Journal of Mathematical Sciences, Vol. 184, No. 4, July, 2012

APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS OF FUNCTIONS HAVING (α, ψ)- DERIVATIVES IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

R. Akgün

Balikesir University
10145, Balikesir, Turkey
rakgun@balikesir.edu.tr

V. Kokilashvili *

A. Razmadze Mathematical Institute
1, M. Aleksidze Str., Tbilisi 0193, Georgia
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186, Georgia
kokil@rmi.ge

UDC 517.9

We prove direct simultaneous and converse approximation theorems by trigonometric polynomials for functions f and (α, ψ)-derivatives of f in weighted Lebesgue spaces with variable exponent. Bibliography: 11 titles.

1 Introduction

Let $\boldsymbol{T} := [0, 2\pi]$, and let $\mathscr{P}(\boldsymbol{T})$ be the class of Lebesgue measurable functions $p(x) : \boldsymbol{T} \to (1, \infty)$ such that

$$1 < p_*(\boldsymbol{T}) := \operatorname*{ess\,inf}_{x \in \boldsymbol{T}} p(x) \leqslant p^* := \operatorname*{ess\,sup}_{x \in \boldsymbol{T}} p(x) < \infty.$$

A function $\omega : \boldsymbol{T} \to [0, \infty]$ is called a *weight* on $\boldsymbol{T}$ if it is a 2π-periodic, a.e. positive, and Lebesgue measurable function. We define the weighted variable exponent Lebesgue space $L_\omega^{p(\cdot)}$ as the collection of 2π-periodic Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ with the finite norm

$$\|f\|_{p(\cdot),\omega} := \inf \left\{ \alpha > 0 : \int\limits_{\boldsymbol{T}} |(f(x)/\alpha)\,\omega(x)|^{p(x)}\, dx \leqslant 1 \right\},$$

where $p \in \mathscr{P}(\boldsymbol{T})$. The space $L_{2\pi}^{p(\cdot)}$ is a Banach space.

For given $p \in \mathscr{P}(\boldsymbol{T})$ the class of weights ω satisfying the condition [1]

$$\|\omega \chi_Q\|_{p(\cdot),1} \left\|\omega^{-1} \chi_Q\right\|_{p'(\cdot),1} \leqslant C\,|Q|$$

* To whom the correspondence should be addressed.

Translated from *Problems in Mathematical Analysis* **65**, May, 2012, pp. 3-12.

for all balls Q in $\boldsymbol{T}$ is denoted by $A_{p(\cdot)}(\boldsymbol{T})$. Here, $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$. The variable exponent $p(x)$ is said to be *log-Hölder continuous* on $\boldsymbol{T}$ if there exists a constant $c \geqslant 0$ such that

$$|p(x_1) - p(x_2)| \leqslant \frac{c}{\log(e + 1/|x_1 - x_2|)} \quad \forall\, x_1, x_2 \in \boldsymbol{T}. \tag{1.1}$$

We denote by $\mathscr{P}^{\log}(\boldsymbol{T})$ the class of exponents $p \in \mathscr{P}(\boldsymbol{T})$ such that $1/p : \boldsymbol{T} \to [0,1]$ is log-Hölder continuous on $\boldsymbol{T}$.

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $f \in L_\omega^{p(\cdot)}$, then, as was proved in [1], the Hardy–Littlewood maximal function $\mathscr{M}$ is bounded in $L_\omega^{p(\cdot)}$ if and only if $\omega \in A_{p(\cdot)}(\boldsymbol{T})$.

Let $f \in L_\omega^{p(\cdot)}$, and let

$$\mathscr{A}_h f(x) := \frac{1}{h} \int\limits_{x-h/2}^{x+h/2} f(t)\, dt, \quad x \in \boldsymbol{T},$$

be the Steklov mean operator. If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, then $\mathscr{A}_h$ is bounded in $L_\omega^{p(\cdot)}$.

For $x, h \in \boldsymbol{T}$ and $0 \leqslant r$ we define

$$\sigma_h^r f(x) := (I - \mathscr{A}_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} (\mathscr{A}_h)^k,$$

where $f \in L_\omega^{p(\cdot)}$, Γ is the Gamma function, and I is the identity operator.

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, and $f \in L_\omega^{p(\cdot)}$, then

$$\|\sigma_h^r f\|_{p(\cdot),\omega} \leqslant c\, \|f\|_{p(\cdot),\omega}. \tag{1.2}$$

For $0 \leqslant r$ we can define the *fractional moduli of smoothness* for $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, and $f \in L_\omega^{p(\cdot)}$ by the formula

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{0<h_i,t\leqslant\delta} \left\| \prod_{i=1}^{[r]} (I - \mathscr{A}_{h_i})\, \sigma_t^{\{r\}} f \right\|_{p(\cdot),\omega}, \quad r \geqslant 1, \quad \delta \geqslant 0,$$

where

$$\Omega_0(f,\delta)_{p(\cdot),\omega} := \|f\|_{p(\cdot),\omega},$$

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{0<t\leqslant\delta} \|\sigma_t^r f\|_{p(\cdot),\omega}, \quad 0 < r < 1,$$

$[r]$ denotes the integer part of the nonnegative real number r and $\{r\} := r - [r]$.

