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Random evolutions with locally independent increments on increasing time intervals

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Abstract. Three main schemes of limit theorems for random evolutions are discussed: averaging, diffusion approximation, and the asymptotics of large deviations. Markov stochastic evolutions with locally independent increments on increasing time intervals $T_\varepsilon = t/\varepsilon \to \infty$, $\varepsilon \to 0$, are considered. The asymptotic behavior of random evolutions is investigated with the use of solutions of the singular perturbation problems for reducibly invertible operators.

Keywords. Markov random evolutions, averaging, diffusion approximation, asymptotics of large deviations.

Introduction

Studies of the asymptotic behavior of random evolutions on increasing time intervals are based on the use of solutions of the problem of singular perturbation for a reducibly invertible operator [4, Chap. 5]. In this case, the substantiation of the limiting transition is realized with the use of a martingale characterization of Markov processes and the conditions of relative compactness of probability measures [1].

The main object of the asymptotic analysis of a random evolution is the generator of a corresponding two-component Markov process [4, Chap. 2].

The asymptotic analysis is performed for random evolutions in the scheme of series with small parameter $\varepsilon \to 0$ ($\varepsilon > 0$) of a series [4, Chap. 3].

Results of the asymptotic analysis depend essentially on the conditions of normalization of a random evolution with the small parameter of a series.

The theory of random processes includes three main schemes of limit theorems:

1. The law of large numbers or the averaging.

2. The central limit theorem or the diffusion approximation.

3. The asymptotics of large deviations or the estimation of exponentially small probabilities.

Each of the main schemes involves the own conditions of normalization with the parameter $\varepsilon \to 0$ of a series.

The available literature on limit theorems for random processes and, in particular, for random evolutions, is vast.

Most simply, one should consider the main monographs devoted to the theory of limit theorems.

To study the asymptotic analysis of random evolutions on increasing time intervals, it is sufficient to take the monographs in the list of references and references therein.

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1. Random evolutions with locally independent increments

A stochastic additive functional [4, § 2.6] is set by the relations

$$\xi(t) = \xi_0 + \int_0^t \eta(ds; x(s)), \quad t \geq 0,\ \xi_0 \in R^d. \tag{1.1}$$

The Markov switching process $x(t),\ t \geq 0$, in the standard phase space $(E, \mathcal{E})$ is set by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \quad x \in E, \tag{1.2}$$

on test-functions $\varphi(x) \in B_E$, where B_E is the Banach space of bounded functions with sup-norm: $\|\varphi(x)\| := \sup_{x \in E} |\varphi(x)|$.

A random evolution in the Euclidean space R^d is set by the totality of processes with locally independent increments $\eta(t; x),\ t \geq 0,\ x \in E$, determined by the generators

$$\mathbf{\Gamma}(x)\varphi(u) = \int_{R^d} [\varphi(u+v) - \varphi(u)]\Gamma(u, dv, x), \quad u \in R^d,\ x \in E. \tag{1.3}$$

The Markov random evolution (1.1) is characterized by the generator [1, § 2.6]

$$\mathbb{L}\varphi(u, x) = Q\varphi(\cdot, x) + \mathbf{\Gamma}(x)\varphi(u, \cdot) \tag{1.4}$$

of the two-component Markov process $\xi(t), x(t),\ t \geq 0$.

2. Processes with locally independent increments in the scheme of series

The following three main schemes of series are considered: averaging, diffusion approximation, and large deviations in the scheme of asymptotically low diffusion.

2.1. Averaging

Processes with locally independent increments in the averaging scheme are set by the generator

$$\mathbf{\Gamma}^\varepsilon\varphi(u) = \varepsilon^{-1} \int_{R^d} [\varphi(u + \varepsilon v) - \varphi(u)]\Gamma(u, dv), \quad u \in R^d, \tag{2.1}$$

which corresponds to the normalization

$$\eta^\varepsilon(t) = \varepsilon\eta(t/\varepsilon), \quad t \geq 0.$$

In this case, the processes with locally independent increments $\eta^\varepsilon(t),\ t \geq 0$, are considered on increasing time intervals $T^\varepsilon = t/\varepsilon \to \infty,\ \varepsilon \to 0$.

On sufficiently smooth test-functions $\varphi(u) \in C^2(R^d)$, generator (2.1) admits the asymptotic representation

$$\mathbf{\Gamma}^\varepsilon \varphi(u) = b(u)\varphi'(u) + \delta^\varepsilon \varphi(u), \tag{2.2}$$

where

$$b(u)\varphi'(u) := \sum_{k=1}^{d} b_k(u)\varphi'_k(u),$$

$$\varphi'_k(u) := \partial\varphi(u)/\partial u_k, \quad b_k(u) := \int_{R^d} v_k \Gamma(u, dv), \quad 1 \le k \le d,$$

with an insignificant term

$$\|\delta^\varepsilon \varphi(u)\| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^2(R^d). \tag{2.3}$$

Here, $b(u) := \int_{R^d} v\Gamma(u, dv)$ is the vector of the first moments of the process.

The asymptotic representation (2.2) of the generator of a process with locally independent increments serves as the base for the proof of the convergence

$$\eta^\varepsilon(t) = \varepsilon\eta(t/\varepsilon) \Rightarrow \eta^0(t). \quad \varepsilon \to 0,$$

Here, the limit process is set by a solution of the evolution equation [4, § 3.3.1]

$$d\eta^0(t)/dt = b(\eta^0(t)), \quad \eta^0(0) = \eta_0.$$

2.2. Diffusion approximation

The processes with locally independent increments in the scheme of diffusion approximation are set by the normalization

$$\eta^\varepsilon(t) = \varepsilon\eta(t/\varepsilon^2), \quad t \ge 0, \tag{2.4}$$

under the additional *balance condition*:

$$b(u) = \int_{R^d} v\Gamma(u, dv) \equiv 0. \tag{2.5}$$

The generator of the normed process (2.4) takes the form

$$\mathbf{\Gamma}^\varepsilon \varphi(u) = \varepsilon^{-2} \int_{R^d} [\varphi(u + \varepsilon v) - \varphi(u)]\Gamma(u, dv), \quad u \in R^d. \tag{2.6}$$

