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ON THE OSCILLATION OF HIGHER-ORDER DELAY DIFFERENTIAL EQUATIONS

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The aim of this paper is to study the asymptotic properties and oscillation of the nth-order delay differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t) f\left(x(\tau(t))\right) = 0. \qquad (E)$$

The results obtained are based on some new comparison theorems that reduce the problem of oscillation of an nth-order equation to the problem of oscillation of one or more first-order equations. We handle both cases

$$\int^{\infty} r^{-1/\gamma}(t)\, dt = \infty \qquad \text{and} \qquad \int^{\infty} r^{-1/\gamma}(t)\, dt < \infty.$$

The comparison principles simplify the analysis of Eq. (E).

1. Introduction

In this paper, we examine the asymptotic and oscillatory behavior of solutions of the nth-order $(n \geq 3)$ delay differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t) f\left(x(\tau(t))\right) = 0. \qquad (E)$$

We assume that $q, \tau \in C([t_0, \infty))$, $r \in C^1([t_0, \infty))$, $f \in C((-\infty, \infty))$, and the following conditions are satisfied:

(H_1) γ is the ratio of two odd positive integers;

(H_2) $r(t) > 0$, $r'(t) > 0$, and $q(t) > 0$;

(H_3) $\tau(t) \leq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$, and $\tau(t)$ is nondecreasing;

(H_4) $x f(x) > 0$ for $x \neq 0$, $f(x)$ is nondecreasing, and

$$-f(-xy) \geq f(xy) \geq f(x) f(y) \quad \text{for} \quad xy > 0.$$

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By a *solution* of Eq. (E) we mean a function $x(t) \in C^{n-1}[T_x, \infty)$, $T_x \geq t_0$, for which $r(t)(x^{(n-1)}(t))^\gamma \in C^1[T_x, \infty)$ and $x(t)$ satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of Eq. (E) that satisfy the condition

$$\sup\{|x(t)| : t \geq T\} > 0 \quad \text{for all} \quad T \geq T_x,$$

and we tacitly assume that Eq. (E) possesses such solutions. A solution of Eq. (E) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)$, and it is said to be *nonoscillatory* otherwise. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Equation (E) and its special cases, especially for $n = 2$, were studied by many authors (see, e.g., [2–19]), mainly under the condition

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\, ds = \infty. \tag{1.1}$$

There are comparatively fewer results (see, e.g., [1] and [20]) for Eq. (E) in the case where

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\, ds < \infty. \tag{1.2}$$

In this paper, we consider both possibilities.

If the gap between t and $\tau(t)$ is small, then there exists a nonoscillatory solution of Eq. (E), and so in this case our goal is to prove that every nonoscillatory solution of Eq. (E) tends to zero as $t \to \infty$. On the other hand, if the difference $t - \tau(t)$ is large enough, then we study the oscillation of Eq. (E). Our aim in this paper is to study both of these cases as well.

Various techniques have been used in investigating higher-order differential equations. Our method here is based on establishing new comparison theorems that compare the nth-order equation (E) with one or a couple of first-order delay differential equations in the sense that the oscillation of these first-order equations implies the oscillation of Eq. (E). These comparison theorems greatly simplify the analysis of Eq. (E).

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all sufficiently large t.

2. Main Results

Our results make use of the following estimate, which is due to Philos and Staikos (see [17, 18]):

Lemma A. *Let $z \in C^k([t_0, \infty))$ and assume that $z^{(k)}$ is of fixed sign and not identically zero on a subray of $[t_0, \infty)$. If, moreover, $z(t) > 0$, $z^{(k-1)}(t) z^{(k)}(t) \leq 0$, and*

$$\lim_{t \to \infty} z(t) \neq 0,$$

then, for every $\delta \in (0, 1)$, there exists $t_\delta \geq t_0$ such that

$$z(t) \geq \frac{\delta}{(k-1)!}\, t^{k-1}\, |z^{(k-1)}(t)| \quad \text{on} \quad [t_\delta, \infty). \tag{2.1}$$

The positive solutions of Eq. (E) have the following structure:

Lemma 1. *If $x(t)$ is a positive solution of Eq. (E), then $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is decreasing, all derivatives $x^{(i)}(t),\ 1 \le i \le n-1,$ are of constant signs, and $x(t)$ satisfies either*

$$x^{(n-1)}(t) > 0, \quad x^{(n)}(t) < 0 \tag{C_1}$$

or, if (1.2) holds,

$$x^{(n-2)}(t) > 0, \quad x^{(n-1)}(t) < 0. \tag{C_2}$$

Proof. Since $x(t)$ is a positive solution of Eq. (E), it follows from Eq. (E) that

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = -q(t) f\left(x(\tau(t))\right) < 0.$$

Thus, $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is decreasing, which implies that either $x^{(n-1)}(t) > 0$ or $x^{(n-1)}(t) < 0$. Note that the second case may occur only if (1.2) holds. Moreover, since $x(t) > 0$, it follows from $x^{(n-1)}(t) < 0$ that $x^{(n-2)}(t) > 0$.

On the other hand, if $x^{(n-1)}(t) > 0$, then, using the fact that $r'(t) > 0$ in the expression

$$0 > \left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = r'(t)\left[x^{(n-1)}(t)\right]^{\gamma} + r(t)\gamma\left[x^{(n-1)}(t)\right]^{\gamma-1} x^{(n)}(t),$$

we conclude that $x^{(n)}(t) < 0$. This completes the proof of the lemma.

We next give some criteria for excluding the possibility that cases (C_1) and (C_2) occur.

Theorem 1. *Let (1.1) hold. If, for some constant $\delta \in (0,1)$, the first-order delay differential equation*

$$y'(t) + q(t) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(t)}{r^{1/\gamma}(\tau(t))}\right) f\left(y^{1/\gamma}(\tau(t))\right) = 0 \tag{E_1}$$

is oscillatory, then

(i) for even n, Eq. (E) is oscillatory,

(ii) for odd n, every nonoscillatory solution $x(t)$ of Eq. (E) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Assume that $x(t)$ is a nonoscillatory solution of Eq. (E), say, $x(t) > 0$. It follows from Lemma 1 that $x(t)$ satisfies (C_1).

If n is even, then it is clear from (C_1) that

$$\lim_{t\to\infty} x(t) \ne 0.$$

Thus, it follows from Lemma A that, for every $\delta \in (0,1)$,

$$x(\tau(t)) \geq \frac{\delta}{(n-1)!} \frac{\tau^{n-1}(t)}{r^{1/\gamma}(t)} \left(r^{1/\gamma}(t) x^{(n-1)}(\tau(t)) \right) \tag{2.2}$$

eventually. Using (2.2) in Eq. (E), we see that $y(t) = r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is a positive solution of the delay differential inequality

$$y'(t) + q(t) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(t)}{r^{1/\gamma}(\tau(t))} \right) f\left(y^{1/\gamma}(\tau(t)) \right) \leq 0.$$

By Theorem 1 in [16], we conclude that the corresponding equation (E) also has a positive solution. This contradiction proves part (i) of the theorem.

Now assume that n is odd. We claim that

$$\lim_{t\to\infty} x(t) = 0.$$

If this is not the case, then, proceeding exactly as in the proof of part (i), we again obtain that Eq. (E_1) has a positive solution. This contradiction proves part (ii) of the theorem.

Remark 2. It follows from the proof of Theorem 1 that the oscillation of (E_1) prevents case (C_1) of Lemma 1 from occurring provided that

$$\lim_{t\to\infty} x(t) \neq 0.$$

Applying the criteria for the oscillation of (E_1), we immediately obtain sufficient conditions for cases (i) and (ii) of Theorem 1 to hold. We offer two such results.