In this case, for $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, and $f \in L_\omega^{p(\cdot)}$ we have

$$\Omega_r(f,\delta)_{p(\cdot),\omega} \leqslant c\, \|f\|_{p(\cdot),\omega},$$

where the constant $c > 0$ depends only on r and p.

Remark 1.1. The modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot),\omega}$, $r \in \mathbb{R}^+$ has the following properties for $p \in P^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, and $f \in L_\omega^{p(\cdot)}$:

(i) $\Omega_r(f,\delta)_{p(\cdot),\omega}$ is a nonnegative and nondecreasing function of $\delta \geqslant 0$,

(ii) $\Omega_r(f_1+f_2,\cdot)_{p(\cdot),\omega} \leqslant \Omega_r(f_1,\cdot)_{p(\cdot),\omega} + \Omega_r(f_2,\cdot)_{p(\cdot),\omega}$,

(iii) $\lim\limits_{\delta\to 0^+} \Omega_r(f,\delta)_{p(\cdot),\omega} = 0$.

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, then $\omega^{p(x)} \in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense [2] in $L_\omega^{p(\cdot)}$. This allows us to consider approximation problems in $L_\omega^{p(\cdot)}$. Approximation by trigonometric polynomials in $L_\omega^{p(\cdot)}$ was considered in [3]–[8] In [9, 10], on the basis of the transformed Fourier series, the so-called lambda derivatives were introduced and inequalities are obtained in a refined form like the Besov and Timan inequalities.

On the other hand, if $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $L_\omega^{p(\cdot)} \subset L^1(\boldsymbol{T})$. For given $f \in L_\omega^{p(\cdot)}$ we introduce the *Fourier series* and the *conjugate Fourier series* of f by the formulas

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx\right) \tag{1.3}$$

and

$$\widetilde{f}(x) \backsim \sum_{k=1}^{\infty} \left(a_k(f)\sin kx - b_k(f)\cos kx\right).$$

We say that a function $f \in L_\omega^{p(\cdot)}$, $p \in \mathscr{P}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, has a (α,ψ)-derivative f_α^ψ if for a given sequence $\psi(k), k = 1,2,\ldots$ and a number $\alpha \in \mathbb{R}$ the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f)\cos k\left(x + \frac{\alpha\pi}{2k}\right) + b_k(f)\sin k\left(x + \frac{\alpha\pi}{2k}\right)\right) \tag{1.4}$$

is the Fourier series of the function f_α^ψ. Taking $\psi(k) = k^{-\alpha}$, $k = 1,2,\ldots$, $\alpha \in \mathbb{R}^+$, we have the fractional derivative $f^{(\alpha)}$ of f in the sense of Weyl. Taking $\psi(k) = k^{-\alpha}\ln^{-\beta} k$, $k = 1,2,\ldots$, $\alpha,\beta \in \mathbb{R}^+$, we have the power logarithmic-fractional derivative $f^{(\alpha,\beta)}$ of f.

Let $\mathfrak{M}$ be the set of functions $\psi(v)$ that are convex downwards for any $v \geqslant 1$ and satisfy the condition $\lim_{v\to\infty}\psi(v) = 0$.

We associate every function $\psi \in \mathfrak{M}$ with a pair of functions $\eta(t) = \psi^{-1}(\psi(t)/2)$ and $\mu(t) = t/(\eta(t) - t)$.

We set

$$\mathfrak{M}_0 := \{\psi \in \mathfrak{M} : 0 < \mu(t) \leqslant K\}.$$

We define

$$E_n(f)_{p(\cdot),\omega} := \inf_{T\in\mathscr{T}_n} \|f - T\|_{p(\cdot),\omega}, \quad n = 0,1,2,\ldots, \quad f \in L_\omega^{p(\cdot)},$$

where $\mathscr{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

Theorem 1.1. *Let $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in \mathbb{R}$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \to 0$ as $k \to \infty$, then for every $n = 0,1,2,3,\ldots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n+1)\, E_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega}.$$

Corollary 1.1. *Under the assumptions of Theorem* 1.1*,*

$$E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n+1)\left\|f_\alpha^\psi\right\|_{p(\cdot),\omega}$$

with a constant $c>0$ independent of n.

Using Theorem 1.1 and Theorem 1.4 in [4], we have the following Jackson type direct theorem.

Theorem 1.2. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in \mathbb{R}$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \to 0$ as $k \to \infty$, then for every $n = 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n+1)\,\Omega_r\left(f_\alpha^\psi, \frac{1}{n}\right)_{p(\cdot),\omega}.$$

Theorem 1.3. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, and*

$$\sum_{\nu=1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} < \infty,$$

then $f_\alpha^\psi \in L_\omega^{p(\cdot)}$ and

$$E_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega} \leqslant c\left((\psi(n))^{-1} E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega}\right),$$

where the constant $c > 0$ depends only on α and p.