On sufficiently smooth test-functions $\varphi(u) \in C^3(R^d)$, generator (2.6) admits the asymptotic representation

$$\mathbf{\Gamma}^\varepsilon \varphi(u) = \frac{1}{2} B(u)\varphi''(u) + \delta^\varepsilon \varphi(u), \tag{2.7}$$

where

$$B(u)\varphi''(u) := \sum_{k,r=1}^{d} B_{kr}(u)\partial^2\varphi(u)/\partial u_k \partial u_r,$$

with an negligible term

$$\|\delta^\varepsilon \varphi(u)\| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^3(R^d).$$

The principal terms in (2.7) sets the diffusion process with the covariance matrix

$$B(u) = [B_{kr}(u); \quad 1 \le k, r \le d],$$

$$B_{kr}(u) = \int_{R^d} v_k v_r \Gamma(u, dv), \quad 1 \le k, r \le d.$$

The asymptotic representation (2.7) serves as the base for the proof of the convergence of processes [4, Chap. 6]

$$\eta^\varepsilon(t) = \varepsilon \eta(t/\varepsilon^2) \Rightarrow \zeta(t), \quad \varepsilon \to 0.$$

Here, the limit diffusion process $\zeta(t),\ t \ge 0$, is set by the stochastic equation

$$d\zeta(t) = \sigma(\zeta(t)) dw(t), \quad \sigma^*(u)\sigma(u) = B(u).$$

2.3. Asymptotically small diffusion

The processes with locally independent increments in the scheme of asymptotically small diffusion are set by the normalization [5]

$$\eta^\varepsilon(t) = \varepsilon^2 \eta(t/\varepsilon^3), \quad t \ge 0. \tag{2.8}$$

Under the additional condition of balance

$$b(u) = \int_{R^d} v \Gamma(u, dv) \equiv 0,$$

the relevant generator of process (2.8) takes the form

$$\mathbf{\Gamma}^\varepsilon \varphi(u) = \varepsilon^{-3} \int_{R^d} [\varphi(u + \varepsilon^2 v) - \varphi(u)] \Gamma(u, dv), \quad u \in R^d. \tag{2.9}$$

On sufficiently smooth test-functions $\varphi(u) \in C^3(R^d)$, generator (2.9) admits the asymptotic representation

$$\mathbf{\Gamma}^\varepsilon \varphi(u) = \varepsilon \frac{1}{2} B(u) \varphi''(u) + \varepsilon \delta^\varepsilon \varphi(u) \tag{2.10}$$

with the negligible term

$$\|\delta^\varepsilon \varphi(u)\| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^3(R^d).$$

The asymptotic representation (2.10) of generator (2.9) of process (2.8) means that the asymptotic relation

$$\begin{aligned} &\eta^\varepsilon(t) = \varepsilon^2 \eta(t/\varepsilon^3) \simeq \varepsilon \zeta(t), \quad \varepsilon \to 0, \\ &d\zeta(t) = \sigma(\zeta(t))\, dw(t) \end{aligned} \tag{2.11}$$

is satisfied. The normed process with locally independent increments (2.8) can be asymptotically represented as a small diffusion $\varepsilon\zeta(t),\ t \ge 0$, with the covariance matrix $B(u) = \sigma^*(u)\sigma(u)$.

3. Random evolutions in the scheme of series

Like the previous item, we consider three main schemes of the asymptotic analysis: averaging, diffusion approximation, and asymptotically small diffusion.

3.1. Averaging

A random evolution in the averaging scheme is set by the generator

$$\mathbb{L}^{\varepsilon}\varphi(u,x) = [\varepsilon^{-1}Q + \mathbf{\Gamma}^{\varepsilon}(x)]\varphi(u,x). \tag{3.1}$$

Here, by definition (see i. 2.1),

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(u) = \varepsilon^{-1}\int_{R^d}[\varphi(u+\varepsilon v) - \varphi(u)]\Gamma(u,dv;x). \tag{3.2}$$

The representation of the generator of a random evolution (3.1)–(3.2) means that the two-component Markov process defining a random evolution in the averaging scheme has the following normalization:

$$\varepsilon\xi(t/\varepsilon), \quad x_t^{\varepsilon} := x(t/\varepsilon), \quad t \geq 0, \ \varepsilon \to 0. \tag{3.3}$$

Hence, a random evolution is considered on increasing time intervals $T^{\varepsilon} = t/\varepsilon \to \infty, \ \varepsilon \to 0$.

The asymptotic analysis of a behavior of the random evolution (3.3) given by generator (3.1)–(3.2), is performed by applying a *solution of the problem of singular perturbation for the reducibly invertible operator* Q defining a switching Markov process $x(t), \ t \geq 0$ (see [4, Chaps. 3, 5, and 6]).

The main assumption is as follows:

P1: A Markov process $x(t), \ t \geq 0$, defined by generator (1.2) in the standard phase space $(E, \mathcal{E})$ is *uniformly ergodic with stationary distribution* $\pi(A), \ A \in \mathcal{E}$.

Under this condition, the generator Q is reducibly invertible with the zero-space defined by the projector [4, Chap. 5]

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x) =: \widehat{\varphi}\mathbf{1}(x), \quad \mathbf{1}(x) := 1, \ x \in E. \tag{3.4}$$

In this case, there exists a bounded *potential* R_0 prescribed by a solution of the equation

$$QR_0 = R_0Q = \Pi - I. \tag{3.5}$$

Hence, the Poisson equation

$$Q\varphi(x) = \psi(x), \quad \Pi\psi(x) = 0, \tag{3.6}$$

has the unique solution in a subspace of values of the generator Q, which is set by the equality

$$\varphi(x) = -R_0\psi(x). \tag{3.7}$$

The asymptotic representation of generator (3.1)–(3.2) of the random evolution (3.3) is realized on the perturbed test-function [4, Chap. 5]

$$\varphi^{\varepsilon}(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x).$$

