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ON THE SOLVABILITY OF THE JUMP PROBLEM IN CLIFFORD ANALYSIS

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ABSTRACT. Let Ω be a bounded open and oriented connected subset of $\mathbb{R}^n$ which has a compact topological boundary Γ, let $\mathcal{C}$ be the Dirac operator in $\mathbb{R}^n$, and let $\mathbb{R}_{0,n}$ be the Clifford algebra constructed over the quadratic space $\mathbb{R}^n$. An $\mathbb{R}_{0,n}$-valued smooth function $f : \Omega \to \mathbb{R}_{0,n}$ in Ω is called monogenic in Ω if $\mathcal{D}f = 0$ in Ω. The aim of this paper is to present the most general condition on Γ obtained so far for which a Hölder continuous function f can be decomposed as $F^+ - F^- = f$ on Γ, where the components $F^\pm$ are extendable to monogenic functions in $\Omega^\pm$ with $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^n \setminus (\Omega \cup \Gamma)$, respectively.

CONTENTS

1. Introduction

The Clifford algebra is an associative and noncommutative algebraic structure that was introduced in the middle of the 19th century, which is regarded as a generalization of a complex number and Hamilton's quaternions (see [9]).

Clifford analysis, which is thorough treated in [8], is a higher-dimensional function theory offering a successful generalization of the theory of holomorphic functions in the complex plane and, at the same time, a refinement of classical harmonic analysis. It involves the study of functions on Euclidean space with values in a Clifford algebra and has important applications in a variety of fields, including geometry and theoretical physics.

As one of the most striking examples of the applicability of Clifford analysis, we can refer the reader to the paper [17] on the full nonlinear, three-dimensional water wave problem by Sijue Wu, for which she was awarded in 2001 the Satter Prize of the American Mathematical Society and where Clifford analysis played an important role in proving her results.

Let us outline the main problem treated in this paper. If Ω is a bounded open and oriented connected subset of $\mathbb{R}^n$ with boundary Γ given by a compact topological surface, let $\Omega^\pm$ with $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^n \setminus (\Omega \cup \Gamma)$ denote the complementary connected domains separated by Γ on $\mathbb{R}^n$, and suppose that f is any continuous $\mathbb{R}_{0,n}$-valued function defined on Γ; then the question is:

Can we decompose f additively into two components

$$F^+(x) - F^-(x) = f(x), \quad x \in \Gamma, \tag{1}$$

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such that $F^\pm$ are extendable to monogenic functions in $\Omega^\pm$?

It is a well-known fact that in proving the existence of the boundary value of the required Cauchy transform via the Plemelj–Sokhotski formulas, the solvability of the above jump problem (1) is an easy task whenever f is a Hölder continuous function and Γ is assumed sufficiently smooth.

Properties of the boundary values of the Clifford–Cauchy transform were examined for compact Lyapunov surfaces by Iftimie [13] who proved in 1965 that it had Hölder-continuous limit values for any Hölder continuous densities and who obtained Plemelj–Sokhotski-type formulas; this was one of the first papers on the subject.

A great deal of research in the framework of Clifford analysis has been devoted to the study of the existence of the continuous boundary values of the Clifford–Cauchy transform over surfaces in Euclidean spaces of higher dimensions, assuming rougher conditions on the smoothness of surfaces (see [1, 5] and the references therein).

The question of the existence of the continuous extension of the Clifford–Cauchy transform on a rectifiable surface in $\mathbb{R}^n$ which at the same time satisfies the so-called Ahlfors–David regular condition is optimally answered in [7].

By far much more subtle is the case where Γ can be thought of as a non-rectifiable or even a fractal surface. A deeper discussion and a fine work here is to be found in [2, 3, 6]. To achieve this, an alternative way of defining the Clifford–Cauchy transform, where a central role is played by the Théodoresco transform type involving fractal dimensions, is described.

