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ON GAPS IN THE SPECTRUM OF THE OPERATOR OF ELASTICITY THEORY ON A HIGH CONTRAST PERIODIC STRUCTURE

V. V. Zhikov*

Vladimir State University
87, ul. Gor'kogo, Vladimir 600000, Russia
zhikov@vlsu.ru

S. E. Pastukhova

Moscow State Institute of Radio Engineering, Electronics
and Automation (Technical University)
78, pr. Vernadskogo, Moscow 119454, Russia
pas-se@yandex.ru

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We study the spectrum of a periodic problem of elasticity theory such that the coefficients of the equation are high contrast dependent on a small parameter ε. We prove that for sufficiently small ε there are gaps in the continuous spectrum, the number of gaps unboundedly increases, and the limit set for the spectrum can be exactly described. The proof is based on the two-scale averaging principle for an ε-periodic two-phase elastic medium with the contrast coefficient $1 : \varepsilon^2$ between hard and soft phases in moduli of elasticity. Bibliography: 11 titles.

1 Statement of the Problem and the Main Results

We consider a strongly inhomogeneous two-phase periodic medium occupying the entire space $\mathbb{R}^d$, $d \geqslant 2$, and depending on a small parameter ε, with the periodicity cell $\varepsilon\square = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^d$. There is a hard phase F_1^ε and a soft phase F_0^ε so that $\mathbb{R}^d = F_1^\varepsilon \cup F_0^\varepsilon$, where $F_1^\varepsilon = \varepsilon F_1$ and $F_0^\varepsilon = \varepsilon F_0$ are obtained by contraction from 1-periodic sets F_1 and $F_0 = \mathbb{R}^d \setminus F_1$ with the periodicity cell $\square = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$. Further we use the average over the cell

$$\langle \cdot \rangle = \int\limits_{\square} \cdot \, dy.$$

We impose the following conditions on the geometry of the sets F_1 and F_0:

(i) F_1 is connected,

(ii) F_0 is disperse,

* To whom the correspondence should be addressed.

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(iii) the boundary ∂F_0 is Lipschitz.

The dispersion of the set F_0 means that F_0 is divided into disjoint components ("granules" or "inclusions") such that each component is a bounded domain and the periodicity cell $\square$ intersects finitely many components.

For a model case we can consider the case where the inclusion $F_0 \cap \square = B$ is the ball $B = \{|y| < r\}$, $r < 1/2$. The strongly inhomogeneous ε-periodic elasticity tensor is defined in the entire space $\mathbb{R}^d$ by the formula

$$a^\varepsilon(x) = \begin{cases} a^{(1)}(x/\varepsilon) & \text{on } F_1^\varepsilon, \\ \varepsilon^2 a^{(0)}(x/\varepsilon) & \text{on } F_0^\varepsilon. \end{cases} \tag{1.1}$$

Here, $a^{(l)}(y)$, $l = 0$ or $l = 1$, is a 1-periodic tensor satisfying the standard conditions of symmetry and positive definiteness. The tensor $a^{(l)}$ acts on the matrix of the gradient of displacements $u = (u_1, \dots, u_d)$ by the rule

$$a^{(l)} \nabla u = a^{(l)}_{ijpq} \frac{\partial u_p}{\partial x_q} = a^{(l)}_{ijpq} e_{pq}(u), \quad e_{pq}(u) = \frac{1}{2}\Big(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\Big),$$

where the passage to the symmetric gradient was realized in view of the symmetry of $a^{(l)}$. In the model case, for $a^{(l)}$, $l = 0, 1$, one can take any constant isotropic tensor

$$\begin{gathered} a_{ijpq} = k_1 \delta_{ij} \delta_{pq} + k_0 (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}), \\ k_0 > 0, \ k_1 \geqslant 0 \text{ are the Lamé coefficients,} \end{gathered} \tag{1.2}$$

δ_{ij} is the Kronecker symbol.

The quadratic form

$$\int_{\mathbb{R}^d} a^\varepsilon \nabla u \cdot \nabla u \, dx = \int_{\mathbb{R}^d} a^\varepsilon e(u) \cdot e(u) \, dx,$$

corresponding to the elastic strain energy defines in $L^2(\mathbb{R}^d)^d$ the operator of elasticity theory

$$\mathscr{A}_\varepsilon = -\operatorname{div} a^\varepsilon(x) \nabla. \tag{1.3}$$

By the general theory of operators with periodic coefficients, the spectrum of the operator $\mathscr{A}_\varepsilon$ is the union of segments

$$\operatorname{Sp} \mathscr{A}_\varepsilon = \bigcup_{j \geqslant 1} [\alpha_j^\varepsilon, \beta_j^\varepsilon], \tag{1.4}$$

where neighboring segments can overlap each other. If these segments do not intersect, then gaps (intervals $(\beta_j^\varepsilon, \alpha_{j+1}^\varepsilon)$) arise in the spectrum. The presence or absence of interior gaps is the important characteristic of the operator. In applications, the presence of gaps in the spectrum is of a particular interest since for certain frequencies of perturbations in the form of harmonic oscillations in time, the periodic medium does not propagate elastic waves. We recall that an elastic medium with the constant nondegenerate elasticity tensor propagates d waves along each direction for any perturbation frequencies. For example, for the isotropic tensor (1.2) there are two such plane waves, longitudinal and transverse, whose lengths are found from the dispersion relation connecting oscillation frequencies relative to time and space, which reflects the fact that there are no interior gaps in the spectrum of the operator.