Lemma 3.1. *There exists the asymptotic representation*

$$\mathbb{L}^\varepsilon \varphi^\varepsilon(u,x) = \widehat{\mathbf{L}}\varphi(u) + \delta_L^\varepsilon(x)\varphi(u) \tag{3.8}$$

with the negligible term

$$\|\delta_L^\varepsilon(x)\varphi(u)\| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^2(R^d).$$

The limit operator is prescribed by the relations

$$\begin{aligned} \widehat{\mathbf{L}}\varphi(u) &= \widehat{b}(u)\varphi'(u), \\ \widehat{b}(u) = \Pi b(u;x) &= \int_E \pi(dx) b(u;x), \\ b(u;x) &:= \int_{R^d} v\Gamma(u,dv;x). \end{aligned} \tag{3.9}$$

Hence, the limit operator (3.9) sets a determinate evolution $\widehat{u}(t),\ t \geq 0$, that is described by a solution of the evolution equation

$$d\widehat{u}(t)/dt = \widehat{b}(\widehat{u}(t)), \quad \widehat{u}(0) = u_0 \in R^d. \tag{3.10}$$

The asymptotic representation (3.8)–(3.9) serves as the base for the proof of the convergence of random evolutions in the averaging scheme [4, Chap. 6]:

$$\xi^\varepsilon(t) := \varepsilon\xi(t/\varepsilon) \Rightarrow \widehat{u}(t), \quad \varepsilon \to 0, \tag{3.11}$$

under the condition of convergence of the initial values

$$\varepsilon\xi^\varepsilon(0) \Rightarrow u_0, \quad \varepsilon \to 0. \tag{3.12}$$

3.2. Diffusion approximation

A random evolution in the scheme of diffusion approximation is considered under the additional *condition of balance*. As distinct from the situation concerning with the analysis of the process with locally independent increments in Section 2, we distinguish two conditions of balance: *local* (LB) and *total* (TB) ones.

3.2.1. Diffusion approximation under the condition of local balance

$$\text{LB:} \qquad b(u;x) := \int_{R^d} v\Gamma(u,dv;x) \equiv 0. \tag{3.13}$$

A random evolution is considered at the following normalization:

$$\xi^\varepsilon(t) = \varepsilon\xi(t/\varepsilon^2), \quad x_t^\varepsilon = x(t/\varepsilon^2). \tag{3.14}$$

The generator of the random evolution (3.14) takes the form

$$\mathbb{L}^\varepsilon\varphi(u,x) = [\varepsilon^{-2}Q + \mathbf{\Gamma}^\varepsilon(x)]\varphi(u,x), \tag{3.15}$$

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(u) = \varepsilon^{-2}\int\limits_{R^d}[\varphi(u+\varepsilon v)-\varphi(u)]\Gamma(u,dv;x). \tag{3.16}$$

On sufficiently smooth functions $\varphi(u) \in C^3(R^d)$, generator (3.16) admits the asymptotic representation

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(u) = \frac{1}{2}B(u;x)\varphi''(u) + \delta_{\Gamma}^{\varepsilon}(u;x)\varphi(u), \tag{3.17}$$

with the negligible term

$$\|\delta_{\Gamma}^{\varepsilon}(u;x)\varphi(u)\| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^3(R^d).$$

The principal term in (3.17) takes the form

$$\begin{gathered} B(u;x)\varphi''(u) := \sum_{k,r}^{d} B_{kr}(u;x)\varphi''_{kr}(u), \\ B_{kr}(u;x) := \int\limits_{R^d} v_k v_r \Gamma(u,dv;x), \quad \varphi''_{kr}(u) := \partial^2\varphi(u)/\partial u_k \partial u_r. \end{gathered} \tag{3.18}$$

The asymptotic representation of generator (3.15)–(3.16) of random evolution (3.14) is realized on the perturbed test-function

$$\varphi^{\varepsilon}(u,x) = \varphi(u) + \varepsilon^2\varphi_1(u,x).$$

Lemma 3.2. *Under the conditions of local balance, there exists the asymptotic representation of generator* (3.15)–(3.16) *of the random evolution* (3.14)

$$\mathbb{L}^{\varepsilon}\varphi^{\varepsilon}(u,x) = \widehat{\mathbf{L}}\varphi(u) + \delta_L^{\varepsilon}(u;x)\varphi(u), \tag{3.19}$$

with the negligible term

$$\|\delta_L^{\varepsilon}(u;x)\varphi(u)\| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^3(R^d).$$

The limit generator

$$\begin{gathered} \widehat{\mathbf{L}}\varphi(u) = \frac{1}{2}\widehat{B}(u)\varphi''(u), \\ \widehat{B}(u) = [\widehat{B}_{kr}(u);\ 1 \le k,r \le d], \\ \widehat{B}_{kr}(u) = \int\limits_{E}\pi(dx)B_{kr}(u;x), \quad B_{kr}(u;x) = \int\limits_{R^d} v_k v_r \Gamma(u,dv;x). \end{gathered} \tag{3.20}$$

Hence, the limit generator (3.20) *sets the process of diffusion*

$$\begin{aligned} &d\zeta(t) = \sigma(\zeta(t))\,dw(t), \quad t \ge 0,\ \zeta(0) = \zeta_0, \\ &\sigma^*(u)\sigma(u) = \widehat{B}(u). \end{aligned} \tag{3.21}$$

The asymptotic representation (3.19)–(3.20) of the generator of the random evolution (3.14) serves as the base for the proof of the convergence of the random evolution in the scheme of diffusion approximation

$$\xi^{\varepsilon}(t) = \varepsilon\xi(t/\varepsilon^2) \Rightarrow \zeta(t), \quad \varepsilon \to 0, \tag{3.22}$$

under the additional condition of convergence of the initial values

$$\varepsilon\xi^{\varepsilon}(0) \Rightarrow \zeta_0, \quad \varepsilon \to 0.$$

3.2.2. Diffusion approximation under the condition of total balance

$$\text{TB:} \qquad \begin{aligned} b(u;x) &= \int_{R^d} v\Gamma(u,dv;x) \not\equiv 0, \\ \widehat{b}(u) &= \int_E \pi(dx) b(u;x) \equiv 0. \end{aligned} \tag{3.23}$$