For example, if $f \in C^{0,\alpha}(\Gamma)$ (i.e., the function f satisfies the Hölder condition on Γ with exponent $\alpha, 0 < \alpha \leq 1$), then the solution of the jump problem (1) can be obtained under the condition that

$$\alpha > \frac{\operatorname{Dm}\Gamma}{n}, \tag{2}$$

where $\operatorname{Dm}\Gamma$ is the upper Minkowski (also referred as box) fractal dimension (to be defined later) of the set Γ.

The condition (2) cannot be weakened on the whole class of surfaces with fixed box dimension (see [4] for more details).

It is worth noting that the uniqueness of the aforementioned decomposition, if it exists, will depend on having discussed here the removable singularities phenomenon. It is an idea handled with great care in [6].

We shall discuss solvability conditions of the jump problem for monogenic functions, reproving earlier results and going deeper. For this to happen, we turn our attention to estimating a new metric characteristic of the surface instead of using the accustomed box dimension.

2. Preliminaries. Clifford Algebras and Monogenic Functions

The real Clifford algebra associated with $\mathbb{R}^n$ endowed with the Euclidean metric is the minimal enlargement of $\mathbb{R}^n$ to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^n$.

Thus, it follows that if $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$, then we have that

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

Every element $a \in \mathbb{R}_{0,n}$ is of the form

$$a = \sum_{A \subseteq N} a_A e_A, \quad N = \{1, \dots, n\}, \quad a_A \in \mathbb{R},$$

where $e_\emptyset = e_0 = 1$, $e_{\{j\}} = e_j$, and

$$e_A = e_{\alpha_1} \cdots e_{\alpha_k} \quad \text{for} \quad A = \{\alpha_1, \dots, \alpha_k\}, \quad \alpha_j \in \{1, \dots, n\}, \quad \alpha_1 < \dots < \alpha_k.$$

The conjugation is defined by

$$\overline{a} := \sum_A a_A \overline{e_A},$$

where

$$\overline{e_A} = (-1)^k e_{i_k} \cdots e_{i_2} e_{i_1} \quad \text{if} \quad e_A = e_{i_1} e_{i_2} \cdots e_{i_k}.$$

Note that for $x \in \mathbb{R}^n$, we thus have that

$$x\,\overline{x} = \overline{x}\,x = |x|^2.$$

By means of the conjugation, the norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,n}$ by setting

$$|a|^2 = \sum_A |a_A|^2.$$

A function defined in some subset $\mathbf{E}$ of $\mathbb{R}^n$ with values in the Clifford algebra $\mathbb{R}_{0,n}$ is a map $f : \mathbf{E} \to \mathbb{R}_{0,n}$ of the form

$$f(x) = \sum_A f_A(x) e_A, \quad x \in \mathbf{E},$$

where f_A are real components of f; then the notions of continuity and differentiability of f are introduced by means of those corresponding to its real components.

In particular, if $\mathbf{E} \subset \mathbb{R}^n$ is a compact set, then $C^{0,\alpha}(\mathbf{E})$, $0 < \alpha \leq 1$ denotes the class of all Hölder-continuous $\mathbb{R}_{0,n}$-valued functions f for which

$$|f|_{\alpha,\mathbf{E}} := \sup_{\substack{x,y \in \mathbf{E} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

Introducing the metric

$$\|f\|_{\alpha,\mathbf{E}} = \max_{x \in \mathbf{E}} |f(x)| + |f|_{\alpha,\mathbf{E}},$$

one can prove that $C^{0,\alpha}(\mathbf{E})$ is a real Banach space (see [11, p. 358]).

Define the so-called Dirac operator by

$$\mathcal{D} = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$

It is a first-order elliptic operator whose fundamental solution is given by

$$\underline{E}(\underline{x}) = \frac{1}{\sigma_n} \frac{\overline{\underline{x}}}{|\underline{x}|^n}, \quad \underline{x} \in \mathbb{R}^n \setminus \{0\},$$

where σ_n is the area of the unit sphere in $\mathbb{R}^n$.

If Ω is open in $\mathbb{R}^n$ and $f \in C^1(\Omega)$, then f is said to be *monogenic* if $\mathcal{D}f = 0$ in Ω. The best general reference here is [8].

