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SPECTRAL ESTIMATES FOR SCHRÖDINGER OPERATORS WITH SPARSE POTENTIALS ON GRAPHS

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A construction of "sparse potentials," suggested by the authors for the lattice $\mathbb{Z}^d$, $d > 2$, is extended to a large class of combinatorial and metric graphs whose global dimension is a number $D > 2$. For the Schrödinger operator $-\Delta - \alpha V$ on such graphs, with a sparse potential V, we study the behavior (as $\alpha \to \infty$) of the number $N_-(-\Delta - \alpha V)$ of negative eigenvalues of $-\Delta - \alpha V$. We show that by means of sparse potentials one can realize any prescribed asymptotic behavior of $N_-(-\Delta - \alpha V)$ under very mild regularity assumptions. A similar construction works also for the lattice $\mathbb{Z}^2$, where $D = 2$. Bibliography: 13 *titles.*

1 Introduction

There exists a far-reaching parallelism between the theory of the Schrödinger operator on $\mathbb{R}^d$ and its discrete analogue on $\mathbb{Z}^d$. Still, there are issues where this parallelism is violated. Certainly, the most known examples of such a violation come from the fact that, unlike the case of $\mathbb{R}^d$, the Laplacian on $\mathbb{Z}^d$ is a bounded operator. However, there are also examples of a different origin. One of them concerns estimates for the number $N_-(-\Delta - \alpha V)$ of the negative eigenvalues for the operator $-\Delta - \alpha V$, where $V \geqslant 0$ is a potential and $\alpha > 0$ is a large parameter (the "coupling constant"). Both for $\mathbb{R}^d$ and $\mathbb{Z}^d$ the "Rozenblum–Lieb–Cwikel estimate" is satisfied: namely,

$$N_-(-\Delta - \alpha V) \leqslant C\alpha^{d/2}\|V\|_{d/2}^{d/2}, \quad C = C(d), \quad d \geqslant 3. \tag{1.1}$$

Here, $\|\cdot\|_q$ stands for the L^q-norm in the continuous case and for the ℓ^q-norm in the discrete case. In both cases, the inequality (1.1) is valid for any potential V such that the norm on the

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right-hand side is finite. The difference between these cases is as follows:

$$\text{on } \mathbb{R}^d : N_-(-\Delta - \alpha V) = o(\alpha^{d/2}) \ \text{ if and only if } \ V \equiv 0,$$

while

$$\text{on } \mathbb{Z}^d : N_-(-\Delta - \alpha V) = o(\alpha^{d/2}) \ \text{ for any } \ V \in \ell^{d/2}. \tag{1.2}$$

This latter effect was, probably, observed for the first time in [1]. The same effect manifests itself if, instead of the Schrödinger operator on $\mathbb{Z}^d$, we consider the Schrödinger operator on an arbitrary combinatorial or metric graph whose "global dimension" is greater than 2; this was shown in [2]. Here, the global dimension is defined as the exponent D such that the heat kernel $P(t; x, y)$ corresponding to the Laplacian on the given graph satisfies the estimate

$$\|P(t; \cdot, \cdot)\|_{L^\infty} = O(t^{-D/2}), \quad t \to \infty. \tag{1.3}$$

For the graph $\mathbb{Z}^d$ one has $D = d$.

For the discrete case, in view of (1.2), even a single example of a potential such that

$$N_-(-\Delta - \alpha V) \asymp \alpha^{d/2}$$

was unknown for some time. Only in [3], the authors suggested a general scheme that allowed us (for $G = \mathbb{Z}^d$) to construct discrete potentials such that the function $N_-(-\Delta - \alpha V)$ has any prescribed asymptotic behavior as $\alpha \to \infty$ under very mild regularity conditions. This includes also the case

$$N_-(-\Delta - \alpha V) \sim C\alpha^{d/2}$$

with $C > 0$. This scheme was based upon some asymptotic estimates for $N_-(-\Delta - \alpha V)$ whose nature differs drastically from the estimate (1.1) and the majority of other estimates described in [1, 2]. Namely, these latter ones are permutation–invariant with respect to the potential V. In contrast, the new estimates and asymptotic formulas obtained in [3] depend not only on the size of V, but also on the geometry of its support. More exactly, the potential V is supposed to be supported on a sequence of very sparsely placed points. The eigenvalue distribution results obtained for such *sparse potentials* are quite flexible. On the other hand, this class of potentials is rather special.

In the present paper, we show that the construction of sparse potentials *with a prescribed eigenvalue behavior* is not restricted to $G = \mathbb{Z}^d$, $d > 2$, but can be, with minor modifications, extended to a large class of combinatorial graphs with $D > 2$. This will be done in Section 2.

The case $D = 2$ is critical in this topic; the general scheme of the spectral analysis of the Schrödinger operator breaks down at many points (as it also happens to the classical Schrödinger operator in $\mathbb{R}^2$). In Section 3, we show, however, that the approach based upon sparse potentials works in the important case $G = \mathbb{Z}^2$ as well (although using some more specific analytical tools.)