Corollary 1. *Assume that (1.1) holds,*

$$f(u^{1/\gamma})/u \geq 1 \quad \text{for} \quad 0 < |u| \leq 1, \tag{2.3}$$

and, for some $\delta \in (0,1)$,

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))} \right) ds > \frac{1}{\mathrm{e}}. \tag{2.4}$$

Then the following assertions are true:

(i) if n is even, then Eq. (E) is oscillatory;

(ii) if n is odd, then every nonoscillatory solution $x(t)$ of Eq. (E) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. First, note that (2.4) yields

$$\int_{t_0}^{\infty} q(s) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) ds = \infty.$$

By Theorem 1, it is sufficient to show that Eq. (E_1) is oscillatory. Assume to the contrary that Eq. (E_1) has an eventually positive solution $y(t)$. Then $y'(t) < 0$. We claim that

$$\lim_{t\to\infty} y(t) = 0.$$

If this is not the case, then there exists $\ell > 0$ such that $y(\tau(t)) > \ell$. Integrating Eq. (E_1) from t_1 to t, we get

$$y(t_1) = y(t) + \int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) f\left(y^{1/\gamma}(\tau(s))\right) ds$$

$$\geq f\left(\ell^{1/\gamma}\right) \int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) ds.$$

Letting $t \to \infty$, we obtain a contradiction, and so

$$\lim_{t\to\infty} y(t) = 0.$$

Thus, $0 < y(t) \leq 1$ eventually. Using (2.3) in Eq. (E_1), one can easily see that $y(t)$ is a positive solution of the differential inequality

$$y'(t) + q(t) f\left(\frac{\delta}{(n-1)!} \frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) y(\tau(t)) \leq 0. \tag{2.5}$$

However, by Theorem 2.4.1 in [13], condition (2.4) ensures that inequality (2.5) has no positive solutions. This contradiction completes the proof of the theorem.

The second result is contained in the following corollary:

Corollary 2. *Let (1.1) hold and let β be the ratio of two odd positive integers with $\beta < \gamma$. If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s) \frac{\left(\tau^{n-1}(s)\right)^{\beta}}{r^{\beta/\gamma}(\tau(s))} ds > 0, \tag{2.6}$$

then

(i) for even n, the differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t) x^{\beta}(\tau(t)) = 0 \tag{E^{β}}$$

is oscillatory;

(ii) for odd n, every nonoscillatory solution of Eq. (E^β) $x(t)$ satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. First, note that (2.6) yields

$$\int_{t_0}^{\infty} q(s)\frac{\left(\tau^{n-1}(s)\right)^\beta}{r^{\beta/\gamma}(\tau(s))}\,ds = \infty.$$

In view of Theorem 1, it is sufficient to show that Eq. (E_1), which now reduces to

$$y'(t) + \left(\frac{\delta}{(n-1)!}\right)^\beta q(t)\frac{\left(\tau^{n-1}(t)\right)^\beta}{r^{\beta/\gamma}(\tau(t))}\,y^{\beta/\gamma}(\tau(t)) = 0, \tag{E_1^β}$$

is oscillatory. Assume that (E_1^β) has an eventually positive solution $y(t)$. Similarly to the proof of Corollary 1, we can show that $y(t)$ is decreasing and

$$\lim_{t\to\infty} y(t) = 0.$$

Integrating (E_1^β) from $\tau(t)$ to t, we obtain

$$0 = y(t) - y(\tau(t)) + \left(\frac{\delta}{(n-1)!}\right)^\beta \int_{\tau(t)}^{t} q(s)\frac{\left(\tau^{n-1}(s)\right)^\beta}{r^{\beta/\gamma}(\tau(s))}\,y^{\beta/\gamma}(\tau(s))\,ds.$$

From the monotonicity of $y^{\beta/\gamma}(\tau(t))$, we have

$$\left(\frac{(n-1)!}{\delta}\right)^\beta y^{1-\beta/\gamma}(\tau(t)) \ge \int_{\tau(t)}^{t} q(s)\frac{\left(\tau^{n-1}(s)\right)^\beta}{r^{\beta/\gamma}(\tau(s))}\,ds.$$

Taking the limit superior of both sides, we obtain a contradiction to (2.6), and this establishes the desired result.

Next, we turn our attention to the case where n is odd. Employing an additional condition, we are able to ensure the oscillation of all solutions of Eq. (E) for odd n. In other words, we are able to eliminate the possibility that there are nonoscillatory solutions converging to zero. For convenience, we set

$$\xi_1(t) = \xi(t), \quad \xi_{i+1}(t) = \xi_i\big(\xi(t)\big),$$

$$J_1(t) = \xi(t) - t, \quad J_{i+1}(t) = \int_t^{\xi(t)} J_i(s)\,ds,$$

where $\xi(t) \in C([t_0,\infty))$.