Corollary 1.2. *Under the assumptions of Theorem* 1.3*, if $r \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} (\nu\psi(v))^{-1} E_\nu(f)_{p(\cdot),\omega} < \infty,$$

there exist constants $c, C > 0$ depending only on ψ, r, and p such that

$$\Omega_r\left(f_\alpha^\psi, \frac{1}{n}\right)_{p(\cdot),\omega} \leqslant \frac{c}{n^r}\sum_{\nu=0}^{n} \nu^{r-1}(\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} + C\sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega}.$$

Theorem 1.4. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in [0, \infty)$, and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \to 0$ as $k \to \infty$, then there exists $T \in \mathscr{T}_n$, $n = 1, 2, 3, \ldots$ and a constant $c > 0$ depending only on α and p such that*

$$\left\|f_\alpha^\psi - T_\alpha^\psi\right\|_{p(\cdot),\omega} \leqslant cE_n\left(f_\alpha^\psi\right)_{p(\cdot),\omega}.$$

In the particular case $\psi(k) = k^{-\alpha}\ln^{-\beta} k$, $k = 1, 2, \ldots$, $\alpha, \beta \in \mathbb{R}^+$, we have the following new results for power logarithmic-fractional derivatives.

Theorem 1.5. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha, \beta \in \mathbb{R}$, and $f, f^{(\alpha,\beta)} \in L_\omega^{p(\cdot)}$, then for every $n = 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant \frac{c}{n^\alpha \ln^\beta (n+1)} E_n\big(f^{(\alpha,\beta)}\big)_{p(\cdot),\omega}.$$

Corollary 1.3. *Under the assumptions of Theorem* 1.5*,*

$$E_n(f)_{p(\cdot),\omega} \leqslant \frac{c}{n^\alpha \ln^\beta (n+1)} \big\|f^{(\alpha,\beta)}\big\|_{p(\cdot),\omega}$$

with a constant $c > 0$ independent of n.

Theorem 1.6. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha, \beta \in \mathbb{R}$, and $f, f^{(\alpha,\beta)} \in L_\omega^{p(\cdot)}$, then for every $n = 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant \frac{c}{n^\alpha \ln^\beta (n+1)} \Omega_r \left(f^{(\alpha,\beta)}, \frac{1}{n}\right)_{p(\cdot),\omega}.$$

Theorem 1.7. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in \mathbb{R}$, and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega} < \infty,$$

then $f^{(\alpha,\beta)} \in L_\omega^{p(\cdot)}$ and

$$E_n\big(f^{(\alpha,\beta)}\big)_{p(\cdot),\omega} \leqslant c \left(n^\alpha \ln^\beta n E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega} \right),$$

where the constant $c > 0$ depends only on α, β, and p.

Corollary 1.4. *Under the assumptions of Theorem* 1.7*, if $r \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega} < \infty,$$

there exist constants $c, C > 0$ depending only on α,β, r, and p such that

$$\Omega_r \left(f^{(\alpha,\beta)}, \frac{1}{n}\right)_{p(\cdot),\omega} \leqslant \frac{c}{n^r} \sum_{\nu=1}^{n} \nu^{r+\alpha-1} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega} + C \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega}.$$

Theorem 1.8. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in [0, \infty)$, and $f, f^{(\alpha,\beta)} \in L_\omega^{p(\cdot)}$, then there exists $T \in \mathscr{T}_n$, $n = 1, 2, 3, \ldots$ and a constant $c > 0$ depending only on α and p such that*

$$\big\|f^{(\alpha,\beta)} - T^{(\alpha,\beta)}\big\|_{p(\cdot),\omega} \leqslant c E_n\big(f^{(\alpha,\beta)}\big)_{p(\cdot),\omega}.$$

Theorem 1.7 and Corollary 1.4 were proved in L^p ($\omega \equiv 1$, constant $p \in (1, \infty)$) in [11].

2 Auxiliary Results

We define the nth partial sum of (1.3)

$$S_n(f) := S_n(x,f) := \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f)\cos kx + b_k(f)\sin kx), \quad n = 0,1,2,\dots .$$

Lemma 2.1. *[7] If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, and $f \in L_\omega^{p(\cdot)}$, then there are constants $c, C > 0$ such that*

$$\left\|\widetilde{f}\right\|_{p(\cdot),\omega} \leqslant c\|f\|_{p(\cdot),\omega} \tag{2.1}$$

and

$$\|S_n(\cdot,f)\|_{p(\cdot),\omega} \leqslant C\|f\|_{p(\cdot),\omega}, \quad n = 1,2,\dots . \tag{2.2}$$

Remark 2.1. [4] Under the assumptions of Lemma 2.1, there exists a constant $c > 0$ such that

$$\|f - S_n(\cdot,f)\|_{p(\cdot),\omega} \leqslant cE_n(f)_{p(\cdot),\omega} \asymp E_n\left(\widetilde{f}\right)_{p(\cdot),\omega}.$$

Definition 2.1. Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary sequence, and $\alpha \in \mathbb{R}$. We write $(\alpha,\psi) \in B$ if

$$\left\|(T_n)_\alpha^\psi\right\|_{p(\cdot),\omega} \leqslant c(\psi(n))^{-1}\|T_n\|_{p(\cdot),\omega}$$

for any $T_n \in \mathscr{T}_n$, where the constant c is independent of n.