A random evolution is considered at the same normalization:

$$\xi^\varepsilon(t) = \varepsilon\xi(t/\varepsilon^2), \quad x_t^\varepsilon := x(t/\varepsilon^2),\ t \ge 0. \tag{3.24}$$

On sufficiently smooth test-functions $\varphi(u) \in C^3(R^d)$, the generator of the random evolution (3.15)–(3.16) admits the asymptotic decomposition

$$\mathbb{L}^\varepsilon\varphi(u,x) = [\varepsilon^{-2}Q + \varepsilon^{-1}\mathbf{\Gamma}(x) + \mathbf{B}(x)]\varphi(u,x), \tag{3.25}$$

$$\begin{aligned} \mathbf{\Gamma}(x)\varphi(u) &= b(u;x)\varphi'(u), \\ \mathbf{B}(x)\varphi(u) &= \frac{1}{2}B(u;x)\varphi''(u). \end{aligned} \tag{3.26}$$

Now, the asymptotic representation of the generator (3.25)–(3.26) is realized on the perturbed test-functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x).$$

Lemma 3.3. *Under conditions of total balance, there exists the asymptotic representation of generator* (3.25)–(3.26) *of the random evolution* (3.24)

$$\mathbb{L}^\varepsilon\varphi^\varepsilon(u,x) = \widehat{\mathbb{L}}\varphi(u) + \delta_L^\varepsilon(u;x)\varphi(u), \tag{3.27}$$

with the negligible term

$$\|\delta_L^\varepsilon(u,x)\varphi(u)\| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^3(R^d).$$

The limit generator

$$\begin{aligned} \widehat{\mathbb{L}}\varphi(u) &= \frac{1}{2}\widehat{B}(u)\varphi''(u) + \widehat{b}_0(u)\varphi'(u), \\ \widehat{B}(u) &= [\widehat{B}_{kr}(u);\ 1 \le k, r \le d], \end{aligned} \tag{3.28}$$

$$\begin{aligned} \widehat{B}_{kr}(u) &= \widehat{B}_{kr}^{(1)}(u) + \widehat{B}_{kr}^{(2)}(u), \\ \widehat{B}_{kr}^{(i)}(u) &= \int_E \pi(dx) B_{kr}^{(i)}(u;x), \quad i = 1,2, \\ \widehat{B}_{kr}^{(1)}(u;x) &= 2b^*(u;x)R_0 b(u;x), \\ \widehat{B}_{kr}^{(2)}(u;x) &= \int_{R^d} v_k v_r \Gamma(u,dv;x), \end{aligned} \tag{3.29}$$

$$\widehat{b}_0(u) = \int_E \pi(dx) b_0(u;x), \quad b_0(u;x) = 2b^*(u;x)R_0 b'_u(u;x).$$

Hence, the limit generator (3.28)–(3.29) sets the diffusion process with drift

$$d\zeta(t) = \sigma(\zeta(t))dw(t) + \widehat{b}_0(\zeta(t))dt, \quad \zeta(0) = \zeta_0, \tag{3.30}$$

for which the covariance matrix $\widehat{B}(u) = \sigma^*(u)\sigma(u)$ contains two terms:

$\widehat{B}^{(1)}(u) = [\widehat{B}^{(1)}_{kr}(u);\ 1 \le k, r \le d]$ is determined by fluctuations of the first moments of the random evolution;

$\widehat{B}^{(2)}(u) = [\widehat{B}^{(2)}_{kr}(u);\ 1 \le k, r \le d]$ is determined by the second moments of the random evolution.

The asymptotic representation (3.27)–(3.29) serves as the base for the proof of the convergence of random evolutions

$$\varepsilon\xi(t/\varepsilon^2) \Rightarrow \zeta(t), \quad \varepsilon \to 0, \tag{3.31}$$

under the additional condition of convergence of the initial values

$$\varepsilon\xi(0) \Rightarrow \zeta_0, \quad \varepsilon \to 0.$$

3.3. Asymptotically small diffusion

A random evolution in the scheme of asymptotically small diffusion is considered at the following normalization (compare with [5]):

$$\xi^\varepsilon(t) = \varepsilon^2\xi(t/\varepsilon^3), \quad x^\varepsilon_t := x(t/\varepsilon^2), \tag{3.32}$$

at the *local condition of balance*. At the *total condition of balance*, the normalization of the switching Markov process is different:

$$\xi^\varepsilon(t) = \varepsilon^2\xi(t/\varepsilon^3), \quad x^\varepsilon_t := x(t/\varepsilon^3). \tag{3.33}$$

The random evolutions (3.32) and (3.33) are prescribed by the generators

$$\mathbb{L}^\varepsilon_\Lambda\varphi(u, x) = [\varepsilon^{-2}Q + \mathbf{\Gamma}^\varepsilon(x)]\varphi(u, x), \tag{3.34}$$

and

$$\mathbb{L}^\varepsilon_T\varphi(u, x) = [\varepsilon^{-3}Q + \mathbf{\Gamma}^\varepsilon(x)]\varphi(u, x), \tag{3.35}$$

respectively. The generator of the evolutionary component

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = \varepsilon^{-3}\int_{R^d}[\varphi(u + \varepsilon^2 v) - \varphi(u)]\Gamma(u, dv; x) \tag{3.36}$$

admits the asymptotic decompositions

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = \varepsilon\frac{1}{2}B(u; x)\varphi''(u) + \varepsilon\delta^\varepsilon_T(x)\varphi(u) \tag{3.37}$$

and

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = \varepsilon^{-1}b(u; x)\varphi'(u) + \varepsilon\frac{1}{2}B(u; x)\varphi''(u) + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u), \tag{3.38}$$

respectively.

The following *problems of singular perturbation* arise:

$$\begin{aligned}\mathbb{L}^\varepsilon_\Lambda\varphi(u,x) &= [\varepsilon^{-2}Q + \varepsilon\mathbf{B}(x)]\varphi(u,x) + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u,x),\\ \mathbb{L}^\varepsilon_T\varphi(u,x) &= [\varepsilon^{-3}Q + \varepsilon^{-1}\mathbf{\Gamma}(x) + \varepsilon\mathbf{B}(x)]\varphi(u,x) + \varepsilon\delta^\varepsilon_T(x)\varphi(u,x).\end{aligned} \tag{3.39}$$

Here, $\mathbf{\Gamma}(x)\varphi(u) := b(u;x)\varphi'(u)$.