Theorem 2. *Let n be odd and let (1.1) hold. Assume that $\xi(t) \in C([t_0, \infty))$ is such that*

$$\xi(t) \text{ is nondecreasing}, \quad \xi(t) > t, \quad \text{and} \quad \xi_{n-1}(\tau(t)) < t. \tag{2.7}$$

If, for some $\delta \in (0,1)$, Eq. (E_1) is oscillatory and the equation

$$y'(t) + q(t) f\left(r^{-1/\gamma}(\xi_{n-1}(\tau(t)) J_{n-1}(\tau(t))\right) f\left(y^{1/\gamma}(\xi_{n-1}(\tau(t))\right) = 0 \tag{E_2}$$

is also oscillatory, then Eq. (E) is oscillatory.

Proof. Assume to the contrary that $x(t)$ is a positive solution of Eq. (E). Then, by Theorem 1, the oscillation of Eq. (E) implies that

$$\lim_{t \to \infty} x(t) = 0.$$

Thus, in view of Lemma 1, $x(t)$ satisfies

$$(-1)^i x^{(i)}(t) > 0, \quad i = 1, 2, \ldots, n-1. \tag{2.8}$$

Consequently,

$$-x^{(n-2)}(t) \geq x^{(n-2)}(\xi(t)) - x^{(n-2)}(t) = \int\limits_t^{\xi(t)} x^{(n-1)}(s)\, ds$$

$$\geq x^{(n-1)}(\xi(t))(\xi(t) - t) = x^{(n-1)}(\xi(t)) J_1(t).$$

The repeated integration of the previous inequalities from t to $\xi(t)$ yields

$$x(t) \geq x^{(n-1)}(\xi_{n-1}(t)) J_{n-1}(t),$$

or, equivalently,

$$x(\tau(t)) \geq \left[r^{1/\gamma}(\xi_{n-1}(\tau(t))) x^{(n-1)}(\xi_{n-1}(\tau(t)))\right] \frac{J_{n-1}(\tau(t))}{r^{1/\gamma}(\xi_{n-1}(\tau(t)))}.$$

Using the last inequality in Eq. (E), we see that $y(t) = r(t)\left[x^{(n-1)}(t)\right]^\gamma$ is a positive solution of the delay differential inequality

$$y'(t) + q(t) f\left(r^{-1/\gamma}(\xi_{n-1}(\tau(t)) J_{n-1}(\tau(t)))\right) f\left(y^{1/\gamma}(\xi_{n-1}(\tau(t)))\right) \leq 0.$$

It follows from Theorem 1 in [16] that the corresponding equation (E_2) also has a positive solution. This contradiction completes the proof.

Remark 3. Similarly to Remark 2 above, the oscillation of Eq. (E_2) prevents case (C_1) of Lemma 1 from holding provided that

$$\lim_{t\to\infty} x(t) \neq 0.$$

As an application of Theorem 2, we have the following corollary:

Corollary 3. *Let n be odd and let (1.1), (2.3), and (2.4) hold for some $\delta \in (0,1)$. Assume that $\xi(t) \in C([t_0,\infty))$ is such that (2.7) is satisfied. If*

$$\liminf_{t\to\infty} \int_{\xi_{n-1}(\tau(t))}^{t} q(s) f\left(r^{-1/\gamma}\left(\xi_{n-1}(\tau(s))\right) J_{n-1}(\tau(s))\right) ds > \frac{1}{\mathrm{e}}, \tag{2.9}$$

then Eq. (E) is oscillatory.

Proof. By Theorem 2, it is sufficient to show that both equations (E_1) and (E_2) are oscillatory. It follows from the proof of Corollary 1 that the oscillation of (E_1) is due to (2.4). Using arguments similar to those in the proof of Corollary 1, one can show that (2.9) guarantees the oscillation of (E_2). This proves the corollary.

