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INDEPENDENCE OF TEMPERATURES OF PHASE TRANSITIONS OF THE DOMAIN OCCUPIED BY A TWO-PHASE ELASTIC MEDIUM

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We consider a two-phase elastic medium with zero boundary condition on the displacement field and zero force. We show that the temperatures of phase transitions are independent of the domain occupied by the medium. Bibliography: 6 titles.

The strain energy functional of a two-phase elastic medium is defined by the equality

$$I[u,\chi,t,\Omega]=\int\limits_{\Omega}\{\chi(F^{+}(\nabla u)+t)+(1-\chi)F^{-}(\nabla u)\}\,dx-\int\limits_{\Omega}g\cdot u\,dx-\int\limits_{\partial\Omega}f\cdot u\,dS, \tag{1}$$

where $\Omega\subset R^m$ is a bounded domain with Lipschitz boundary, $g\in L_2(\Omega,R^m)$ and $f\in L_2(\partial\Omega,R^m)$ are external force fields, the m-dimensional vector-valued function u corresponds to the displacement field, the characteristic function χ defines the distribution of phases labeled by the superscripts $\pm$, the energy densities $F^{\pm}$ have the form

$$F^{\pm}(M)=a^{\pm}_{ijkl}(e(M)-\zeta^{\pm})_{ij}(e(M)-\zeta^{\pm})_{kl}, \tag{2}$$

$e(M)=(M+M^*)/2$, M belongs to the space $R^{m\times m}$ of $m\times m$ matrices, and the parameter t is interpreted as the temperature. The components of the elasticity modulus tensors $a^{\pm}_{ijkl}$ satisfy the traditional symmetry positive definiteness conditions [1], and the residual strain tensors $\zeta^{\pm}$ are symmetric matrices. In (2) and everywhere below, we adopt the convention regarding the summation with respect to repeated indices from 1 to m.

To describe the set of admissible displacement fields, we fix a function $u_0\in W_2^1(\Omega,R^m)$ and a measurable subset $\Gamma_0\subset\partial\Omega$ (the variants $\Gamma_0=\varnothing$ and $\Gamma_0=\partial\Omega$ are also possible). We consider the set of admissible displacement fields

$$\mathbb{X}(\Omega)=\{u\in W_2^1(\Omega,R^m):(u-u_0)|_{\Gamma_0}=0\}. \tag{3}$$

The set of *admissible phase distributions* is the set of measurable characteristic functions:

$$\mathbb{Z}(\Omega)=\{\chi\in L_\infty(\Omega):\ \chi(x)=\chi^2(x)\,\text{almost everywhere in}\,\Omega\}. \tag{4}$$

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By an *equilibrium state* of a two-phase elastic medium for a given t we mean the pair $\widehat{u}_t$, $\widehat{\chi}_t$ minimizing the energy functional

$$I[\widehat{u}_t, \widehat{\chi}_t, t, \Omega] = \inf_{u \in \mathbb{X}(\Omega), \chi \in \mathbb{Z}(\Omega)} I[u, \chi, t, \Omega], \quad \widehat{u}_t \in \mathbb{X}(\Omega), \quad \widehat{\chi}_t \in \mathbb{Z}(\Omega). \tag{5}$$

We say that an equilibrium state is a *one-phase state* if

$$\widehat{\chi}_t = \chi^+ \equiv 1 \quad \text{or} \quad \widehat{\chi}_t = \chi^- \equiv 0 \tag{6}$$

and a *two-phase state* in the opposite case.

Under some additional conditions, for the variational problem (5) the existence of temperatures of phase transitions $t_\pm = t_\pm(\Omega)$ was established in [2]

$$-\infty < t_- \leqslant t_+ < \infty, \tag{7}$$

which was characterized as follows:

$$\begin{aligned} &\text{for } t < t_- \text{ a unique solution to the problem (5) is the pair } \widehat{u}_t = u^+,\ \widehat{\chi}_t = \chi^+,\\ &\text{for } t > t_+ \text{ a unique solution to the problem (5) is the pair } \widehat{u}_t = u^-,\ \widehat{\chi}_t = \chi^-,\\ &\text{for } t \in (t_-, t_+) \text{ the problem (5) has no one-phase equilibrium states,} \end{aligned} \tag{8}$$

where $\widehat{u}^\pm$ are unique solutions to the variational problems

$$I[\widehat{u}^\pm, \chi^\pm, t, \Omega] = \inf_{u \in \mathbb{X}(\Omega)} I[u, \chi^\pm, t, \Omega], \quad \widehat{u}^\pm \in \mathbb{X}(\Omega). \tag{9}$$

Some sufficient conditions for the coincidence of $t_\pm$ can be found in [3].

