 < \alpha \le 1/6$, all eigenvalues lying in the disk $|z| \le R_1^{2-\alpha}(\alpha)$ are real, simple, and satisfy the inequalities

$$x_{n-1}^{2-\alpha}(\alpha) < \lambda_n(\alpha) < x_n^{2-\alpha}(\alpha). \tag{6.1.7}$$

(6) *As $\alpha \to 0+$ (then $\rho = (2-\alpha)^{-1} \to 1/2+$), the following asymptotics holds:*

$$\mathcal{N}(\rho, 2) = 8\pi^{-2}\alpha^{-1}\ln\left(\frac{1}{\alpha}\right) + O\left(\alpha^{-1}\ln\ln\left(\frac{1}{\alpha}\right)\right).$$

(7) *If $0 < \alpha \le 0.1$, then eigenvalues lying in the disk $|\lambda| \le \alpha^{-2}$ satisfy the following estimate, more exact than (6.1.7):*

$$\begin{aligned} \left(\frac{\pi\,(n+\varepsilon)}{\cos\pi\varepsilon}\right)^{2-\alpha} < \lambda_n(\alpha) < \left(\frac{\pi\,(n+2\varepsilon)}{\cos\pi\varepsilon}\right)^{2-\alpha}, \\ \left(\frac{\pi n}{\cos\pi\varepsilon}\right)^{2-\alpha} < \lambda_n(\alpha) < \left(\frac{\pi\,(n+\varepsilon)}{\cos\pi\varepsilon}\right)^{2-\alpha}. \end{aligned} \tag{6.1.8}$$

In particular, independently of the parity of n, we have the inequality

$$(\pi n)^{2-\alpha} < \lambda_n(\alpha) < (\pi n)^2. \tag{6.1.9}$$

We do not give here the proof of Theorem 6.1.5 (it can be found in [24]). We only prove assertion (1) (in [24], the absence of real eigenvalues for $\alpha \ge 0.5$ was proved). Note that assertion (2) immediately follows from the lower estimate (6.1.4), and assertion (6) is a consequence of Theorem 6.1.4.

For $\alpha = 0$, problem (6.1.6) is a classical Sturm–Liouville problem and all its eigenvalues are real: $\lambda_n = (\pi n)^2$. This fact is confirmed by the identity $E_{1/2}(-z; 2) \equiv \dfrac{\sin\sqrt{z}}{\sqrt{z}}$. Thus, for any fixed n, inequality (6.1.9) implies the limit relation

$$\lim_{n\to\infty} \lambda_n(\alpha) = \lambda_n(0) = (\pi n)^2.$$

A more careful analysis of the double inequality (6.1.9) yields a stronger assertion. If $h(\alpha)$ is an arbitrary positive function increasing on the interval $0 < \alpha < 0.1$ and such that

$$\lim_{\alpha\to 0+} h(\alpha) = 0,$$

then

$$\lim_{\alpha\to 0+} \max\left\{ \left|\lambda_n(\alpha) - (\pi n)^2\right| \;\middle|\; 1 \le n \le h(\alpha)\,(\alpha\ln(1/\alpha))^{-1/2} \right\} = 0. \tag{6.1.10}$$

Indeed, since for $n = O(\exp(1/\alpha))$ we have the relation $n^\alpha - 1 = O(\alpha\ln n)$, $\alpha \to 0+$, and

$$(\pi n)^2 - (\pi n)^{2-\alpha} = (\pi n)^2\left(1 - (\pi n)^{-\alpha}\right) = O\left(n^2\alpha\ln(\pi n)\right) = O\left(\alpha n^2\ln n\right), \tag{6.1.11}$$

from (6.1.11) and the condition

$$n \le h(\alpha)\,(\alpha\ln(1/\alpha))^{-1/2}$$

follows the estimate

$$(\pi n)^2 - (\pi n)^{2-\alpha} = O\left(h(\alpha)\right),$$

which proves (6.1.10).

The following question arises: Is a general theorem on the convergence of appropriately numbered roots of the family of the functions $E_\rho(z;\mu)$ to roots of the function $E_{\rho_0}(z;\mu)$ if the parameter μ is constant and $\rho \to \rho_0$ valid? There exists examples of pairs (ρ_0, μ) for which this assertion is invalid: $\mu > 3$ and $\rho \to 1/2-$ or $\rho_0 = 1$ and $\mu = 1, 0, -1, \ldots$. However, there is reason to believe that in other cases this theorem is valid. This question awaits investigation.

In concluding this section, we note an aspect of Nakhushev's problem that was not reflected in Theorem 6.1.5. What is the number α_0 such that for $\alpha < \alpha_0$, there exists a real eigenvalue and for $\alpha > \alpha_0$, all eigenvalues lie outside $\mathbb{R}$? Such a number exists by the first part of Theorem 6.1.1 owing to the fact that the function f decreases:

$$\alpha_0 = \sup\left\{2 - \frac{1}{\rho} \;\middle|\; f(\rho) > 2\right\}$$

and due to inequality (6.1.4), which shows that the set of values $\rho \in (1/2, 1)$ such that $f(\rho) > 2$ is nonempty. Theorem 6.1.5 shows that $0.36 < \alpha_0 < 0.45$, but we cannot obtain a more exact estimate of α_0 without a computer. Assertion (3) of Theorem 6.1.5 gives an opportunity to find α_0 using computers. Indeed, to find the first two digits of the decimal representation of the number α_0, it suffices to compute with high accuracy values of the Mittag-Leffler function $E_\rho(x;2)$, $\rho = (2-\alpha)^{-1}$, on the segment $-32 \le x \le -3$ for values of the parameter $\alpha \in \{0.37,\ 0.38,\ 0.39,\ 0.40,\ 0.41,\ 0.42,\ 0.43,\ 0.44\}$. Computer simulations gave the estimate $0.4 < \alpha_0 < 0.41$, which, in our opinion, can be proved analytically.

6.2. Proof of Theorem 6.1.2

By Theorem 1.1.3 (we set $m = 0$), the function

$$F_\rho(x;\mu) \equiv x^{\mu-1} E_\rho\big(-x^{1/\rho};\mu\big) \quad \text{for} \quad \frac{1}{3} < \rho < 1, \quad \mu \le \frac{1}{\rho}, \quad x > 0, \tag{6.2.1}$$

admits the representation

$$F_\rho\big(x;\mu\big) = 2\rho\exp(x\cos\pi\rho)\cos\big(x\sin(\pi\rho) - \pi\rho(\mu-1)\big) + \omega_\rho(x;\mu), \tag{6.2.2}$$

where

$$\omega_\rho(x;\mu) = \frac{1}{\pi}\left(I_1(x;\rho,\mu)\sin\pi\left(\mu - \frac{1}{\rho}\right) + I_2(x;\rho,\mu)\sin\pi\mu\right)$$

and

$$I_1(x;\rho,\mu) = \int_0^{+\infty} \frac{t^{s-1}e^{-xt}dt}{t^{2/\rho} + 2t^{1/\rho}\cos(\pi/\rho) + 1}, \quad I_2(x;\rho,\mu) = \int_0^{+\infty} \frac{t^{s-1}e^{-xt}dt}{t^{1/\rho} + 2\cos(\pi/\rho) + t^{-1/\rho}}, \tag{6.2.3}$$

$s = -\mu + 1 + 1/\rho$.

First, we consider the case $1/2 < \rho \le 2/3$. By Theorem 1.5.3 we have the inequality

$$\big|\omega_\rho(x;\mu)\big| < 0.48x^{-s}\Gamma(s). \tag{6.2.4}$$

By (6.1.3), it suffices to prove the theorem for the values of the parameter $\mu \le -3 + 1/\rho$. Then $s \ge 4$ and the length of the segment $\Delta_s = [s, 3s]$ is equal to $2s$ and does not exceed $4\pi/\sqrt{3}$. At one of the points of an arbitrary semi-interval of length $2\pi\operatorname{cosec}(\pi\rho)$, the function $\cos(x\sin\pi\rho - \gamma)$, $\gamma \in \mathbb{R}$, is equal to -1. Since in the case considered $\operatorname{cosec}(\pi\rho) \le 2/\sqrt{3}$, from (6.2.2) and (6.2.4) we conclude that

$$\exists x \in \Delta_s : F_\rho(x;\mu) < -2\rho\exp(\cos\pi\rho) + 0.48x^{-s}\Gamma(s) < 0.5x^{-s}\Gamma(s) - e^{-0.5x}.$$

Here we have used the inequalities $2\rho > 1$ and $-0.5 \le \cos\pi\rho < 0$ for $1/2 < \rho \le 2/3$. For $s \ge 2$, the inequality

$$\Gamma(s) < 2\left(\frac{s}{e}\right)^s$$

holds[11]. Applying it, we can write

$$\exists x \in \Delta_s : F_\rho(x;\mu) < x^{-s}e^{-s}s^s - e^{-0.5x} = x^{-s}\left(e^{-s}s^s - x^s e^{-0.5x}\right).$$

It is easy to prove that the minimum of the function $x^s e^{-0.5x}$ on the segment Δ_s is attained at the point $x = s$; this means that the expression $s^s e^{-s} - x^s e^{-0.5x}$ is not greater than $s^s e^{-s} - s^s e^{-0.5s} < 0$. Thus, the function $F_\rho(x;\mu)$ at some point $x \in \Delta_s \subset (0,+\infty)$ is negative. Therefore, the function $E_\rho(z;\mu)$ takes a negative value on $(-\infty, 0)$. Since

$$\lim_{\substack{z\to+\infty,\\ z\in\mathbb{R}}} E_\rho(z;\mu) = +\infty \quad \forall\mu\in\mathbb{R},$$

this proves that the Mittag-Leffler function has a real root.

Now we consider the case $2/3 < \rho \le 3/4$. Since the minima of the denominators of the integrands in (6.2.3) are equal to $\sin^2(\pi/\rho) \ge 0.5$ and $2 + 2\cos(\pi/\rho) \ge 2 - \sqrt{2}$ respectively and the sum of their reciprocals does not exceed $3 + 1/\sqrt{2}$, we have

$$\left|\omega_\rho(x;\mu)\right| \le x^{-s}\Gamma(s)\left(3 + \frac{1}{\sqrt{2}}\right)\frac{1}{\pi} < 1.186x^{-s}\Gamma(s). \tag{6.2.5}$$

At some point of an arbitrary segment of length $6\,\mathrm{cosec}(\pi\rho)$, the function $\cos(x\sin\pi\rho - \gamma)$, $\gamma\in\mathbb{R}$, takes a value not greater than $\cos 3 < -0.9899$. Since $s \ge 4$ and $\mathrm{cosec}(\pi\rho) \le \sqrt{2}$, the length of the segment $\Delta_s' = [s/\sqrt{2}, 2\sqrt{2}s]$ is equal to $(3/\sqrt{2})s$ and is not less than $6\,\mathrm{cosec}(\pi\rho)$. Therefore,

$$\exists x \in \Delta_s' : F_\rho(x;\mu) \le 1.186x^{-s}\Gamma(s) + (\cos 3)2\rho\exp(x\cos\pi\rho). \tag{6.2.6}$$

It is easy to verify that the function $\rho\exp(x\cos\pi\rho)$ decreases with respect to the variable ρ on the segment $1/2 \le \rho \le 3/4$ for any fixed $x \ge 1$; therefore,

$$2\rho\exp(x\cos\pi\rho) \ge 1.5\exp\left(-\frac{x}{\sqrt{2}}\right) \quad \forall x \ge 1.$$

This and (6.2.6) imply that

$$\begin{aligned}
\exists x \in \Delta_s' : F_\rho(x;\mu) &\le 1.186x^{-s}\Gamma(s) - \cos(\pi-3)1.5\exp\left(-\frac{x}{\sqrt{2}}\right)\\
&= x^{-s}\cos(\pi-3)\left[1.186\sec(\pi-3)\Gamma(s) - 1.5x^s\exp\left(-\frac{x}{\sqrt{2}}\right)\right]\\
&< x^{-s}\cos(\pi-3)\left[1.2\Gamma(s) - 1.5x^s\exp\left(-\frac{x}{\sqrt{2}}\right)\right]\\
&= 1.5x^{-s}\cos(\pi-3)\left[0.8\Gamma(s) - x^s\exp\left(-\frac{x}{\sqrt{2}}\right)\right].
\end{aligned}$$

It remains to prove the negativeness of the function

$$g(s,x) = 0.8\Gamma(s) - x^s\exp\left(-\frac{x}{\sqrt{2}}\right) \quad \text{for} \quad x\in\Delta_s', \quad s \ge 4.$$

[11]It is equivalent to

$$\varphi(s) \equiv \ln\Gamma(s) - s\ln s + s < \ln 2.$$

Since the function φ decreases on $(0,+\infty)$, $\varphi'(s) = \psi(s) - \ln(s) < 0$ for all $s > 0$, it suffices to prove the inequality only for $s = 2$. We have the relation $\varphi(2) = 2 - \ln 4 < \ln 2$.

The relations

$$0.8\Gamma(s) < 1.1 s^s e^{-s} \quad \forall s \geq 4, \quad \min_{x\in\Delta'_s}\left[x^s \exp\left(-\frac{x}{\sqrt{2}}\right)\right] = s^s e^{-s}\left(\frac{\sqrt{8}}{e}\right)^s$$

are proved immediately. They imply the estimate

$$g(s,x) < 1.1 s^s e^{-s} - s^s e^{-s}\left(\frac{\sqrt{8}}{e}\right)^s = s^s e^{-s}\left(1.1 - \left(\frac{\sqrt{8}}{e}\right)^s\right)$$
$$\leq s^s e^{-s}\left(1.1 - \left(\frac{\sqrt{8}}{e}\right)^4\right) = s^s e^{-s}\left(1.1 - 64e^{-4}\right) < 0.$$

Theorem 6.1.2 for the case $1/2 < \rho \leq 3/4$ is proved.

In the case $3/4 < \rho < 1$, we estimate the modulus of the function $\omega_\rho(x;\mu)$; more exactly, we use the presence of the factors $\sin\pi(\mu - 1/\rho)$ and $\sin\pi\mu)$ in the integrals I_1 and I_2 in its representation. We have the inequalities

$$\left|\omega_\rho(x;\mu)\right| \leq \frac{x^{-s}\Gamma(s)}{\pi}\left[\frac{\left|\sin\pi\left(\mu - \dfrac{1}{\rho}\right)\right|}{\sin^2\dfrac{\pi}{\rho}} + \frac{|\sin\pi\mu|}{4\cos^2\dfrac{\pi}{2\rho}}\right] \leq \frac{x^{-s}\Gamma(s)}{\pi\sin^2\dfrac{\pi}{\rho}}\left[\left|\sin\pi\left(\mu - \frac{1}{\rho}\right)\right| + |\sin\pi\mu|\right].$$

It is easy to verify that for $\mu \in (-2m, -2m-1+1/\rho)$, $m \in \mathbb{N}$, both expressions $\sin\pi\mu$ and $\sin\pi(\mu-1/\rho)$ are positive and their sum does not exceed $2\sin(\beta/2)$, where $\beta = 1/\rho - 1$. Therefore,

$$|\omega_\rho(x;\mu)| \leq \frac{x^{-s}\Gamma(s)}{\sin\pi\beta\cos\dfrac{\pi\beta}{2}}.$$

In particular, for $3/4 < \rho \leq 5/6$ we have the estimate

$$|\omega_\rho(x;\mu)| \leq \frac{x^{-s}\Gamma(s)}{\pi\sin\frac{\pi}{5}\cos\frac{\pi}{10}} < 0.6x^{-s}\Gamma(s).$$

Since

$$s = 1 - \mu + \frac{1}{\rho} > |\mu| + 2$$

and

$$|\mu| \geq (1-\rho)^{-2} > 16, \quad \frac{3}{4} < \rho \leq \frac{5}{6},$$

we have

$$s \geq 18 \quad \Longrightarrow \quad s\frac{\sqrt{3}}{2} > 4\pi.$$

Therefore we obtain

$$\exists x \in \left[s\frac{\sqrt{3}}{2}, s\sqrt{3}\right] : \cos\big(x\sin\pi\rho - \pi\rho(\mu-1)\big) = -1.$$

Therefore,

$$\exists x \in \left[s\frac{\sqrt{3}}{2}, s\sqrt{3} \right] : F_\rho(x;\mu) < 0.6x^{-s}\Gamma(s) - 2\rho \exp(x\cos\pi\rho)$$

$$\leq 0.6x^{-s}\Gamma(s) - 1.5\exp\left(-x\frac{\sqrt{3}}{2}\right) = 0.6x^{-s}\left(\Gamma(s) - 2.5x^s \exp\left(-x\frac{\sqrt{3}}{2}\right)\right).$$

Since

$$\Gamma(s)s^s e^{-s} \quad \text{for } s \geq 8, \quad \min\left\{ x^s \exp\left(-x\frac{\sqrt{3}}{2}\right) \Big| s\frac{\sqrt{3}}{2} \leq x \leq s\sqrt{3} \right\}$$
$$= s^s(\sqrt{3})^s \exp\left(-\frac{3s}{2}\right) > s^s e^{-s},$$

we obtain that at some point the function $F_\rho(x;\mu)$ attains a negative value. This implies (see the reasoning above) the presence of a real root of the function $E_\rho(z;\mu)$.