Finally, in Section 4, we show that this approach can be successfully modified to the Schrödinger operator on such metric graphs for which the associated weighted combinatorial graph admits the construction of sparse potentials.

Sparse potentials were introduced by Pearson [4] in the spectral theory of the one-dimensional Schrödinger operator and were considered in higher dimensions by Molchanov and Vainberg [5, 6]. Compared with our papers, these authors analyzed problems of a different type, and they used another definition of sparseness. They did not discuss sparse potentials on graphs different from $\mathbb{Z}^d$.

2 Operators on Combinatorial Graphs

2.1 Operator $\mathbf{B}_{V,G}$

Let G be a combinatorial graph with the set of vertices $\mathscr{V} = \mathscr{V}(G)$ and the set of edges $\mathscr{E} = \mathscr{E}(G)$. We write $e = (v, v')$ for the edge whose endpoints are the vertices v and v'. We then also say that v and v' are *neighboring vertices* and write $v \sim v'$. For the sake of simplicity, we assume that the graph is connected and has no loops, multiple edges, or vertices of degree one. We suppose that $\#\mathscr{V} = \infty$ and degrees of all vertices are finite. On $\mathscr{V}$, we consider the standard counting measure σ. So, $\sigma(v) = 1$ for each vertex v. Our basic Hilbert space is $\ell^2 = \ell^2(G) = L^2(\mathscr{V}; \sigma)$. We also denote $\ell^q = \ell^q(G) = L^q(\mathscr{V}; \sigma)$, $0 < q \leqslant \infty$.

With each edge $e \in \mathscr{E}$ we associate a weight $g_e > 0$. On $\ell^2(G)$, we consider the quadratic form

$$\mathbf{a}_G[f] = \sum_{e \in \mathscr{E}; e=(v,v')} g_e |f(v) - f(v')|^2 \tag{2.1}$$

with the natural domain $\operatorname{Dom} \mathbf{a}_G = \{f \in \ell^2(G) : \mathbf{a}_G[f] < \infty\}$. This quadratic form is nonnegative and closed. Recall that, by definition, the latter means that $\operatorname{Dom} \mathbf{a}_G$ is complete with respect to the norm $(\mathbf{a}_G[f] + \|f\|^2)^{1/2}$. Denote by $-\Delta_G$ the selfadjoint operator on ℓ^2, associated with the quadratic form $\mathbf{a}_G$. It is easy to see that on its domain the operator Δ_G acts according to the formula

$$(\Delta_G f)(v) = \sum_{v' \sim v} f(v') g_{(v,v')} - f(v) \sum_{v' \sim v} g_{(v,v')}. \tag{2.2}$$

If the weights g_e and the degrees $\deg v$ of all vertices are uniformly bounded, then the quadratic form (2.1) and hence the operator Δ_G are bounded.

Due to the embeddings $\ell^1 \subset \ell^2 \subset \ell^\infty$, any operator, bounded in ℓ^2, is bounded also as acting from ℓ^1 to ℓ^∞, with the corresponding estimate for the norms. This applies, in particular, to the operators $\exp(\Delta_G t)$ and, therefore, the heat kernel satisfies the estimate

$$\|P(t; \cdot, \cdot)\|_{L^\infty} \leqslant C < \infty \quad \forall t \in \mathbb{R}.$$

Our main assumption here is that (1.3) is satisfied with some $D > 2$. Then by the Varopoulos theory [7], we have

$$\|f\|^2_{\ell^p} \leqslant C \mathbf{a}_G[f], \quad p = p(D) = D(D-2)^{-1}, \quad \forall f \in \operatorname{Dom} \mathbf{a}_G. \tag{2.3}$$

In particular, (2.3) is valid for any function with finite support. Denote by $\mathscr{H}(G)$ the completion of the set of all such functions in the metric generated by the quadratic form $\mathbf{a}_G$. This is a Hilbert space. The estimate (2.3) implies that $\mathscr{H}(G)$ can be realized as a space of functions embedded into ℓ^p. It follows that any function $f \in \mathscr{H}(G)$ tends to zero "at infinity" (the notation $f \to 0$). More exactly, $f \to 0$ means that for any $\varepsilon > 0$ the set $\{v \in \mathscr{V} : |f(v)| > \varepsilon\}$ is finite.

Below, we systematically use the Birman–Schwinger operator $\mathbf{B}_{V,G}$ that corresponds to the Schrödinger operator $-\Delta - \alpha V$ (cf., for example, [1] and the references therein). Recall that, in our case, $\mathbf{B}_{V,G}$ is the operator in the space $\mathscr{H}(G)$ generated by the quadratic form

$$\mathbf{b}_{V,G}[f] = \sum_{v \in \mathscr{V}} V(v)|f(v)|^2. \tag{2.4}$$

The eigenvalue distribution characteristics for the operators $-\Delta_G - \alpha V$ in $\ell^2(G)$ and $\mathbf{B}_{V,G}$ in $\mathscr{H}(G)$ are connected by the famous Birman–Schwinger principle: for any $\alpha > 0$

$$N_-(-\Delta_G - \alpha V) = n(\alpha^{-1}, \mathbf{B}_{V,G}). \tag{2.5}$$

Here,

$$n(s, \mathbf{T}) = \#\{n \in \mathbb{N} : \lambda_n(\mathbf{T}) > s\}, \quad s > 0,$$

stands for the eigenvalue distribution function of a compact nonnegative operator $\mathbf{T}$. Moreover, the following two properties are equivalent.