Theorem 1.1. *For sufficiently small ε there are gaps in the spectrum* $\operatorname{Sp} \mathscr{A}_\varepsilon$*, and the number of gaps unboundedly increases as $\varepsilon \to 0$.*

Theorem 1.1 is valid not only for the model structure, but also for any other structure invariant under the rotation about the coordinate axes by $\pi/2$. The tensor $a^{(1)}$ can be arbitrary, whereas the tensor $a^{(0)}$ should possess the cubic symmetry (cf. [1, Chapter I]).

We study gaps following the plan below.

1. *We find the limit operator $\mathscr{A}$.* In the case under consideration, we cannot consider the usual strong resolvent convergence [2], but need to use a special two-scale variant of this operator convergence so that we deal with the limit operator which, unlike the original operator, acts not in $L^2(\mathbb{R}^d)^d$, but in the larger space $L^2(\mathbb{R}^d \times \square)^d$. We need a special theory of resolvent convergence adapted to the case under consideration.

2. *We study the spectrum of the limit operator $\mathscr{A}$* by establishing the presence of infinitely many gaps.

3. *We prove the Hausdorff convergence of the spectra* $\operatorname{Sp} \mathscr{A}_\varepsilon$ *to the spectrum* $\operatorname{Sp} \mathscr{A}$*.* This means that

(H_1) if $\lambda \in \operatorname{Sp} \mathscr{A}$, then $\exists \lambda_\varepsilon \in \operatorname{Sp} \mathscr{A}_\varepsilon : \lambda_\varepsilon \to \lambda$,

(H_2) if $\lambda_\varepsilon \in \operatorname{Sp} \mathscr{A}_\varepsilon$ and $\lambda_\varepsilon \to \lambda$, then $\lambda \in \operatorname{Sp} \mathscr{A}$.

These three results imply that the spectrum of the operator $\mathscr{A}_\varepsilon$ contains gaps if ε is sufficiently small and the number of gaps tends to infinity as $\varepsilon \to 0$; moreover, it is possible to localize the gaps with as high accuracy as desired.

We comment the above plan.

The strong two-scale resolvent convergence

$$\mathscr{A}_\varepsilon \xrightarrow{2} \mathscr{A} \tag{1.5}$$

is established in the same way as in [3, 4] (cf. details in Section 5). As in [3, 4], but taking into account the vector case, one can study the spectrum of the limit operator. The first property (H_1) of the Hausdorff convergence is a consequence of the operator convergence (1.5) (cf. [4, 5]). Thus, additional efforts are required only for obtaining the second property (H_2) of the Hausdorff convergence of spectra. This property assumes a certain strong compactness of the family of eigenfunctions of the continuous spectrum of the operator $\mathscr{A}_\varepsilon$. Indeed, with a point $\lambda_\varepsilon \in \operatorname{Sp} \mathscr{A}_\varepsilon$ one can associate the Bloch eigenfunction u_ε which is a solution to the eigenvalue problem

$$\mathscr{A}_\varepsilon u_\varepsilon = \lambda_\varepsilon u_\varepsilon$$

with quasiperiodic boundary conditions on the periodicity cell $\varepsilon\square$. Let $\lambda_\varepsilon \to \lambda$. Then, passing to the limit in this equation in the sense of the two-scale convergence, we find

$$\mathscr{A} u = \lambda u,$$

where u is a two-scale function of $x \in \mathbb{R}^d$ and $y \in \square$. It will be an eigenfunction of the operator $\mathscr{A}$ if, in addition, we show that u does not vanish identically. The last condition requires a certain a priori compactness of the family u_ε. This property is rigorously formulated in Section 7 (cf. Lemma 7.1).

2 The Limit Operator

We introduce the set V of functions of the form

$$u(x,y) = u_1(x) + u_0(x,y), \quad u_1 \in H^1(\mathbb{R}^d)^d,$$
$$u_0 \in L^2(\mathbb{R}^d, H^1_{\rm per}(\square)^d), \quad u_0(x,\cdot)|_{F^1\cap\square} = 0.$$

Since the interface separating the phases F_1 and F_0 is Lipschitz and the inclusion $B = F_0 \cap \square$ is disperse (cf. Assumptions (ii) and (iii) in Section 1), we can represent V as

$$V = H^1(\mathbb{R}^d)^d + L^2(\mathbb{R}^d, H^1_0(B)^d),$$

where $H^1_0(B)$ is the Sobolev space defined as the closure of $C_0^\infty(B)$ in the norm

$$\Big(\int\limits_B |\nabla\varphi|^2 dx \Big)^{1/2}.$$

In the Hilbert space $L^2(\mathbb{R}^d, L^2(\square, dy)^d) = L^2(\mathbb{R}^d \times \square, dx \times dy)^d$, we extract the subspace

$$H = L^2(\mathbb{R}^d)^d + L^2(\mathbb{R}^d, L^2(B)^d) = L^2(\mathbb{R}^d, \mathbb{R}^d + L^2(B)^d).$$

The space H is closed, which follows from the elementary inequality for functions $f(x,y)=f_1(x)+f_0(x,y)$, $f_1{\in}L^2(\mathbb{R}^d)$, $f_0{\in}L^2(\mathbb{R}^d, L^2(B))$:

$$\int\limits_\square \int\limits_{\mathbb{R}^d} |f|^2 dydx \geqslant (1-|B|^{\frac12}) \Bigg[\int\limits_{\mathbb{R}^d} |f_1|^2 dx + \int\limits_\square \int\limits_{\mathbb{R}^d} |f_0|^2 dydx \Bigg]. \tag{2.1}$$

It is clear that V belongs to H and is dense there.