Let $\varepsilon = 1-\rho$. For $5/6 < \rho < 1$ (or, equivalently, $0 < \varepsilon < 1/6$ and $0 < \beta < 1/5$), by the relations

$$\sin\pi\beta)\cos\frac{\pi\beta}{2} > \sin\frac{\pi\beta}{\beta+1} = \sin\pi\varepsilon > 3\varepsilon \tag{6.2.7}$$

(explained below) we have the inequalities

$$|\omega_\rho(x;\mu)| \leq \frac{x^{-s}\Gamma(s)}{3\pi\varepsilon} < \frac{x^{-s}s^s e^{-s}}{\pi\sqrt{s}\varepsilon}.$$

At the end of this section, we prove the inequality

$$\Gamma(s) < es^{s-0.5}e^{-s} \quad \forall s > 1.$$

Since

$$s > |\mu| \geq \varepsilon^{-2},$$

we obtain the estimate

$$|\omega_\rho(x;\mu)| < \pi^{-1}s^s(xe)^{-s}. \tag{6.2.8}$$

We also have

$$\sqrt{s} > 3\operatorname{cosec}\pi\varepsilon.$$

Therefore,

$$\exists x \in [s-\sqrt{s}, s+\sqrt{s}] : \cos\Big(x\sin\pi\rho - \pi\rho(\mu-1)\Big) \leq \cos 3 < -0.989.$$

This and (6.2.8) and (6.2.2) imply

$$\exists x \in [s-\sqrt{s}, s+\sqrt{s}] : F_\rho(x;\mu) \ll \pi^{-1}s^s(xe)^{-s} - 1.97\rho\exp(x\cos\pi\rho)$$
$$< \pi^{-1}s^s(xe)^{-s} - 1.97\left(\frac{5}{6}\right)e^{-x} = \pi^{-1}x^{-s}\left(s^s e^{-s} - \left(5\pi\cdot\frac{1.97}{6}\right)x^s e^{-x}\right)$$
$$< \pi^{-1}x^{-s}\left(s^s e^{-s} - 4x^s e^{-x}\right). \tag{6.2.9}$$

It is easy to verify that the minimum of the function $x^s e^{-x}$ on the segment $[s-\sqrt{s}, s+\sqrt{s}]$ is attained at the point $x = s - \sqrt{s}$ and is equal to

$$(s-\sqrt{s})^s \exp(-s+\sqrt{s}) = s^s e^{-s} \exp\left[\sqrt{s} + s\ln\left(1 - \frac{1}{\sqrt{s}}\right)\right]$$
$$= s^s e^{-s} \exp\left[-\frac{1}{2} - \sum_{k=3}^{\infty} \frac{s^{1-k/2}}{k}\right] > s^s e^{-s-1}. \quad (6.2.10)$$

From (6.2.9) and (6.2.10) we see that on the segment $[s-\sqrt{s}, s+\sqrt{s}]$, there exists a point at which the function F attains a negative value. Therefore, the Mittag-Leffler function also attains a negative value on $(-\infty, 0)$, which was required. Theorem 6.1.2 is completely proved.

In concluding this section, we prove several inequalities related to the gamma-function, which were used in the proof of Theorem 6.1.2. By the Stirling formula, as $s \to +\infty$, we have the inequality

$$\Gamma(s) \sim \left(\frac{s}{e}\right)^s \sqrt{\frac{2\pi}{s}} \quad \Longleftrightarrow \quad \ln\Gamma(s) = \left(s - \frac{1}{2}\right)\ln s - s + \frac{1}{2}\ln 2\pi + o(1). \quad (6.2.11)$$

We need a nonasymptotic upper estimate of the gamma-function. Asymptotics (6.2.11) yields the following lower estimate of the Γ-function:

$$\Gamma(s) > \left(\frac{s}{e}\right)^s \sqrt{\frac{2\pi}{s}} \quad \Longleftrightarrow \quad \ln\Gamma(s) > \left(s - \frac{1}{2}\right)\ln s - s + \frac{1}{2}\ln 2\pi \quad \forall s > 0. \quad (6.2.12)$$

For completeness, we present the proof of this inequality; the concept of this proof (the decreasing of the function $g(s) = \ln\Gamma(s) - (s - 1/2)\ln s - s$) will be used below. It is easy to prove that the decreasing of $g(s)$ together with the limit relation

$$\lim_{s\to+\infty} g(s) = 0.5\ln 2\pi$$

proves the inequality

$$g(s) > 0.5\ln 2\pi \quad \forall s > 0,$$

which is equivalent to (6.2.12). We have the relation

$$g'(s) = \psi(s) - \ln s + \frac{1}{2s}.$$

To prove the negativeness of g' we note that

$$\lim_{s\to+\infty} g'(s) = \lim_{s\to+\infty} (\psi(s) - \ln s) = 0$$

(see Lemma 3.4.1). Therefore, the increasing of $g'(s)$ (in particular, the positiveness of $g''(s)$) implies the required assertion. We have the relation

$$g''(s) = \psi'(s) - \frac{1}{s} - \frac{1}{2s^2} = \sum_{k=0}^{\infty} \frac{1}{(k+s)^2} - \frac{1}{s} - \frac{1}{2s^2}.$$

Clearly, the positiveness of $g''(s)$ follows from the following lemma applied to the function $\varphi_s(t) = (s+t)^{-2}$.

Lemma 6.2.1. *Let $\varphi \in C^1[0, +\infty)$ be a positive, decreasing, and convex function,*

$$I = \int_0^{+\infty} \varphi(t)dt < +\infty.$$

Then

$$\sum_{k=0}^{\infty} \varphi(k) > I + \frac{\varphi(0)}{2}.$$

This lemma follows from the integral representation of the series

$$\sum_{k=0}^{\infty} \varphi(k) = I + \frac{\varphi(0)}{2} + \int_0^{+\infty} \sigma(x) d\varphi'(x), \quad \text{where} \quad \sigma(x) = \frac{\{x\} - \{x\}^2}{2}.$$

Now we obtain an upper estimate of $\Gamma(s)$. By the decreasing of $g(s)$, we have the inequality $g(s) < g(y)$, $0 < y < s$. Exponentiating this inequality, we have

$$\Gamma(s) < \frac{\Gamma(y)e^y}{y^{y-1/2}} s^{s-1/2} e^{-s}, \quad 0 < y < s. \tag{6.2.13}$$

Setting in (6.2.13) $y = 1$, $y = 2$, $y = 4$, and $y = 8$, we obtain all upper estimates of $\Gamma(s)$ used in the proof of Theorem 6.1.2. The greater y, the "closer" estimate (6.2.13) asymptotics (6.2.11), since

$$\lim_{y \to +\infty} \Gamma(y) e^y y^{1/2-y} = \sqrt{2\pi}.$$

Finally, we prove relations (6.2.7). The inequality

$$\sin \pi\varepsilon > 3\varepsilon \quad \Longleftrightarrow \quad \frac{\sin \pi\varepsilon}{\pi\varepsilon} > \frac{3}{\pi}, \quad 0 < \varepsilon < \frac{1}{6},$$

follows from the decreasing on $(0, \pi)$ of the function $t^{-1} \sin t$, which is equal to $3/\pi$ at the point $t = \pi/6$. The inequality

$$\begin{aligned} \sin \frac{\pi\beta}{\beta+1} < \sin \pi\beta \cos \frac{\pi\beta}{2} \quad &\Longleftrightarrow \quad \sin \pi\beta \left(1 - \cos \frac{\pi\beta}{2}\right) < \sin \pi\beta - \sin \frac{\pi\beta}{\beta+1} \\ &\Longleftrightarrow \quad \sin \pi\beta \sin^2 \frac{\pi\beta}{4} < \sin\left(\frac{\pi}{2}\left(\beta - \frac{\beta}{\beta+1}\right)\right) \cos\left(\frac{\pi}{2}\left(\beta + \frac{\beta}{\beta+1}\right)\right) \end{aligned}$$

can be strengthened if we replace $\sin^2 \dfrac{\pi\beta}{4}$ by a larger value $\pi^2\beta^2/16$ and $\cos\left(\dfrac{\pi}{2}\left(\beta + \dfrac{\beta}{\beta+1}\right)\right)$ by a smaller value $1/2$. Thus, it remains to prove that

$$\frac{5\beta^2}{8} \sin \pi\beta < \frac{1}{2} \sin \frac{\pi\beta^2}{2(\beta+1)}.$$

We strengthen this inequality replacing $\sin \dfrac{\pi\beta^2}{2(\beta+1)}$ by a smaller value $\dfrac{3\beta^2}{2(\beta+1)}$. We obtain the inequality

$$\frac{5}{8} \sin \pi\beta < \frac{3}{4(\beta+1)} \quad \Longleftrightarrow \quad (\beta+1) \sin \pi\beta < \frac{6}{5}.$$

The last inequality for $0 < \beta < 1/5$ is obvious.

6.3. Auxiliary Inequalities

In this section, we prove several inequalities needed in the sequel. The first lemma is the most cumbersome.

Lemma 6.3.1. *Introduce the notation*

$$v(\alpha) = 2\alpha - 0.7\alpha^2, \quad \varepsilon(\alpha) = \frac{\alpha}{4 - 2\alpha}.$$

Then the function

$$g(\alpha) = v(\alpha)\ln 2\pi + \ln\Gamma(1 - v(\alpha)) + \ln(1 + 2\varepsilon(\alpha)) - 2\pi \sin \pi\varepsilon(\alpha)$$

is positive on the semi-interval $0 < \alpha \le 1/2$.

For simplicity, we used a computer to prove the following four numerical inequalities:

$$g(0.3) > 0.075, \quad g(0.35) > 0.075, \quad g(0.4) > 0.13, \quad g(0.45) > 0.13. \tag{6.3.1}$$

Of course, (6.3.1) can be proved without a computer by using the Taylor expansions for the sine and the logarithm and the relation

$$\ln\Gamma(1 - z) = \gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} z^k, \quad |z| < 1, \tag{6.3.2}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and γ is the Euler constant. Formula (6.3.2) follows from the relations

$$\psi^{(m)}(1) = (-1)^{m-1} m!\zeta(m + 1), \quad m \in \mathbb{N}, \quad \psi(1) = -\gamma$$

(see [25, p. 775]).

Proof of Lemma 6.3.1. We have the equality

$$\begin{aligned} g'(\alpha) &= v'(\alpha)\Big(\ln 2\pi - \psi\big(1 - v(\alpha)\big)\Big) + \varepsilon'(\alpha)\left(\frac{2}{1 + 2\varepsilon(\alpha)} - 2\pi^2 \cos \pi\varepsilon(\alpha)\right) \\ &= (2 - 1.4\alpha)\Big(\ln 2\pi - \psi\big(1 - v(\alpha)\big)\Big) + \frac{1}{2}\left(1 - \frac{\alpha}{2}\right)^{-2}\left(\frac{1}{1 + 2\varepsilon(\alpha)} - \pi^2 \cos \pi\varepsilon(\alpha)\right). \end{aligned} \tag{6.3.3}$$

We outline our plan. From (6.3.1) we see (values of the function are estimated on a net with step 0.05) that to prove the positiveness of $g(\alpha)$ on the semi-interval $0.4 < \alpha \le 0.5$, it suffices to verify the inequality

$$g'(\alpha) > -2.6, \quad 0.4 < \alpha < 0.5, \tag{6.3.4}$$

and the positiveness of the function g on the interval $0.3 < \alpha < 0.4$ follows from the inequality

$$g'(\alpha) > -1.5, \quad 0.3 < \alpha < 0.4. \tag{6.3.5}$$

Further, we show that the polynomial

$$g_1(\alpha) = 0.359 - 3.42\alpha + 3.99\alpha^2$$

is a minorant of $g'(\alpha)$ on the interval $0 < \alpha < 0.3$. Since $g(0) = 0$, we have the relation

$$g(\alpha) \ge \int_0^{\alpha} g_1(t)dt = 0.395\alpha - 1.71\alpha^2 + 1.33\alpha^3 = 0.1\alpha P(\alpha),$$

where

$$P(\alpha) = 3.95 - 17.1\alpha + 13.3\alpha^2.$$

It is easy to verify that the polynomial P decreases on the segment $0 \le \alpha \le 0.3$ and hence on the interval $0 < \alpha < 0.3$ it exceeds its value 0.17 at the point 0.3. This concludes the proof of the positiveness of the function g. Thus, it remains to verify inequalities (6.3.4), (6.3.5), and $g'(\alpha) \geq g_1(\alpha)$, $0 < \alpha < 0.3$.

Inequality (6.3.4), due to (6.3.3), can be rewritten in the form

$$\left(\pi^2 \cos \pi\varepsilon(\alpha) - \frac{1}{1+2\varepsilon(\alpha)}\right)\left(1 - \frac{\alpha}{2}\right)^{-2} < 5.2 + (4 - 2.8\alpha)\Big[\ln 2\pi - \psi\big(1 - v(\alpha)\big)\Big]. \tag{6.3.6}$$

It is easy to prove that the function $\pi^2 \cos \pi\varepsilon - (1+2\varepsilon)^{-1}$ decreases on the segment $1/46 \leq \varepsilon \leq 1/2$ and the function $-\psi(1 - v(\alpha))$ increases on the interval $0 < \alpha < 1/2$. Since $\varepsilon(0.4) = 1/8$, we can substitute on the left-hand side of (6.3.6) the number 1/8 instead of $\varepsilon(\alpha)$ and on the right-hand side $\psi(1 - v(0.4)) = \psi(0.312)$ and obtain a stronger but simpler inequality than (6.3.6)

$$\left(\pi^2 \cos\frac{\pi}{8} - \frac{1}{1.25}\right)\left(1 - \frac{\alpha}{2}\right)^{-2} < 5.2 + (4 - 2.8\alpha)\big(\ln 2\pi - \psi(0.312)\big), \tag{6.3.7}$$

which will be proved for $0/4 < \alpha < 0.5$. The following numerical estimates hold:

$$\pi^2 < 9.87, \quad \cos\frac{\pi}{8} < 0.93, \quad \ln 2\pi > 1.8378, \quad -\psi(0.312) > -\psi(1/3) > 3.$$

By the identity

$$\psi(z) + \frac{1}{z} = \psi(z+1), \quad z \in \mathbb{C}\backslash\{0, -1, -2, \ldots\},$$

and the negativeness of the function ψ on the semi-interval $(0, 1.46]$, we have the relations

$$-\psi(t) = \frac{1}{t} - \psi(t+1) > \frac{1}{t}, \quad 0 < t \leq 0.46.$$

Using these estimates, we simplify and strengthen (6.3.7):

$$8.4\left(1 - \frac{\alpha}{2}\right)^{-2} < 5.2 + 4.8(4 - 2.8\alpha) \quad \Longleftrightarrow \quad 13.44\alpha + 8.4\left(1 - \frac{\alpha}{2}\right)^{-2} < 24.4, \quad 0.4 < \alpha < 0.5.$$

Since the left-hand side of the last inequality is an increasing function, it is less that its value at the point $\alpha = 0.5$, and this value is less than 22. Thus, inequality (6.3.4) is proved.

Inequality (6.3.5) is proved similarly. Rewrite it in the equivalent form:

$$\left(\pi^2 \cos \pi\varepsilon(\alpha) - \frac{1}{1+2\varepsilon(\alpha)}\right)\left(1 - \frac{\alpha}{2}\right)^{-2} < 3 + (4 - 2.8\alpha)\Big(\ln 2\pi - \psi\big(1 - v(\alpha)\big)\Big).$$

As above, using the monotonicity of the corresponding functions and the relations

$$\varepsilon(0.3) = \frac{3}{34}, \quad \cos\frac{3\pi}{34} < 0.962, \quad v(0.3) = 0.537, \quad -\psi(0.463) > 2,$$

we strengthen and simultaneously simplify this inequality:

$$8.65\left(1 - \frac{\alpha}{2}\right)^{-1} < 3 + 3.83(4 - 2.8\alpha) \quad \Longleftrightarrow \quad 10.724\alpha + 8.65\left(1 - \frac{\alpha}{2}\right)^{-1} < 18.32, \quad 0.3 < \alpha < 0.4.$$

The left-hand side of the last inequality is an increasing function of the variable α, which is less than 18 at the point $\alpha = 0.4$. Inequality (6.3.5) is proved.