3. Approximate Dimension and d-Summability

In the remainder of the paper, coinciding with our previously introduced terminology, if Ω is a bounded oriented connected open subset of $\mathbb{R}^n$ whose boundary Γ is a compact topological surface, we will use the temporary notation $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^n \setminus (\Omega \cup \Gamma)$.

The closure, diameter and boundary of a set $\mathbf{E} \subset \mathbb{R}^n$ will be denoted by $\overline{\mathbf{E}}$, $|\mathbf{E}|$ and $\partial\mathbf{E}$, respectively.

Let $\mathcal{H}$ denote the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. The perimeter of a finite polygonal domain $P \subset \mathbb{R}^n$, denoted by $\lambda(P)$, is defined to be $\mathcal{H}(\partial P)$. On the other hand, we will use the symbol $w(P)$ to denote the diameter of the largest ball contained in P.

Let $\mathcal{P}^+ = \{P_k,\ k = 1, 2, \dots\}$ be a polygonal decomposition of Ω described by a sequence of nonoverlapping polygonal domains such that $\overline{P_k} \subset \overline{\Omega}$, $k = 1, 2, \dots,$ and

$$\overline{\bigcup_{1 \le k} P_k} = \overline{\Omega}.$$

Every decomposition of Ω is required to intersect any closed subset $\mathbf{E} \subset \mathbb{R}^n$ satisfying $\overline{\mathbf{E}} \subset \Omega$, with a finite number of P_k's.

There is no loss of generality in assuming that P_{k+1} has a common side with the union $\bigcup\limits_{1 \le j \le k} \overline{P_j}$ for any k.

Hence, the sequence of polygons

$$\Gamma_k^+ := \partial \left(\bigcup_{1 \le j \le k} \overline{P_j} \right)$$

converges to Γ for $k \to \infty$ from Ω.

Throughout the paper, the convergence of a sequence of polygons will be with respect to the Hausdorff metric.

The sum

$$\mathcal{M}_q(\mathcal{P}^+) := \sum_{k \ge 1} \lambda(P_k) w^{q+1-n}(P_k)$$

is called the (refined) q-mass of the decomposition $\mathcal{P}^+$.

Definition 1. Let $N^+(\Gamma)$ be the set of all values q such that there exists a decomposition $\mathcal{P}^+$ with finite $q-$mass $\mathcal{M}_q(\mathcal{P}^+)$. Then $\operatorname{Dma}^+ \Gamma := \inf N^+(\Gamma)$ is called the inner approximate dimension of Γ.

Proceeding analogously, let

$$\mathcal{P}^- = \{P_k,\ k = 0, 1, 2, \dots\}$$

be a polygonal decomposition of Ω^-, where the polygonal domain P_0 contains ∞ and all other are bounded domains. This decomposition generates polygons Γ_k^-, converging to Γ from D^-. We set

$$\mathcal{M}_q(\mathcal{P}^-) := \sum_{k \ge 1} \lambda(P_k) w^{q+1-n}(P_k)$$

to state the following definition.

Definition 2. Let $N^-(\Gamma)$ be the set of all values q such that there exists a decomposition $\mathcal{P}^-$ with finite $q-$mass $\mathcal{M}_q(\mathcal{P}^-)$. Then $\operatorname{Dma}^- \Gamma := \inf N^-(\Gamma)$ is called the outer approximate dimension of Γ.

In this way, we obtain what we shall call the approximate dimension of Γ, which is, by definition, the number

$$\operatorname{Dma} \Gamma := \min\{\operatorname{Dma}^+ \Gamma, \operatorname{Dma}^- \Gamma\}.$$

Recall that the box dimension of a bounded set $\mathbf{E} \subset \mathbb{R}^n$ is given by

$$\operatorname{Dm} \mathbf{E} := \varlimsup_{\epsilon \to 0} \frac{\log N_{\mathbf{E}}(\epsilon)}{-\log \epsilon},$$

where $N_{\mathbf{E}}(\varepsilon)$ denotes the minimal number of ε-balls needed to cover $\mathbf{E}$.