The following quadratic form is defined in the space V:

$$Q(u,u) = \int\limits_{\mathbb{R}^d} a^{\rm hom} \nabla u_1 \cdot \nabla u_1 dx + \int\limits_{\mathbb{R}^d} \int\limits_\square a^{(0)} \nabla_y u_0 \cdot \nabla_y u_0 dxdy, \tag{2.2}$$

where $a^{\rm hom}$ is the so-called averaging tensor defined by the equality

$$a^{\rm hom}\xi\cdot\xi = \inf \Bigg\{ \int\limits_{\square\cap F_1} a^{(1)}(\xi + e(\varphi))\cdot(\xi+e(\varphi))dy : \ \varphi \in C^\infty_{per}(\square)^d \Bigg\},$$

where ξ is any symmetric $d \times d$-matrix. It is known that the tensor $a^{\rm hom}$ is symmetric and positive definite. The last property is a consequence of the connectivity of the phase F_1.

The form $Q(u,u)$ is closed in H in view of the inequality (2.1) and defines the nonnegative adjoint operator $\mathscr{A}$ as the sum $\mathscr{A} = A_1 + A_0$.

The *spatial* operator

$$A_1 = -\operatorname{div} a^{\rm hom}\nabla \tag{2.3}$$

acts in $L^2(\mathbb{R}^d)^d$ and has the same structure as the original operator (1.3), but it is essentially simpler because the tensor $a^{\rm hom}$ is constant.

The *Bloch* operator A_0 acts in the space $L^2(\mathbb{R}^d, L^2(B, dy)^d) \subset L^2(\mathbb{R}^d, L^2(\square, dy)^d)$ and is given by the formula

$$q(u_0, u_0) = \int\limits_{\mathbb{R}^d} \int\limits_{B} a^{(0)} \nabla_y u_0 \cdot \nabla_y u_0 dx dy$$

with the domain dom $q = L^2(\mathbb{R}^d, H_0^1(B)^d)$.

The equation

$$\mathscr{A} u = Pf, \tag{2.4}$$

where $f \in L^2(\mathbb{R}^d \times \square)^d$ and $P : L^2(\mathbb{R}^d \times \square)^d \to H$ is the orthogonal projection, means the validity of the integral identity

$$\int\limits_{\mathbb{R}^d} a^{\text{hom}} \nabla u_1 \cdot \nabla \varphi_1 dx + \int\limits_{\mathbb{R}^d} \int\limits_{\square} a^{(0)} \nabla u_0 \cdot \nabla \varphi_0 dx dy = \int\limits_{\mathbb{R}^d} \int\limits_{\square} f \cdot \varphi dx dy \tag{2.5}$$

for any test function $\varphi = \varphi_1 + \varphi_0 \in V$. It is useful to note that

$$Pf = g(x,y) = \begin{cases} f(x,y), & y \in B, \\ \dfrac{1}{|\square \setminus B|} \displaystyle\int\limits_{\square \setminus B} f(x,y) dy, & y \in \square \setminus B; \end{cases}$$

moreover,

$$\int\limits_{\square} g(x,y) dy = \int\limits_{B} g(x,y) dy + \int\limits_{\square \setminus B} g(x,y) dy = \int\limits_{\square} f(x,y) dy = \langle f \rangle.$$

Hence the right-hand side of (2.5) is represented as the sum

$$\int\limits_{\mathbb{R}^d} \int\limits_{\square} f \cdot \varphi dx dy = \int\limits_{\mathbb{R}^d} \langle f \rangle \cdot \varphi_1 dx + \int\limits_{\mathbb{R}^d} \int\limits_{\square} f \cdot \varphi_0 dx dy.$$

Setting $\varphi_0 = 0$ and then $\varphi_1 = 0$ in (2.5), we find

$$\begin{aligned} &A_1 u_1 = \langle f \rangle \quad \text{in } L^2(\mathbb{R}^d)^d, \\ &A_0 u_0 = f \quad \text{in } L^2(\mathbb{R}^d \times B)^d. \end{aligned} \tag{2.6}$$

We preserve the notation A_0 for the operator in $L^2(B)^d$ given by

$$\int\limits_{B} a^{(0)} \nabla v \cdot \nabla v dy \tag{2.7}$$

with the domain $H_0^1(B)^d$. The last operator has compact resolvent, its spectrum is discrete and consists of finite-multiple eigenvalues, which can be divided

$$\{\omega_1, \omega_2, \dots\} \cup \{\omega_1', \omega_2', \dots\}, \tag{2.8}$$

in such a way that the second part contains those eigenvalues for which all the corresponding eigenfunctions have zero average over B by inclusion. We denote by $\{\varphi_n\}$ the orthonormal system of eigenfunctions corresponding to the eigenvalues $\omega_1, \omega_2, \dots$.