We obtain a lower estimate for $g'(\alpha)$, $0 < \alpha < 0.3$. By (6.3.2) we have the relation

$$-\psi(1 - v) = \gamma + \sum_{k=1}^{\infty} \zeta(k+1)v^k, \quad |v| < 1.$$

From this and the inequalities

$$\gamma > 0.5772, \quad \zeta(2) = \frac{\pi^2}{6} > 1.644, \quad \zeta(3) > 1.2, \quad \zeta(s) > 1 \ \forall s > 1,$$
$$v(\alpha) = \alpha(2 + 0.7\alpha) > 1.79\alpha \text{ for } 0 < \alpha < 0.3,$$

we obtain the lower estimate

$$\ln 2\pi - \psi\big(1 - v(\alpha)\big) > 2.415 + 1.644 \cdot 1.79\alpha + 1.2 \cdot 1.79^2\alpha^2 + \sum_{k=3}^{\infty}(1.79\alpha)^k. \qquad (6.3.8)$$

Since

$$\cos t < 1 - \frac{t^2}{2} + \frac{t^4}{24}, \quad \forall t > 0,$$

we have the estimate

$$\cos t < 1 - t^2\left(\frac{1}{2} + \frac{t^2}{24}\right) < 1 - t^2\left(\frac{1}{2} - \frac{9\pi^2}{24 \cdot 34^2}\right) < 1 - 0.496t^2$$

for $0 < t < \pi\varepsilon(0.3) = 3\pi/34$.

Therefore, taking into account the inequality $\varepsilon > \alpha/4$, we have the estimate

$$-\pi^2 \cos \pi\varepsilon(\alpha) > \pi^2\big(-1 + 0.496\pi^2\varepsilon^2(\alpha)\big) > -\pi^2 + 0.496\pi^4\frac{\alpha^2}{16} > -\pi^2 + 3\alpha^2.$$

This implies

$$\frac{1}{2}\left(\frac{1}{1+2\varepsilon(\alpha)} - \pi^2\cos\pi\varepsilon(\alpha)\right) > \frac{1}{2}\left(1 - \pi^2 + 3\alpha^2 + \frac{1}{1+2\varepsilon(\alpha)} - 1\right)$$
$$= \frac{1}{2}\left(1 - \pi^2 + 3\alpha^2 - \frac{\alpha}{2}\right) > -4.435 + \frac{3\alpha^2}{2} - \frac{\alpha}{4}. \qquad (6.3.9)$$

From (6.3.3), (6.3.8), and (6.3.9) we deduce the lower estimate for the derivative

$$g'(\alpha) > (2 - 1.4\alpha)\left[2.415 + 2.942\alpha + 3.844\alpha^2 + \sum_{k=3}^{\infty}(1.79)^k\alpha^k\right]$$
$$- 4.435\left(1 - \frac{\alpha}{2}\right)^{-2} + \left(\frac{3\alpha^2}{2} - \frac{\alpha}{4}\right)\left(1 - \frac{\alpha}{2}\right)^{-2}.$$

Since

$$2 - 1.4\alpha > 2 - 1.4 \cdot 0.3 = 1.58, \quad \left(1 - \frac{\alpha}{2}\right)^{-2} = \sum_{k=0}^{\infty}(k+1)2^{-k}\alpha^k,$$

we have

$$g'(\alpha) > 0.395 - 3.381\alpha + 1.58\left(2.942\alpha + 3.844\alpha^2 + \sum_{k=3}^{\infty}(1.79)^k\alpha^k\right)$$
$$- 4.435\left(\alpha + 0.75\alpha^2 + \sum_{k=3}^{\infty}(k+1)2^{-k}\alpha^k\right) - 0.25\alpha + 1.25\alpha^2 + \sum_{k=3}^{\infty}(11k-12)2^{-k-1}\alpha^k. \qquad (6.3.10)$$

Since the sequence $a_k = 1.58 \cdot 1.79^k$ increases and $b_k = 4.435(k+1)2^{-k}$ decreases for $k \geq 1$, the coefficients of the expansion of the right-hand side of (6.3.10) in powers of α are positive starting from the second. Therefore, rejecting all powers of α starting from the third, we obtain the lower estimate

$$g'(\alpha) > 0.395 + \big(-3.381 + 1.58 \cdot 2.942 - 4.435 - 0.25\big)\alpha$$
$$+ \big(1.58 \cdot 3.844 - 4.435 \cdot 0.75 + 1.25\big)\alpha^2 > 0.395 - 3.42\alpha + 3.99\alpha^2,$$

which was required. The lemma is completely proved. □

Lemma 6.3.2. *Let*

$$h(\rho) = \exp\big[-\pi\cot\pi(1-\rho)\big].$$

The following inequalities hold:

$$h(\rho) < \frac{1}{6}, \quad \frac{2}{3} < \rho < \frac{3}{4}, \quad h(\rho) < \frac{1}{23}, \quad \frac{3}{4} \le \rho < 1, \quad 2.042h(\rho) < \frac{1}{\rho} - 1, \quad \frac{2}{3} < \rho < 1. \tag{6.3.11}$$

Proof. The formula for $h(\rho)$ shows that this function decreases. Therefore,

$$h(\rho) < h\left(\frac{2}{3}\right) = \exp\left(-\pi\cot\frac{\pi}{3}\right) = \exp\left(-\frac{\pi}{\sqrt{3}}\right) < \frac{1}{6}, \quad \frac{2}{3} < \rho < \frac{3}{4},$$
$$h(\rho) \le h\left(\frac{3}{4}\right) = \exp\left(-\pi\cot\frac{\pi}{4}\right) = e^{-\pi} < \frac{1}{23}, \quad \frac{3}{4} \le \rho < 1.$$

To prove the last inequality of the lemma we set $t = 1-\rho$ and apply the estimate $1/\rho = (1-t)^{-1} > 1+t$. It remains to verify that

$$2\exp\big(-\pi\pi t\big) < t \quad \Longleftrightarrow \quad 2 < t\exp\big(\pi\cot\pi t\big), \quad 0 < t < \frac{1}{3}.$$

Since the function $\varphi(t) = t\exp(\pi\cot\pi t)$ decreases on the interval $0 < t < 1$, for $0 < t < 1/3$ we have the relations

$$\varphi(t) > \varphi\left(\frac{1}{3}\right) = \frac{1}{3}\exp\left(\frac{\pi}{\sqrt{3}}\right) > 2.042,$$

which was required. The lemma is completely proved. □

Lemma 6.3.3. *For any $\alpha \in (0, 1/2]$, the inequality*

$$\Gamma(1-\alpha) < e^{1.15\alpha}$$

holds. If $0 < \alpha \le 1/4$, then

$$\Gamma(1-\alpha) < e^{0.9\alpha}.$$

Proof. Consider the function

$$\varphi_c(\alpha) = \ln\Gamma(1-\alpha) - c\alpha,$$

where c is a constant. Since

$$\varphi_c''(\alpha) = \psi'(1-\alpha) > 0,$$

the maximum of this function on any segment $\subset [0,1)$ is attained at one of endpoints of this segment. Since

$$\varphi_{0.9}\left(\frac{1}{4}\right) < 0, \quad \varphi_{1.15}\left(\frac{1}{2}\right) < 0, \quad \varphi_c(0) = 0,$$

the function $\varphi_{0.9}(\alpha)$ is negative on the semi-interval $0 < \alpha \le 1/4$ and the function $\varphi_{1.15}(\alpha)$ is negative on the semi-interval $0 < \alpha \le 1/2$. This implies the assertion of the lemma. □

Lemma 6.3.4. *If $\rho \in [1/2, 1]$, $\mu < 0$, then*

$$E_\rho(x;\mu) > 0 \quad \forall x \ge (1-\mu)^2.$$

Proof. First, let $-1 < \mu < 0$. Since $\rho \le 1$, all terms of the Maclaurin series of $E_\rho(x;\mu)$, except for the constant term, are positive. Therefore, for $x \ge 1$ we have the estimate

$$E_\rho(x;\mu) > \frac{1}{\Gamma(\mu)} + \frac{x}{\Gamma(\mu+1/\rho)} \ge \frac{1}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu+1/\rho)} \ge \frac{1}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu+1)} = \frac{1+\mu}{\Gamma(\mu+1)} > 0.$$

The inequality

$$\frac{1}{\Gamma(\mu+1/\rho)} \ge \frac{1}{\Gamma(\mu+1)}$$

can be proved as follows. We set $\mu + 1 = t$ and $h = 1/\rho - 1$. Then $t \in (0,1)$ and $h \in [0,1]$. We must prove that

$$\frac{1}{\Gamma(t)} \leq \frac{1}{\Gamma(t+h)}.$$

If $t + h \in (1,2)$, then

$$\frac{1}{\Gamma(t+h)} > 1, \quad \frac{1}{\Gamma(t)} < 1.$$

If $t + h \leq 1$, then the required inequality follows from the fact that the function $1/\Gamma(t)$ increases on the segment $0 \leq t \leq 1$.

Now let $\mu \leq -1$. Denote by N the minimal natural number such that $\mu + N/\rho \geq 0$. Then all terms of the Maclaurin series of the function E_ρ, starting from $\dfrac{x^N}{\Gamma(\mu + N/\rho)}$, are positive and hence the following estimate holds:

$$E_\rho(x;\mu) > \frac{x^N}{\Gamma\left(\mu + \frac{N}{\rho}\right)} + \frac{x^{N+1}}{\Gamma\left(\mu + \frac{N+1}{\rho}\right)} - \sum_{k=0}^{N-1} \frac{x^k}{\left|\Gamma(\mu) + \frac{k}{\rho}\right|}.$$

By the identity

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{\sin \pi s}{\pi}$$

we have the inequality

$$\frac{1}{|\Gamma(s)|} \leq \frac{\Gamma(1-s)}{\pi} \quad \forall s < 0.$$

Therefore, having introduced the notation

$$B_k(x) = x^k \Gamma\left(1 - \left(\mu + \frac{k}{\rho}\right)\right),$$

we find

$$E_\rho(x;\mu) > \frac{x^N}{\Gamma\left(\mu + \frac{N}{\rho}\right)} + \frac{x^{N+1}}{\Gamma\left(\mu + \frac{N+1}{\rho}\right)} - \frac{N}{\pi} \max_{0 \leq k \leq N-1} B_k(x). \tag{6.3.12}$$

We prove that the sequence

$$B_k(x) = \exp\left(\varphi_x(k)\right)$$

increases, where

$$\varphi_x(t) = t \ln x + \ln \Gamma\left(1 - \left(\mu + \frac{t}{\rho}\right)\right).$$

We have the formula

$$\varphi_x'(t) = \ln x - \frac{1}{\rho}\psi\left(1 - \left(\mu + \frac{t}{\rho}\right)\right).$$

Since the function ψ increases and the estimate $\psi(t) < \ln(t)$ holds, we obtain the inequality

$$\frac{1}{\rho}\psi\left(1 - \left(\mu + \frac{t}{\rho}\right)\right) \leq \frac{1}{\rho}\psi(1-\mu) < \frac{1}{\rho}\ln(1-\mu) \leq 2\ln(1-\mu) \leq \ln x.$$

Thus, $\varphi_x'(t) > 0$ and the fact that $B_k(x)$ increases is proved. By the choice of N, we have the relation

$$-2 \leq -\frac{1}{\rho} \leq \mu + \frac{N-1}{\rho} < 0,$$

which implies

$$B_k(x) \leq x^{N-1}\Gamma\left(1 - \left(\mu + \frac{N-1}{\rho}\right)\right) \leq 2x^{N-1}. \tag{6.3.13}$$

From (6.3.12) and (6.3.13) we deduce the estimate

$$E_\rho(x;\mu) > \frac{x^N}{\Gamma\left(\mu+\frac{N}{\rho}\right)} + \frac{x^{N+1}}{\Gamma\left(\mu+\frac{N+1}{\rho}\right)} - \frac{2Nx^{N-1}}{\pi}.$$

Since $\mu + \dfrac{N-1}{\rho} < 0$, we have $\mu + \dfrac{N}{\rho} < 2$. If $1 \le \mu + \dfrac{N}{\rho} < 2$, then $\dfrac{1}{\Gamma\left(\mu+\frac{N}{\rho}\right)} \ge 1$ and

$$E_\rho(x;\mu) > x^N - \frac{2Nx^{N-1}}{\pi} \ge x^N - \frac{2(2-\mu)x^{N-1}}{\pi} = x^N\left(1 - \frac{4-2\mu}{\pi x}\right). \qquad (6.3.14)$$

We have the relation

$$x \ge (1-\mu)^2 \quad \Longrightarrow \quad 2x \ge 2 - 4\mu + 2\mu^2 \ge 4 - 4\mu > 4 - 2\mu$$

(recall that $\mu \le -1$). Therefore,

$$\frac{4-2\mu}{\pi x} < 0$$

and the positiveness of $E_\rho(x;\mu)$ is proved. If

$$0 \le \mu + \frac{N}{\rho} < 1,$$

then

$$1 \le \mu + \frac{N+1}{\rho} \le 3, \quad \frac{x^{N+1}}{\Gamma\left(\mu+\frac{N+1}{\rho}\right)} \ge \frac{x^{N+1}}{2} > x^N,$$

and we again obtain (6.3.14). The lemma is proved. □

6.4. Existence of Real Roots of the Mittag-Leffler Function for Particular Values of the Parameter $\mu > 1/\rho$

In this section, we consider the following dependence of the parameter μ on the order ρ of the Mittag-Leffler function (as in Sec. 6.1, $\alpha = 2 - 1/\rho$):

$$\mu(\rho) = \begin{cases} 3 - 3\alpha + 0.7\alpha^2, & \dfrac{1}{2} < \rho \le \dfrac{2}{3}, \\ \dfrac{1}{\rho} + h(\rho), & \dfrac{2}{3} < \rho < 1, \quad \text{where } h(\rho) = \exp\left(-\pi \cot \pi(1-\rho)\right). \end{cases} \qquad (6.4.1)$$

We prove that the function $E_\rho(z;\mu(\rho))$ attains a negative value at some point on $(-\infty,0)$. Since $E_\rho(z;0) = 1/\Gamma(\mu) > 0$ for $\mu > 0$ and, by the asymptotics

$$E_\rho(z;\mu) \sim \frac{-1}{z\Gamma(\mu - 1/\rho)}, \quad z \in \mathbb{R}, \quad z \to -\infty, \quad \rho > 1/2, \quad \mu \notin \left\{\frac{1}{\rho},\ \frac{1}{\rho}-1,\ \frac{1}{\rho}-2, \dots\right\},$$

the function $E_\rho(z;\mu(\rho))$ attains positive values for all sufficiently large (in modulus) negative z, we conclude that $E_\rho(z;\mu(\rho))$ has no less than two real roots. Note that in the case $1/2 < \rho \le 2/3$ (or, equivalently, $0 < \alpha \le 1/2$), from (6.4.1) follows the relation

$$\mu(\rho) - \frac{1}{\rho} = 1 - 2\alpha + 0.7\alpha^2 \qquad (6.4.2)$$

(this immediately implies that $\mu(\rho) > 1/\rho$).

We also obtain a boundary for possible real roots of the function $E_\rho(z;\mu)$ when μ is greater than $\mu(\rho)$.

It would be interesting to prove the following assertion (which seems quite plausible). If the set of real roots of the function $E_\rho(z;\mu)$ is nonempty for some values of the parameters $\rho \in [2/3, 1)$ and $\mu > 1/\rho$, then the number of these roots, with account of their multiplicities, is exactly two. If the parameter ρ "is close" to $1/2$, then, as Theorem 6.1.4 shows, this assertion, in general, is invalid: the number of real roots of $E_\rho(z;\mu)$ may be sufficiently large.

Assertion 6.4.1. *We set*

$$x_\rho = \pi(1+\rho(\mu(\rho)-1))\operatorname{cosec}(\pi\rho).$$

Then for any $\rho \in (1/2,1)$, the following inequality holds:

$$E_\rho\big(-x_\rho^{1/\rho};\mu(\rho)\big) < 0 \quad \Longleftrightarrow \quad F_\rho\big(x_\rho;\mu(\rho)\big) < 0. \tag{6.4.3}$$

Proof. It is easy to verify that

$$\cos\big(x\sin(\pi\rho) - \pi\rho(\mu-1)\big) = -1 \quad \text{for} \quad x = \pi\big(1+\rho(\mu-1)\big)\operatorname{cosec}(\pi\rho).$$

Therefore, by Theorem 1.1.3, we have the formula

$$F_\rho\big(x_\rho;\mu(\rho)\big) = -2\rho\exp(x_\rho\cos\pi\rho) + \frac{x_\rho^{\mu(\rho)-1-1/\rho}}{\Gamma(\mu(\rho)-1/\rho)} - \frac{1}{\pi}\left[I_1\sin\left(\pi\left(\mu(\rho)-\frac{2}{\rho}\right)\right) + I_2\sin\left(\pi\left(\mu(\rho)-\frac{1}{\rho}\right)\right)\right], \tag{6.4.4}$$

where I_1 and I_2 are positive functions of the variable x and the parameters ρ and μ.