The following question arises: whether the phase transition temperatures are determined by the characteristics of a two-phase medium ($a^\pm_{ijkl}$, $\zeta^\pm_{ij}$) and external actions (Γ_0, u_0, g, f). How the size of Ω affects the phase transition temperatures?

As was established in [4], in the case

$$\Gamma_0 = \varnothing, \quad g = 0, \quad f = pn \tag{10}$$

(here, n is the unit outward normal to $\partial\Omega$ and p is a parameter) the temperatures $t_\pm$ depend only on $a^\pm_{ijkl}$, $\zeta^\pm_{ij}$, p, but are independent of Ω.

It is reasonable to suggest that the phase transition temperatures are independent of Ω under the following conditions:

$$u_0 = 0, \quad \Gamma_0 = \partial\Omega, \quad g = 0 \tag{11}$$

since for an isotropic two-phase media the phase transition temperatures are explicitly expressed and are independent of Ω (cf. [5]). We confirm this suggestion in this paper.

Theorem. *Under the conditions* (11) *for the variational problem* (5), *there exist phase transition temperatures* $t_\pm$ *that are independent of the domain* Ω.

Proof. Under the condition (11), the set of admissible displacement fields (3) has the form

$$\mathbb{X}(\Omega) = \overset{\circ}{W}{}^1_2(\Omega, R^m). \tag{12}$$

Therefore, the solutions to the problem (9) are given by the equalities $\widehat{u}^{\pm} = 0$. Since these functions belong to the space $W^1_\infty(\Omega, R^m)$, according to the results of [2] there exit the phase transition temperatures $t_\pm$.

We introduce the functions

$$
\begin{aligned}
&i^+(t,\Omega) = \inf_{u\in\mathbb{X}(\Omega)} I[u,\chi^+,t,\Omega] = |\Omega|(a^+_{ijkl}\zeta^+_{ij}\zeta^+_{kl} + t),\\
&i^-(t,\Omega) = \inf_{u\in\mathbb{X}(\Omega)} I[u,\chi^-,t,\Omega] = |\Omega| a^-_{ijkl}\zeta^-_{ij}\zeta^-_{kl},\\
&i_{\min}(t,\Omega) = \min\{i^+(t,\Omega), i^-(t,\Omega)\} = |\Omega| \begin{cases} a^+_{ijkl}\zeta^+_{ij}\zeta^+_{kl} + t, & t \leqslant t^*,\\ a^-_{ijkl}\zeta^-_{ij}\zeta^-_{kl}, & t \geqslant t^*,\end{cases}\\
&t^* = a^-_{ijkl}\zeta^-_{ij}\zeta^-_{kl} - a^+_{ijkl}\zeta^+_{ij}\zeta^+_{kl}, \qquad i(t,\Omega) = \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}(\Omega)} I[u,\chi,t,\Omega],
\end{aligned}
\tag{13}
$$

where $|\Omega|$ is the measure of Ω. By the definitions (13), we have

$$i(t,\Omega) \leqslant i_{\min}(t,\Omega). \tag{14}$$

We set

$$L(\Omega) = \{t \in R^1 : i(t,\Omega) < i_{\min}(t,\Omega)\}. \tag{15}$$

The inclusion $t \in L(\Omega)$ is equivalent to the fact that for a given t the functional $I[u,\chi,t,\Omega]$ has no one-phase equilibrium states. By the definition (8), the existence of the temperatures $t_\pm$ imply

$$L(\Omega) = (t_-(\Omega), t_+(\Omega)). \tag{16}$$

In the case $t_-(\Omega) = t_+(\Omega)$, the set $L(\Omega)$ is empty.