Proposition 2.1. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, and ψ satisfies*

$$\sup_q \sum_{k=2^q}^{2^{q+1}} \left|(\psi_n(k+1))^{-1} - (\psi_n(k))^{-1}\right| \leqslant C\lambda_n, \tag{2.3}$$

where

$$(\psi_n(k))^{-1} = \begin{cases} (\psi(k))^{-1}, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

and

$$\lambda_n = \max_k \left|(\psi_n(k))^{-1}\right| = \max_{k \leqslant n} |\psi(k)|^{-1}. \tag{2.4}$$

Then

$$\left\|(T_n)_\alpha^\psi\right\|_{p(\cdot),\omega} \leqslant c\lambda_n \left\|T_n\right\|_{p(\cdot),\omega},$$

where the constant c depends only on ψ and p.

Proof. We can write

$$\begin{aligned}(T_n)_\alpha^\psi &= \sum_{k=1}^{n} \frac{1}{\psi(k)}\left(a_k(f)\cos k\left(x + \frac{\alpha\pi}{2k}\right) + b_k(f)\sin k\left(x + \frac{\alpha\pi}{2k}\right)\right) \\ &= \sum_{k=1}^{n} \frac{1}{\psi(k)} A_k\left(T_n, x + \frac{\alpha\pi}{2k}\right) = \sum_{k=1}^{n} \frac{1}{\psi(k)}\left\{\cos\frac{\alpha\pi}{2} A_k(T_n,x) - \sin\frac{\alpha\pi}{2} A_k\left(\widetilde{T}_n, x\right)\right\}.\end{aligned}$$

We define the multipliers

$$\mu_k = \begin{cases} (\psi(k))^{-1} \cos \frac{\alpha\pi}{2}, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \quad k = 0, \end{cases}$$

$$\widetilde{\mu}_k = \begin{cases} (\psi(k))^{-1} \sin \frac{\alpha\pi}{2}, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \quad k = 0, \end{cases}$$

and operators

$$(AT_n)(x) = \sum_{k=1}^{n} \frac{1}{\psi(k)} \cos \frac{\alpha\pi}{2} A_k(T_n, x),$$

$$\left(\widetilde{A\widetilde{T}_n}\right)(x) = \sum_{k=1}^{n} \frac{1}{\psi(k)} \sin \frac{\alpha\pi}{2} A_k\left(\widetilde{T}_n, x\right).$$

Therefore,

$$(T_n)_\alpha^\psi(\cdot) = (AT_n)(\cdot) - \left(\widetilde{A\widetilde{T}_n}\right)(\cdot).$$

Using (2.3) and (2.4) we get

$$\sup_k |\mu_k| \leqslant \lambda_n, \quad \sup_k |\overline{\mu}_k| \leqslant \lambda_n,$$

$$\sup_q \sum_{k=2^q}^{2^{q+1}} |\mu(k+1) - \mu(k)| \leqslant C\lambda_n,$$

$$\sup_q \sum_{k=2^q}^{2^{q+1}} |\overline{\mu}(k+1) - \overline{\mu}(k)| \leqslant C\lambda_n.$$

Applying the Marcinkiewicz multiplier theorem for weighted variable exponent Lebesgue spaces [7], we find

$$\begin{aligned} \left\|(T_n)_\alpha^\psi\right\|_{p(\cdot),\omega} &= \left\|(AT_n) - (\widetilde{A\widetilde{T}_n})\right\|_{p(\cdot),\omega} \leqslant \left\|AT_n\right\|_{p(\cdot),\omega} + \left\|\widetilde{A\widetilde{T}_n}\right\|_{p(\cdot),\omega} \\ &\leqslant C\lambda_n \Big(\Big\| \sum_{k=1}^{n} A_k(T_n, x) \Big\|_{p(\cdot),\omega} + \Big\| \sum_{k=1}^{n} A_k\big(\widetilde{T}_n, x\big) \Big\|_{p(\cdot),\omega} \Big). \end{aligned}$$

By the boundedness (2.1) of the conjugate operator, we obtain the desired inequality

$$\left\|(T_n)_\alpha^\psi\right\|_{p(\cdot),\omega} \leqslant C\lambda_n \Big\| \sum_{k=1}^{n} A_k(T_n, x) \Big\|_{p(\cdot),\omega} = C\lambda_n \left\|T_n\right\|_{p(\cdot),\omega}. \qquad \square$$

Proposition 2.1 yields the following corollary which, in fact, is a generalized Bernstein inequality.