First, we consider the case $1/2 < \rho \le 2/3$. We express $\mu(\rho) - 1/\rho - 1$ (see (6.4.2)) and $\mu(\rho) - 2/\rho$ through the parameter α:

$$\mu(\rho) - \frac{1}{\rho} - 1 = -2\alpha + 0.7\alpha^2, \quad \mu(\rho) - \frac{2}{\rho} = -1 - \alpha + 0.7\alpha^2.$$

Substituting these expressions in (6.4.4) and denoting $v(\alpha)$ by $2\alpha - 0.7\alpha^2$, we obtain

$$F_\rho\big(x_\rho;\mu(\rho)\big) = -2\rho\exp(x_\rho\cos\pi\rho) + \frac{x_\rho^{-v(\alpha)}}{\Gamma(1-v(\alpha))} - \frac{I_1\sin\pi\left(\alpha-0.7\alpha^2\right) + I_2\sin\left(\pi v(\alpha)\right)}{\pi}.$$

Since $0 < \alpha - 0.7\alpha^2 < 1/2$ and $0 < v(\alpha) < 1$ for $0 < \alpha \le 0.5$, we have

$$\sin\pi\left(\alpha - 0.7\alpha^2\right) > 0, \quad \sin\pi v(\alpha) > 0 \quad \text{for} \quad 0 < \alpha \le 0.5,$$

and we arrive at the following upper estimate:

$$F_\rho\big(x_\rho;\mu(\rho)\big) < -2\rho\exp(x_\rho\cos\pi\rho) + \frac{x_\rho^{-v(\alpha)}}{\Gamma(1-v(\alpha))}.$$

Thus, it remains to prove the inequality

$$\frac{x_\rho^{-v(\alpha)}}{\Gamma(1-v(\alpha))} \le (1+2\varepsilon)\exp\big(-x_\rho\sin\pi\varepsilon\big)$$
$$\Longleftrightarrow \quad 0 \le v(\alpha)\ln x_\rho + \ln\Gamma(1-v(\alpha)) + \ln(1+2\varepsilon) - x_\rho\sin(\pi\varepsilon), \quad 0 < \alpha \le 1/2 \tag{6.4.5}$$

(here $\varepsilon = \rho - 1/2$).

Obviously, $x_\rho > \pi$. We show that $x_\rho < 2\pi$. Since, owing to (6.4.2),

$$\mu(\rho) - 1 = \frac{1}{\rho} - v(\alpha) \quad \Longrightarrow \quad \rho(\mu(\rho)-1) = 1 - \rho v(\alpha) = 1 - \frac{2\alpha - 0.7\alpha^2}{2-\alpha},$$

we have

$$x_\rho = \pi\left(2 - \frac{2\alpha - 0.7\alpha^2}{2-\alpha}\right)\operatorname{cosec}(\pi\rho) = \pi\left(2 - \alpha - \frac{0.3\alpha^2}{2-\alpha}\right)\sec(\pi\varepsilon)$$
$$= \pi\left(2 - \alpha - 0.6\varepsilon\alpha\right)\sec(\pi\varepsilon) < \pi\left(2-\alpha\right)\sec(\pi\varepsilon).$$

Thus, it remains to prove that

$$2-\alpha < 2\cos(\pi\varepsilon) \quad \Longleftrightarrow \quad 4\sin^2(\pi\varepsilon/2) < \alpha.$$

Since

$$\varepsilon = \frac{\alpha}{4-2\alpha} \le \frac{\alpha}{3} \quad \text{for} \quad 0<\alpha\le \frac{1}{2},$$

we have

$$4\sin^2\frac{\pi\varepsilon}{2} < \pi^2\varepsilon^2 < \frac{\pi^2\alpha^2}{9} < \frac{10}{9}\alpha^2 \le 0.5\alpha\frac{10}{9} < \alpha,$$

which was required.

Further, we note that if on the right-hand side of (6.4.5) we replace x_ρ by x, we obtain a function that decreases on the ray $3 \le x < +\infty$. Indeed, its derivative with respect to x is equal to

$$\frac{v(\alpha)}{x} - \sin\pi\varepsilon < \frac{2\alpha}{x} - 3\varepsilon \le \frac{2}{3}\alpha - 3\varepsilon < 0,$$

since $\varepsilon > \alpha/4$. Therefore, the right-hand side of (6.4.5) exceeds the value obtained after replacing x_ρ by 2π. This leads to the problem of the proof of a simpler but stronger inequality than (6.4.5):

$$0 \le v(\alpha)\ln 2\pi + \ln\Gamma(1-v(\alpha)) + \ln(1+2\varepsilon) - 2\pi\sin\pi\varepsilon, \quad 0<\alpha\le\frac{1}{2}.$$

This has been proved in Lemma 6.3.1, and we have proved assertion 6.4.1 in the case $1/2 < \rho \le 2/3$. □

Consider the case $2/3 < \rho < 1$. We verify that both sines in the representation (6.4.4) are positive and hence the following inequality holds:

$$F_\rho\big(x_\rho;\mu(\rho)\big) < -2\rho\exp\big(x_\rho\cos\pi\rho\big) + \frac{x^{\mu(\rho)-1-1/\rho}}{\Gamma\big(\mu(\rho)-1/\rho\big)}.$$

Indeed,

$$\sin\left(\pi\left(\mu(\rho)-\frac{1}{\rho}\right)\right) = \sin(\pi h(\rho)) > 0$$

since $0<h(\rho)<1$, and

$$\sin\left(\pi\left(\mu(\rho)-\frac{2}{\rho}\right)\right) = \sin\left(\pi\left(h(\rho)-\frac{1}{\rho}\right)\right) = \sin\pi\left(\frac{1}{\rho}-1-h(\rho)\right) > 0$$

since

$$0 < \frac{1}{\rho} - 1 - h(\rho) < 1$$

(the left-hand side inequality was proved in Lemma 6.3.2, and the right-hand side inequality is obvious since $1/\rho - 1 < 1$ for $\rho > 1/2$ and, moreover, $1/\rho - h(\rho) - 1 < 1$).

Thus, we have reduced the problem to the inequality (here $1-\rho = t$)

$$\frac{x_\rho^{h(\rho)-1}}{\Gamma(h(\rho))} < 2\rho\exp(-x_\rho\cos\pi t), \quad \frac{2}{3} < \rho < 1. \tag{6.4.6}$$

We have the expression

$$\begin{aligned} x_\rho &= \pi\left(1+\rho\left(\frac{1}{\rho}+h(\rho)-1\right)\right)\operatorname{cosec}(\pi\rho) \\ &= \pi(2+\rho h(\rho)-\rho)\operatorname{cosec}(\pi t) = \pi(1+t+\rho h(\rho))\operatorname{cosec}(\pi t). \end{aligned} \tag{6.4.7}$$

By Lemma 6.3.2, we have the estimate

$$\rho h(\rho) < \frac{t}{2.042} < 0.49t.$$

Therefore,

$$\begin{aligned} x_\rho \cos \pi t &< \pi(1+1.49t)\cot \pi t = -\ln h(\rho) + 1.49\pi t \cot \pi t, \\ \exp(-x_\rho \cos \pi t) &> h(\rho)\exp(-1.49\pi t \cot \pi t) > h(\rho) e^{-1.49}. \end{aligned} \tag{6.4.8}$$

From (6.4.6) and (6.4.8) we see that it remains to prove the inequality

$$\frac{x_\rho^{h(\rho)-1}}{\Gamma(h(\rho))} < 2\rho e^{-1.49} h(\rho) \iff e^{1.49} x_\rho^{h(\rho)-1} < 2\rho\Gamma(1+h(\rho)), \quad \frac{2}{3} < \rho < 1. \tag{6.4.9}$$

Lemma 6.3.2 implies that $0 < h(\rho) < 1/6$ and hence

$$x_\rho^{h(\rho)-1} < x_\rho^{-5/6}, \quad \Gamma(1+h(\rho)) > \Gamma(1+1/6), \quad 2\rho > \frac{4}{3} \tag{6.4.10}$$

(since $\Gamma(s)$ decreases on the interval $1 < s < 1.4$ and the inequality $x_\rho > 1$). Replacing in (6.4.9) the right-hand side by a smaller value and the left-hand side by a greater value in accordance with (6.4.10), we obtain the inequality

$$e^{1.49} x_\rho^{-5/6} < \frac{4}{3}\Gamma\left(1+\frac{1}{6}\right),$$

which after taking the logarithm becomes

$$1.49 - \frac{5}{6}\ln x_\rho < \ln\frac{4}{3} + \ln\Gamma\left(1+\frac{1}{6}\right). \tag{6.4.11}$$

We estimate the right-hand side of (6.4.11) from below and the left-hand side from above. Since $\ln(4/3) > 0.287$, $\ln\Gamma(1+y) = y\psi(\xi)$, $1 < \xi < 1+y$ $(y > 0)$, and the function ψ increases and $\psi(1) = -\gamma > -0.58$, we have

$$\ln\frac{4}{3} + \ln\Gamma\left(1+\frac{1}{6}\right) > 0.287 - \frac{0.58}{6} > 0.19. \tag{6.4.12}$$

On the other hand, from (6.4.7) and the fact that the function $(1+t)\operatorname{cosec}(\pi t)$ decreases on the interval $0 < t < 0.4$ we obtain

$$x_\rho > \pi(1+t)\operatorname{cosec}\pi t > \pi\left(1+\frac{1}{3}\right)\operatorname{cosec}\frac{\pi}{3} = \frac{8\pi}{3\sqrt{3}} > 4.8, \quad 0 < t < \frac{1}{3}.$$

Therefore,

$$1.49 - \frac{5}{6}\ln x_\rho < 1.49 - \frac{5}{6}\ln 4.8 < 0.185. \tag{6.4.13}$$

From (6.4.13) and (6.4.12) we obtain (6.4.11). This completes the proof of assertion (1).

Assertion 6.4.2. *We set*

$$M(\rho) = \begin{cases} 36\cos^{-4}\pi\rho, & \dfrac{1}{2} < \rho \le \dfrac{3}{4}, \\ \dfrac{6}{h(\rho)}, & \dfrac{3}{4} < \rho < 1. \end{cases}$$

Then for any $\mu \in [\mu(\rho), 1.5/\rho)$, the function $E_\rho(z;\mu)$ is positive on $\mathbb{R}\backslash(-M(\rho),-1)$. In particular, its real roots (if they exist) lie only in the interval $(-M(\rho),-1)$.

In assertion 6.4.2, we need not consider values of the parameter $\mu \ge 1.5/\rho$: in the following section we prove that in the case where $\mu \ge 1.5/\rho$ and $1/2 < \rho < 1$, the function $E_\rho(z;\mu)$ is positive on the whole real axis. Further, this bound for the parameter μ will be lowered.

We also note that the two-sided estimate of possible roots in assertion (2) is sufficiently rough; its refinement required a specific examination.

Proof of assertion 6.4.2. The positiveness of the function $E_\rho(z;\mu)$ for $z \in \mathbb{R}$ such that

$$z \geq -\frac{\Gamma\left(\mu + \frac{1}{\rho}\right)}{\Gamma(\mu)}$$

was proved in Chap. 3 (Corollary 3.3.1 of Lemma 3.3.1). If $\mu \geq q \geq 1$, then

$$\frac{\Gamma(\mu+q)}{\Gamma(q)} \geq \frac{\Gamma(q+1)}{\Gamma(q)} = q \geq 1.$$

This implies the positiveness of $E_\rho(z;\mu)$ on the ray $[-1,+\infty)$ (recall that $\mu(\rho) > 1/\rho$).

The proof of the positiveness of $E_\rho(z;\mu)$ for $z \in \mathbb{R}$, $z \leq -M(\rho)$, is more difficult. First, we consider the case where $1/2 < q \leq 2/3$. We must verify the positiveness of the function $F_\rho(x;\mu) = x^{\mu-1}E_\rho(-x^{1/\rho};\mu)$ on the ray $x \geq M^\rho(\rho)$. From Theorem 1.5.2 we deduce the estimate

$$F_\rho(x;\mu) \geq -2\rho\exp(x\cos\pi\rho) + \frac{x^{\mu-1-1/\rho}}{\Gamma(\mu-1/\rho)} + \omega_\rho(x;\mu),$$

where

$$\left|\omega_\rho(x;\mu)\right| \leq \frac{3\Gamma(1-\mu+2/\rho)}{2\pi}x^{\mu-1-2/\rho} < \frac{\Gamma(1-\mu+2/\rho)}{2}x^{\mu-1-2/\rho}.$$

Since $\mu > 1/\rho$, we have

$$1-\mu+\frac{2}{\rho} < 1+\frac{1}{\rho} < 3 \quad\Longrightarrow\quad \Gamma\left(1-\mu+\frac{2}{\rho}\right) < 2$$

(obviously, $1-\mu+2/\rho > 1$ since $\mu < 1.5/\rho$). Therefore,

$$\left|\omega_\rho(x;\mu)\right| \leq x^{\mu-1-2/\rho} \leq x^{\mu-1-1/\rho}/M(\rho)$$

and we arrive at the estimate

$$F_\rho(x;\mu) > -2\rho\exp(x\cos\pi\rho) + x^{\mu-1-1/\rho}\left[\frac{1}{\Gamma(\mu-1/\rho)} - \frac{1}{M(\rho)}\right]. \tag{6.4.14}$$

We present a lower numerical estimate of the expression

$$\frac{1}{\Gamma(\mu-1/\rho)} - \frac{1}{M(\rho)}, \quad \mu(\rho) \leq \mu \leq \frac{1.5}{\rho}, \quad \frac{1}{2} < \rho \leq \frac{2}{3}.$$

Since the function $1/\Gamma(t)$ increases on the interval $0 < t < 1$ and is greater than t, we have

$$\frac{1}{\Gamma(\mu-1/\rho)} \geq \frac{1}{\Gamma(\mu(\rho)-1/\rho)} = \frac{1}{\Gamma(1-2\alpha+0.7\alpha^2)} \geq \frac{1}{\Gamma(7/40)} > \frac{7}{40}.$$

Since

$$M(\rho) = 36\cos^{-4}\pi\rho \geq 36\cdot 16,$$

we have

$$\frac{1}{\Gamma(\mu-1/\rho)} - \frac{1}{M(\rho)} > \frac{7}{40} - \frac{1}{36\cdot 16} > 0.17. \tag{6.4.15}$$

From (6.4.14) and (6.4.15) we conclude that to prove the positiveness of $F_\rho(x;\mu)$ on the ray $x \geq M^\rho(\rho)$, it suffices to verify the inequality

$$2\rho\exp(x\cos\pi\rho) < 0.17x^{\mu-1-1/\rho} \quad\Longleftrightarrow\quad x^{v(\alpha)}\exp(x\cos\pi\rho) < \frac{0.17}{2\rho},$$

$$\frac{1}{2} < q \leq \frac{2}{3}, \quad x \geq M^\rho(\rho) \geq M^{1/2}(\rho) = 6\cos^{-2}(\pi\rho).$$

We strengthen and simultaneously simplify the last inequality. Since $v(\alpha) < 1$, the left-hand side does not exceed $x\exp(x\cos\pi\rho)$ and the right-hand side is greater than $0.17/(4/3) > 0.1$. Therefore, it remains to verify that

$$x\exp(x\cos\pi\rho) < 0.1 \quad \text{for} \quad x \geq 6\cos^{-2}(\pi\rho), \quad \frac{1}{2} < \rho \leq \frac{2}{3}.$$

The function $x\exp(x\cos\pi\rho)$ decreases on the ray $x \geq |\sec(\pi\rho)|$ and hence for the considered values x it does not exceed its value at the point $x = 6\cos^{-2}\pi\rho$:

$$x\exp(x\cos\pi\rho) \leq 6\sec^2\pi\rho\exp(6\sec\pi\rho) = \frac{1}{6}u^2e^{-u} \leq \frac{1}{6}\max_{u\geq 0}(u^2e^{-u}) = \frac{2}{3}e^{-2} < 0.1.$$

The required inequality is proved.