(A) The operator $\mathbf{B}_{V,G}$ is compact.

(B) $N_-(-\Delta_G - \alpha V) < \infty$ for all $\alpha > 0$.

The equality (2.5) allows one to formulate spectral estimates and asymptotic formulas for $-\Delta_G - \alpha V$ in the terms of the operator $\mathbf{B}_{V,G}$. For instance, the estimate (1.1) means that for $V \in \ell^{d/2}(\mathbb{Z}^d)$ the operator $\mathbf{B}_{V,G}$ belongs to the "weak Neumann–Schatten ideal" $\Sigma_{d/2}$ and its norm in this ideal satisfies the estimate

$$\|\mathbf{B}_{V,G}\|_{\Sigma_{d/2}} \leqslant C\|V\|_{d/2}$$

(cf. [8, § 11.6] for the definition and basic properties of the classes Σ_q).

Below, we describe estimates of a completely different nature. They are valid for a rather large class of graphs and a special, but rather restricted class of potentials. To describe these estimates, we need some properties of the Green function for the Laplacian Δ_G. Recall that we always assume $D > 2$.

We denote by $(\cdot, \cdot)_{\mathscr{H}(G)}$ the scalar product in $\mathscr{H}(G)$ so that

$$(f, g)_{\mathscr{H}(G)} = \mathbf{a}_G[f, g].$$

Up to the end of this section, we drop the subscript $\mathscr{H}(G)$ in this notation and in the notation of the corresponding norm.

2.2 The Green function

For a fixed $v \in \mathscr{V}$ we consider the linear functional on $\mathscr{H}(G)$:

$$\varphi_v(f) = f(v).$$

In view of (2.3), this functional is continuous on $\mathscr{H}(G)$, and therefore there exists a unique function $h_v \in \mathscr{H}(G)$ such that

$$(f, h_v) = f(v) \quad \forall f \in \mathscr{H}(G). \tag{2.6}$$

Thus, the function h_v satisfies the equation

$$-\Delta h_v(w) = \delta_w^v \tag{2.7}$$

and is, obviously, real-valued. It is natural to call it the *Green function* for the operator Δ_G. Taking $f = h_w$ in (2.6), we obtain

$$(h_w, h_v) = h_w(v) = h_v(w) \quad \forall v, w \in \mathscr{V}. \tag{2.8}$$

By (2.3), this implies that

$$h_w(v) \to 0 \quad \text{as } w \text{ is fixed.} \tag{2.9}$$

For $w = v$ the equality (2.8) gives

$$(h_v, h_v) = h_v(v).$$

Hence all the numbers $\mu_v^2 = h_v(v)$ are positive. We normalize (in $\mathscr{H}(G)$) the functions h_v, i.e., we set $\widetilde{h}_v = \mu_v^{-1} h_v$.

The function $\mu(v)^{-1}\widetilde{h}_v$ can be also defined as a *unique* function $f \in \mathscr{H}(G)$ minimizing the quadratic functional $\|f\|^2$ under the condition $f(v) = 1$. Note also that the number μ_v is nothing but the capacity of the one-point set $\{v\}$ with respect to the quadratic form $\mathbf{a}_G$.

2.3 Sparse potentials

For a graph G, we introduce the notion of a *sparse subset* $Y \subset \mathscr{V}$ in the same way as this was done for $G = \mathbb{Z}^d$ in [3, Section 6.2]. Namely, let $Y \subset G$ be an infinite set, and let $\mathscr{H}_Y$ stand for the subspace in $\mathscr{H}(G)$ spanned by the functions $\{h_y : y \in Y\}$. We say that Y is *sparse* if there exists a compact linear operator $\mathbf{T}$ in $\mathscr{H}(G)$ such that the operator $\mathbf{I} - \mathbf{T}$ has bounded inverse and the functions $e_v = (\mathbf{I} - \mathbf{T})^{-1}\widetilde{h}_v$ form an orthonormal (not necessarily complete) system in $\mathscr{H}(G)$. We do not discuss here the weakly sparse subsets also introduced in [3].

We cannot claim that every graph G with $D > 2$ contains a sparse subset. In Subsection 2.4 we give a simple condition of a geometric nature that guarantees the existence of such subsets.

We say that a function V on G (the discrete potential) is *sparse* if its support $Y_V = \{v \in \mathscr{V} : V(v) \neq 0\}$ is a sparse subset. Due to this definition, the quadratic form (2.4) can be written as

$$\mathbf{b}_{V,G}[f] = \sum_{v \in Y_V} V(v)|(f, \widetilde{h}_v)|^2 = \sum_{v \in Y_V} V(v)|(f, (\mathbf{I} - \mathbf{T})e_v)|^2.$$

If, in addition, $V \to 0$, we always enumerate the points $v \in Y_V$, $v = v_n$, in such a way that the sequence $V_n = V(v_n)$ is monotone.