Lemma 3.4. *Generators* (3.39) *admit the following asymptotic representations:*

$$\mathbb{L}^\varepsilon_\Lambda\varphi^\varepsilon_\Lambda(u,x) = \varepsilon\frac{1}{2}\widehat{B}(u)\varphi''(u) + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u) \tag{3.40}$$

on the perturbed test-functions

$$\varphi^\varepsilon_\Lambda(u,x) = \varphi(u) + \varepsilon^2\varphi_1(u,x)$$

and

$$\mathbb{L}^\varepsilon_T\varphi^\varepsilon(u,x) = \varepsilon\left[\frac{1}{2}\widehat{B}_T(u)\varphi''(u) + \widehat{b}_0(u)\varphi'(u)\right] + \varepsilon\delta^\varepsilon_T(x)\varphi(u) \tag{3.41}$$

on the perturbed test-functions

$$\varphi^\varepsilon_\Lambda(u,x) = \varphi(u) + \varepsilon^2\varphi_1(u,x) + \varepsilon^3\varphi_2(u,x).$$

Here,

$$\begin{gathered}\widehat{B}(u) = \int_E \pi(dx)B(u;x), \quad B(u;x) = \int_{R^d} v^*v\Gamma(u,dv;x),\\ \widehat{B}_T(u) = \int_E \pi(dx)B_T(u;x), \quad B_T(u;x) = B(u;x) + B_0(u;x),\\ B_0(u;x) = 2b^*(u;x)R_0b(u;x),\\ \widehat{b}_0(u) = \int_E \pi(dx)b_0(u;x), \quad b_0(u;x) = b(u;x)R_0b'_u(u;x).\end{gathered} \tag{3.42}$$

Proof. The proof of Lemma 3.4 is based on the use of solutions of the problems of singular perturbation for the reducibly invertible operator Q [4, Chap. 5].

Below, we present the results of relevant necessary calculations.

For the operator $\mathbb{L}^\varepsilon_\Lambda$ in (3.39), we have

$$\begin{aligned}\mathbb{L}^\varepsilon_\Lambda\varphi^\varepsilon(u,x) &= [\varepsilon^{-2}Q + \varepsilon\mathbf{B}(x)][\varphi(u) + \varepsilon^2\varphi_1(u,x)] + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u)\\ &= \varepsilon^{-2}Q\varphi(u) + [Q\varphi_1 + \varepsilon\mathbf{B}(x)\varphi(u)] + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u)\\ &= [Q\varphi_1 + \varepsilon\mathbf{B}(x)\varphi(u)] + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u) = \varepsilon\frac{1}{2}\widehat{B}(u)\varphi''(u) + \varepsilon\delta^\varepsilon_\Lambda(x)\varphi(u),\end{aligned}$$

which coincides with (3.40).

Then, for the operator $\mathbb{L}^\varepsilon_T$ in (3.39), we have

$$\begin{aligned}\mathbb{L}^\varepsilon_T\varphi^\varepsilon(u,x) &= [\varepsilon^{-3}Q + \varepsilon^{-1}\mathbf{\Gamma}(x) + \varepsilon\mathbf{B}(x)][\varphi(u) + \varepsilon^2\varphi_1(u,x) + \varepsilon^3\varphi_2(u,x)]\\ &= \varepsilon^{-3}Q\varphi(u) + \varepsilon^{-1}[Q\varphi_1 + \mathbf{\Gamma}(x)\varphi(u)] + [Q\varphi_2 + \varepsilon[\mathbf{\Gamma}(x)\varphi_1 + \mathbf{B}(x)\varphi(u)]]\end{aligned}$$

$$= [Q\varphi_2 + \varepsilon \left[\frac{1}{2} B_T(u;x)\varphi''(u) + b_0(u;x)\varphi'(u)\right] + \varepsilon\delta_T^\varepsilon(x)\varphi(u)$$

$$= \varepsilon \left[\frac{1}{2}\widehat{B}_T(u)\varphi''(u) + \widehat{b}_0(u)\varphi'(u)\right] + \varepsilon\delta_T^\varepsilon(x)\varphi(u),$$

which coincides with (3.41). □

The asymptotic representations (3.40) and (3.41) serve as the base for the proof of the asymptotic relations

$$\xi_T^\varepsilon(t) \simeq \varepsilon\zeta_T(t), \quad \xi_\Lambda^\varepsilon(t) \simeq \varepsilon\zeta_\Lambda(t), \quad t \geq 0, \ \varepsilon \to 0,$$

$$d\zeta_\Lambda(t) = \sigma_\Lambda(\zeta_\Lambda(t))\, dw(t), \quad d\zeta_T(t) = \sigma_T(\zeta_T(t))\, dw(t) + \widehat{b}_0(\zeta_T(t))\, dt,$$

$$\sigma_T^*(u)\sigma_T(u) = B_T(u), \quad \sigma_\Lambda^*(u)\sigma_\Lambda(u) = B_\Lambda(u).$$

4. Large deviations for random evolutions in the scheme of asymptotically small diffusion

Random evolutions with locally independent increments are considered in the scheme of series with the small parameter $\varepsilon \to 0$ ($\varepsilon > 0$) of a series on increasing time intervals with the normalization admitting the approximation with asymptotically small diffusion (see Subsection 3.3). Large deviations for random evolutions with locally independent increments are studied within the method of asymptotic analysis of an *exponential generator* of large deviations, which was developed in [2]. The exponential generator $\mathbf{H}$ of large deviations of the Markov process prescribed by the generator $\mathbf{L}^\varepsilon$ is determined by the relation [2, Part I]:

$$\mathbf{H}^\varepsilon\varphi(u) := e^{-\varphi(u)/\varepsilon}\varepsilon\mathbf{L}^\varepsilon e^{\varphi(u)/\varepsilon}. \tag{4.1}$$

As shown in Subsection 3.3, the approximation of random evolution in the case of the asymptotically small diffusion depends on the additional *condition of balance* (*total* or *local*).