Now let $2/3 < \rho \leq 3/4$. Similarly to (6.2.5), we have the estimate

$$|\omega_\rho(x;\mu)| \leq 1.19\Gamma\left(1-\mu+\frac{2}{\rho}\right)x^{\mu-1-2/\rho} < 1.19\Gamma(2.5)x^{1-\mu-2/\rho} < 1.6x^{1-\mu-1/\rho}/M(\rho).$$

Therefore, we obtain the estimate

$$F_\rho(x;\mu) > -2\rho\exp(x\cos\pi\rho) + x^{\mu-1-1/\rho}\left[\frac{1}{\Gamma(\mu-1/\rho)} - \frac{1.6}{M(\rho)}\right].$$

As above, we see that the right-hand side increases with respect to μ for the considered values of the parameter and obtain the estimate

$$F_\rho(x;\mu) > -2\rho\exp(x\cos\pi\rho) + x^{-1}\left[\frac{1}{\Gamma(h(\rho))} - \frac{1.6}{M(\rho)}\right].$$

Similarly to the above reasoning (here $h(\rho) \geq h(3/4) = e^{-\pi}$ and $M(\rho) \geq 36\cos^{-4}(3\pi/4) = 36.4$), we have the relations

$$\frac{1}{\Gamma(h(\rho))} - \frac{1.6}{M(\rho)} \geq h(\rho) - \frac{1.6}{9\cdot 16} \geq e^{-\pi} - \frac{1}{90} > \frac{1}{24} - \frac{1}{90} > \frac{1}{36}.$$

Thus, we must prove the inequality

$$2\rho\exp(x\cos\pi\rho) < \frac{1}{36x} \quad\Longleftrightarrow\quad 6x\exp(x\cos\pi\rho) < \frac{1}{12\rho}, \quad x \geq \frac{6}{\cos^2(\pi\rho)}, \quad \frac{2}{3} < \rho \leq \frac{3}{4}.$$

Since the function $x\exp(x\cos\pi\rho)$ decreases for $x \geq |\sec\pi\rho|$, we have the estimate

$$6x\exp(x\cos\pi\rho) \leq 36\sec^2\pi\rho\exp(6\sec\pi\rho) = u^2e^{-u} < 36e^{-6} < 0.1, \quad x \geq 6\sec^2\pi\rho.$$

However, we have the inequality

$$\frac{1}{12\rho} \geq \frac{1}{12}\cdot\frac{4}{3} = \frac{1}{9} > 0.1,$$

and we have obtained what was required.

Finally, $3/4 < \rho < 1$. By Theorem 1.4.2 ($m = 1$), we have the formula

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)}\exp(z^\rho) - \frac{1}{z\Gamma(\mu-1/\rho)} + R(z;\rho,\mu), \tag{6.4.16}$$

where

$$|R(z;\rho,\mu)| \leq 2^{1+b/2}\Gamma(b+1)|z|^{-2}, \quad b = \frac{2}{\rho} - \mu.$$

Recall that in this representation of the Mittag-Leffler function, we take the principal branch of the argument in the definition of noninteger powers of z, and on the ray $(-\infty, 0)$ we can set either $z = |z|e^{\pi i}$ or $z = |z|e^{-\pi i}$. A discontinuity of an exponentially small term in the approximate representation of

E_ρ does not introduce any contradiction. From (6.4.16), taking into account the realness of the Mittag-Leffler function on $\mathbb{R}$ ($\mu \in \mathbb{R}$), we obtain the following lower estimate:

$$E_\rho(z;\mu) > \frac{1}{|z|\Gamma(h(\rho))} - \exp\big(|z|^\rho \cos\pi\rho\big) - 2^{1+b/2}\Gamma(b+1)|z|^{-2}, \quad z \in \mathbb{R}, \quad z < -1.$$

Further, by the restrictions $\rho > 3/4$, $\cos\pi\rho < -1/\sqrt{2}$, and $b \le 1/\rho < 4/3$, we have the estimate

$$E_\rho(z;\mu) > \frac{h(\rho)}{|z|} - \exp\left(-\frac{|z|^{3/4}}{\sqrt{2}}\right) - 2^{5/3}\Gamma(1+4/3)|z|^{-2},$$

$$|z|E_\rho(z;\mu) > h(\rho) - |z|\exp\left(-\frac{|z|^{3/4}}{\sqrt{2}}\right) - 4.2|z|^{-1}.$$

This shows that to prove the positiveness of $E_\rho(z;\mu)$ for $z \in \mathbb{R}$, $z \le -6/h(\rho)$, it suffices to verify the inequality

$$|z|\exp\left(-\frac{|z|^{3/4}}{\sqrt{2}}\right) < \frac{1.8}{|z|}, \quad |z| \le -6/h(\rho).$$

Denoting $|z|^{3/4}/\sqrt{2} = u$, we obtain

$$|z|^2\exp\left(-\frac{|z|^{3/4}}{\sqrt{2}}\right) = (u\sqrt{2})^{8/3}e^{-u} = \left(\frac{16}{u}\right)^{1/3} u^3e^{-u} < \max_{u\ge 0}(u^3e^{-u}) = 27e^{-u} < 1.8,$$

which was required (we have used the fact that $h(\rho) < 1/23$ for $3/4 < \rho < 1$ by Lemma 6.3.2 and hence $|z| > 100$ and $u > 16$). Assertion 6.4.2 is completely proved. ☐

6.5. Proof of Theorem 6.1.1

In this section, we take up one more special function, the Wright function (see [45, 46]):

$$e_\beta(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!\,\Gamma(-\beta n)}, \quad 0 < \beta < 1.$$

We need the following of its properties obtained by Wright:

$$e_\beta(y) > 0 \quad \forall y \in (-\infty;0), \qquad e_\beta(y) = O(e^{Ay}), \quad y \to -\infty \quad \forall A > 0, \tag{6.5.1}$$

$$\int_0^{+\infty} s^{\nu-1}e_\beta(-s)\,ds = \frac{\Gamma(\nu)}{\Gamma(\beta\nu)} \quad \forall \nu > 0. \tag{6.5.2}$$

From (6.5.2) by using the substitution $s = tx^{-\beta}$ for any $x > 0$ we obtain

$$\int_0^{+\infty} t^{\nu-1}e_\beta(-tx^{-\beta})\,dt = x^{-\beta\nu}\frac{\Gamma(\nu)}{\Gamma(\beta\nu)}. \tag{6.5.3}$$

The key point of the proof of the theorem is the following.

Lemma 6.5.1. *If $\rho \ge 1/2$, $\mu > 0$, and the function $E_\rho(z;\mu)$ is nonnegative on $\mathbb{R}$, then the following assertions hold*:

(1) *for any $\lambda > \mu$, the function $E_\rho(z;\lambda)$ is positive on $\mathbb{R}$;*
(2) *for any $\beta \in (0,1)$, the function $E_{\rho/\beta}(z;\beta\mu)$ is positive on $\mathbb{R}$.*

Proof. For any $\lambda > 0$, the function $E_\rho(z;\lambda)$ is positive on $[0,+\infty)$ and no difficulties arise. We must prove its positiveness on the ray $(-\infty,0)$. To test the function $E_\rho(z;\lambda)$ on this ray, it is convenient to pass to the function

$$F_\rho(x;\lambda) \equiv x^{\lambda-1}E_\rho(-x^{1/\rho};\lambda).$$

Consider the operator $D^{-\varepsilon}$, $\varepsilon > 0$, defined by the formula

$$D^{-\varepsilon}g(x) = \frac{1}{\Gamma(\varepsilon)}\int_0^x \frac{g(t)dt}{(x-t)^{1-\varepsilon}},$$

defined, at least, on $C[0,+\infty)$ and called the integral of order ε. We need the result of the action of this operator on the function F_ρ:

$$D^{-\varepsilon}F_\rho(x;\mu) = F_\rho(x;\mu+\varepsilon). \tag{6.5.4}$$

Prove Eq. (6.5.4). The power-series expansion of the Mittag-Leffler function yields (here B is the beta-function)

$$\int_0^x \frac{F_\rho(t;\mu)}{(x-t)^{1-\varepsilon}}dt = \int_0^x \sum_{k=0}^\infty \frac{(-1)^k t^{\mu-1+k/\rho}dt}{\Gamma(\mu+k/\rho)(x-t)^{1-\varepsilon}} = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\mu+k/\rho)}\int_0^1 \frac{(ux)^{\mu-1+k/\rho}d(ux)}{(x)^{1-\varepsilon}(1-u)^{1-\varepsilon}}$$

$$= \sum_{k=0}^\infty \frac{(-1)^k x^{\mu+\varepsilon-1+k/\rho}}{\Gamma(\mu+k/\rho)}B\left(\mu+\frac{k}{\rho},\varepsilon\right) = x^{\mu+\varepsilon-1}\sum_{k=0}^\infty \frac{(-1)^k x^{k/\rho}\Gamma(\varepsilon)}{\Gamma(\mu+\varepsilon+k/\rho)} = \Gamma(\varepsilon)F_\rho(x;\mu+\varepsilon).$$

Since the operator $D^{-\varepsilon}$ (for all $\varepsilon > 0$) maps nonnegative functions that do not vanish identically in any right-hand side semi-neighborhood of the origin to positive functions, we obtain the first assertion of the lemma by setting $\varepsilon = \lambda - \mu > 0$.

To prove the second assertion, we use properties (6.5.1) and (6.5.3) of the Wright function. From (6.5.3) we have

$$\int_0^{+\infty} F_\rho(t;\mu)e^0_\beta(-tx^{-\beta})dt = \int_0^{+\infty}\sum_{k=0}^\infty \frac{(-1)^k t^{\mu-1+k/\rho}}{\Gamma(\mu+k/\rho)}e^0_\beta(-tx^{-\beta})dt$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\mu+k/\rho)}\int_0^{+\infty} t^{\mu-1+k/\rho}e^0_\beta(-tx^{-\beta})dt = \sum_{k=0}^\infty \frac{(-1)^k x^{\beta\mu+\beta k/\rho}}{\Gamma(\beta\mu+\beta k/\rho)} = xF_{\rho/\beta}(x;\beta\mu). \tag{6.5.5}$$

The validity of swapping summation and integration must be justified. The series representing $F_\rho(t;\mu)$ converges to this function everywhere on $(0,+\infty)$, but it is also required that a majorant of the modulus of partial sums of this series multiplied by $e^0_\beta(-tx^{-\beta})$ is summable on $\mathbb{R}$ for any $x > 0$. The modulus of any partial sum of this series does not exceed

$$t^{\mu-1}\sum_{k=0}^\infty \frac{t^{k/\rho}}{\Gamma(\mu+k/\rho)} = t^{\mu-1}E_\rho(t^{1/\rho};\mu) = O(e^t), \quad t > 0.$$

The Wright function $e^0_\beta(-t)$ tends to zero faster than e^{-At} as $t \to +\infty$ for all $A > 0$. This implies the summability. From (6.5.5) and the positiveness of the Wright function on $(-\infty,0)$ we obtain the second assertion of the lemma. Lemma 6.5.1 is completely proved. □

Taking $\rho = 1/2$ and $\mu = 3$ (then $E_{1/2}(z;3) = z^{-1}(\cosh\sqrt{z}-1) \geq 0$ on $\mathbb{R}$), we obtain the following consequence of Lemma 6.5.1.

Corollary 6.5.1. *The function $E_{1/2}(z;\lambda)$ is positive on $\mathbb{R}$ for any $\lambda > 3$. The function $E_\rho(z;\lambda)$ is positive on $\mathbb{R}$ for any $\rho > 1/2$ and $\lambda \geq 1.5/\rho$.*

Now we can prove Theorem 6.1.1. We introduce the set

$$B_\rho = \{\mu > 0 \mid \exists z \in \mathbb{R} : E_\rho(z;\mu) < 0\}.$$

The fact that the set B_ρ is nonempty follows from assertion (1) in Sec. 6.4 and the simple inclusion $1/\rho \in B_\rho$, $1/2 < \rho < 1$. Indeed, for any $\rho > 1/2$ and $\mu \in \mathbb{C}$, by Theorem 1.2.1, we have the asymptotics

$$E_\rho(z;\mu) = \frac{-1}{z\Gamma(\mu - 1/\rho)} - \frac{1}{z^2\Gamma(\mu - 2/\rho)} - O\left(\frac{1}{z^3}\right), \quad z \in \mathbb{R}, \quad z \to -\infty. \tag{6.5.6}$$

Therefore, for $\mu = 1/\rho$ and $\rho > 1/2$, $\rho \neq 1$, the following equivalence holds:

$$E_\rho\left(z;\frac{1}{\rho}\right) \sim \frac{-1}{z^2\Gamma(-1/\rho)}, \quad z \in \mathbb{R}, \quad z \to -\infty. \tag{6.5.7}$$

From (6.5.7) and the positiveness of the Γ-function on the interval $(-2,-1)$ we obtain that for all sufficiently large (in modulus) negative z, the function $E_\rho(z;1/\rho)$ is negative. Therefore, $1/\rho \in B_\rho$.

Thus, the set B_ρ is nonempty, and Corollary 6.5.1 shows the boundedness of B_ρ. Therefore, there exists a finite positive function $f(\rho) = \sup B_\rho$, $1/2 < \rho < 1$. From Lemma 6.5.1 we obtain the implication

$$\mu \notin B_\rho \quad \Longrightarrow \quad [\mu, +\infty) \cap B_\rho = \emptyset, \quad \mu > 0,$$

which proves the inclusion $(0, f(\rho)) \subset B_\rho$. By the continuity (and even holomorphicity) of the Mittag-Leffler function with respect to the parameter μ, the inequality

$$E_\rho(z_0;\mu_0) < 0, \quad z_0, \mu_0 \in \mathbb{R},$$

implies the inequality $E_\rho(z_0;\mu) < 0$ for all μ lying in some neighborhood U of the point μ_0. This means that $U \subset B_\rho$ and hence $f(\rho)$ is a boundary point of B_ρ and does not belong to the set B_ρ.

Thus, for $\mu \in (0, f(\rho))$, the function $E_\rho(z;\mu)$ takes at some points of the real axis negative values, the function $E_\rho(z;f(\rho))$ is nonnegative on $\mathbb{R}$, and for $\lambda > f(\rho)$, we have the inequality $E_\rho(z;\lambda) > 0$ for all $z \in \mathbb{R}$.

We prove that $E_\rho(z;f(\rho))$ has a real root. By assertion (2) from Sec. 6.4, for $\mu \geq \mu(\rho)$, all negative values of $E_\rho(z;\mu)$ (if they exist) are located on the segment $[-M(\rho),-1]$. For $\mu(\rho) \leq \mu < f(\rho)$, they exist and, therefore, denoting $m_\rho(\mu)$ by $\min\{E_\rho(z;\mu) \mid -M(\rho) \leq z \leq -1\} < 0$, $\mu(\rho) \leq \mu < f(\rho)$, we obtain that

$$m_\rho(f(\rho)) = \lim_{\mu \to f(\rho)-0} m_\rho(\mu) \leq 0.$$

This proves the existence of a real root of the function $E_\rho(z;f(\rho))$.

It remains to deduce the two-sided estimate (6.1.1) and prove that $f(\rho)$ decreases. The inclusion $1/\rho \in B_\rho$ means the validity of the inequality $1/\rho < f(\rho)$ and Corollary 6.5.1 of Lemma 6.5.1 immediately implies the inequality $f(\rho) \leq 1.5/\rho$. But in the case where $1/2 < \rho < 1$, the function $E_\rho(z;f(\rho))$ has a root and hence for these ρ, the inequality is strong: $f(\rho) < 1.5/\rho$. The second part of Lemma 6.5.1 means the validity of the inequality

$$f(\rho) \leq \frac{\rho_0}{\rho} f(\rho_0), \quad \frac{1}{2} \leq \rho_0 < \rho, \tag{6.5.8}$$

which implies the fact that f decreases. Theorem 6.1.1 is completely proved.

These arguments allow one to give another proof of the theorem, which asserts that the function $E_\rho(z;\mu)$ has no real roots for $\rho > 1$ and $\mu \geq 1/\rho$ but has for $0 < \mu < 1 < \rho$. First, we prove that $f(1) = 1$. Indeed, if $\mu \in (0,1)$, then from (6.5.6) we see that

$$E_1(z;\mu) \sim -(z\Gamma(\mu-1))^{-1}, \quad z \to -\infty, \quad z \in \mathbb{R}.$$

This implies the negativeness of $E_1(z;\mu)$ for all sufficiently large (in modulus) negative z. However, the function $E_1(z;\mu) \equiv e^z$ is positive on $\mathbb{R}$. Here, in contrast to the case $1/2 < \rho < 1$, where the second parameter of the Mittag-Leffler function is equal to $\sup B_1$, there are no real roots. The reason is that a real root of $E_1(z;\mu)$ tends to $-\infty$ as μ tends to 1 from the left. The same effect also appears if $\rho > 1$.

Taking $\rho_0 = 1$, we find from (6.5.8) that $f(\rho) \le 1/\rho$. On the other hand, for $0 < \mu < 1/\rho$, from asymptotics (6.5.6) we obtain the equivalence

$$E_\rho(z;\mu) \sim -\left(z\Gamma\left(\mu - \frac{1}{\rho}\right)\right)^{-1}, \quad z \in \mathbb{R}, \quad z \to -\infty,$$

which implies the existence of negative values of the function $E_\rho(z;\mu)$. Therefore, $f(\rho) = 1/\rho$ for $\rho \ge 1$. In concluding this section, we note that, in our opinion, the function $f(\rho)$ for $1/2 < \rho < 1$ is unlikely to be elementary.