Due to the sparseness of Y_V, the system of elements $\widetilde{h}_v$, $v \in Y_V$, is close to an orthonormal system in $\mathscr{H}(G)$. Hence the properties of the operator $\mathbf{B}_{V,G}$ should be close to those of the diagonal operator with the entries $V(v)$, $v \in Y_V$. More precisely, consider the operator

$$\mathbf{N}_V = \sum_{v \in Y_V} \sqrt{V(v)}(\cdot, e_v)e_v.$$

We have

$$\mathbf{N}_V(\mathbf{I} - \mathbf{T}^*) = \sum_{v \in Y_V} \sqrt{V(v)}(\cdot, \widetilde{h}_v)e_v.$$

Hence

$$\|\mathbf{N}_V(\mathbf{I} - \mathbf{T}^*)f\|^2_{\mathscr{H}(G)} = \sum_{v \in Y_V} V(v)|(f, \widetilde{h}_v)|^2_{\mathscr{H}(G)} = \mathbf{b}_{V,G}[f].$$

This means that

$$\mathbf{B}_{V,G} = (\mathbf{I} - \mathbf{T})\mathbf{N}_V^2(\mathbf{I} - \mathbf{T}^*). \tag{2.10}$$

The first property of $\mathbf{B}_{V,G}$ follows from (2.10) immediately (cf. [3, Theorems 6.3 and 6.4]).

Theorem 2.1. *Let $V \geqslant 0$ be a sparse potential on a weighted graph G with global dimension $D > 2$. Then the corresponding Birman–Schwinger operator $\mathbf{B}_{V,G}$ is bounded if and only if the function V is bounded, and is compact if and only if $V \to 0$. Moreover,*

$$C\|V\|_\infty \leqslant \|\mathbf{B}_{V,G}\| \leqslant C'\|V\|_\infty,$$

where $C = \|(\mathbf{I}-\mathbf{T})^{-1}\|^{-2}$ and $C' = \|\mathbf{I}-\mathbf{T}\|^2$. In the case of compactness, the following two–sided estimate, with the same constants C and C', is valid for the eigenvalues $\lambda_n(\mathbf{B}_{V,G})$:

$$CV_n \leqslant \lambda_n(\mathbf{B}_{V,G}) \leqslant C'V_n.$$

Recall that, in the case of compactness, we have $V_n \searrow 0$ just due to the way of enumeration.

The following result is more advanced. It extends Theorem 6.6 in [3] (where $G = \mathbb{Z}^d$) to the case of general graphs. The proof survives. Still, we reproduce it here to make it possible to read the present paper independently.

We derive this result under an additional condition on the potential V: the sequence $\{V_n\}$ is *moderately varying.* This means that $V_n \searrow 0$ and $V_{n+1}/V_n \to 1$. We use the result due to M. G. Krein [9, Theorem 5.11.3]. Below, we present its formulation, restricting ourselves to the situation we need.

Proposition 2.2. *Let $\mathbf{H} \geqslant 0$ and $\mathbf{S}$ be selfadjoint and compact operators. Suppose that* $\operatorname{rank}\mathbf{H} = \infty$ *and the sequence of nonzero eigenvalues $\lambda_n(\mathbf{H})$ is moderately varying. Then for the operator $\mathbf{M} = \mathbf{H}(\mathbf{I}+\mathbf{S})\mathbf{H}$*

$$\lambda_n(\mathbf{M}) \sim \lambda_n^2(\mathbf{H}).$$

Now we are in a position to prove the following result.

Theorem 2.3. *Let $V \geqslant 0$ be a sparse potential such that the numbers V_n form a moderately varying sequence. Then*

$$\lambda_n(\mathbf{B}_{V,G}) \sim V_n.$$

Proof. In view of (2.10), the nonzero spectrum of the operator $\mathbf{B}_{V,G}$ coincides with that of the operator

$$\mathbf{M}_V := \mathbf{N}_V(\mathbf{I} - \mathbf{T}^*)(\mathbf{I} - \mathbf{T})\mathbf{N}_V = \mathbf{N}_V(\mathbf{I} + \mathbf{S})\mathbf{N}_V, \tag{2.11}$$

where $\mathbf{S} = -\mathbf{T} - \mathbf{T}^* + \mathbf{T}^*\mathbf{T}$. Now, we apply Proposition 2.2 to the operators $\mathbf{H} = \mathbf{N}_V$ and $\mathbf{M}_V$ given by (2.2) and (2.11) respectively. All the assumptions of the proposition are evidently satisfied, and we get the desired result. □

In particular, taking $V_n = n^{-2/D}$, we obtain a potential V such that the eigenvalues $\lambda_n(\mathbf{B}_{V,G})$ asymptotically behave as $n^{-2/D}$. This solves a problem discussed in Section 1.