4.1. Large deviations under the condition of local balance

Random evolutions in the scheme of series are considered at normalization (3.32). The generator of a random evolution is given by relations (3.34) and (3.36).

Theorem 4.1. *Under the condition of local balance and condition* **P1** (*see Subsection* 3.1), *the exponential generator of large deviations for the random evolutions* (3.32) *is determined by the relations*

$$H\varphi(u) = \frac{1}{2}B(u)[\varphi'(u)]^2, \tag{4.2}$$

$$B(u) = \int_E \pi(dx)B(u;x), \ B(u;x) = \int_{R^d} v^* v\Gamma(u, dv; x). \tag{4.3}$$

Proof. Consider the exponential generator (4.1) on the perturbed test-function

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\log[1 + \varepsilon\varphi_1(u,x)]. \tag{4.4}$$

Lemma 4.1. *The exponential generator* (4.1) *on the perturbed test-function* (4.4) *admits the asymptotic representation*

$$H^\varepsilon \varphi^\varepsilon(u,x) = Q\varphi_1 + \widetilde{\mathbf{B}}(x)\varphi(u) + \delta_H^\varepsilon(x)\varphi(u), \tag{4.5}$$

with the negligible term

$$\|\delta_H^\varepsilon(x)\varphi(u)\| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^3(R^d).$$

The operator

$$\widetilde{\mathbf{B}}(x)\varphi(u) = \frac{1}{2}B(u;x)[\varphi'(u)]^2. \tag{4.6}$$

Proof. The proof of Lemma 4.1 is based on the asymptotic analysis of the terms

$$\begin{aligned} H_Q^\varepsilon \varphi^\varepsilon(u,x) &= e^{-\varphi(u)/\varepsilon}[1+\varepsilon\varphi_1]^{-1}\varepsilon^{-1}Q[1+\varepsilon\varphi_1]e^{\varphi(u)/\varepsilon} \\ &= e^{-\varphi(u)/\varepsilon}[1-\varepsilon\varphi_1]\varepsilon^{-1}Q[1+\varepsilon\varphi_1]e^{\varphi(u)/\varepsilon} + \delta_Q^\varepsilon(x)\varphi(u) \\ &= Q\varphi_1 + \delta_Q^\varepsilon(x)\varphi(u) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} H_\Gamma^\varepsilon \varphi^\varepsilon(u,x) &= e^{-\varphi(u)/\varepsilon}[1+\varepsilon\varphi_1]^{-1}\varepsilon\mathbf{\Gamma}^\varepsilon(x)[1+\varepsilon\varphi_1]e^{\varphi(u)/\varepsilon} \\ &= e^{-\varphi(u)/\varepsilon}[1-\varepsilon\varphi_1]^{-1}\varepsilon\mathbf{\Gamma}^\varepsilon(x)[1+\varepsilon\varphi_1]e^{\varphi(u)/\varepsilon} + \delta_\Gamma^\varepsilon(x)\varphi(u) \\ &= \varepsilon^{-2}\int_{R^d}[e^{\Delta_v^\varepsilon\varphi(u)}-1]\Gamma(u,dv;x) + \delta_\Gamma^\varepsilon(x)\varphi(u). \end{aligned}$$

Here,

$$\Delta_v^\varepsilon\varphi(u) = \varepsilon^{-1}[\varphi(u+\varepsilon^2 v) - \varphi(u)] = \varepsilon v\varphi'(u) + \varepsilon^3\widetilde{\varphi}_v''(u).$$

Hence,

$$H_\Gamma^\varepsilon \varphi^\varepsilon(u,x) = \widetilde{\mathbf{B}}(x)\varphi(u) + \delta_\Gamma^\varepsilon(x)\varphi(u) \tag{4.8}$$

with the principal term (4.6). □

The completion of the proof of Theorem 4.1 is realized with the use of a solution of the problem of singular perturbation

$$Q\varphi_1(u,x) + \widetilde{\mathbf{B}}(x)\varphi(u) = \widetilde{\mathbf{B}}\varphi(u) \tag{4.9}$$

in which the condition of solvability of Eq. (4.9) means

$$\widetilde{\mathbf{B}} = \int_E \pi(dx)\widetilde{\mathbf{B}}(x). \tag{4.10}$$

Finally, the relation

$$H^\varepsilon\varphi^\varepsilon(u,x) = H\varphi(u) + \delta_H^\varepsilon(x)\varphi(u) \tag{4.11}$$

completes the proof of Theorem 4.1. □

4.2. Large deviations under the condition of total balance

Consider random evolutions in the scheme of series at normalization (3.33). The generator of a random evolution is given by relations (3.35)–(3.36).

Theorem 4.2. *Under the condition of total balance and condition* **P1**, *the exponential generator of large deviations for the random evolution* (3.33) *is given by the relations*

$$H\varphi(u) = \frac{1}{2}B_T(u)[\varphi'(u)]^2, \quad B_T(u) = B(u) + B_0(u), \tag{4.12}$$

$$\begin{aligned} B(u) &:= \int_E \pi(dx)B(u;x), \quad B_0(u) := \int_E \pi(dx)B_0(u;x), \\ B(u;x) &= \int_{R^d} v^* v\Gamma(u, dv; x), \quad B_0(u;x) = 2b^*(u;x)R_0 b(u;x). \end{aligned} \tag{4.13}$$

Proof. The exponential generator (4.1) is considered on the perturbed test-function

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \log[1 + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)]. \tag{4.14}$$

Lemma 4.2. *The exponential generator* (4.1) *on the perturbed test-function* (4.14) *admits the asymptotic representation*

$$H^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-1}[Q\varphi_1 + \mathbf{\Gamma}(x)\varphi(u)] + [Q\varphi_2 - \varphi_1 Q\varphi_1 + \widetilde{\mathbf{B}}(x)\varphi(u)] + \delta_H^\varepsilon(x)\varphi(u), \tag{4.15}$$

with the negligible term

$$\|\delta_H^\varepsilon(x)\varphi(u)\| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^3(R^d),$$
$$\mathbf{\Gamma}(x)\varphi(u) := b(u;x)\varphi'(u).$$