6.6. Proof of Theorem 6.1.3

Lower estimates of the function $f(\rho)$ follow from assertion (1) of Sec. 6.4. To prove upper estimates, it suffices to verify the positiveness of the function $F_\rho(x;3-2\alpha)$ on the ray

$$x > \left(\frac{\Gamma(3-2\alpha+1/\rho)}{\Gamma(3-2\alpha)}\right)^\rho.$$

Recall that, by Lemma 3.1.1, the function $E_\rho(z;\mu)$ is positive on the ray

$$z \ge -\frac{\Gamma(\mu+1/\rho)}{\Gamma(\mu)}$$

of the real axis for $\mu > 0$. Thus, we verify the positiveness of $E_\rho(z;3-2\alpha)$ for all $z \in \mathbb{R}$ and hence prove the estimate

$$f(\rho) < 3 - 2\alpha, \quad \frac{1}{2} < \rho \le \frac{2}{3}.$$

In particular, $f(2/3) < 2$. This and (6.5.8) imply

$$f(\rho) < \frac{\frac{2}{3} f\left(\frac{2}{3}\right)}{\rho} < \frac{4/3}{\rho}, \quad \rho > \frac{2}{3},$$

which proves the upper estimate (6.1.5).

Obtaining lower estimate of the variable x, which is equal to

$$\left(\frac{\Gamma(3-2\alpha+1/\rho)}{\Gamma(3-2\alpha)}\right)^\rho,$$

is rather difficult. We deduce the inequality

$$\left(\frac{\Gamma(\mu+1/\rho)}{\Gamma(\mu)}\right)^\rho > 2, \quad \mu = 3-2\alpha, \quad \frac{1}{2} < \rho \le \frac{2}{3}, \tag{6.6.1}$$

and then prove the positiveness of the function $F_\rho(x;3-2\alpha)$ for $x > \pi/2$. Taking the logarithm of (6.6.1), we obtain the equivalent inequality

$$\ln 2 < \rho\left(\ln\Gamma\left(\mu + \frac{1}{\rho}\right) - \ln\Gamma(\mu)\right) = \rho \int\limits_{\mu}^{\mu+1/\rho} \psi(t)dt.$$

Since ψ increases, the last integral mean is no less than its value corresponding to the minimal value of the parameter $\mu = 2$:

$$\rho \int_2^{2+1/\rho} \psi(t)dt.$$

We have the relation

$$\frac{d}{d\rho}\left(\rho \int_2^{2+1/\rho} \psi(t)dt\right) = \int_2^{2+1/\rho} \psi(t)dt - \frac{1}{\rho}\psi\left(2+\frac{1}{\rho}\right) < 0$$

since ψ increases. Thus, the minimum of the left-hand side of (6.6.1) is equal to

$$\left(\frac{\Gamma(2+3/2)}{\Gamma(2)}\right)^{2/3} = (\Gamma(3.5))^{2/3} > 3^{2/3} > 2,$$

which was required.

To prove the positiveness of the function $F_\rho(x; 3-2\alpha)$ for $x \geq 2$, we use Theorem 1.5.2 with $m = 1$ and $\mu = 3 - 2\alpha$. Taking into account the relations

$$\rho(\mu-1) = \rho(2-2\alpha) = \rho(2-\alpha) - \rho\alpha = 1 - \rho\alpha, \quad \mu - \frac{1}{\rho} = 1 - \alpha, \quad \mu - \frac{2}{\rho} = -1$$

and setting $\varepsilon = \rho - 1/2$, we obtain the representation

$$F_\rho(x; 3-2\alpha) = -2\rho \exp\big(-x\sin\pi\varepsilon\big)\cos\big(x\cos\pi\varepsilon + \pi\rho\alpha\big) + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \omega_\rho(x), \tag{6.6.2}$$

in which the remainder $\omega_\rho(x)$ is estimated as follows:

$$\big|\omega_\rho(x)\big| \leq \frac{\Gamma(2)}{\pi x^2}\frac{\sin\pi\alpha}{4\cos^2\dfrac{\pi}{2\rho}} < \frac{\alpha}{2x^2}. \tag{6.6.3}$$

We deduce a two-sided estimate of the operand of the cosine in (6.6.2) for $2 \leq x \leq 4$. Consider the function of two variables

$$\varphi(x, \rho) = x\cos\pi\varepsilon + \pi\rho\alpha.$$

Since $\partial\varphi/\partial x > 0$, the maximum of this function in the rectangle

$$\left\{(x,\rho)\;\middle|\; 2 \leq x \leq 4,\ \frac{1}{2} \leq \rho \leq \frac{2}{3}\right\}$$

is attained at $x = 4$ and the minimum at $x = 2$. Further, we have the relations

$$\frac{\partial\varphi(x,\rho)}{\partial\rho} = \pi(2 - x\sin\pi\varepsilon) \quad \Longrightarrow \quad \frac{\partial\varphi(2,\rho)}{\partial\rho} > 0, \quad \frac{\partial\varphi(4,\rho)}{\partial\rho} \geq 0,$$

$$\text{for} \quad \frac{1}{2} \leq \rho \leq \frac{2}{3} \quad \Longleftrightarrow \quad 0 \leq \varepsilon \leq \frac{1}{6}.$$

Therefore, in this rectangle the following inequality holds:

$$2 = \varphi\left(2, \frac{1}{2}\right) \leq \varphi(x;\rho) \leq \varphi\left(4, \frac{2}{3}\right) = 4\cos\frac{\pi}{6} + \frac{\pi}{3} < \frac{3\pi}{2}.$$

From this we conclude that the first term on the right-hand side of (6.6.2), which is equal to $-2\rho e^{-x\sin\pi\varepsilon}\cos\varphi(x,\rho)$, is positive and the following inequality holds:

$$F_\rho(x; 3-2\alpha) > \frac{x^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\alpha}{2x^2} \geq \frac{1}{\sqrt{\pi x}} - \frac{1}{4x^2} > 0, \quad 2 \leq x \leq 4.$$

For $x \geq 4$ from (6.6.2) and (6.6.3) we obtain

$$F_\rho(x; 3-2\alpha) > \frac{x^{-\alpha}}{\Gamma(1-\alpha)} - 2\rho \exp\big(-x\sin\pi\varepsilon\big) - \frac{\alpha}{2x^2}.$$

Therefore,

$$x^\alpha \Gamma(1-\alpha) F_\rho(x; 3-2\alpha) > 1 - 2\rho\Gamma(1-\alpha)x^\alpha \exp\big(-x\sin\pi\varepsilon\big) - \frac{\alpha\Gamma(1-\alpha)}{2x^{2-\alpha}}.$$

We roughen this estimate slightly and simplify it at the same time. Since

$$2\rho = 1 + 2\varepsilon < e^{2\varepsilon}, \quad \Gamma(1-\alpha) \leq \sqrt{\pi},$$
$$\sin\pi\varepsilon \geq 3\varepsilon \quad 0 < \varepsilon \leq \frac{1}{6},$$
$$x^{2-\alpha} \geq x^{1.5} \geq 8, \quad x \geq 4,$$

we have

$$x^\alpha \Gamma(1-\alpha) F_\rho(x; 3-2\alpha) \geq 1 - \Gamma(1-\alpha)x^\alpha e^{2\varepsilon - 3\varepsilon x} - 0.12\alpha.$$

The function $x^\alpha e^{-3\varepsilon x}$ decreases on the ray

$$x \geq \frac{\alpha}{3\varepsilon} = \frac{4-2\alpha}{3};$$

therefore, for $x \geq 4$ we have the inequality

$$x^\alpha e^{-3\varepsilon x} \leq 4^\alpha e^{-12\varepsilon} < e^{1.4\alpha - 12\varepsilon}.$$

This means that it remains to prove the positiveness of the function

$$g(\alpha) = 1 - \Gamma(1-\alpha)e^{1.4\alpha - 10\varepsilon} - 0.12\alpha, \quad 0 < \alpha \leq \frac{1}{2},$$

where

$$\varepsilon = \frac{\alpha}{4-2\alpha}.$$

For $0 < \alpha \leq 1/4$, using the estimates

$$\varepsilon > \frac{\alpha}{4}, \quad \Gamma(1-\alpha) < e^{0.9\alpha}$$

(see Lemma 6.3.3), we obtain

$$g(\alpha) > 1 - e^{-0.2\alpha} - 0.12\alpha.$$

Since

$$1 - e^{-t} > t - \frac{t^2}{2} \quad \forall t > 0,$$

we see that

$$g(\alpha) > 0.08\alpha - 0.02\alpha^2 > 0, \quad 0 < \alpha \leq \frac{1}{4}.$$

In the case $1/4 < \alpha \leq 1/2$, applying the estimates

$$\varepsilon > \frac{\alpha}{3.5}, \quad \Gamma(1-\alpha) < e^{1.15\alpha}$$

(see Lemma 6.3.3), we also obtain that

$$g(\alpha) > 1 - e^{-0.2\alpha} - 0.12\alpha > 0, \quad \frac{1}{4} < \alpha \leq \frac{1}{2},$$

which was required. Theorem 6.1.3 is proved.

6.7. Sketch of the Proof of Theorem 6.1.4

In this section, we describe the main steps of the proof of Theorem 6.1.4. We denote $1-\mu$ by s and consider the sequence

$$x_n = x_n(\rho,\mu) = \pi(n-\rho s)\sec(\pi\varepsilon), \quad n\in\mathbb{N}, \tag{6.7.1}$$

which is useful for the localization of real roots and counting of their number. The proof is based on the representation of the function

$$F_\rho(z;\mu) = z^{\mu-1}E_\rho(-z^{1/\rho};\mu)$$

in the half-plane $\operatorname{Re} z>0$ by Theorem 1.5.2. Namely, setting in this theorem $m=1$, we obtain the equality

$$F_\rho(x;\mu) = 2\rho\exp\big(-z\sin\pi\varepsilon\big)\cos\big(z\cos\pi\varepsilon+\pi\rho s\big)+\frac{z^{-s-1/\rho}}{\Gamma(\mu-1/\rho)}+\omega_\rho(z;\mu), \tag{6.7.2}$$

where $\varepsilon=\rho-1/2$ and $\omega_\rho(z;\mu)$ is a function of the variable z holomorphic in the half-plane $\operatorname{Re} z>0$:

$$\big|\omega_\rho(z;\mu)\big| \le \frac{(|z|/x)^s\Gamma(s+2/\rho)}{\pi x^{s+2/\rho}}\left[\left|\sin\left(\pi\left(\mu-\frac{2}{\rho}\right)\right)\right|+\frac12\left|\sin\left(\pi\left(\mu-\frac{1}{\rho}\right)\right)\right|\right]. \tag{6.7.3}$$

We see from (6.7.2) that the power of z in the second term is equal to $-s-1/\rho=\mu-3+\alpha$ (where $\alpha=2-1/\rho\to0$ as $\rho\to1/2$) and is negative and is separated from zero if $\mu<3$. Therefore, the exponent (although its operand is also negative but tends to zero as $\rho\to1/2+$) prevails over $\dfrac{z^{-s-1/\rho}}{\Gamma(\mu-1/\rho)}$, at least when $x=\Re z$ is less than

$$R_1(\varepsilon) = b\operatorname{cosec}\pi\varepsilon\ln\operatorname{cosec}\pi\varepsilon, \tag{6.7.4}$$

where

$$b=\begin{cases}3-\mu & \text{if } \mu\notin\mathbb{Z},\\ 4-\mu & \text{if } \mu\in\mathbb{Z}.\end{cases}$$

Conversely, if $x\ge R(\varepsilon)$,

$$R(\varepsilon) = b\operatorname{cosec}\pi\varepsilon\Big(\ln\operatorname{cosec}\pi\varepsilon+2\ln\ln\operatorname{cosec}\pi\varepsilon\Big), \tag{6.7.5}$$

then the first term becomes substantially less than the second, and real roots are absent.

The asymptotics of the number of real roots is obtained as follows. Denote by $n(\varepsilon)$ the largest number n such that $x_n(\rho,\mu)\le R_1(\varepsilon)$; a number n_0 depending only on μ but independent of ρ will be chosen later. We will prove in the next section that on the sides of the trapezium

$$T_n=\Big\{z=x+iy \;\Big|\; x_n\le x\le x_{n+1},\ -x\le y\le x\Big\}, \quad n_0\le n\le n(\varepsilon)-1, \tag{6.7.6}$$

the following inequality holds:

$$\left|\frac{z^{-s-1/\rho}}{\Gamma(\mu-1/\rho)}\right|+\big|\omega_\rho(z;\mu)\big|<2\rho\Big|\exp\big(-z\sin\pi\varepsilon\big)\cos\big(z\cos\pi\varepsilon+\pi\rho s\big)\Big|. \tag{6.7.7}$$

It implies the coincidence of the number of roots (with account of their multiplicities) of the function $F_\rho(z;\mu)$ and $\exp\big(-z\sin\pi\varepsilon\big)\cos\big(z\cos\pi\varepsilon+\pi\rho s\big)$ inside any such trapezium. Roots of the latter function can be easily found:

$$\cos\big(z\cos\pi\varepsilon+\pi\rho s\big)=0 \quad\Longleftrightarrow\quad z=\sec\pi\varepsilon\left(\pi n+\frac{\pi}{2}-\pi\rho s\right)=\frac{x_n+x_{n+1}}{2}, \quad n\in\mathbb{Z}.$$

All these roots are simple. Thus, inside each trapezium T_n, $n_0\le n\le n(\varepsilon)-1$, the function $F_\rho(z;\mu)$ has a unique root. Since the function F_ρ is real-valued on $\mathbb{R}$ and the trapeziums T_n are symmetric with respect to $\mathbb{R}$, this root is real. Thus, the segment $x_{n_0} \le x \le x_{n(\varepsilon)}$ contains exactly $n(\varepsilon) - n_0$ roots of the function $F_\rho(x;\mu)$. We see from (6.7.1) and (6.7.4) that

$$n(\varepsilon) = \frac{R_1(\varepsilon)}{\pi} \cos \pi\varepsilon + O(1) = \frac{b \cot(\pi\varepsilon)}{\pi} \ln \operatorname{cosec} \pi\varepsilon + O(1) = \frac{b}{\pi^2 \varepsilon} \ln \frac{1}{\varepsilon} + O\left(\frac{1}{\varepsilon}\right)$$

is the principal term of the asymptotics for the number of real roots of $E_\rho(z;\mu)$.

We prove in Sec. 6.8 that for $x \ge R(\varepsilon)$, the function $F_\rho(x;\mu)$ has constant sign for all ρ sufficiently close to 1/2 from the right and, therefore, has no roots. The following step of the derivation of the asymptotics for $\mathcal{N}(\rho;\mu)$ is the estimate of the number of roots of $F_\rho(x;\mu)$ on the segment $x_{n(\varepsilon)} = R_1(\varepsilon) + O(1) \le x \le R(\varepsilon)$. The length of this segment is $O(\varepsilon^{-1} \ln\ln \varepsilon^{-1})$, $\varepsilon \to 0+$. We prove that on the sides of the rectangle

$$\Pi(\varepsilon) = \left\{ x + iy \;\middle|\; x_{n(\varepsilon)} \le x \le x_{m(\varepsilon)},\ |y| \le \operatorname{cosec} \pi\varepsilon \right\}$$

(the segment considered bisects this rectangle), the modulus of the logarithmic derivative $F'_\rho(z;\mu)/F_\rho(z;\mu)$ does not exceed 2 (for all ρ sufficiently close to 1/2 and fixed μ). Therefore, the number of roots of the function $F_\rho(z;\mu)$ in $\Pi(\varepsilon)$ and, moreover, on the center line of this rectangle does not exceed the perimeter of $\Pi(\varepsilon)$ divided by π, i.e., $O(\varepsilon^{-1} \ln\ln \varepsilon^{-1})$; this is the remainder of the asymptotics.