We illustrate our results by the following examples:

Example 1. Consider the nth-order nonlinear differential equation

$$\left(t^3\left(x^{(n-1)}(t)\right)^3\right)' + \frac{b}{t^{3n-5}}\, x^3(\lambda t) = 0 \tag{2.10}$$

with $b > 0$ and $0 < \lambda < 1$. Condition (2.4) reduces to

$$\delta^3 b\lambda^{3n-6} \ln\frac{1}{\lambda} > \frac{\left((n-1)!\right)^3}{\mathrm{e}} \quad \text{for some} \quad \delta \in (0,1), \tag{2.11}$$

or, simply,

$$b\lambda^{3n-6} \ln\frac{1}{\lambda} > \frac{\left((n-1)!\right)^3}{\mathrm{e}}, \tag{2.12}$$

since (2.12) implies (2.11). Hence, Corollary 1 guarantees that if (2.12) holds, then

(i) for even n, (2.10) is oscillatory,

(ii) for odd n, every nonoscillatory solution $x(t)$ of (2.10) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

For $n = 3$ and $\beta > 0$ such that $3\beta^3(\beta+1)^4 = b\lambda^{-3\beta}$, one such solution is $x(t) = t^{-\beta}$.

Moreover, if n is odd, we set $\xi(t) = \alpha t$, where

$$\alpha = \frac{1 + \lambda^{-1/(n-1)}}{2}.$$

Then condition (2.9) takes the form

$$\frac{b}{\alpha^{3n-3}\lambda^3}\left(\frac{(\lambda-1)(\lambda^2-1)\,\ldots\,(\lambda^{n-1}-1)\lambda^{n-1}}{(n-1)!}\right)^3 \ln\frac{1}{\alpha^{n-1}\lambda} > \frac{1}{\mathrm{e}}. \tag{2.13}$$

It follows from Corollary 3 that (2.10) is oscillatory even if n is odd, provided that both conditions (2.12) and (2.13) are satisfied.

We now turn our attention to the case where (1.2) holds. It is useful to note that, in this case, Eq. (E) may have a solution $x(t)$ with property $x(t)x'(t) < 0$ no matter if n is even or odd.

Theorem 3. *Let (1.2) hold. If, for some constant $\delta \in (0,1)$ and every $t_1 \geq t_0$, both the first-order delay differential equations (E_1) and the equation*

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(s)\right) ds\right]^{1/\gamma} f^{1/\gamma}\left(y(\tau(t))\right) = 0 \tag{E_3}$$

are oscillatory, then every nonoscillatory solution of Eq. (E) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E) such that

$$\lim_{t\to\infty} x(t) \neq 0.$$

We may assume that $x(t) > 0$. Lemma 1 implies that $x(t)$ satisfies either (C_1) or (C_2). On the other hand, it follows from the proof of Theorem 1 that the oscillation of (E_1) implies that case (C_1) is impossible. We shall show that the oscillation of (E_3) excludes case (C_2).

Lemma 1 gives the estimate

$$x(\tau(t)) \geq \frac{\delta}{(n-2)!}\tau^{n-2}(t)\, x^{(n-2)}(\tau(t)). \tag{2.14}$$

Using (2.14) in Eq. (E), we get

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t) f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(t)\right) f\left(x^{(n-2)}(\tau(t))\right) \leq 0.$$

Integrating, we obtain

$$-r(t)\left[x^{(n-1)}(t)\right]^{\gamma} \geq \int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(s)\right) f\left(x^{(n-2)}(\tau(s))\right) ds,$$

which, in view of the monotonicity of $f\left(x^{(n-2)}(\tau(t))\right)$, gives

$$-x^{(n-1)}(t) \geq r^{-1/\gamma}(t) f^{1/\gamma}\left(x^{(n-2)}(\tau(t))\right)\left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(s)\right) ds\right]^{1/\gamma}.$$

Consequently, $y(t) = x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(s)\right) ds\right]^{1/\gamma} f^{1/\gamma}\left(y(\tau(t))\right) \leq 0.$$

By Theorem 1 in [16], the corresponding equation (E_3) also has a positive solution. This contradiction shows that

$$\lim_{t\to\infty} x(t) = 0$$

and completes the proof of the theorem.