2.4 On the existence of sparse subsets

Given a vertex $v \in \mathscr{V}$, let us consider a function f_v that vanishes at all vertices $w \neq v$ and $f_v(v) = 1$. By the extremal property of the Green function h_v, we have

$$\mu_v^{-2} = \mu_v^{-2}\|\widetilde{h}_v\|^2 \leqslant \|f_v\|^2 = \mathbf{a}_G[f_v] = \sum_{w\sim v} g_{(v,w)}.$$

Hence

$$\mu_v^2 = h_v(v) \geqslant \left(\sum_{w\sim v} g_{(v,w)}\right)^{-1}.$$

Given a number $R > 0$, we say that a vertex v is *R–mild* if

$$\sum_{w\sim v} g_{(v,w)} \leqslant R.$$

Proposition 2.4. *Suppose that a graph G is such that $D > 2$ and for some $R > 0$ the graph G contains infinitely many R-mild vertices. Then G contains a sparse subset.*

Proof. We fix an infinite subset $G' \subset G$ consisting of R-mild vertices. Choose any double sequence $\{\varepsilon_{mn}\}$, $m, n \in \mathbb{N}$, $m \neq n$, of positive numbers such that $\varepsilon_{mn} = \varepsilon_{nm}$ and $\sum \varepsilon_{mn}^2 < R^{-2}$. Suppose that the vertices $v_1, \dots, v_{n-1}$ are already chosen (here, v_1 is arbitrary). Then, in view of (2.9), we can choose a point $v_n \in G'$ in such a way that $|(h_{v_k}, h_{v_n})| < \varepsilon_{kn}$ for $k = 1, \dots, n-1$. As a result of this inductive procedure, we get a sequence of vertices $\{v_n\} \subset G'$ such that $(h_{v_n}, h_{v_n}) = \mu_{v_n}^2$ and $|(h_{v_m}, h_{v_n})| < \varepsilon_{mn}$ if $m < n$. By (2.8), the same inequality holds for $m > n$. Since all the vertices v_n are R-mild, we have $|(\widetilde{h}_{v_n}, \widetilde{h}_{v_n})| < R\varepsilon_{mn}$.

It follows that the Gram matrix $\mathbf{M} = \{(\widetilde{h}_{v_n}, \widetilde{h}_{v_n})\}$ is such that $\mathbf{M} - \mathbf{I}$ is Hilbert–Schmidt; moreover, $\|\mathbf{M} - \mathbf{I}\|_{\mathfrak{S}_2} < 1$. By Theorem VI.3.3 in [9], this implies the sparseness of the set Y with $\mathbf{T} \in \mathfrak{S}_2$ (cf. [3, Section 6.2], where the case of $G = \mathbb{Z}^d$ was analyzed in more detail). ☐

3 Operators on $\mathbb{Z}^2$

It is well known that a lot of complications arise in the study of the classical Schrödinger operator in $\mathbb{R}^2$, compared to the higher-dimensional case. They are related to the nonsigndefiniteness of the fundamental solution, its logarithmic singularity, the absence of the proper sharp Sobolev embedding theorem, and so on. One of the important consequences is the absence of a direct analogy of the Birman–Schwinger principle (2.5) since the closure of the space of compactly supported functions in the metric of the Dirichlet integral is not a space of functions any more.

We encounter the same complication if we try to extend the general reasoning in the previous section to the case of operators on combinatorial graphs with global dimension $D = 2$. Similar to the case of $\mathbb{R}^2$, much more specific instruments are needed.

In this section we present an approach to the eigenvalue analysis of the Schrödinger operator on $\mathbb{Z}^2$.

We consider the graph G whose set of vertices $\mathscr{V}$ is the lattice $\mathbb{Z}^2$ which we understand as embedded in the natural way into $\mathbb{R}^2$. Two vertices are said to be *connected with an edge* if they are the closest neighbors in $\mathbb{R}^2$ and the weight of any such edge is taken as 1. It is convenient to identify a vertex $v \in \mathbb{Z}^2$ with its Cartesian coordinates $x = (x_1, x_2) \in \mathbb{Z}^2 \subset \mathbb{R}^2$.

3.1 The Birman–Schwinger principle, a Hardy type inequality, and the space $\mathscr{H}_0$

Following Section 2, we introduce the quadratic form

$$\mathbf{a}_{\mathbb{Z}^2}[f] = \sum_{x,x' \in \mathbb{Z}^2, x \sim x'} |f(x) - f(x')|^2. \tag{3.1}$$

This quadratic form is bounded in $\ell^2 = \ell^2(\mathbb{Z}^2)$ and defines the selfadjoint operator $-\Delta$. For a bounded real-valued function $V(x)$ on $\mathbb{Z}^2$ and a coupling constant $\alpha \geqslant 0$, we consider the Schrödinger operator $-\Delta - \alpha V$, and our concern is in the study of the behavior of $N_-(-\Delta - \alpha V)$ as $\alpha \to \infty$.