We study the spectral problem

$$\mathscr{A}u = \lambda u, \quad u(x,y){=}u_1(x){+}u_0(x,y). \tag{2.9}$$

By the description of the operator $\mathscr{A}$, the equality (2.9) is equivalent to the system of coupled equations (cf. (2.4)–(2.6))

$$\begin{aligned} A_1 u_1 &= \lambda(u_1 + \langle u_0 \rangle), \\ A_0 u_0 &= \lambda(u_1 + u_0). \end{aligned} \tag{2.10}$$

If $\lambda \bar{\epsilon} \{\omega_1, \omega_2, \ldots\}$, then Equation $(2.10)_2$ can be solved relative to u_0, namely,

$$u_0(y,\lambda) = \lambda b(y,\lambda) u_1, \tag{2.11}$$

where

$$b(y,\lambda) = \Sigma_n \frac{\varphi_n \times \langle \varphi_n \rangle}{\omega_n - \lambda}. \tag{2.12}$$

Substituting (2.12) into $(2.10)_1$, we get

$$\begin{aligned} A_1 u_1 &= \beta(\lambda) u_1, \\ \beta(\lambda) &= \lambda E + \lambda^2 \langle b(\cdot,\lambda) \rangle, \end{aligned} \tag{2.13}$$

where E is the identity matrix. Using (2.12), we find

$$\beta(\lambda) = \lambda E + \lambda^2 \Sigma_n \frac{\langle \varphi_n \rangle \times \langle \varphi_n \rangle}{\omega_n - \lambda}. \tag{2.14}$$

Thus, the validity of Equation (2.9) for u means the validity of (2.13) and (2.11) for the components of u.

Let us find eigenvalues of the operator $\mathscr{A}$. We suppose that $A_0 u_0 = \lambda u_0$ and $\langle u_0 \rangle = 0$. For example, this is possible if λ coincides with one of the eigenvalues ω'_j (cf. (2.8)) or λ is equal to some multiple eigenvalue ω_j. Then $u(x,y) = l(x) u_0(y)$ (i.e., $u_1 = 0$) satisfies Equation (2.10), i.e., we have an eigenfunction of the operator $\mathscr{A}$; moreover, the multiplicity of the eigenvalue λ is equal to infinity. There are no other eigenvalues and eigenfunctions, except for the abovementioned Bloch ones.

The resolvent equation for the operator $\mathscr{A}$ is studied in a similar way. As a result, we obtain the following assertion.

Lemma 2.1. *A point λ belongs to the resolvent set of the operator $\mathscr{A}$ if and only if $\lambda \bar{\epsilon}\, \mathrm{Sp}\, A_0$ and the matrix $\beta(\lambda)$ is negative definite, which can be shortly written as*

$$\rho(\mathscr{A}) = \rho(A_0) \cap \{\lambda : \beta(\lambda) < 0\}. \tag{2.15}$$

To describe the boundary of the resolvent set $\rho(\mathscr{A})$, we need, in accordance with (2.15), to study points λ where the matrix $\beta(\lambda)$ is singular. With such a point one can associate an eigenvalue problem. We consider such problems below.

3 Analog of Electrostatic Problem

We consider the eigenvalue problem: find $\lambda \in [0, \infty)$ and $c + v \in \mathbb{R}^d + H_0^1(B)^d$ such that

$$\int_B a^{(0)} \nabla v \cdot \nabla w dy = \lambda \int_\square (c+v) \cdot (t+w) dy \quad \forall t + w \in \mathbb{R}^d + H_0^1(B)^d. \tag{3.1}$$

Here, we deal with the operator defined by formula (2.7) with the domain extended to $\mathbb{R}^d + H_0^1(B)^d$. The operator acts in $\mathbb{R}^d + L^2(B)^d$. In the case of scalar elliptic equations of the second order, a similar problem is known as an electrostatic problem.

For $\lambda \neq 0$ the problem (3.1) can be written as the identity

$$\int_B a^{(0)} \nabla v \cdot \nabla w dy = \lambda \int_\square (v - \langle v \rangle) \cdot w dy \quad \forall w \in H_0^1(B)^d$$

or, in terms of the operator A_0,

$$A_0 v = \lambda (v - \langle v \rangle), \quad \lambda \neq 0. \tag{3.2}$$

We describe the *trivial part* of the spectrum of the problem (3.1). This part involves

- $\lambda = 0$ corresponding to eigenfunctions that are constant vectors,
- $\lambda \in \mathrm{Sp}\, A_0$ such that there exists an eigenfunction φ such that $\langle \varphi \rangle = 0$.

We find the *nontrivial spectrum* of the problem (3.1). Since $\langle v \rangle \neq 0$, we have $\lambda \bar{\epsilon} \{\omega_1, \omega_2, \ldots\}$. Then from Equation (3.2) we have $v = -\lambda b \langle v \rangle$, which implies

$$v + \lambda b \langle v \rangle = 0 \Longrightarrow (E + \lambda \langle b \rangle) \langle v \rangle = 0$$

or $\beta(\lambda)\langle v \rangle = 0$ since $\lambda \neq 0$. Thus, $\beta(\lambda)$ is singular and $\langle v \rangle \in \mathrm{Ker}\, \beta(\lambda)$.