Proof. The proof of Lemma 4.2 is based on the asymptotic analysis of the terms

$$\begin{aligned} H_Q^\varepsilon\varphi^\varepsilon(u,x) &= e^{-\varphi/\varepsilon}[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2]^{-1}\varepsilon^{-2}Q[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2]e^{\varphi/\varepsilon} \\ &= e^{-\varphi/\varepsilon}[1 - \varepsilon\varphi_1 - \varepsilon^2\varphi_2]\varepsilon^{-2}Q[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2]e^{\varphi/\varepsilon} + \delta_Q^\varepsilon(x)\varphi(u) \\ &= \varepsilon^{-1}Q\varphi_1 - \varphi_1 Q\varphi_1 + \delta_Q^\varepsilon(x)\varphi(u), \end{aligned}$$

$$\begin{aligned} H_\Gamma^\varepsilon\varphi^\varepsilon(u,x) &= e^{-\varphi/\varepsilon}[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2]^{-1}\varepsilon\mathbf{\Gamma}^\varepsilon(x)[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2]e^{-\varphi/\varepsilon} \\ &= e^{-\varphi/\varepsilon}[1 - \varepsilon\varphi_1]\varepsilon\mathbf{\Gamma}^\varepsilon(x)[1 + \varepsilon\varphi_1]e^{-\varphi/\varepsilon} + \delta_\Gamma^\varepsilon(x)\varphi(u) \\ &= \varepsilon^{-1}\mathbf{\Gamma}(x)\varphi(u) + \widetilde{\mathbf{B}}(x)\varphi(u) + \delta_\Gamma^\varepsilon(x)\varphi(u). \end{aligned}$$

Hence,

$$H^\varepsilon\varphi^\varepsilon(u,x) = H_Q^\varepsilon\varphi^\varepsilon(u,x) + H_\Gamma^\varepsilon\varphi^\varepsilon(u,x),$$

so that

$$H^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-1}[Q\varphi_1 + \mathbf{\Gamma}(x)\varphi(u)] + [Q\varphi_2 - \varphi_1 Q\varphi_1 + \widetilde{\mathbf{B}}(x)\varphi(u)] + \delta_H^\varepsilon(x)\varphi(u),$$

which coincides with (4.15). □

The completion of the proof of Theorem 4.2 is realized with the use of solutions of the problems of singular perturbation [4, Chap. 5]:

$$Q\varphi_1 + b(u;x)\varphi'(u) = 0, \quad \Pi b(u;x) \equiv 0, \tag{4.16}$$

$$\varphi_1(u,x) = R_0 b(u;x)\varphi'(u). \tag{4.17}$$

Then we have

$$Q\varphi_2 - \varphi_1 Q\varphi_1 + \widetilde{\mathbf{B}}(x)\varphi(u) = \mathbf{B}\varphi(u). \tag{4.18}$$

The condition of solvability of Eq. (4.18) with regard for solution (4.17) yields assertion (4.12)–(4.13) of Theorem 4.2. □

4.3. Large deviations with shift

Large deviations under the condition of balance (local or total) are characterized by the evolutionary operator that is generated by the quadratic form (4.2)–(4.3) in Subsection 4.1 and (4.12)–(4.13) in Subsection 4.2.

At the same time, the scheme of asymptotically small diffusion can involve "small jumps" generating the linear term in the exponential generator (see [3]). In this case, the linear generator of a random evolution must contain the component of a slow shift generated by small jumps (compare with (3.38)):

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = [\varepsilon^{-1}b(u;x) + b_1(u;x)]\varphi'(u) + \varepsilon\frac{1}{2}B(u;x)\varphi''(u) + \varepsilon\delta_\Gamma^\varepsilon(x)\varphi(u). \tag{4.19}$$

In this case, it is sufficient to assume that the Lévy measure $\Gamma^\varepsilon(u, dv; x)$ depends on the parameter ε of a series so that the following asymptotic representation of the first moments of an evolution is valid:

$$b^\varepsilon(u;x) := \int_{R^d} v\Gamma^\varepsilon(u, dv; x) = b(u;x) + \varepsilon b_1(u;x) + \varepsilon\delta_b^\varepsilon(u;x). \tag{4.20}$$

Under the additional *condition of small shift* (4.20), the generator of the evolutionary component admits the asymptotic decomposition (compare with (3.39))

$$\begin{aligned}\mathbf{\Gamma}^\varepsilon(x)\varphi(u) &= \varepsilon^{-3}\int_{R^d}[\varphi(u+\varepsilon^2 v) - \varphi(u)]\Gamma^\varepsilon(u, dv; x)\\ &= [\varepsilon^{-1}b(u;x) + b_1(u;x)]\varphi'(u) + \varepsilon\frac{1}{2}B(u;x)\varphi''(u) + \varepsilon\delta_\Gamma^\varepsilon(x)\varphi(u).\end{aligned} \tag{4.21}$$

There arises the *problem of singular perturbation* for the truncated linear generator of a random evolution

$$\mathbb{L}^\varepsilon\varphi(u,x) = [\varepsilon^{-3}Q + \varepsilon^{-1}\mathbf{\Gamma}(x) + \mathbf{\Gamma}_1(x) + \varepsilon\mathbf{B}(x)]\varphi(u,x). \tag{4.22}$$

Lemma 4.3. *Generator* (4.22) *admits the asymptotic representation*

$$\mathbb{L}^\varepsilon\varphi^\varepsilon(u,x) = [\widehat{b}_1(u) + \varepsilon\widehat{b}_0(u)]\varphi'(u) + \varepsilon\frac{1}{2}\widehat{B}(u)\varphi''(u) + \varepsilon\delta_L^\varepsilon(x)\varphi(u) \tag{4.23}$$

on the perturbed test-functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon^2\varphi_1(u,x) + \varepsilon^3\varphi_2(u,x).$$

Here, the function of variations $\widehat{B}(u)$ *is given by formulas* (3.42), *the coefficient of shift is given by formulas*

$$\begin{aligned} \widehat{b}_1(u) &= \int_E \pi(dx) b_1(u;x), \\ \widehat{b}_0(u) &= \int_E \pi(dx) b^*(u;x) R_0 b'_u(u;x). \end{aligned} \tag{4.24}$$

The slow shift in the asymptotic representation (4.23) *prescribed by the operator*

$$\widehat{\mathbf{\Gamma}}_1 \varphi(u) := \widehat{b}_1(u)\varphi'(u) \tag{4.25}$$

arises also in the exponential generator of large deviations for random evolutions with shift.