So far so good, however, if we try to apply the Birman–Schwinger principle, as we did it above, we fail. The reason for this lies in the fact that the space $\mathscr{H}$, the closure of the space of finitely supported functions in the metric (3.1), is not a space of functions any more. In fact, it is easy to construct a family of functions u_n on $\mathbb{Z}^2$ converging to a nonzero constant pointwise, so that $\mathbf{a}_G[u_n] \to 0$. This is also related to the behavior of the heat kernel for $-\Delta$ on $\mathbb{Z}^2$. Standard calculations by means of the Fourier series show that the heat kernel $P(t; \cdot, \cdot)$ decays exactly as t^{-1} as $t \to \infty$, so the global dimension is equal to 2 and the Varopoulos theory does not apply.

In order to handle these inconveniences, it is sufficient to impose the Dirichlet boundary condition at one point, say at $x = \mathbf{0} = (0, 0)$.

So, denote by ℓ^2_0 the (obviously closed) subspace in ℓ^2 consisting of functions with zero value at $\mathbf{0}$. The quadratic form (3.1) defines a selfadjoint operator $-\Delta_0$ in ℓ^2_0, and we denote by $-\Delta_0 - \alpha V$ the corresponding Schrödinger operator.

Since the subspace ℓ^2_0 has codimension 1 in ℓ^2, the standard application of the variational principle gives

$$N_-(-\Delta_0 - \alpha V) \leqslant N_-(-\Delta - \alpha V) \leqslant N_-(-\Delta_0 - \alpha V) + 1. \tag{3.2}$$

Therefore, for our goal it suffices to study the operator $-\Delta_0 - \alpha V$.

The minor reduction above brings a great advantage: applying the Birman–Schwinger principle becomes possible now. To explain this, we start by establishing a Hardy type inequality in ℓ^2_0.

Proposition 3.1. *For some constant C and any function $f \in \ell^2_0$ with compact support*

$$\sum_{x \in \mathbb{Z}^2 \setminus \mathbf{0}} |f(x)|^2 |x|^{-2} (\log|x| + 2)^{-2} \leqslant C \mathbf{a}_{\mathbb{Z}^2}[f]. \tag{3.3}$$

Proof. The inequality (3.3) is proved in a way similar to the discrete Hardy type inequality in [3, Section 4] by interpolating a function on $\mathbb{Z}^2$ to a function on $\mathbb{R}^2$. In all squares in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$, we apply the same piecewise linear interpolation as in [3]. Additionally, for the four central squares, the ones having the vertex $\mathbf{0}$, we multiply the interpolating function by the cut-off $\varphi(|x|)$, which vanishes for $|x| < \frac{1}{2}$ and is equal to 1 for $|x| \geqslant \frac{3}{4}$. The inequality (3.3) follows now by applying the classical Hardy inequality in $\mathbb{R}^2$ with log term (valid for functions vanishing in a fixed neighborhood of the origin) to the interpolant. ◻

We define the space $\mathscr{H}_0$ as the closure of the set of finitely supported functions in ℓ^2_0 in the metric (3.1). It follows from (3.3) that, on this set, the convergence with respect to (3.1) implies the convergence in ℓ^2 with the weight as in (3.3). Therefore, $\mathscr{H}_0$ is a space of functions.

Now we are able to define the Birman–Schwinger operator $\mathbf{B}^{(0)}_{V,\mathbb{Z}^2}$ in the space $\mathscr{H}_0$ by means of the quadratic form (2.4), obviously, closed. By the general Birman–Schwinger principle and (3.2), we have

$$n(\alpha^{-1}, \mathbf{B}^{(0)}_{V,\mathbb{Z}^2}) \leqslant N_-(-\Delta_0 - \alpha V) \leqslant n(\alpha^{-1}, \mathbf{B}^{(0)}_{V,\mathbb{Z}^2}) + 1.$$

3.2 The Green function

It follows from (3.3) that the linear functional $\varphi_x(f) = f(x)$ is continuous in $\mathscr{H}_0$ and defines a unique function $h_x \in \mathscr{H}_0$ such that

$$(f, h_x) = f(x) \quad \forall f \in \mathscr{H}_0. \tag{3.4}$$

Thus, the function h_x satisfies (2.7); it is real-valued and will be called, similar to the considerations in Section 2, the *Green function* for the operator Δ on $\mathbb{Z}^2$. The relation (2.8) holds too, but, at this point, the analogy with Section 2 stops. The reason for this is that we do not have the inequality (2.3) any more and the Hardy inequality (3.3) does not imply that $h_x(y) \to 0$ as $|x-y| \to \infty$. Moreover, the asymptotic formula (3.9) below shows that this property fails in the $\mathbb{Z}^2$-setting. Nevertheless, we again define $\widetilde{h}_x = \mu_x^{-1} h_x$, where $\mu_x^2 = (h_x, h_x) = h_x(x)$. Again, as in the case of graphs with $D > 2$ (cf. the end of Subsection 2.2), the function $\mu_x^{-1}\widetilde{h}_x$ is a unique minimizer (in the space $\mathscr{H}_0(\mathbb{Z}^2)$) of the functional (3.1) under the condition $f(x) = 1$.

We are going to find an explicit expression for h_x. To do this, we first consider the fundamental solution for the Laplacian on $\mathbb{Z}^2$ *without the boundary condition* at $\mathbf{0}$, i.e., the function $\mathscr{G}(x)$, $x \in \mathbb{Z}^2$, such that $\Delta \mathscr{G}(x) = \delta^x_{\mathbf{0}}$. Such a function, if exists, is not unique; it is defined up to a $\mathbb{Z}^2$-harmonic additive term. We are going to deal with a specific function $\mathscr{G}$.