In the notation of [12], the surface Γ is said to be d-summable if the improper integral

$$\int_0^1 N_\Gamma(\tau) \tau^{d-1} d\tau$$

converges.

It is easy to verify that any d-summable surface Γ has box dimension $\mathrm{Dm}(\Gamma) \le d$. Moreover, if $\mathrm{Dm}(\Gamma) = d$ then Γ is $d+\delta$-summable for any $\delta > 0$.

We introduce the notation

$$\mathcal{W} = \bigcup_k \mathcal{W}^k$$

for the Whitney decomposition of Ω by k-cubes, following Harrison and Norton [12].

Lemma 2 in [12] shows that the d-sum defined by $\sum_{Q\in\mathcal{W}} |Q|^d$ is finite provided Γ is d-summable.

Let Γ be d-summable; we set $\Gamma_k^+ = \partial P_k$, where

$$P_k = \{\underline{x} \in Q : Q \in \mathcal{W}^j \text{ for some } j \le k\}.$$

Then the polygons Γ_k^+ converges to Γ from Ω. Moreover, we have

$$\sum_{k\ge 1} \lambda(P_k) w^{d+1-n}(P_k) \le c \sum_{k\ge 1} (\lambda(P_k))^{\frac{d}{n-1}} \le c \sum_{Q\in\mathcal{W}} |Q|^d < +\infty.$$

Here and in the sequel, c is a positive constant not necessarily the same in different occurrences.

The preceding observation leads to the following result.

Proposition 1. *If Γ is d-summable, then* $\mathrm{Dma}\,\Gamma \le d$.

The following result ensures the existence of one d-summable surface Γ in $\mathbb{R}^n$ with $\mathrm{Dma}\,\Gamma < d$, for $d \in (n-1, n)$.

Proposition 2. *Given any $d \in (n-1, n)$, then there exists a d-summable surface Γ with* $\mathrm{Dma}\,\Gamma < d$.

Proof. Let $\delta > 0$ be sufficiently small such that $d - \delta \in (n-1, n)$. Following the techniques used in [4] we can extend the construction likewise to that in the proof of [14, Theorem 2] to produce a surface Γ in $\mathbb{R}^n$ with box dimension $\mathrm{Dm}\,\Gamma = d - \delta$ (hence d-summable!) and with $\mathrm{Dma}\,\Gamma < d - \delta < d$. □

Geometrically speaking, an important special case is when Γ is left Ahlfors–David regular ((l)AD-regular for brevity), i.e., $\mathcal{H}(\Gamma) < \infty$ and there exists a constant c_Γ such that

$$c_\Gamma\, r^{n-1} \le \mathcal{H}(\Gamma \cap B(x, r)) \quad \text{for} \quad x \in \Gamma,\ 0 < r \le |\Gamma|, \tag{3}$$

where $B(x, r)$ denotes the closed ball with center x and radius r (see [2, 3]).

A nice link between this geometric notion and the d-summable condition is given by the following lemma.

Lemma 1. *Let Γ satisfy the* (l)AD*-regular condition. Then it is also $(n-1+\epsilon)$-summable for any $\epsilon > 0$ and*

$$\sum_{Q\in\mathcal{W}} |Q|^{n-1+\epsilon} \le c\frac{\mathcal{H}(\Gamma)}{\epsilon}, \tag{4}$$

where c depends only on n and c_Γ.

Roughly speaking, this lemma asserts that the $(n-1+\epsilon)$-sum is less than a constant times $\mathcal{H}(\Gamma)/\epsilon$.

Proof. The proof of $(n-1+\epsilon)$-summability easily follows from the fact that

$$N_\Gamma(\tau) \le P_\Gamma(\frac{\tau}{2}),$$

where $P_\Gamma(\tau/2)$ is the so-called packing number: the greatest number of disjoint $\tau/2$-balls with centers in Γ (see [15]).