Corollary 2.1. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*(\boldsymbol{T}))$, $\alpha \in \mathbb{R}$, $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers, and $T_n \in \mathscr{T}_n$, then $(\alpha, \psi) \in B$.*

The proof follows from $\lambda_n = (\psi(n))^{-1}$.

3 Proofs

Proof of Theorem 1.1. We set

$$A_k(x,f) := a_k \cos kx + b_k \sin kx.$$

Since the set of trigonometric polynomials is dense in $L_\omega^{p(\cdot)}$, for given $f \in L_\omega^{p(\cdot)}$ we have

$$E_n(f)_{p(\cdot),\omega} \to 0 \quad \text{as } n \to \infty.$$

From the first inequality in Remark 2.1 we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x,f)$$

in the norm $\|\cdot\|_{p(\cdot),\omega}$. For $k = 1,2,3,\ldots$ we know that

$$\begin{aligned} A_k(x,f) &= a_k \cos k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) + b_k \sin k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) \\ &= A_k\left(x + \frac{\alpha\pi}{2k}, f\right)\cos\frac{\alpha\pi}{2} + A_k\left(x + \frac{\alpha\pi}{2k}, \widetilde{f}\right)\sin\frac{\alpha\pi}{2} \end{aligned}$$

and

$$A_k\left(x, f_\alpha^\psi\right) = \frac{1}{\psi(k)} A_k\left(x + \frac{\alpha\pi}{2k}, f\right).$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{\infty} A_k(x,f) &= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, f\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, \widetilde{f}\right) \\ &= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\psi(k) A_k(x, f_\alpha^\psi) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\psi(k) A_k(x, (\widetilde{f})_\alpha^\psi). \end{aligned}$$

Hence

$$f(\cdot) - S_n(\cdot, f) = \cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\psi(k)A_k(\cdot, f_\alpha^\psi) + \sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\psi(k)A_k(\cdot, (\widetilde{f})_\alpha^\psi).$$

Since

$$\begin{aligned} \sum_{k=n+1}^{\infty}\psi(k)A_k(\cdot, f_\alpha^\psi) &= \sum_{k=n+1}^{\infty}\psi(k)[(S_k(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) - (S_{k-1}(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot))] \\ &= \sum_{k=n+1}^{\infty}(\psi(k) - \psi(k+1))(S_k(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) - \psi(n+1)(S_n(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty}\psi(k)A_k(\cdot, (\widetilde{f})_\alpha^\psi) &= \sum_{k=n+1}^{\infty}(\psi(k) - \psi(k+1))(S_k(\cdot, (\widetilde{f})_\alpha^\psi) - (\widetilde{f})_\alpha^\psi(\cdot)) \\ &\quad - \psi(n+1)(S_n(\cdot, (\widetilde{f})_\alpha^\psi) - (\widetilde{f})_\alpha^\psi(\cdot)) \end{aligned}$$

we obtain

$$\begin{aligned}\|f(\cdot)-S_n(\cdot,f\|_{p(\cdot),\omega} &\leqslant \sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))\|S_k(\cdot,f_\alpha^\psi)-f_\alpha^\psi(\cdot)\|_{p(\cdot),\omega}\\ &\quad+\psi(n+1)\|S_n(\cdot,f_\alpha^\psi)-f_\alpha^\psi(\cdot)\|_{p(\cdot),\omega}+\sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))\\ &\quad\times\|S_k(\cdot,(\widetilde{f})_\alpha^\psi)-(\widetilde{f})_\alpha^\psi(\cdot)\|_{p(\cdot),\omega}+\psi(n+1)\|S_n(\cdot,(\widetilde{f})_\alpha^\psi)-(\widetilde{f})_\alpha^\psi(\cdot)\|_{p(\cdot),\omega}\\ &\leqslant c\left[\sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))E_k(f_\alpha^\psi)_{p(\cdot),\omega}+\psi(n+1)E_n(f_\alpha^\psi)_{p(\cdot),\omega}\right]\\ &\quad+c\left[\sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))E_k((\widetilde{f})_\alpha^\psi)_{p(\cdot),\omega}+\psi(n+1)E_n((\widetilde{f})_\alpha^\psi)_{p(\cdot),\omega}\right].\end{aligned}$$

Consequently, from the equivalence in Remark 2.1 we have

$$\begin{aligned}&\|f(\cdot)-S_n(\cdot,f)\|_{p(\cdot),\omega}\\ &\quad\leqslant c\left[\sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))+\psi(n+1)\right]\{E_k(f_\alpha^\psi)_{p(\cdot),\omega}+E_n(\widetilde{f}_\alpha^\psi)_{p(\cdot),\omega}\}\\ &\quad\leqslant cE_n(f_\alpha^\psi)_{p(\cdot),\omega}\left[\sum_{k=n+1}^{\infty}(\psi(k)-\psi(k+1))+\psi(n+1)\right]\leqslant c\psi(n+1)E_n(f_\alpha^\psi)_{p(\cdot),\omega}. \qquad \square\end{aligned}$$