The converse assertion is proved in the same simple way: if $c \in \mathrm{Ker}\, \beta(\lambda) \setminus 0$, then the pair λ, $v - c$, where $v = -\lambda b(y, \lambda) c$, is a solution to the problem (3.1).

Thus we have proved the following assertion.

Lemma 3.1. *At all points of the nontrivial spectrum of the problem* (3.1) *and only at such points, the matrix* $\beta(\lambda)$ *is singular.*

4 Symmetry and Its Consequences

We assume that there are symmetries in the geometry of structure and the tensor $a^{(0)}$ in the operator A_0 is isotropic. We study properties of the matrices $b(y, \lambda)$ and $\beta(\lambda)$ defined by (2.12) and (2.13). The matrix $b(y)$=$b(y, \lambda)$ is solving for the equation

$$A_0 u_0 - \lambda u_0 = c, \quad c \in \mathbb{R}^d.$$

We denote by $u_0(y, c)$ the solution to this equation. Then $u_0(y, c) = b(y) c$.

Suppose that an orthogonal transformation S sends the set $B = F_0 \cap \square$ into itself. For example, for such a transformation one can take a rotation about a coordinate axis by $\pi/2$ or a mirror reflection relative to a coordinate plane. Then

i) the mapping $g(y) \to S^{-1}g(Sy)$ realizes an isomorphism of $H_0^1(B)^d$.

Hence it is easy to obtain the invariance property:

ii) $u_0(y, S^{-1}c) = S^{-1}u_0(Sy, c)$,

which can be written in terms of the matrix $b(y)$ in view of the equality $u_0(y, c) = b(y)c$:

iii) $b(y) = S^{-1}b(Sy)S$.

Taking the average over the cell $\langle b(\cdot)\rangle$ and recalling the orthogonality of S, we have

iv) $\langle b(\cdot)\rangle = S^{-1}\langle b(\cdot)\rangle S$.

For special symmetries of S we show by direct computations:

v) if S is a reflection relative to the vertical or horizontal axis in the plane (the case $d = 2$), then the matrix $\langle b(\cdot)\rangle$ is diagonal,

vi) if S is the rotation about any coordinate axis by $\pi/2$ (here, $d \geqslant 2$ is arbitrary), then $\langle b(\cdot)\rangle$ is isotropic, i.e., proportional to the identity: $\langle b(\cdot)\rangle = \gamma E,\ \gamma = \text{const}$.

The last two properties are extended to the matrix $\beta(\lambda)$ by definition. In particular, for the model structure we have

$$\beta(\lambda) = \beta_{11}(\lambda)E, \quad \beta_{11} = \lambda + \lambda^2\Sigma_n\frac{|\langle\varphi_n^1\rangle|^2}{\omega_n - \lambda},$$

where $\varphi_n = (\varphi_n^1, \dots, \varphi_n^d)$ is the coordinate representation of the vector φ_n. This function is meromorphic with poles at the points $\omega_1, \omega_2, \dots$ and monotonically increases from $-\infty$ to $+\infty$ on (ω_n, ω_{n+1}), $n\geqslant 1$. The zeros $\nu_1, \nu_2, \dots$ of this function are nontrivial eigenvalues of the problem (3.1).

Summarizing the above results, we can assert that

$$Sp\, A = (\cup_{n\geqslant 0}[\nu_n, \omega_{n+1}]) \cup \{\omega_1', \omega_2', ...\}.$$

Then the interval (ω_n, ν_n), $n\geqslant 1$, is a gap if it does not contain points of the set $\{\omega_1', \omega_2', ...\}$. In the opposite case, this interval is divided into several gaps. In any case, in the spectrum of the operator A there are infinitely many gaps.

5 Resolvent Convergence

We recall the definition of the two-scale convergence in $L^2(\Omega) = L^2(\Omega, dx)$, where Ω is a domain in $\mathbb{R}^d$ (cf. [6]–[8]). Let a sequence $u_\varepsilon \in L^2(\Omega)$ be bounded, i.e.,

$$\limsup_{\varepsilon\to 0} \int\limits_\Omega |u_\varepsilon|^2 dx < \infty.$$

Definition 5.1. We say that $u_\varepsilon \in L^2(\Omega)$ *weakly two-scale converges* to a function $u(x, y) \in L^2(\Omega \times \square, dx \times dy)$ and write $u_\varepsilon(x) \stackrel{2}{\rightharpoonup} u(x, y)$ if

$$\lim_{\varepsilon\to 0} \int\limits_\Omega u_\varepsilon(x)\varphi(x)b(\varepsilon^{-1}x)dx = \int\limits_\Omega \int\limits_\square u(x, y)\varphi(x)b(y)dxdy$$

for any $\varphi \in C_0^\infty(\Omega)$ and $b \in C_{\text{per}}^\infty(\square)$.