Remark 4. The oscillation of Eq. (E_3) prevents case (C_2) in Lemma 1 from occurring provided that

$$\lim_{t\to\infty} x(t) \neq 0.$$

Next, we eliminate the possibility that

$$\lim_{t\to\infty} x(t) = 0$$

from Theorem 3 even if (1.2) holds. We consider another first-order delay differential equation, namely,

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s)\, ds\right]^{1/\gamma} f^{1/\gamma}\left(J_{n-2}(\tau(t))\right) f^{1/\gamma}\left(y\left(\xi_{n-2}(\tau(t))\right)\right) = 0. \qquad (E_4)$$

Theorem 4. *Let (1.2) hold. Assume that, for some* $\delta \in (0,1)$ *and every* $t_1 \geq t_0$, *both* (E_1) *and* (E_3) *are oscillatory. Assume further that there exists* $\xi(t) \in C([t_0,\infty))$ *such that*

(i) for odd n, *(2.7) holds and* (E_2) *is oscillatory;*

(ii) for even n, (E_4) *is oscillatory for every* $t_1 \geq t_0$ *and*

$$\xi(t) \text{ is nondecreasing}, \quad \xi(t) > t, \quad \text{and} \quad \xi_{n-2}(\tau(t)) < t. \qquad (2.15)$$

Then Eq. (E) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of Eq. (E). It follows from the proofs of Theorems 1 and 3 that

$$\lim_{t\to\infty} x(t) = 0.$$

Then, in view of Lemma 1, $x(t)$ must satisfy (2.8).

(i) If n is odd, then it follows from the proof of Theorem 2 that Eq. (E) is oscillatory due to the oscillation of (E_2).

(ii) Assume that n is even. We shall show that (2.8) cannot hold. Proceeding exactly as in the proof of Theorem 2, we obtain

$$x(t) \ge x^{(n-2)}(\xi_{n-2}(t))J_{n-2}(t). \tag{2.16}$$

On the other hand, the integration of Eq. (E) yields

$$-r(t)\left[x^{(n-1)}(t)\right]^{\gamma} \ge \int_{t_1}^{t} q(s)f(x(\tau(s)))\,ds \ge f(x(\tau(t)))\int_{t_1}^{t} q(s)\,ds.$$

In other words,

$$-x^{(n-1)}(t) \ge r^{-1/\gamma}(t)f^{1/\gamma}(x(\tau(t)))\left[\int_{t_1}^{t} q(s)\,ds\right]^{1/\gamma},$$

which, combined with (2.16), implies that $y(t) = x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s)\,ds\right]^{1/\gamma} f^{1/\gamma}\Big(J_{n-2}(\tau(t))\Big)f^{1/\gamma}\Big(y\big(\xi_{n-2}(\tau(t))\big)\Big) \le 0.$$

Again by Theorem 1 in [16], the corresponding equation (E_2) must have a positive solution.

This completes the proof of the theorem.

Remark 5. The oscillation of (E_4) prevents case (C_2) in Lemma 1 from holding provided that

$$\lim_{t\to\infty} x(t) \ne 0.$$

Corollary 4. *Let (1.2) and (2.3) hold and assume that, for some $\delta \in (0,1)$ and every $t_1 \ge t_0$, both (2.4) and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} r^{-1/\gamma}(u)\left[\int_{t_1}^{u} q(s)f\left(\frac{\delta}{(n-2)!}\tau^{n-2}(s)\right)ds\right]^{1/\gamma} du > \frac{1}{\mathrm{e}} \tag{2.17}$$

are satisfied. Then every nonoscillatory solution of Eq. (E) tends to zero as $t\to\infty$.

Assume, in addition, that there exists $\xi(t) \in C([t_0,\infty))$ such that *(i) for odd n, (2.7) and (2.9) hold,*

(ii) for even n, (2.15) and the following relation hold:

$$\liminf_{t\to\infty} \int_{\xi_{n-2}(\tau(t))}^{t} r^{-1/\gamma}(u) \left[\int_{t_1}^{u} q(s)\, ds \right]^{1/\gamma} f^{1/\gamma}\big(J_{n-2}(\tau(u))\big)\, du > \frac{1}{\mathrm{e}}. \tag{2.18}$$

Then Eq. (E) is oscillatory.