Then, in accordance with (3), we have

$$c_\Gamma \frac{\tau^{n-1}}{2^{n-1}} P_\Gamma\left(\frac{\tau}{2}\right) \leq \mathcal{H}(\Gamma)$$

and hence

$$N_\Gamma(\tau) \leq c\mathcal{H}(\Gamma)\tau^{1-n}.$$

Consequently,

$$\int_0^1 N_\Gamma(\tau)\tau^{n-1+\epsilon-1}d\tau \leq c\mathcal{H}(\Gamma)\int_0^1 \tau^{\epsilon-1}d\tau = c\frac{\mathcal{H}(\Gamma)}{\epsilon}, \tag{5}$$

which establishes the $(n-1+\epsilon)$-summability of Γ.

Now we can proceed similarly to the proof of [12, Lemma 2], which together with (5) yields (4). □

The following lemma reveals a basic result, which will be needed in Sec. 4.

Lemma 2. *Under the hypotheses of Lemma 1, let $f \in C^{0,\alpha}(\Gamma)$ and $\tilde{f}$ be its Whitney extension. Then for any $p < \frac{1}{1-\alpha}$,*

$$\int_\Omega |\mathcal{D}\tilde{f}(y)|^p\, dy < c|f|_{\alpha,\Gamma}\mathcal{H}(\Gamma)w(\Omega)^{\frac{1}{2}(1-p(1-\alpha))},$$

where c is a constant depending only on n, α, p, and c_Γ.

Before starting the proof, we recall that the Whitney extension of a function $f \in C^{0,\alpha}(\mathbf{E})$, where $\mathbf{E}$ is compact, is a compactly supported one, denoted by $\tilde{f} \in C^\infty(\mathbb{R}^n \setminus \mathbf{E}) \cap C^{0,\alpha}(\mathbb{R}^n)$, such that $\tilde{f}|_\mathbf{E} = f$ and satisfying

$$|\mathcal{D}\tilde{f}(x)| \leq |f|_{\alpha,\mathbf{E}}\operatorname{dist}(x,\mathbf{E})^{\alpha-1} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \mathbf{E}.$$

This construction is adapted from [16].

Proof. We have

$$\begin{aligned}\int_\Omega |\mathcal{D}\tilde{f}(y)|^p dy = \sum_{Q\in\mathcal{W}}\int_Q |\mathcal{D}\tilde{f}(y)|^p dy &\leq |f|_{\alpha,\Gamma}\sum_{Q\in\mathcal{W}}\int_Q \operatorname{dist}(y,\Gamma)^{p(\alpha-1)}dy \\ &\leq c|f|_{\alpha,\Gamma}\sum_{Q\in\mathcal{W}}|Q|^{n-p(1-\alpha)} = c|f|_{\alpha,\Gamma}\sum_{Q\in\mathcal{W}}|Q|^{1-p(1-\alpha)}|Q|^{n-1}.\end{aligned}$$

Obviously, for $Q \in \mathcal{W}$ we have $|Q| \leq w(\Omega)$ and hence that

$$|Q| \leq w(\Omega)^{1/2}|Q|^{1/2}.$$

Consequently,

$$\int_\Omega |\mathcal{D}\tilde{f}(y)|^p dy \leq c|f|_{\alpha,\Gamma}w(\Omega)^{\frac{1}{2}(1-p(1-\alpha))}\sum_{Q\in\mathcal{W}}|Q|^{n-1+\epsilon},$$

where $\epsilon = (1-p(1-\alpha))/2$.

The proof is completed by using Lemma 1 with $\epsilon = (1-p(1-\alpha))/2$. □

4. The Jump Problem for Monogenic Functions

The main result of our paper, to be stated and proved in this section, provides a criterion for the solution of the jump problem (1) in the more general setting obtained so far.