Definition 5.2. We say that $u_\varepsilon \in L^2(\Omega)$ *strongly two-scale converges* to a function $u(x,y) \in L^2(\Omega \times \square, dx \times dy)$ and write $u_\varepsilon(x) \xrightarrow{2} u(x,y)$ if

$$\lim_{\varepsilon\to 0} \int\limits_\Omega u_\varepsilon(x) v_\varepsilon(x) dx = \int\limits_\Omega \int\limits_\square u(x,y) v(x,y) dx dy$$

provided that $v_\varepsilon(x) \overset{2}{\rightharpoonup} v(x,y)$.

We also indicate some properties of the two-scale convergence in $L^2(\Omega)$:

(i) a bounded sequence is compact in the sense of the weak two-scale convergence,

(ii) if $u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y)$, then

$$\liminf_{\varepsilon\to 0} \int\limits_\Omega |u_\varepsilon|^2 dx \geqslant \int\limits_\Omega \int\limits_\square |u(x,y)|^2 dx dy,$$

(iii) $u_\varepsilon(x) \xrightarrow{2} u(x,y) \Longleftrightarrow u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y)$ and

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |u_\varepsilon|^2 dx = \int\limits_\Omega \int\limits_\square |u(x,y)|^2 dx dy,$$

(iv) if $\varphi \in C_0^\infty(\Omega)$ and $b \in C_{\mathrm{per}}^\infty(\square)$, then

$$\varphi(x) b(\varepsilon^{-1} x) \xrightarrow{2} \varphi(x) b(y).$$

Let H be a subspace of $L^2(\mathbb{R}^d \times \square, dx \times dy)$, and let P be the orthogonal projection onto H. Suppose that $\mathscr{A}_\varepsilon$ and $\mathscr{A}$ are selfadjoint operators in $L^2(\mathbb{R}^d, dx)$ and H respectively.

Definition 5.3. The *strong two-scale resolvent convergence* of $\mathscr{A}_\varepsilon$ to $\mathscr{A}$, denoted by $\mathscr{A}_\varepsilon \xrightarrow{2} \mathscr{A}$, means

$$(\mathscr{A}_\varepsilon + 1)^{-1} f_\varepsilon \xrightarrow{2} (\mathscr{A} + 1)^{-1} P f \quad \text{if } f_\varepsilon(x) \xrightarrow{2} f(x,y). \tag{5.1}$$

This operator convergence implies important consequences (cf. [4, 5, 9, 10]), for example, the convergence of parabolic and hyperbolic semigroups and the convergence of spectral projections, In particular, the following general assertion holds.

Proposition 5.1. *If $\mathscr{A}_\varepsilon \xrightarrow{2} \mathscr{A}$, then Property (H_1) of the Hausdorff convergence of spectra holds.*

An averaging principle for the scalar problem of double porosity was proved in [3]. This principle served as the start point for the further study of spectral aspects for the corresponding operator of double porosity (cf. [3] and [4]). The problem of elasticity theory under consideration is similar to the problem of double porosity and can be regarded as its vector version. As in [3], we can prove an averaging principle for a high-contrast elastic composite. We present one of possible settings.

Theorem 5.1. *For the operator of elasticity theory $\mathscr{A}_\varepsilon$ introduced in Section* 1 (*cf.* (1.3)) *the convergence* (5.1) *holds, where the operator $\mathscr{A}$ is defined in* (2.2).

6 Description of the Spectrum of the Operator $\mathscr{A}_\varepsilon$

Suppose that a 1-periodic composite consists of hard and soft phases F_1 and F_0, as above. In $L^2(\mathbb{R}^d)^d$, we introduce the operator

$$\mathscr{B}_\varepsilon = -\varepsilon^{-2} \operatorname{div}_y a_\varepsilon(y)\nabla_y, \tag{6.1}$$

where

$$a_\varepsilon(y) = \begin{cases} a^{(1)}(y) & \text{on } F_1, \\ \varepsilon^2 a^{(0)}(y) & \text{on } F_0; \end{cases} \tag{6.2}$$

the tensors $a^{(l)}(y)$, $l=0$ or $l=1$, are the same as in (1.1), $\varepsilon \in (0, \varepsilon_0]$. More exactly, the operator $\mathscr{B}_\varepsilon$ is given by the quadratic form

$$\int\limits_{\mathbb{R}^d} \varepsilon^{-2} a_\varepsilon \nabla_y v \cdot \nabla_y v dy$$

whose domain is the Sobolev space $H^1(\mathbb{R}^d)^d$. We note that the tensor

$$\varepsilon^{-2} a_\varepsilon(y) = \begin{cases} \varepsilon^{-2} a^{(1)}(y) & \text{on } F_1, \\ a^{(0)}(y) & \text{on } F_0, \end{cases} \tag{6.3}$$

is uniformly separated from zero and unboundedly increases on the hard phase as $\varepsilon \to 0$.

We recall the description of the spectrum of an operator with periodic coefficients. For the operator (6.1) we introduce a family of eigenvalue problems with quasiperiodic boundary conditions on the cell $\square = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

$$\begin{aligned} &\mathscr{B}_\varepsilon v = \lambda_\varepsilon v, \quad v(y) = e^{ik\cdot y} w(y), \\ &w(y) \in H^1_{\text{per}}(\square)^d, \end{aligned} \tag{6.4}$$

where $k \in [-\pi, \pi)^d$ is a vector parameter (quasimomentum) and $H^1_{\text{per}}(\square)^d$ is the Sobolev space of 1-periodic functions, complex-valued in the case under consideration.