Let us prove the following relations:

$$
\begin{aligned}
&L(\Omega_e) = L(\Omega), \quad \Omega_e = \{x + e : x \in \Omega,\ e \text{ is a fixed vector in } R^m\},\\
&L(\Omega^\lambda) = L(\Omega), \quad \Omega^\lambda = \{\lambda x : x \in \Omega,\ \lambda \text{ is a fixed number in } (0,\infty)\},\\
&L(\Omega') \supset L(\Omega) \text{ for an arbitrary bounded domain } \Omega' \subset R^m,\ \Omega' \supset \Omega.
\end{aligned}
\tag{17}
$$

Making the change of variables, we find

$$
\begin{aligned}
&\int_{\Omega_e} \{\chi(F^+(\nabla u) + t) + (1-\chi)F^-(\nabla u)\}\, dx = \int_{\Omega} \{\widetilde{\chi}(F^+(\nabla \widetilde{u}) + t) + (1-\widetilde{\chi})F^-(\nabla \widetilde{u})\}\, d\widetilde{x}\\
&x \in \Omega_e, \quad \widetilde{x} \in \Omega, \quad x = \widetilde{x} + e, \quad \chi(x) = \widetilde{\chi}(\widetilde{x}), \quad u(x) = \widetilde{u}(\widetilde{x}), \quad \widetilde{u} \in \mathbb{X}(\Omega), \quad \widetilde{\chi} \in \mathbb{Z}(\Omega).
\end{aligned}
\tag{18}
$$

Since any pair $u \in \mathbb{X}(\Omega_e)$, $\chi \in \mathbb{Z}(\Omega_e)$ can be obtained from the pair $\widetilde{u}$, $\widetilde{\chi}$ according to (18), we have $i(t,\Omega_e) = i(t,\Omega)$. By the definition (13) of $i_{\min}$, we have a similar equality $i_{\min}(t,\Omega_e) = i_{\min}(t,\Omega)$. Then

$$i_{\min}(t,\Omega_e) - i(t,\Omega_e) = i_{\min}(t,\Omega) - i(t,\Omega), \tag{19}$$

which implies the first assertion in (17).

Making the change of variables, we get

$$\int_{\Omega^\lambda} \{\chi(F^+(\nabla u) + t) + (1-\chi)F^-(\nabla u)\}\, dx = \lambda^m \int_{\Omega} \{\widetilde{\chi}(F^+(\nabla \widetilde{u}) + t) + (1-\widetilde{\chi})F^-(\nabla \widetilde{u})\}\, d\widetilde{x} \tag{20}$$

$$x \in \Omega^\lambda, \quad \widetilde{x} \in \Omega, \quad x = \lambda \widetilde{x}, \quad \chi(x) = \widetilde{\chi}(\widetilde{x}), \quad u(x) = \lambda \widetilde{u}(\widetilde{x}), \quad \widetilde{u} \in \mathbb{X}(\Omega), \quad \widetilde{\chi} \in \mathbb{Z}(\Omega).$$

Since any pair $u \in \mathbb{X}(\Omega^\lambda)$, $\chi \in \mathbb{Z}(\Omega^\lambda)$ can be obtained from the pair $\widetilde{u}$, $\widetilde{\chi}$ according to (20), we have $i(t, \Omega^\lambda) = \lambda^m i(t, \Omega)$. By the definition (13) of $i_{\min}$, we have the similar equality $i_{\min}(t, \Omega^\lambda) = \lambda^m i_{\min}(t, \Omega)$. Then

$$i_{\min}(t, \Omega^\lambda) - i(t, \Omega^\lambda) = \lambda^m (i_{\min}(t, \Omega) - i(t, \Omega)), \tag{21}$$

which implies the second assertion in (17).