Theorem 1. *For any $f \in C^{0,\alpha}(\Gamma)$, the jump problem* (1) *has a solution whenever*

$$\alpha > \frac{\operatorname{Dma}\Gamma}{n}. \tag{6}$$

Let $\mathcal{P} = \{P_k, k = 1, 2, \dots\}$ be a polygonal decomposition of Ω^+ (inner or outer) with boundaries $\Gamma_k = \partial P_k$. Then we set

$$\Lambda = \cup_{k\geq 1}\Gamma_k.$$

After applying once more the Whitney extension to the restriction of $\tilde{f}$ on $\overline{\Lambda}$, we obtain the function f^* with the following additional property: the restriction of f^* to any connected component O of $\mathbb{R}^n \setminus \overline{\Lambda}$ is equal there to the Whitney extension of $\tilde{f}|_{\partial O}$.

Denote by Δ_j, $j = 1, \dots k$, the connected components of P_k and define

$$\Phi_k(x) = \int_{\Gamma_k} E(y-x)\nu(y)f^*(y)d\mathcal{H}(y) = \sum_{j=1}^{k} \int_{\partial\Delta_j} E(y-x)\nu(y)f^*(y)d\mathcal{H}(y).$$

Since $\partial\Delta_j$ is (1)AD-regular for every $j = 1, \dots k$, by Lemma 2 we have that $\mathcal{D}f^*$ is integrable on each Δ_j. Consequently, by the Borel–Pompeiu formula we have

$$\Phi_k(x) = \sum_{j=1}^{k} \left(\chi_{\Delta_j} f^*(x) + \int_{\Delta_j} E(y-x)\mathcal{D}f^*(y)dy \right)$$

or, equivalently,

$$\Phi_k(x) = \chi_{P_k} f^*(x) + \int_{P_k} E(y-x)\mathcal{D}f^*(y)dy.$$

Proof. It suffices to show that, under the condition (6), the function given by

$$\Phi_*(x) = \lim_{k\to\infty} \Phi_k(x)$$

is well defined and satisfies (1).

First, we prove that Φ_* is well defined. To do this, it suffices to show that $\mathcal{D}f^*$ is integrable in Ω^+. Let us denote $w_k = w(P_k)$ and $\lambda_k = \mathcal{H}(\Gamma_k)$. Then, by Lemma 2 with $p = 1$,

$$\int_{\Omega^+} |\mathcal{D}f^*(y)|dy \leq c\sum_{k\geq 1} \lambda_k w_k^{\frac{\alpha}{2}} \leq c\sum_{k\geq 1} \lambda_k w_k^{\alpha}, \tag{7}$$

which is due to the fact that $0 < w_1 \leq w_2 \leq \dots$.

Let $q \in (\operatorname{Dma}\Gamma, \operatorname{Dma}\Gamma + \delta)$ with sufficiently small $\delta > 0$ such that

$$\sum_{k\geq 1} \lambda_k w_k^{q+1-n} < \infty$$

and

$$\alpha > \frac{q}{n}.$$

After choosing such a q and by (7), it is easy to show that

$$\int\limits_{\Omega^+} |\mathcal{D}f^*(y)|dy \leq c \sum_{k\geq 1} \lambda_k w_k^{\alpha} \leq c \sum_{k\geq 1} \lambda_k w_k^{q+1-n} < \infty,$$

which proves our first claim.

Consequently,

$$\Phi_*(x) = \lim_{k\to\infty} \Phi_k(x) = \chi_{\Omega^+} f^*(x) + \int\limits_{\Omega^+} E(y-x)\mathcal{D}f^*(y)dy.$$

It is easy to verify that if we take $p = \dfrac{n-q}{1-\alpha}$, then $p > n$ and $\mathcal{D}f^* \in L^p(\Omega^+)$. Combining these assertions with the basic properties of the right-inverse Theodoresco operator (in the sense of Clifford analysis; see, e.g., [11]), we can assert that the restrictions $\Phi_*|_{\Omega^\pm}$ are Hölder continuous (with exponent $(p-n)/p$) in $\overline{\Omega^\pm}$, $\Phi_*(x)$ is, moreover, a monogenic function in $\mathbb{R}^n \setminus \Gamma$, and the usual continuous boundary values of these restrictions are connected by relation (1). □

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