In accordance with (6.4), for $w(y)$ we have the following family of periodic problems:

$$\varepsilon^{-2} A(k) w = \lambda_\varepsilon w, \quad A(k) = -(\nabla_y + ik)^* a_\varepsilon (\nabla_y + ik).$$

By definition, a solution to this problem is a function $w \in H^1_{\text{per}}(\square)^d$ satisfying the integral identity

$$\varepsilon^{-2} \int\limits_{\square} a_\varepsilon (\nabla w + ikw) \cdot (\nabla \overline{\varphi} - ik\overline{\varphi}) dy = \lambda \int\limits_{\square} w \cdot \overline{\varphi} dy, \quad \varphi \in H^1_{\text{per}}(\square)^d,$$

where the bar means the complex conjugation. Setting $\varphi = w = e^{-ik\cdot y} v$, we find the energy equality

$$\varepsilon^{-2} \int\limits_{\square} a_\varepsilon \nabla v \cdot \nabla \overline{v} dy = \lambda \int\limits_{\square} |v|^2 dy. \tag{6.5}$$

For every k the operator $\varepsilon^{-2}A(k)$ is selfadjoint in $L^2_{\text{per}}(\square)^d$, has compact resolvent, and its spectrum consists of countably many finite-multiple eigenvalues

$$0 \leqslant \lambda_1^\varepsilon(k) \leqslant \lambda_2^\varepsilon(k) \leqslant \dots$$

arranged in nondescending order. The functions $\lambda_j^\varepsilon(k)$ are continuous and 2π-periodic in k. The spectrum $\text{Sp}\ \mathscr{B}_\varepsilon$ has the zone structure

$$\text{Sp}\ \mathscr{B}_\varepsilon = \cup_{j\geqslant 1}[\alpha_j^\varepsilon, \beta_j^\varepsilon], \quad \alpha_j^\varepsilon = \min_k \lambda_j^\varepsilon(k), \quad \beta_j^\varepsilon = \max_k \lambda_j^\varepsilon(k). \tag{6.6}$$

The change of variables

$$v(y) = u(\varepsilon y), \quad u(x) = v(\varepsilon^{-1}x), \tag{6.7}$$

where $\nabla_y v(y) = \varepsilon(\nabla_x u(x))|_{x=\varepsilon y}$, sends the problem (6.4) to the problem with quasiperiodic boundary conditions on the cell $\varepsilon\square$:

$$\begin{aligned} &\mathscr{A}_\varepsilon u = \lambda_\varepsilon u, \\ &u(x) = e^{ik\cdot x}z(x), \quad z(x)\ \varepsilon\text{-periodic}, \end{aligned} \tag{6.8}$$

where the operator $\mathscr{A}_\varepsilon$ is defined in (1.3). Moreover, the eigenfunction $v(y)$ is transformed to the eigenfunction $u(x)$, whereas the eigenvalue λ_ε remains unchanged. Thus, we have shown that

$$\text{Sp}\ \mathscr{A}_\varepsilon = \text{Sp}\ \mathscr{B}_\varepsilon, \tag{6.9}$$

and, by (6.6), the representation in (1.4) holds.

The spectrum of a scalar operator, similar to the operator $\mathscr{B}_\varepsilon$, was studied in [11] on the basis of variational arguments and properties of monotone increasing forms. In our case, in view of (6.9), the asymptotic properties of the spectrum $\text{Sp}\ \mathscr{A}_\varepsilon$ indicated in Theorem 1.1 are generalized to $\text{Sp}\ \mathscr{B}_\varepsilon$. In particular, the following assertion holds.

Theorem 6.1. *In the spectrum of the operator $\mathscr{B}_\varepsilon$ defined in* (6.1)–(6.3) *there are gaps for sufficiently small ε, and the number of gaps unboundedly increases as $\varepsilon \to 0$.*

7 Hausdorff Convergence of Spectra

In this section, we prove the second property of the Hausdorff convergence of spectra (cf. Section 1). Namely, assuming that $\lambda_\varepsilon \in \text{Sp}\ \mathscr{A}_\varepsilon$ and $\lambda_\varepsilon \to \lambda$, we show that $\lambda \in \text{Sp}\ \mathscr{A}$. Since it is already known that $\text{Sp}\ A_0 \subset \text{Sp}\ \mathscr{A}$ (cf. (2.15)), we assume that $\lambda \overline{\in} \text{Sp}\ A_0$.

By the description of the spectrum $\text{Sp}\ \mathscr{A}_\varepsilon$ (cf. Section 6) for $\lambda_\varepsilon \in \text{Sp}\ \mathscr{A}_\varepsilon$, there are quasiperiodic eigenfunctions u_ε of the problem (6.8) normalized by the condition

$$\fint_{\varepsilon\square} |u_\varepsilon|^2 dx = \varepsilon^{-d} \int_{\varepsilon\square} |u_\varepsilon|^2 dx = 1. \tag{7.1}$$

From the energy equality for u_ε (cf. (6.5) which is an analog for the problem (6.4)) we have

$$\fint_{\varepsilon\square} a^\varepsilon \nabla u_\varepsilon \cdot \nabla \overline{u}_\varepsilon dx = \lambda_\varepsilon.$$

Then for any cube $\Omega = (-t,t)^d$ we obtain the boundedness properties

$$1/2 \leqslant \fint_\Omega |u_\varepsilon|^2 dx \leqslant 2, \qquad \fint_\Omega a^\varepsilon \nabla u_\varepsilon \cdot \nabla \overline{u}_\varepsilon dx \leqslant 2\lambda_\varepsilon \tag{7.2}$$

for sufficiently small ε.