Proof of Theorem 1.3. Let T_n be the best approximating polynomial for $f\in L_\omega^{p(\cdot)}$. We set $n_0=n, n_1:=[\eta(n)]+1,\ldots,n_k:=[\eta(n_{k-1})]+1,\ldots$. In this case, the series

$$T_{n_0}(\cdot)+\sum_{k=1}^{\infty}\left(T_{n_k}(\cdot)-T_{n_{k-1}}(\cdot)\right)$$

converges to f in the $L_\omega^{p(\cdot)}$-norm. We consider the series

$$(T_{n_0}(\cdot))_\alpha^\psi+\sum_{k=1}^{\infty}\left(T_{n_k}(\cdot)-T_{n_{k-1}}(\cdot)\right)_\alpha^\psi. \tag{3.1}$$

Applying the generalized Bernstein inequality (Corollary 2.1) to the difference $u_k(\cdot):=T_{n_k}(\cdot)-T_{n_{k-1}}(\cdot)$, we get

$$\begin{aligned}\|(u_k)_\alpha^\psi\|_{p(\cdot),\omega}\leqslant c\frac{\|u_k\|_{p(\cdot),\omega}}{\psi(n_k)} &\leqslant c\frac{(\|T_{n_k}-f\|_{p(\cdot),\omega}+\|T_{n_{k-1}}-f\|_{p(\cdot),\omega})}{\psi(n_k)}\\ &\leqslant cE_{n_{k-1}+1}(f)_{p(\cdot),\omega}(\psi(n_k))^{-1}.\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty}\left\|(u_k)_\alpha^\psi\right\|_{p(\cdot),\omega}\leqslant c\left(E_{n+1}(f)_{p(\cdot),\omega}(\psi(n))^{-1}+\sum_{k=1}^{\infty}E_{n_k+1}(f)_{p(\cdot),\omega}(\psi(n_k))^{-1}\right).$$

Let $z = \overline{\eta}(\tau) := \psi^{-1}(2\psi(t))$ for $\tau \geqslant \eta(1)$. Since $\psi \in \mathfrak{M}_0$ we get

$$\frac{\tau}{\tau - \overline{\eta}(\tau)} = \frac{\eta(z)}{\eta(z) - z} = 1 + \frac{z}{\eta(z) - z} = 1 + \mu(\psi, z) \leqslant c,$$

and for $\tau \in [t, \eta(t)]$, $\tau \geqslant \eta(1)$

$$\frac{\tau - \overline{\eta}(\tau)}{\eta(t) - t} \leqslant \frac{\tau\eta(t)}{\eta(t)(\eta(t) - t)} = \frac{\tau}{\eta(t)}\left(1 + \frac{t}{\eta(t) - t}\right)$$
$$\leqslant \frac{\tau}{\eta(t)}(1 + \mu(\psi, t)) \leqslant c\frac{\tau}{\eta(t)} \leqslant c.$$

Then $\psi(\tau) \geqslant \psi(\eta(t)) > \psi(\tau)/2$ for any $\tau \in [t, \eta(t)], \tau \geqslant \eta(1)$. Without loss of generality one can assume that $\eta(t) - t > 1$. In this case, we get

$$\frac{E_{n_k+1}(f)_{p(\cdot),\omega}}{\psi(n_k)} \leqslant C \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{\psi(v)} \frac{1}{\eta(n_{k-1}) - n_{k-1}}$$
$$\leqslant c \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{v\psi(v)} \frac{v}{(v - \overline{\eta}(v))} \frac{v - \overline{\eta}(v)}{\eta(n_{k-1}) - n_{k-1}} \leqslant \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{v\psi(v)}.$$

Therefore,

$$\sum_{k=1}^{\infty} \|(u_k)_\alpha^\psi\|_{p(\cdot),\omega} \leqslant c\left(E_{n+1}(f)_{p(\cdot),\omega}(\psi(n))^{-1} + \sum_{v=n+1}^{\infty} E_v(f)_{p(\cdot),\omega}(v\psi(v))^{-1}\right).$$

The right-hand side of the last inequality converges and, consequently, the series (3.1) converges in the norm to some function $S(\cdot)$ from $L_\omega^{p(\cdot)}$. Let $a_k^{(n)} := a_k(T_n)$ and $b_k^{(n)} := b_k(T_n)$, $k = 0, 1, 2, \ldots$, be coefficients of polynomials T_n. The corresponding coefficients $\alpha_k^{(n)}$, $\beta_k^{(n)}$ of the polynomials $(T_n)_\alpha^\psi$ have the form

$$\alpha_k^{(n)} = \frac{1}{\psi(k)}\left\{\cos\frac{\alpha\pi}{2}a_k^{(n)} + \sin\frac{\alpha\pi}{2}b_k^{(n)}\right\},$$

$$\beta_k^{(n)} = \frac{1}{\psi(k)}\left\{\cos\frac{\alpha\pi}{2}b_k^{(n)} - \sin\frac{\alpha\pi}{2}a_k^{(n)}\right\}.$$