Theorem 4.3. *The exponential generator of large deviations for random evolutions with locally independent increments prescribed by the linear generator*

$$\mathbb{L}^\varepsilon \varphi(u,x) = [\varepsilon^{-3} Q + \mathbf{\Gamma}^\varepsilon(x)]\varphi(u,x), \tag{4.26}$$

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = \varepsilon^{-3} \int_{R^d} [\varphi(u+\varepsilon^2 v) - \varphi(u)]\Gamma^\varepsilon(u, dv; x),$$

under the additional condition of small shift (4.20) *is prescribed by the relation*

$$\mathbb{H}\varphi(u) = \frac{1}{2}\widehat{B}(u)[\varphi'(u)]^2 + \widehat{b}_1(u)\varphi'(u). \tag{4.27}$$

Here, the function of variations $\widehat{B}(u)$ *is given by formulas* (4.3) *under the condition of local balance and by formulas* (4.13) *under the condition of total balance.*

The proof of Theorem 4.3 is based on the asymptotic representation of the exponential generator of a random evolution on the perturbed test-functions (4.14).

Lemma 4.4. *The exponential generator* (4.1) *for a random evolution that is prescribed by generator* (4.19) *admits the asymptotic representation* (*compare with* (4.15))

$$\mathbf{H}^\varepsilon \varphi^\varepsilon(u,x) = \varepsilon^{-1}[Q\varphi_1 + \mathbf{\Gamma}(x)\varphi(u)] + [Q\varphi_2 - \varphi_1 Q\varphi_1 + [\mathbf{\Gamma}_1(x) + \widetilde{\mathbf{B}}(x)]]\varphi(u) \tag{4.28}$$

on the perturbed test-functions

$$\varphi^\varepsilon(u) + \varepsilon \log[1 + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)].$$

The proof of Lemma 4.4 coincides, in essence, with that of Lemma 4.2.

The exponential generator of a random evolution under the condition of local balance (3.13) is calculated analogously.

It is worth noting that the algorithms of calculation of exponential generators under the conditions of local balance (3.13) and total balance (3.23) are significantly different. It is sufficient to compare Lemmas 4.1 and 4.2.

5. Concluding remarks

Remark 5.1. The exponential generators of large deviations (4.2)–(4.3) and (4.12)–(4.13) in the Euclidean space R^d, $d \geq 1$, can be represented by the quadratic form

$$H\varphi(u) = \frac{1}{2}\sum_{k,r=1}^{d} B_{kr}(u)\varphi_k'(u)\varphi_r'(u), \tag{5.1}$$

$$\varphi_k'(u) := \partial\varphi(u)/\partial u_k, \quad 1 \leq k \leq d.$$

In addition, the exponential generator (5.1) can be extended on the space of absolutely continuous functions [2]

$$C_b^1(R^d) = \Big\{\varphi : \ \exists \lim_{|u|\to\infty} \varphi(u) = \varphi(\infty), \ \lim_{|u|\to\infty} \varphi'(u) = 0\Big\}. \tag{5.2}$$

The variational representation of the compensating function of deviations is realized in the space $C_b^1(R^d)$.

Remark 5.2. The problems of large deviations for random processes are realized in four stages [2, Part I].

Stage 1. Limit behavior of an exponential operator in the scheme of series.

Stage 2. Exponential density of Markov processes.

Stage 3. The principle of comparison for a limit exponential generator.

Stage 4. Variational representation of the functional of action.

The exponential generator of large deviations for random evolutions in the scheme of asymptotically small diffusion in Theorems 4.1–4.3 is given by the univalent function

$$H(u;p) = \frac{1}{2}\sum_{k,r=1}^{d} B_{kr}(u)p_k p_r + \sum_{k=1}^{d} b_k p_k \tag{5.3}$$

for which stages 2–4 are considered in monograph [2] (see also [3]). Hence, the function of deviations is prescribed by the Frechét–Legendre transformation

$$L(u;q) = \sup_{p\in R^d}\{p\cdot q - H(u;p)\}. \tag{5.4}$$

The functional of action is determined by the relation

$$J_T(u) = \int_0^T L(u(t),\dot{u}(t))\,dt. \tag{5.5}$$

Remark 5.3. The exponential generators for *random evolutions* with locally independent increments in the scheme of asymptotically small diffusion that are presented by Theorems 4.1–4.3 coincide with the exponential generators for *random processes* with locally independent increments in the same scheme of asymptotically small diffusion. This natural fact can be formulated with the use of heuristic arguments. However, the problem of substantiation of heuristic arguments remains open. The simplest is the procedure of averaging of the measure Lévy of a random evolution over the stationary distribution of the switching Markov process (see [4, Chap. 3])

$$\widehat{\Gamma}(u,dv) := \int_E \pi(dx)\Gamma(u,dv;x).$$

In this case, the problem of connection of a random evolution characterized by the Lévy measure $\Gamma(u, dv; x)$ with the random process characterized by the averaged Lévy measure $\widehat{\Gamma}(u, dv)$ remains open. With the same success, it is possible to assert (on the heuristic level) that the principle of averaging for a random evolution prescribed by the evolution equation is valid. However, in order to substantiate the principle of averaging, we need to use a solution of the problem of singular perturbation (see [4, Chap. 5]) or at least the ergodic Birkhoff–Khinchin theorem. The seeming complexity of efforts aimed at the substantiation of the principle of averaging at the heuristic obviousness of the principle only confirms the presence of heuristically simple assertions in mathematics, which require a complicated mathematical apparatus for their substantiation.

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