The final step of the proof of Theorem 6.1.4 is the upper estimate of the number of roots of $E_\rho(z;\mu)$ that are "nonlarge" in modulus, i.e., roots lying on the ray $[-x_{n_0}^{1/\rho}, +\infty)$. Actually (see Lemma 6.3.4) they lie on the segment $[-x_{n_0}^{1/\rho}, s^2] \subset [-x_{n_0}^2, s^2]$. As n_0, we take the minimum of the numbers n such that $x_n \ge A^2(s)$, where

$$A(s) = 2^{s+4}(s+3)^2, \quad s \ge 0, \quad A(s) = \frac{2A(0)}{b} \; -2 < s < 0$$

(recall that $s = 1 - \mu$). Since the distance between two neighboring elements of the sequence x_n is equal to $\pi \sec \pi\varepsilon$, we have

$$x_{n_0} \le A^2(s) + 4 < \sqrt{2} A(s) \quad \Longrightarrow \quad x_{n_0}^2 < 2A^4(s),$$

and it remains to obtain an upper estimate for the number of roots of the function $E_\rho(z;\mu)$ on the segment $[-2A^4(s), s^2]$. It is intuitively clear that the number of roots on this segment is bounded from above by a value depending only on s (i.e., on μ) but independent of ρ. The strong proof is as follows. Since the number of roots of an arbitrary entire function, which does not vanish identically, on any compact is finite (or roots are absent), we see that for all $\delta \in (0,1]$, except for, perhaps, a finite number of values, the function $E_{1/2}(z;\mu)$ has no roots on the sides of the rectangle

$$K_\delta = \left\{ z \in \mathbb{C} \;\middle|\; -2A^4(s) - \delta \le \operatorname{Re} z \le s^2 + \delta,\ \left|\operatorname{Im} z\right| \le \delta \right\}.$$

Since the family of functions $E_\rho(z;\mu)$ converges to $E_{1/2}(z;\mu)$ as $\rho \to 1/2$ uniformly on any compact in $\mathbb{C}$ (in particular, in the rectangle K_δ), the numbers of roots of the functions $E_\rho(z;\mu)$ and $E_{1/2}(z;\mu)$ in K_δ coincide for all ρ sufficiently close to 1/2, Thus, the number of roots of the function $E_\rho(z;\mu)$ lying on the ray $[-x_{n_0}^{1/\rho}, +\infty)$ does not exceed the number of roots of the function $E_{1/2}(z;\mu)$ lying in the rectangle K_1, i.e., $O(1)$ for fixed μ and $\rho \to 1/2+$.

The above considerations lead to the following conclusion. Theorem 6.1.4 will be completely proved if we can perform the following:

(1) prove the fact that the function $F_\rho(x;\mu)$ has constant sign for $x \ge R(\varepsilon)$;
(2) deduce the inequality (6.7.7) on the sides of the trapeziums T_n defined in (6.7.6);

(3) prove the estimate

$$\left|\frac{F_\rho'(z;\mu)}{F_\rho(z;\mu)}\right| \le 2$$

on the sides of the rectangle $\Pi(\varepsilon)$.

This will be done in the following section.

In concluding this section, we explain how to choose n_0 (or, equivalently, $A(s)$). The value $A(s)$ is chosen so that the following inequality holds:

$$\left|\omega_\rho(z;\mu)\right| \le \frac{|z|^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|}, \quad \operatorname{Re} z \ge A^2(s), \quad |\operatorname{Im} z| \le \operatorname{Re} z. \tag{6.7.8}$$

We prove this inequality in the case $\mu \le 1$ ($s \ge 0$); for $\mu \in (1,3)$, the proof is similar. First, we note that, by the identity

$$\frac{1}{\Gamma(w)\Gamma(1-w)} = \frac{\sin \pi w}{\pi}$$

and the notation $\alpha = 2 - 1/\rho$ and $s = 1-\mu$, we have

$$\frac{1}{\Gamma(\mu-1/\rho)} = \frac{\sin \pi(\mu-1/\rho)}{\pi}\Gamma(s+2-\alpha). \tag{6.7.9}$$

It is easy to verify that the inequality

$$\left|\sin\left(\pi\left(\mu-\frac{2}{\rho}\right)\right)\right| \le 2\left|\sin\left(\pi\left(\mu-\frac{1}{\rho}\right)\right)\right| \tag{6.7.10}$$

holds both for $\mu \in \mathbb{Z}$ (for all $\alpha \in (0,1/2]$) and for $\mu \notin \mathbb{Z}$ (for any ρ sufficiently close to $1/2$). Relations (6.7.3), (6.7.9), and (6.7.10) together with the restriction $|z| \le x\sqrt{2}$ yields the relations

$$\begin{aligned}
\left|\omega_\rho(z;\mu)\right| &\le \frac{2^{s/2}\Gamma(s+4-2\alpha)}{\pi x^{s+2/\rho}}\cdot\frac{5}{2}\left|\sin\left(\pi\left(\mu-\frac{1}{\rho}\right)\right)\right| \\
&= \frac{2.5\cdot 2^{s/2}(s+3-2\alpha)(s+2-2\alpha)\Gamma(s+2-2\alpha)}{\pi x^{s+2/\rho}}\left|\sin\left(\pi\left(\mu-\frac{1}{\rho}\right)\right)\right| \\
&< \frac{2^{2+s/2}(s+3)^2\Gamma(s+2-\alpha)}{\pi x^{s+2/\rho}}\left|\sin\left(\pi\left(\mu-\frac{1}{\rho}\right)\right)\right| = \frac{2^{2+s/2}(s+3)^2}{x^{s+2/\rho}\left|\Gamma\left(\mu-\frac{1}{\rho}\right)\right|} \\
&= \frac{2^{2+s/2}(s+3)^2}{|z|^{s+2/\rho}\left|\Gamma\left(\mu-\frac{1}{\rho}\right)\right|}\left|\frac{z}{x}\right|^{s+2/\rho} \le \frac{2^{s+4}(s+3)^2}{|z|^{s+2/\rho}\left|\Gamma\left(\mu-\frac{1}{\rho}\right)\right|} \equiv \frac{A(s)|z|^{-s-2/\rho}}{\left|\Gamma\left(\mu-\frac{1}{\rho}\right)\right|}.
\end{aligned}$$

Since $|z| \ge A^2(s)$ and $1/\rho > 7/4$ (we consider only values of ρ from the interval $1/2 < \rho < 4/7$ since $\rho \to 1/2+$), we arrive at (6.7.8) replacing $A(s)$ by $|z|^{1/2}$ in the last expression.

We show that the same inequality as (6.7.8) holds for the derivative $\omega_\rho'(z;\mu)$, but in a narrower domain. Namely,

$$\left|\omega_\rho'(z;\mu)\right| \le \frac{|z|^{-s-1/\rho-s/4}}{\left|\Gamma\left(\mu-\frac{1}{\rho}\right)\right|}, \quad \operatorname{Re} z \ge A^2(s)+2, \quad |\operatorname{Im} z| \le \operatorname{Re} z - 3. \tag{6.7.11}$$

The definition of $A(s)$ implies the inequality

$$2(s+4)A^{-2}(s) < \ln 2. \tag{6.7.12}$$

We use the fact that if $z \in \mathbb{C}$, $R > 0$, the function $\varphi(\zeta)$ is holomorphic in the disk

$$K = \left\{\zeta \in \mathbb{C} \;\middle|\; |\zeta - z| \le R\right\},$$

and

$$\max_{\zeta\in K}|\varphi(\zeta)| = M,$$

then

$$\varphi'(z) = \frac{1}{2\pi i}\int\limits_{|\zeta-z|=R}\frac{\varphi(\zeta)}{(\zeta-z)^2}\,d\zeta \quad\Longrightarrow\quad |\varphi'(z)| \le \frac{M}{R}.$$

Taking $R = 2$, we obtain the inequality

$$\bigl|\omega_\rho'(z;\mu)\bigr| \le \frac{1}{2}\max_{|\zeta-z|=2}\bigl|\omega_\rho(\zeta;\mu)\bigr|. \tag{6.7.13}$$

From (6.7.8), (6.7.13), and (6.7.12) we obtain

$$\begin{aligned}\bigl|\omega_\rho'(z;\mu)\bigr| &\le \frac{1}{2}\frac{(|z|-2)^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|}\\ &= \frac{1}{2}\frac{|z|^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|}\left(\frac{|z|}{|z|-2}\right)^{s+1/\rho+5/4} \le \frac{1}{2}\frac{|z|^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|}\exp\left(\frac{2(s+1/\rho+5/4)}{|z|-2}\right)\\ &\le \frac{1}{2}\frac{|z|^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|}\exp\left(\frac{2(s+4)}{A^2(s)}\right) < \frac{|z|^{-s-1/\rho-5/4}}{|\Gamma(\mu-1/\rho)|},\end{aligned}$$

which was required.

6.8. Completion of the Proof of Theorem 6.1.4

In this and following sections, we use the notation $v = \operatorname{cosec}\pi\varepsilon$.

Representation (6.7.2) shows that the fact of constant sign of the function $F_\rho(x;\mu)$ for $x \ge R(\varepsilon)$ follows from the inequality

$$2\rho\exp\left(-\frac{x}{v}\right) < \frac{x^{-s-1/\rho}}{|\Gamma(\mu-1/\rho)|} - \bigl|\omega_\rho(x;\mu)\bigr|, \quad x \ge R(\varepsilon). \tag{6.8.1}$$

Note that

$$\frac{1}{\Gamma(\mu-1/\rho)} = \frac{1}{\Gamma(\mu-2+\alpha)}$$

preserves its sign for fixed $\mu\in\mathbb{R}$ and all sufficiently small positive α. From (6.7.8) we see that inequality (6.8.1) can be replaced by the following stronger inequality:

$$2\exp\left(-\frac{x}{v}\right) < \left(1-\frac{1}{x}\right)\frac{x^{-s-1/\rho}}{|\Gamma(\mu-2+\alpha)|}, \quad x \ge R(\varepsilon).$$

If μ is noninteger, then, as $\rho\to 1/2+$, the fraction $1/|\Gamma(\mu-2+\alpha)|$ is separated from zero. This reduces the problem to the inequality

$$Cx^{s+2}\exp\left(-\frac{x}{v}\right) < 1, \quad x \ge R(\varepsilon), \quad C = C(s), \tag{6.8.2}$$

when the parameter ε is sufficiently close to zero. We have the relation

$$R(\varepsilon) \sim bv\ln v, \quad \varepsilon\to 0+, \tag{6.8.3}$$

and the function $x^{s+2}\exp(-x/v)$ decreases on the ray $x \ge (s+2)v$. Therefore (if the parameter ε is small), it suffices to prove inequality (6.8.2) only for $x = R(\varepsilon)$. Recall that $b = s+2$ (in the case $\mu\notin\mathbb{Z}$) and

$$\frac{R(\varepsilon)}{v} = b(\ln v + 2\ln\ln v) \quad\Longrightarrow\quad \exp\left(-\frac{R(\varepsilon)}{v}\right) = v^{-b}(\ln v)^{-2b}.$$

Therefore, the left-hand side of inequality (6.8.2) for $x = R(\varepsilon)$ is equal to $CR^b(\varepsilon)v^{-b}(\ln v)^{-2b} \to 0$ as $\varepsilon \to 0+$ by (6.8.3), and hence is less than 1.

If $\mu \in \mathbb{Z}$, then

$$\big|\Gamma(\mu - 2 + \alpha)\big| \asymp \alpha^{-1} \asymp v, \quad \varepsilon \to 0+,$$

and we must prove the inequality

$$C_1 v x^{s+2} \exp(-x/v) < 1, \quad x \geq R(\varepsilon), \quad C_1 = C_1(s). \tag{6.8.4}$$

As in the previous case, it suffices to prove (6.8.4) only for $x = R(\varepsilon)$. But here $b = s+3$ and, after increasing the left-hand side of (6.8.4) (replacing v by $R(\varepsilon)$) we arrive at the inequality

$$C_1 R^b(\varepsilon) \exp\left(-\frac{R(\varepsilon)}{v}\right) < 1,$$

which has been proved above.

Prove inequality (6.7.7). As above, $x = \operatorname{Re} z$. Since on the straight lines

$$\operatorname{Re} z = \pi(n - \gamma)\sec\theta, \quad \theta, \gamma \in \mathbb{R}, \quad n \in \mathbb{Z},$$

the modulus of the function $\cos(z\cos\theta + \pi\gamma)$ is not less than 1, the relations

$$2\rho > 1, \quad -s - \frac{1}{\rho} = \mu - 3 + \alpha < 0$$

hold (the parameter $\mu < 3$ is fixed, $\alpha \to 0+$), and $|z| \geq x$, inequality (6.7.7), with account of (6.7.8) and bounds for $\operatorname{Re} z$, can be replaced by the following stronger inequality:

$$\left(1 + \frac{1}{x}\right)\frac{x^{-s-1/\rho}}{|\Gamma(\mu - 1/\rho)|} < \exp\left(-\frac{x}{v}\right), \quad A^2(s) \leq x \leq R_1(\varepsilon). \tag{6.8.5}$$

This inequality guarantees that (6.7.7) also holds on the lateral sides of trapeziums since the modulus of the cosine exceeds 1 on them by the estimates

$$\big|\cos(\xi + i\eta)\big| \geq |\sinh\eta| \geq |\eta|, \quad \xi, \eta \in \mathbb{R}, \quad \operatorname{Im}\big(z\cos\pi\varepsilon\big) = x\cos\pi\varepsilon > \frac{x}{2} \geq \frac{A^2(0)}{2} \geq \frac{144^2}{2}.$$

First, we prove (6.8.5) for $\mu \in (2,3)$; then $b = 3 - \mu \in (0,1)$. In this situation, if the parameter ρ is close to $1/2$ from the right, we have

$$0 < \mu - \frac{1}{\rho} < 1 \quad \Longrightarrow \quad 0 < \frac{1}{\Gamma(\mu - 1/\rho)} < 1.$$

Since $x \geq 4A^2(0)b^{-2} = 288^2 b^{-2}$, we have

$$1 + \frac{1}{x} < \exp(10^{-4}b^2)$$

and hence, for $\mu \in (2,3)$, inequality (6.8.5) can be replaced by the following stronger but simpler inequality:

$$\exp\big(10^{-4}b^2\big)x^{\alpha-b}\exp\frac{x}{v} < 1 \quad \Longleftrightarrow \quad 10^{-4}b^2 + (\alpha - b)\ln x + \frac{x}{v} < 0, \quad A^2(s) \leq x \leq R_1(\varepsilon). \tag{6.8.6}$$

It is easy to verify the convexity of the function $-b\ln x + x/v$ and the inequality

$$\alpha\ln x \leq \alpha\ln R_1(\varepsilon) \leq 4v^{-1}\ln v, \qquad v > b \tag{6.8.7}$$

(recall that $R_1(\varepsilon) = bv \ln v$ and $v = \operatorname{cosec} \pi\varepsilon$). As is known, the maximum of a convex function on a segment is attained at one of the endpoints of this segment. Therefore, to prove (6.8.6) it suffices to verify the validity of the following two inequalities:

$$10^{-4}b^2 + 4v^{-1} \ln v - 2b \ln A(s) + \frac{A^2(s)}{v} < 0, \tag{6.8.8}$$

$$10^{-4}b^2 + 4v^{-1} \ln v - b \ln R_1(\varepsilon) + \frac{R_1(\varepsilon)}{v} < 0. \tag{6.8.9}$$

Since

$$\lim_{\varepsilon \to 0+} \left(4v^{-1} \ln v + \frac{A^2(s)}{v} \right) = 0,$$

we see that for all sufficiently small ε, the left-hand side of (6.8.8) is less than

$$10^{-3}b^2 - 2b \ln A(s) < 10^{-3}b^2 - 2b \ln A(0) < 10^{-3}b^2 - 8b < 0.$$

Thus, inequality (6.8.8) holds. Further, we have the relation

$$-b \ln R_1(\varepsilon) + \frac{R_1(\varepsilon)}{v} = -b \ln(b \ln v) \to -\infty \quad \text{as } \varepsilon \to 0+,$$

and hence inequality (6.8.9) also holds.

Prove that (6.8.5) in the case $\mu < 2$, $\mu \notin \mathbb{Z}$. From (6.7.9) we obtain the inequality

$$\frac{1}{|\Gamma(\mu - 1/\rho)|} \le \frac{\Gamma(s + 2 - \alpha)}{\pi}.$$

This shows that inequality (6.8.5) can be replaced by the following stronger inequality:

$$x^{\alpha - b} \Gamma(b - \alpha) \exp \frac{x}{v} < \frac{\pi x}{x + 1}, \quad A^2(s) \le x \le R_1(\varepsilon). \tag{6.8.10}$$

Taking into account the lower bound for x and the upper estimate (6.8.7) for x^α, we obtain the following, stronger but simpler than (6.8.10), inequality:

$$x^{-b} \exp \frac{x}{v} < e, \quad 1 < b < 2, \quad x^{-b} \Gamma(b) \exp \frac{x}{v} < e, \quad b > 2, \quad A^2(s) \le x \le R_1(\varepsilon). \tag{6.8.11}$$

Prove it. Taking the logarithm of (6.8.11) and using the convexity of the function $x/v - b \ln x$, we reduce the problem to the following inequalities:

$$\begin{gathered} -b \ln x + \frac{x}{v} < 1, \quad 1 < b < 2, \\ -b \ln x + \ln \Gamma(b) + \frac{x}{v} < 1, \quad b > 2, \\ x = A^2(s), \quad x = R_1(\varepsilon). \end{gathered} \tag{6.8.12}$$

As was noted above,

$$\lim_{\varepsilon \to 0+} \left(-b \ln R_1(\varepsilon) + \frac{R_1(\varepsilon)}{v} \right) = - \lim_{\varepsilon \to 0+} b \ln(b \ln v) = -\infty$$

independently of $b > 0$. Therefore, inequalities (6.8.12) at the point $x = R_1(\varepsilon)$ for sufficiently small positive ε hold. For $x = A^2(s)$, the first inequality (6.8.12) for small ε is obvious:

$$\lim_{\varepsilon \to 0+} \frac{A^2(s)}{v} = 0, \quad -b \ln x < 0.$$

We prove the validity of the inequality

$$-b \ln A^2(s) + \ln \Gamma(b) < 0;$$

after this, it suffices to prove the second inequality for $x = A^2(s)$. For $b > 2$, by the choice of $A(s)$, we have the inequality $A^2(s) > 4^b$. Therefore,

$$-b \ln A^2(s) + \ln \Gamma(b) < -b^2 + \ln \Gamma(b) = -b^2 + b \ln b = b(-b + \ln b) \le -b.$$

Thus, inequalities (6.8.12) are completely proved.