Since $(T_n(\cdot))_\alpha^\psi \to S(\cdot)$ as $n \to \infty$, we have $\alpha_k^{(n)} \to a_k(S)$ and $\beta_k^{(n)} \to b_k(S)$ as $n \to \infty$ for $k = 0, 1, 2, \ldots$. Since $\alpha_k^{(n)} \to a_k(f)$ and $\beta_k^{(n)} \to b_k(f)$ as $n \to \infty$ for $k = 0, 1, 2, \ldots$, we have

$$a_k(S) = \frac{1}{\psi(k)}\left\{\cos\frac{\alpha\pi}{2}a_k^{(n)} + \sin\frac{\alpha\pi}{2}b_k^{(n)}\right\},$$

$$b_k(S) = \frac{1}{\psi(k)}\left\{\cos\frac{\alpha\pi}{2}b_k^{(n)} - \sin\frac{\alpha\pi}{2}a_k^{(n)}\right\}.$$

We conclude that the Fourier series of S has the form (1.4). This means that the function f has a (ψ, α)-derivative f_α^ψ of class $L_\omega^{p(\cdot)}$ and

$$f_\alpha^\psi = (T_n)_\alpha^\psi + \sum_{k=1}^{\infty}(u_k)_\alpha^\psi \tag{3.2}$$

in the $L^{p(\cdot)}_\omega$-norm. Therefore, from (3.2) it follows that

$$E_n(f^\psi_\alpha)_{p(\cdot),\omega} \leqslant c\left((\psi(n))^{-1}E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty}(\nu\psi(v))^{-1}E_\nu(f)_{p(\cdot),\omega}\right). \qquad \square$$

Proof of Corollary 1.2. Since [4]

$$\Omega_r\left(f,\frac{1}{n}\right)_{p(\cdot),\omega} \leqslant \frac{c}{n^r}\sum_{\nu=1}^{n}\nu^{r-1}E_\nu\left(f\right)_{p(\cdot),\omega},$$

using Theorem 1.3, we have

$$\Omega_r\left(f^\psi_\alpha,\frac{1}{n}\right)_{p(\cdot),\omega} \leqslant \frac{c}{n^r}\sum_{\nu=1}^{n}\nu^{r-1}E_\nu(f^\psi_\alpha)_{p(\cdot),\omega} \leqslant \frac{c}{n^r}\left\{\sum_{\nu=1}^{n}\nu^{r-1}\left(\psi\left(v\right)\right)^{-1}E_v\left(f\right)_{p(\cdot),\omega}\right.$$
$$\left.+\sum_{\nu=1}^{n}\nu^{r-1}\sum_{m=v+1}^{\infty}\left(m\psi\left(m\right)\right)^{-1}E_m\left(f\right)_{p(\cdot),\omega}\right\}.$$

Using the equality

$$\sum_{\nu=1}^{n}b_v\sum_{m=v}^{n}a_m = \sum_{m=1}^{n}a_m\sum_{v=1}^{m}b_v,$$

we get

$$\Omega_r\left(f^\psi_\alpha,\frac{1}{n}\right)_{p(\cdot),\omega} \leqslant \frac{c}{n^r}\sum_{\nu=0}^{n}\nu^{r-1}\left(\psi\left(v\right)\right)^{-1}E_\nu\left(f\right)_{p(\cdot),\omega} + C\sum_{\nu=n+1}^{\infty}\left(\nu\psi\left(v\right)\right)^{-1}E_\nu\left(f\right)_{p(\cdot),\omega}. \qquad \square$$

Proof of Theorem 1.4. We set

$$W_n(f) := W_n(\cdot,f) := \frac{1}{n+1}\sum_{\nu=n}^{2n}S_\nu(\cdot,f)$$

for $n = 0,1,2,\ldots$. Since $W_n(\cdot,f^\psi_\alpha) = (W_n(\cdot,f))^\psi_\alpha$, we have

$$\|f^\psi_\alpha(\cdot)-(S_n(\cdot,f))^\psi_\alpha\|_{p(\cdot),\omega} \leqslant \|f^\psi_\alpha(\cdot)-W_n(\cdot,f^\psi_\alpha)\|_{p(\cdot),\omega} + \|(S_n(\cdot,W_n(f)))^\psi_\alpha-(S_n(\cdot,f))^\psi_\alpha\|_{p(\cdot),\omega}$$
$$+\|(W_n(\cdot,f))^\psi_\alpha-(S_n(\cdot,W_n(f)))^\psi_\alpha\|_{p(\cdot),\omega} := I_1+I_2+I_3.$$