The averaging principle for the problem of elasticity theory on a high-contrast composite can be formulated as the following assertion, which is necessary for passing to the limit in (6.8).

Theorem 7.1. *Let Ω be a Lipschitz domain, and let a sequence u_ε be such that*

$$\limsup_{\varepsilon \to 0} \int_\Omega (|u_\varepsilon|^2 + |e(u_\varepsilon)|^2) dx < \infty,$$

$$\int_\Omega a^\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi dx = \int_\Omega g_\varepsilon \cdot \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega)^d,$$

where $g_\varepsilon \in L^2(\Omega)^d$ is bounded and $g_\varepsilon(x) \xrightarrow{2} g(x,y)$. Then, at least, on a subsequence,

$$\begin{gathered} u_\varepsilon(x) \xrightarrow{2} u(x,y) = u_1(x) + u_0(x,y), \\ u_1 \in H^1(\mathbb{R}^d)^d, \quad u_0 \in L^2(\Omega, H_0^1(B)^d) \end{gathered} \tag{7.3}$$

and the limit function satisfies the integral identity

$$\int_\Omega a^{\mathrm{hom}} \nabla u_1 \cdot \nabla \varphi_1 dx + \int_\Omega \int_\square a^{(0)} \nabla_y u_0 \cdot \nabla_y \varphi_0 dxdy = \int_\Omega \int_\square g \cdot (\varphi_1 + \varphi_0) dxdy$$

for any $\varphi_1 \in C_0^\infty(\Omega)^d$ and $\varphi_0 \in L^2(\Omega, H_0^1(B)^d)$.

By the boundedness properties (7.2) and Theorem 7.1, passing to the limit in the equation $\mathscr{A}_\varepsilon u_\varepsilon = \lambda_\varepsilon u_\varepsilon$, we get the relations

$$u_\varepsilon \xrightarrow{2} u(x,y), \quad u(x,y) = u_1(x) + u_0(x,y), \quad \mathscr{A} u = \lambda u. \tag{7.4}$$

Here, the weak convergence can be replaced with the strong convergence owing to the following key result about compactness.

Lemma 7.1. *Let $\{u_\varepsilon\}$ be a sequence of quasiperiodic eigenfunctions of the operator $\mathscr{A}_\varepsilon$ on the cell $\varepsilon\square$, normalized by the condition (7.1), and let the corresponding eigenvalues λ_ε converge to $\lambda \overline{\in} \operatorname{Sp} A_0$. Then $\{u_\varepsilon\}$ is compact in the sense of the strong two-scale convergence in any bounded domain.*

Similar variants of the compactness principle for the scalar problem of double porosity are proved in [3, 4]. The methods proposed there are suitable for "double porosity in elasticity theory," Moreover, the dispersion of the soft phase remains an essential fact in this situation.

By Lemma 7.1 and properties of the strong two-scale convergence (cf. Section 5), from $(7.2)_1$ it follows that

$$1/2 \leqslant \fint_\Omega \int_\square |u|^2 dxdy \leqslant 2. \tag{7.5}$$

Hence $u \not\equiv 0$; moreover, $u_1 \not\equiv 0$. Otherwise, from (7.4) it follows that $u(x, y) = 0 + u_0(x, y)$ is an eigenfunction of the operator A_0 corresponding to the eigenvalue λ, which is impossible by assumption.

As was shown in Section 2, from Equation (6.8) we obtain (cf. (2.13))

$$A_1 u_1 = \beta(\lambda) u_1, \quad A_1 = -\operatorname{div} a^{\mathrm{hom}} \nabla. \tag{7.6}$$

Now, we prove that $\lambda \in \operatorname{Sp} \mathscr{A}$. Suppose the opposite. By Lemma 2.1, the matrix $\beta(\lambda)$ is negative definite. From the estimate (7.5) and the inequality (2.1) we have

$$\fint_{\Omega} |u_1|^2 dx \leqslant c_1.$$

Consequently, each component of the vector $u_1(x)$ is a tempered distribution, i.e., belongs to the space $\mathscr{S}'(\mathbb{R}^d)$ of functionals on the Schwartz space $\mathscr{S}(\mathbb{R}^d)$. Then for $u_1(x)$ we can define the Fourier transform which, in view of (7.6), satisfies the equality

$$[a^{\mathrm{hom}}(\xi \times \xi) - \beta(\lambda)] \widehat{u}_1(\xi) = 0. \tag{7.7}$$

Since the tensor a^{hom} is symmetric and positive definite, the matrix $a^{\mathrm{hom}}(\xi \times \xi)$ is negative definite, which by the properties of $\beta(\lambda)$, means the positive definiteness of the entire matrix in the square brackets in (7.7). Consequently, $u_1 \equiv 0$, which contradicts the above-proved assertion.

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