Proof. Conditions (2.4), (2.9), (2.17), and (2.18) ensure that (E_1), (E_2), (E_3), and (E_4), respectively, are oscillatory. The assertion now follows from Theorems 3 and 4.

Example 2. Consider the nth-order nonlinear differential equation

$$\left(t^6 \left(x^{(n-1)}(t)\right)^3\right)' + \frac{b}{t^{3n-8}} x^3(\lambda t) = 0 \tag{2.19}$$

with $b > 0$ and $0 < \lambda < 1$. Conditions (2.4) and (2.17) reduce to

$$b\lambda^{3n-9} \ln \frac{1}{\lambda} > \frac{\big((n-1)!\big)^3}{\mathrm{e}} \tag{2.20}$$

and

$$b^{1/3}\lambda^{n-2} \ln \frac{1}{\lambda} > \frac{3^{1/3}(n-2)!}{\mathrm{e}}, \tag{2.21}$$

respectively. Corollary 4 guarantees that every nonoscillatory solution $x(t)$ of (2.19) tends to zero as $t \to \infty$ provided that both conditions (2.20) and (2.21) are satisfied.

On the other hand, if n is odd, then we set $\xi(t) = \alpha t$, where

$$\alpha = \frac{1 + \lambda^{-1/(n-1)}}{2}.$$

Then condition (2.9) takes the form

$$\frac{b}{\alpha^{6n-6}\lambda^6} \left(\frac{(\lambda-1)(\lambda^2-1)\,\ldots\,(\lambda^{n-1}-1)\lambda^{n-1}}{(n-1)!} \right)^3 \ln \frac{1}{\alpha^{n-1}\lambda} > \frac{1}{\mathrm{e}}. \tag{2.22}$$

Conditions (2.20)–(2.22) imply that Corollary 4 holds, and so all solutions of Eq. (2.19) with odd n are oscillatory.

For even n, if we set $\xi(t) = \alpha t$, where

$$\alpha = \frac{1 + \lambda^{-1/(n-2)}}{2},$$

then condition (2.18) takes the form

$$b(\lambda - 1)(\lambda^2 - 1) \ldots (\lambda^{n-2} - 1)\lambda^{n-2} \ln \frac{1}{\alpha^{n-2}\lambda} > \frac{(n-2)!(3n-9)^{1/3}}{\mathrm{e}}. \tag{2.23}$$

It follows from Corollary 4 that Eq. (2.10) with even n is oscillatory if conditions (2.20), (2.21), and (2.23) are satisfied.

Example 3. Consider the fourth-order delay differential equation

$$\left(\mathrm{e}^t x'''(t)\right)' + \frac{\mathrm{e}^{t-1/2}}{16} x(t-1) = 0, \quad t \geq 1. \tag{2.24}$$

This equation was studied by Zhang et al. in [20]. They showed that every nonoscillatory solution of (2.24) tends to zero as $t \to \infty$ (this conclusion also follows from our Corollary 4). In particular, $x(t) = \mathrm{e}^{-t/2}$ is a solution of (2.24). We now consider the more general differential equation

$$\left(\mathrm{e}^t x'''(t)\right)' + b\,\mathrm{e}^t x(t-1) = 0, \quad t \geq 1. \tag{2.25}$$

It is not difficult to verify that both (2.4) and (2.17) hold. If we set

$$\xi(t) = \frac{t}{4},$$

then (2.18) takes the form

$$b > \frac{2^5}{\mathrm{e}},$$

which, according to Corollary 4, yields the oscillation of (2.25). This is a new phenomenon, which does not appear to have been studied previously.

3. Summary

In this paper, we have presented new comparison theorems for studying the asymptotic behavior and oscillation of Eq. (E) from the oscillation of a set of suitable first-order delay differential equations. Thus, our method substantially simplifies the examination of higher-order equations, and what is more, it supports the value of continued research on first-order delay differential equations. Our results here extend and complement many recent ones in the literature. Suitable illustrative examples have also been provided.

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