For $\mathbb{Z}^d$ with $d > 2$ such a solution can be easily found by means of the Fourier series (cf. [3, formula (6.1)]). In the 2-dimensional case, however, the integral in formula (6.1) in [3] diverges, so a different approach is needed.

The first formula for $\mathscr{G}(x)$ was found in [10]. We do not reproduce it since it is not convenient for further calculations. We will use another formula for the fundamental solution presented, for example, in [11, Chapter 15, formula (1)]:

$$\mathscr{G}(x) = (2\pi)^{-2} \int\limits_{(-\pi,\pi)^2} \frac{\cos(x\theta) - 1}{Z(\theta)} d\theta, \tag{3.5}$$

where

$$Z(\theta) = Z(\theta_1, \theta_2) = 1 - \frac{1}{2}(\cos\theta_1 + \cos\theta_2), \quad x\theta = x_1\theta_1 + x_2\theta_2.$$

For our goal, it is more convenient to use the exponential form of (3.5):

$$\mathscr{G}(x) = (2\pi)^{-2} \int\limits_{(-\pi,\pi)^2} \frac{\exp(ix\theta) - 1}{Z(\theta)} d\theta. \tag{3.6}$$

So, $\mathscr{G}(\mathbf{0}) = 0$. It is also known [11, Chapter 15] that $\mathscr{G}(x)$ has the asymptotics

$$\mathscr{G}(x) \sim \frac{1}{\pi}\big(2\log|x| + \log 8 + 2\gamma\big), \quad |x| \to \infty, \tag{3.7}$$

uniformly in all directions (γ is the Euler constant).

We construct the function $H_x(y)$ as follows:

$$H_x(y) = -\mathscr{G}(x-y) + \mathscr{G}(x) + \mathscr{G}(y). \tag{3.8}$$

It is easy to check that $H_x(y) = H_y(x)$, $H_x(\mathbf{0}) = 0$, and $-\Delta_y H_x(y) = \delta_x^y$. It will be shown below that $H_x = h_x$.

By the asymptotics (3.7),

$$H_x(y) \sim \frac{1}{\pi}\big(2\log|x| + \log 8 + 2\gamma\big), \quad |y| \to \infty, \tag{3.9}$$

and

$$H_x(x) = 2\mathscr{G}(x) \sim \frac{2}{\pi}\big(2\log|x| + \log 8 + 2\gamma\big), \quad |x| \to \infty. \tag{3.10}$$

The following property is the most important one.

Proposition 3.2. *For any $x \in \mathbb{Z}^2 \setminus \mathbf{0}$ the function $H_x(y)$ belongs to $\mathscr{H}_0$.*

Proof. We denote $\mathbf{1}_2 = (0,1)$, fix $x \in \mathbb{Z}^2 \setminus \mathbf{0}$, and consider the expression

$$d_2 H_x(y) = H_x(y + \mathbf{1}_2) - H_x(y).$$

By (3.6),

$$\begin{aligned} d_2 H_x(y) &= (2\pi)^{-2} \int\limits_{(-\pi,\pi)^2} \frac{1}{Z(\theta)}\Big(e^{i(y+\mathbf{1}_2)\theta} - e^{iy\theta} - e^{i(y-x+\mathbf{1}_2)\theta} + e^{i(y-x)\theta}\Big)d\theta \\ &= -(2\pi)^{-2} \int\limits_{(-\pi,\pi)^2} \frac{(e^{i\mathbf{1}_2\theta} - 1)(e^{-ix\theta} - 1)}{Z(\theta)} \exp(iy\theta)d\theta. \end{aligned} \tag{3.11}$$

Considered as a function on the torus $(-\pi,\pi)^2$, the denominator in (3.11) has the only zero at the point $\theta = 0$ and $Z(\theta) \asymp |\theta|^2$ near $\theta = 0$. Both functions $e^{i\mathbf{1}_2\theta} - 1$ and $e^{-ix\theta} - 1$ vanish at $\theta = 0$, the main parts being degree one homogeneous in θ. Therefore, the function

$$K(\theta) = \frac{(e^{i\mathbf{1}_2\theta} - 1)(e^{-ix\theta} - 1)}{Z(\theta)}$$

is bounded on $(-\pi,\pi)^2$, in particular, it belongs to $L^2((-\pi,\pi)^2)$. So, the expression (3.11) represents the Fourier coefficient of the function $K(\theta)$. By the Plancherel equality, we have

$$d_2 H_x(\cdot) \in \ell^2(\mathbb{Z}^2).$$

In a similar way, the function $d_1 H_x(y) = H_x(y + \mathbf{1}_1) - H_x(y)$ (where $\mathbf{1}_1 = (1,0)$) belongs to $\ell^2(\mathbb{Z}^2)$. These two facts mean that $\mathbf{a}_{\mathbb{Z}^2}[H_x] < \infty$ for any x.