Consider integer values of $\mu \le 2$. Then

$$\left| \sin\left(\pi\left(\mu - \frac{1}{\rho}\right)\right)\right| = \left| \sin \pi(\mu - 2 + \alpha)\right| = \sin \pi\alpha \le \pi\alpha,$$

and (6.7.9) implies the inequality

$$\frac{1}{|\Gamma(\mu - 1/\rho)|} \le \alpha\Gamma(s + 2 - \alpha). \tag{6.8.13}$$

In this case, $b = s + 3$ and, owing to (6.8.13), inequality (6.8.5) can be replaced by the following:

$$\left(1 + \frac{1}{x}\right) \alpha\Gamma(b - 1 - \alpha) x^{1+\alpha-b} \exp\frac{x}{v} < 1, \quad A^2(s) \le x \le R_1(\varepsilon). \tag{6.8.14}$$

Taking the logarithm of (6.8.14), applying (6.8.7) and estimate $\ln(1 + 1/x) < 1/x$, and using the fact that the function $(1 - b)\ln x + x/v$ is convex, we reduce the problem to the inequality

$$\begin{gathered} \frac{1}{x} + 4v^{-1}\ln v + \ln\alpha + \ln\Gamma(b - 1 - \alpha) + (1 - b)\ln x + \frac{x}{v} < 0; \\ x = A^2(s), \quad x = R_1(\varepsilon), \quad b \in \mathbb{Z}, \quad b \ge 2. \end{gathered} \tag{6.8.15}$$

For $x = A^2(s)$, the validity of inequality (6.8.15) for all sufficiently small positive ε is obvious since its left-hand side tends to $-\infty$ as $\varepsilon \to 0+$ on account of the term $\ln\alpha$. For $x = R_1(\varepsilon)$, we have the expression

$$\ln\alpha + (1 - b)\ln x + \frac{x}{v} = -\ln v + O(1) + (1 - b)\ln(v \ln v) + b \ln v = (1 - b)\ln\ln v + O(1).$$

This expression tends to $-\infty$ as $\varepsilon \to 0+$ and other terms of the left-hand side of (6.8.15) are bounded for $x = R_1(\varepsilon)$ and $\varepsilon \to 0+$. Inequality (6.8.15) is proved.

The final part of the proof of Theorem 6.1.4 is the estimate of the modulus of the logarithmic derivative of the function $F_\rho(z;\mu)$ on the sides of the rectangle $\Pi(\varepsilon)$:

$$\left|\frac{F'_\rho(z;\mu)}{F_\rho(z;\mu)}\right| \le 2.$$

We set

$$\Phi_\rho(z;\mu) = 2\rho \exp\big(-z \sin\pi\varepsilon\big) \cos\big(z \cos\pi\varepsilon + \pi\rho s\big).$$

Then

$$F_\rho(z;\mu) = \Phi_\rho(z;\mu) + \frac{z^{-s-1/\rho}}{\Gamma(\mu - 1/\rho)} + \omega_\rho(z;\mu),$$

$$F'_\rho(z;\mu) = \Phi'_\rho(z;\mu) - \frac{(s + 1/\rho)z^{-s-1/\rho-1}}{\Gamma(\mu - 1/\rho)} + \omega'_\rho(z;\mu),$$

$$\Phi'_\rho(z;\mu) = -2\rho \exp\big(-z\sin\pi\varepsilon\big) \sin\big(z\cos\pi\varepsilon + \pi\rho s + \pi\varepsilon\big).$$

The plan of further action is as follows. On the left vertical side $\operatorname{Re} z = x_{n(\varepsilon)}$ of the rectangle $\Pi(\varepsilon)$ we prove the inequality

$$\frac{|z|^{-s-1/\rho}}{|\Gamma(\mu - 1/\rho)|} + \big|\omega_\rho(z;\mu)\big| + \big|\omega'_\rho(z;\mu)\big| < \frac{1}{3}\exp\left(-\frac{x}{v}\right). \tag{6.8.16}$$

The sequence x_n is such that on the straight lines $\operatorname{Re} z = x_n$ we have the inequality

$$\left|\Phi'_\rho(z;\mu)\right| \le \left|\Phi_\rho(z;\mu)\right| = \exp\left(-\frac{x}{v}\right)\cosh\left(y\cos\pi\varepsilon\right), \quad z = x+iy.$$

This and (6.8.16) immediately imply the inequalities

$$\left|F_\rho(z;\mu)\right| \ge \frac{2}{3}\left|\Phi_\rho(z;\mu)\right|, \quad \left|F'_\rho(z;\mu)\right| \le \frac{4}{3}\left|\Phi_\rho(z;\mu)\right|,$$

which yield the required upper estimate for the modulus of the logarithmic derivative. The inequality

$$\frac{|z|^{-s-1/\rho}}{|\Gamma(\mu-1/\rho)|} + \left|\omega_\rho(z;\mu)\right| + \left|\omega'_\rho(z;\mu)\right| < \frac{1}{4}\exp\left(-\frac{x}{v}\right)\sinh\left(v\cos\pi\varepsilon\right), \tag{6.8.17}$$

similar to (6.8.16), is proved on the horizontal sides $x_{n(\varepsilon)} \le \operatorname{Re} z \le x_{m(\varepsilon)}$, $\operatorname{Im} z = \pm v$ of the rectangle $\Pi(\varepsilon)$. As above, taking into account the estimates

$$\left|\Phi_\rho(z;\mu)\right| \ge 2\rho\exp\left(-\frac{x}{v}\right)\left|\sinh\left(y\cos\pi\varepsilon\right)\right|, \quad \left|\Phi'_\rho(z;\mu)\right| \le \left|\coth\left(y\cos\pi\varepsilon\right)\Phi_\rho(z;\mu)\right| \tag{6.8.18}$$

(here $z = x+iy$), from (6.8.17) we deduce the inequality

$$\left|\frac{F'_\rho(z;\mu)}{F_\rho(z;\mu)}\right| \le 2$$

on the horizontal sides of $\Pi(\varepsilon)$; if $0 < \varepsilon \le 1/6$, then

$$\coth\left(v\cos\pi\varepsilon\right) = \coth\left(\cot\pi\varepsilon\right) \le \coth\sqrt{3} < 1.1.$$

An upper estimate of the modulus of the logarithmic derivative of the function F_ρ on the right vertical side

$$l_\varepsilon = \left\{z\in\mathbb{C} \;\middle|\; \operatorname{Re} z = x_{m(\varepsilon)},\ |\operatorname{Im} z| \le v\right\}$$

of the rectangle $\Pi(\varepsilon)$ is proved differently. Introduce the notation

$$Z_\rho(z;\mu) = \frac{z^{-s-1/\rho}}{|\Gamma(\mu-1/\rho)|}.$$

On $\mathbb{R}_+$, the function $Z_\rho(z;\mu)$ is real-valued and its sign coincides with the sign of $1/\Gamma(\mu-1/\rho)$, which we denote by δ, $\delta\in\{-1,1\}$. Recall that the number $\mu<3$ is fixed, $\alpha = 2-1/\rho \to 0+$, and hence the sign of

$$\frac{1}{\Gamma(\mu-1/\rho)} = \frac{1}{\Gamma(\mu-2+\alpha)}$$

(when the parameter α is positive and sufficiently small) is constant and is equal to 1 if the entire part of μ is even and to -1 in the opposite case. The function $\Phi_\rho(z;\mu)$ is real and preserves its sign on vertical straight lines x_n+iy, $n\in\mathbb{Z}$, $y\in\mathbb{R}$, $\operatorname{sgn}\Phi_\rho(x_n+iy;\mu) = (-1)^n$. The remainder of dividing $m(\varepsilon)$ by 2 is such that

$$\operatorname{sgn}\Phi_\rho(x_{m(\varepsilon)}+iy;\mu) = \delta.$$

Thus,

$$\left|\Phi_\rho(z;\mu)\right| = \delta\left|\Phi_\rho(z;\mu)\right|, \quad z\in l_\varepsilon. \tag{6.8.19}$$

If the variable z "moves" vertically upward or downward from $\mathbb{R}$, the function $Z_\rho(z;\mu)$ is no longer real-valued, but $\left|\operatorname{Re} Z_\rho(z;\mu)\right|$ is insignificantly less than $\left|Z_\rho(z;\mu)\right|$ while $\arg z$ is not large. For $z\in l_\varepsilon$, we have the relation

$$|\arg z| \le \arctan\left(\frac{v}{x_{m(\varepsilon)}}\right) = O\left(\frac{1}{\ln v}\right).$$

Therefore,

$$\operatorname{Re} Z_\rho(z;\mu) = \delta \frac{|z|^{-s-1/\rho}}{|\Gamma(\mu - 1/\rho)|}(1 + o(1)), \quad \varepsilon \to 0+, \quad \text{uniformly with respect to } z \in l_\varepsilon. \tag{6.8.20}$$

Relations (6.8.19), (6.8.20), and (6.7.8) imply the asymptotic relation

$$\big|\operatorname{Re} F_\rho(z;\mu)\big| = \big|\Phi_\rho(z;\mu)\big| + \big|Z_\rho(z;\mu)\big|(1 + o(1)), \quad \varepsilon \to 0+, \quad \text{uniformly with respect to } z \in l_\varepsilon.$$

Since

$$\big|Z'_\rho(z;\mu)\big| + \big|\omega'_\rho(z;\mu)\big| = o\big|Z_\rho(z;\mu)\big|, \quad \varepsilon \to 0+, \quad \text{uniformly with respect to } z \in l_\varepsilon$$

and for $z \in l_\varepsilon$, we have the inequality

$$\big|\Phi'_\rho(z;\mu)\big| \le \big|\Phi_\rho(z;\mu)\big|,$$

we now obtain the required estimate of $|F'_\rho/F_\rho|$ on the side l_ε.

Now we deduce inequalities (6.8.16) and (6.8.17). Since the moduli of the remaining ω_ρ and its derivative by order are less that the modulus of the function Z_ρ, $|z| \ge x$,

$$\lim_{\varepsilon \to 0+} R^\alpha(\varepsilon) = 1,$$

it suffices to prove the inequalities

$$\frac{4x^{-s-2}}{|\Gamma(\mu - 2 + \alpha)|} \exp\frac{x}{v} < 1, \quad x = x_{n(\varepsilon)},$$

$$\frac{x^{-s-2}}{|\Gamma(\mu - 2 + \alpha)|} \exp\frac{x}{v} < 0.1 \exp(\cot \pi\varepsilon), \quad x_{n(\varepsilon)} \le x \le x_{m(\varepsilon)},$$

which after taking the logarithm become

$$-b \ln x_{n(\varepsilon)} + \frac{x_{n(\varepsilon)}}{v} < C, \quad -b \ln x + \frac{x}{v} < C + \cot \pi\varepsilon, \quad x_{n(\varepsilon)} \le x \le x_{m(\varepsilon)}, \quad \mu \notin \mathbb{Z},$$

$$-\ln v + (1-b)\ln x_{n(\varepsilon)} + \frac{x_{n(\varepsilon)}}{v} < C, \quad -\ln v + (1-b)\ln x + \frac{x}{v} < C + \cot \pi\varepsilon,$$

$$x_{n(\varepsilon)} \le x \le x_{m(\varepsilon)}, \quad \mu \in \mathbb{Z}.$$

These inequalities are proved by the same way as above. The proof of Theorem 6.1.4 is complete.

6.9. Proof of Assertion (1) of Theorem 6.1.5

Since the function $f(\rho)$ from Theorem 6.1.1 decreases, it suffices to verify the absence of eigenvalues for $\alpha = 0.45$, i.e., the positiveness of the function $E_\rho(z;2)$, $\rho = (2-\alpha)^{-1} = 1/1.55 = 20/31$, on $\mathbb{R}$. Since there are no roots for $z \ge -\Gamma(4-\alpha)$, it remains to prove the positiveness of the function $F_\rho(x;2)$ for $x \ge (\Gamma(3.35))^\rho$. But $(\Gamma(3.35))^{20/31} > 2\pi/3$ and hence we prove the positiveness of the function $F_{20/31}(x;2)$ for $x \ge 2\pi/3$.

Taking $m = 1$, $\mu = 2$, and $\rho = 20/31$ in Theorem 1.1.3 and introducing the notation $F(x) = F_{20/31}(x;2)$, after transformations we obtain the representation

$$F(x) = \frac{40}{31} \exp\left[-x \sin\left(\frac{9\pi}{62}\right)\right] \sin\left[x \cos\left(\frac{9\pi}{62}\right) - \frac{9\pi}{62}\right] + \frac{x^{-0.55}}{\Gamma(0.45)} - \omega(x), \tag{6.9.1}$$

in which

$$\omega(x) = \frac{1}{\pi}\left[I_1(x) \sin\left(\frac{\pi}{10}\right) + I_2(x) \cos\left(\frac{\pi}{20}\right)\right], \quad 0 < I_k(x) < \frac{\Gamma(2.1)}{kx^{2.1}}, \quad k = 1, 2. \tag{6.9.2}$$

Representation (6.9.1), numerical estimates of quantities in (6.9.2), and estimate $1/\Gamma(0.45) > 0.508$ yield the inequality

$$F(x) > \frac{40}{31}\exp\left[-x\sin\left(\frac{9\pi}{62}\right)\right]\sin\left[x\cos\left(\frac{9\pi}{62}\right) - \frac{9\pi}{62}\right] + 0.508x^{-0.55} - 0.27x^{-2.1}. \tag{6.9.3}$$

From (6.9.3) we immediately obtain the positiveness of $F(x)$ if

$$\sin\left[x\cos\left(\frac{9\pi}{62}\right) - \frac{9\pi}{62}\right] > 0, \quad x > 1.$$

These inequalities hold for $2\pi/3 \le x \le 4$; therefore, $F(x) > 0$ on this segment.

If $4 < x \le 4.5$, then

$$\sin\left(x\cos\left(\frac{9\pi}{62}\right) - \frac{9\pi}{62}\right) > -\frac{1}{2}.$$

From this and (6.9.3), taking into account the numerical estimate

$$\sin\left(\frac{9\pi}{62}\right) > 0.44,$$

we obtain the inequality

$$F(x) > 0.508x^{-0.55} - 0.27x^{-2.1} - \frac{20}{31}\exp(-0.44x).$$

Since

$$e^{-t} < \frac{1}{et} \quad \Longrightarrow \quad \frac{40}{31}\exp(-0.44x) < \frac{20}{31\cdot 0.44ex} < \frac{0.6}{x},$$

for $x \in (4, 4.5]$ we have the inequality

$$\begin{aligned} F(x) > 0.5x^{-0.55} - 0.27x^{-2.1} - 0.6x^{-1} &= 0.5x^{-0.55}\big(1 - 0.54x^{-1.55} - 1.2x^{-0.45}\big) \\ &> 0.5x^{-0.55}\big(1 - 0.54\cdot 4^{-1.55} - 1.2\cdot 4^{-0.45}\big) > 0, \end{aligned}$$

since

$$1 - 0.54\cdot 4^{-1.55} - 1.2\cdot 4^{-0.45} > 0.2.$$

Finally, for $x > 4.5$ we have the relation

$$\begin{aligned} F(x) &> 0.508x^{-0.55} - 0.27x^{-2.1} - \frac{40}{31}\exp(-0.44x) \\ &= 0.508x^{-0.55}\left[1 - \frac{0.27}{0.508}x^{-1.55} - \frac{40}{0.508\cdot 31}x^{0.55}e^{-0.44x}\right]. \end{aligned}$$

This and numerical estimates

$$\frac{0.27}{0.508} < 0.532, \qquad \frac{40}{0.508\cdot 31} < 2.55$$

imply (for $x > 4.5$)

$$F(x) > 0.508x^{-0.55}G(x),$$

where

$$G(x) = 1 - 0.532x^{-1.55} - 2.55x^{0.55}e^{-0.44x}.$$

Since $G(x)$ increases on the ray $x > 4.5$, the following estimate is valid:

$$G(x) > 1 - 0.532\cdot(4.5)^{-1.55} - 2.55\cdot(4.5)^{0.55}e^{-1.98} > 0.1,$$

which proves the positiveness of the function F. Assertion (1) of Theorem 6.1.5 is proved.

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