For an arbitrary bounded domain $\Omega \subset \Omega' \subset R^m$ and functions $u \in \mathbb{X}(\Omega)$, $\chi \in \mathbb{Z}(\Omega)$ we set

$$u'(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \Omega' \setminus \Omega, \end{cases} \qquad \chi'(x) = \begin{cases} \chi(x), & x \in \Omega, \\ 0, & x \in \Omega' \setminus \Omega. \end{cases} \tag{22}$$

Then $u' \in \mathbb{X}(\Omega')$, $\chi' \in \mathbb{Z}(\Omega')$ and

$$\int\limits_{\Omega'} \{\chi'(F^+(\nabla u') + t) + (1 - \chi')F^-(\nabla u')\}\, dx$$
$$= \int\limits_{\Omega} \{\chi(F^+(\nabla u) + t) + (1 - \chi)F^-(\nabla u)\}\, dx + |\Omega' \setminus \Omega| a^-_{ijkl} \zeta^-_{ij} \zeta^-_{kl}. \tag{23}$$

We estimate from below the left-hand side of (23) by $i(t, \Omega')$. Minimizing the right-hand side of the obtained inequality over all $u \in \mathbb{X}(\Omega)$, $\chi \in \mathbb{Z}(\Omega)$, we find

$$i(t, \Omega') \leqslant i(t, \Omega) + |\Omega' \setminus \Omega| a^-_{ijkl} \zeta^-_{ij} \zeta^-_{kl}. \tag{24}$$

Instead of (22), we consider

$$u'(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \Omega' \setminus \Omega, \end{cases} \qquad \chi'(x) = \begin{cases} \chi(x), & x \in \Omega, \\ 1, & x \in \Omega' \setminus \Omega. \end{cases} \tag{25}$$

Then $u' \in \mathbb{X}(\Omega')$, $\chi' \in \mathbb{Z}(\Omega')$ and

$$\int\limits_{\Omega'} \{\chi'(F^+(\nabla u') + t) + (1 - \chi')F^-(\nabla u')\}\, dx$$
$$= \int\limits_{\Omega} \{\chi(F^+(\nabla u) + t) + (1 - \chi)F^-(\nabla u)\}\, dx + |\Omega' \setminus \Omega| (a^+_{ijkl} \zeta^+_{ij} \zeta^+_{kl} + t). \tag{26}$$

Consequently,

$$i(t, \Omega') \leqslant i(t, \Omega) + |\Omega' \setminus \Omega| (a^+_{ijkl} \zeta^+_{ij} \zeta^+_{kl} + t). \tag{27}$$

Combining the inequalities (24) and (27), we conclude that

$$i(t, \Omega') \leqslant i(t, \Omega) + |\Omega' \setminus \Omega| \min\{a^+_{ijkl} \zeta^+_{ij} \zeta^+_{kl} + t, a^-_{ijkl} \zeta^-_{ij} \zeta^-_{kl}\} = i(t, \Omega) + i_{\min}(t, \Omega') - i_{\min}(t, \Omega). \tag{28}$$

From the estimate (28) we find

$$i_{\min}(t, \Omega) - i(t, \Omega) \leqslant i_{\min}(t, \Omega') - i(t, \Omega'), \tag{29}$$

which implies the third equation in (17).

To complete the proof of the theorem, we shift the domain Ω by a vector e in such a way that the origin belongs to Ω_e. We fix two balls $B_r(0)$ and $B_R(0)$ by the condition $B_r(0) \subset \Omega_e \subset B_R(0)$. By (17), we have

$$L(B_r(0)) \subset L(\Omega_e) \subset L(B_R(0)), \quad L(\Omega_e) = L(\Omega), \quad L(B_r(0)) = L(B_R(0)) = L(B),$$

where B is the unit ball in R^m centered at the origin. Then $L(\Omega) = L(B)$. Hence, under the condition (11), the temperatures of phase transitions for the functional (1) in an arbitrary bounded domain Ω coincide with the temperatures of phase transitions for the same functional in B. The theorem is proved. □

Remark. Under the condition (11), the estimates for the phase transition temperatures obtained in [6] imply that the temperatures coincide if and only if

$$a^+_{iikl}\zeta^+_{kl} = a^-_{iikl}\zeta^-_{kl}, \quad a^+_{ijkl}\zeta^+_{kl} = a^-_{ijkl}\zeta^-_{kl} \tag{30}$$

respectively. We note that, under the condition (10), the temperatures of phase transitions always coincide (cf. [4]).

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