In this case, form the boundedness of S_n in $L^{p(\cdot)}_\omega$ we obtain the boundedness of W_n in $L^{p(\cdot)}_\omega$ and

$$I_1 \leqslant \|f^\psi_\alpha(\cdot)-S_n(\cdot,f^\psi_\alpha)\|_{p(\cdot),\omega} + \|S_n(\cdot,f^\psi_\alpha)-W_n(\cdot,f^\psi_\alpha)\|_{p(\cdot),\omega}$$
$$\leqslant cE_n(f^\psi_\alpha)_{p(\cdot),\omega} + \|W_n(\cdot,S_n(f^\psi_\alpha)-f^\psi_\alpha)\|_{p(\cdot),\omega} \leqslant cE_n(f^\psi_\alpha)_{p(\cdot),\omega}.$$

From Lemma 2.1 we get

$$I_2 \leqslant c(\psi(n))^{-1}\|S_n(\cdot,W_n(f))-S_n(\cdot,f)\|_{p(\cdot),\omega},$$
$$I_3 \leqslant c(\psi(n))^{-1}\|W_n(\cdot,f)-S_n(\cdot,W_n(f))\|_{p(\cdot),\omega} \leqslant c(\psi(n))^{-1}E_n(W_n(f))_{p(\cdot),\omega}.$$

Now, we have

$$\|S_n(\cdot,W_n(f))-S_n(\cdot,f)\|_{p(\cdot),\omega} \leqslant \|S_n(\cdot,W_n(f))-W_n(\cdot,f)\|_{p(\cdot),\omega} + \|W_n(\cdot,f)-f(\cdot)\|_{p(\cdot),\omega}$$

$$+\|f(\cdot)-S_n(\cdot,f)\|_{p(\cdot),\omega} \leqslant cE_n(W_n(f))_{p(\cdot),\omega}+cE_n(f)_{p(\cdot),\omega}+cE_n(f)_{p(\cdot),\omega}.$$

Since $E_n(W_n(f))_{p(\cdot),\omega} \leqslant cE_n(f)_{p(\cdot),\omega}$, we get

$$\|f_\alpha^\psi(\cdot)-(S_n(\cdot,f))_\alpha^\psi\|_{p(\cdot),\omega} \leqslant cE_n(f_\alpha^\psi)_{p(\cdot),\omega}+c(\psi(n))^{-1}E_n(W_n(f))_{p(\cdot),\omega}$$
$$+cE_n(f)_{p(\cdot),\omega} \leqslant cE_n(f_\alpha^\psi)_{p(\cdot),\omega}+c(\psi(n))^{-1}E_n(f)_{p(\cdot),\omega}.$$

Since $E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n+1)E_n(f_\alpha^\psi)_{p(\cdot),\omega}$ in view of Theorem 1.1, we obtain

$$\|f_\alpha^\psi(\cdot)-(S_n(\cdot,f))_\alpha^\psi\|_{p(\cdot),\omega} \leqslant cE_n(f_\alpha^\psi)_{p(\cdot),\omega}.$$ □

References

1. D. Cruz-Uribe, L. Diening, and P. Hästö, "The maximal operator on weighted variable Lebesgue spaces," *Fract. Calc. Appl. Anal.* **14**, No. 3, 361–374 (2011).

2. V. Kokilashvili and S. Samko, "Singular integrals in weighted Lebesgue spaces with variable exponent," *Georgian Math. J.* **10**, No. 1, 145–156 (2003).

3. R. Akgün, "Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent," *Ukrainian Math. J.* **63**, No. 1, 1–26 (2011).

4. R. Akgün, "Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth," *Georgian Math. J.* **18**, No. 2, 203–235 (2011).

5. R. Akgün and V. Kokilashvili, "The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces," *Georgian Math. J.* **18**, No. 3, 399–423 (2011).

6. R. Akgün and V. Kokilashvili, "On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces," *Banach J. Math. Anal.* **5**. No. 1 (2011).

7. V. Kokilashvili and S. Samko, "Harmonic analysis in weighted spaces with nonstandard growth," *J. Math. Anal. Appl.* **352**, No. 1, 15–34 (2009).

8. D. M. Israfilov, V. M. Kokilashvili, and S. Samko, "Approximation in weighted Lebesgue and Smirnov spaces with variable exponent," *Proc. A. Razmadze Math. Inst.* **143**, 45–55 (2007).

9. V. Kokilashvili, "On converse theorems of constructive theory of functions in Lp spaces," *Tr. Tbiliss. Mat. Inst. im. Razmadze Akad. Nauk Gruzin. SSR* **29**, 183–189 (1964).

10. V. Kokilashvili, "On estimate of best approximation and modulus of smoothness in Lebesgue spaces of periodic functions with transformed Fourier spaces" [in Russian], *Soobshch. Akad. Nauk Gruzin. SSR* **35**, No. 1, 3–8 (1965).

11. B. V. Simonov and S. Yu. Tikhonov, "On embeddings of function classes defined by constructive characteristics" In: *Approximation and Probability*, pp. 285–307, Banach Center Publ. **72**, Polish Acad. Sci., Warsaw (2006).

Submitted on April 4, 2012