It remains to show that the function H_x can be approximated in the metric $\mathbf{a}_G$ by finitely supported functions. This is done in the standard way by setting $H_x^{(m)}(y) = H_x(y)F_m(|y|)$, where

$$F_m(s) = \begin{cases} 1, & s \leqslant 1, \\ (\log m)^{-1}(\log m - \log s), & s \in (1, m), \\ 0, & s \geqslant m. \end{cases}$$

Using that H_x is bounded and the above properties of $d_j H_x$, it is easy to show that $\mathbf{a}_{\mathbb{Z}^2}[H_x - H_x^{(m)}] \to 0$ as $m \to \infty$. $\square$

So, the functions H_x and h_x, both in $\mathscr{H}_0$, are solutions of the same equation. Due to the uniqueness in the construction of h_x, we have $h_x = H_x$, and all the properties established in this section for the functions H_x hold now for the functions h_x.

3.3 Sparse sets in $\mathbb{Z}^2$

The definition of a sparse set given in Subsection 2.3 requires a minor modification in the case of $\mathbb{Z}^2$: the space $\mathscr{H}$ should be replaced by $\mathscr{H}_0$. The next statement is an analogue of Proposition 2.4. However, the proof is a little bit different, it uses the asymptotic formula (3.7) and its consequences.

Proposition 3.3. *The graph $\mathbb{Z}^2$ contains a sparse set.*

Proof. For the functions $\widetilde{h}_x$ introduced in Subsection 3.2 we have

$$(\widetilde{h}_x, \widetilde{h}_y) = h_x(x)^{-\frac{1}{2}} h_y(y)^{-\frac{1}{2}} h_x(y).$$

For a fixed $x \in \mathbb{Z}^2$ we use the asymptotics (3.9) and (3.10) as $y \to \infty$. This gives

$$(\widetilde{h}_x, \widetilde{h}_y) \sim \mu_x^{-1}(\log|y|)^{-\frac{1}{2}}(2\log|x| + \log 8 + 2\gamma). \tag{3.12}$$

It is clear from (3.12) that $(\widetilde{h}_x, \widetilde{h}_y) \to 0$ as x is fixed and $y \to \infty$.

The rest of the proof uses the same inductive procedure as in Subsection 2.4. Fix a double sequence $\{\varepsilon_{mn}\}$, $m, n \in \mathbb{N}$, $m \neq n$, of positive numbers such that $\varepsilon_{mn} = \varepsilon_{nm}$ and $\sum \varepsilon_{mn}^2 < 1$. Suppose that the points $x_1, \dots, x_{n-1}$ are already chosen (x_1 is arbitrary). In view of (3.12), we can choose a point $x_n \in \mathbb{Z}^2 \setminus \mathbf{0}$ such that

$$|(\widetilde{h}_{x_m}, \widetilde{h}_{x_n})| < \varepsilon_{mn} \quad \forall m < n. \tag{3.13}$$

As a result of this inductive procedure, we obtain an infinite sequence $x_n \in \mathbb{Z}^2 \setminus \mathbf{0}$ such that (3.13) is valid for all $m, n \in \mathbb{N}$, $m < n$. By symmetry, this inequality is satisfied also for $m > n$.

It follows (exactly as it was in Subsection 2.4) that the Gram matrix $\mathbf{M} = \{(\widetilde{h}_{x_m}, \widetilde{h}_{x_n})\}$ is such that $\|\mathbf{M} - \mathbf{I}\|_{\mathfrak{S}_2} < 1$, and hence the set $Y = \{x_n\}$ is sparse by [9, Theorem VI.3.3]. $\square$

We define a *sparse potential* on $\mathbb{Z}^2$ as such whose support is a sparse subset. The spectral properties of the corresponding operator $\mathbf{B}^{(0)}_{V,\mathbb{Z}^2}$ and those of the operator $-\Delta_0 - \alpha V$ are studied in the same way as in Section 2. This leads to analogues of Theorems 2.1 and 2.3. The next statement is a combination of these both results. The proof remains the same and we skip it. We only remind that the operator $\mathbf{T}$ in the formulation is the one, appearing in the definition of a sparse subset (cf. Subsection 2.3).

Theorem 3.4. *Let $V \geqslant 0$ be a sparse potential on $\mathbb{Z}^2$. Then the following assertions hold.*

1. *The corresponding Birman–Schwinger operator $\mathbf{B}^{(0)}_{V,\mathbb{Z}^2}$ is bounded if and only if the function V is bounded, and is compact if and only if $V \to 0$. Moreover,*

$$C\|V\|_{\ell^\infty} \leqslant \|\mathbf{B}^0_{V,\mathbb{Z}^2}\| \leqslant C'\|V\|_{\ell^\infty},$$

where $C = \|(\mathbf{I}-\mathbf{T})^{-1}\|^{-2}$ and $C' = \|\mathbf{I}-\mathbf{T}\|^2$. In the case of compactness, the following two–sided estimate, with the same constants C and C', is valid for the eigenvalues $\lambda_n(\mathbf{B}^{(0)}_{V,\mathbb{Z}^2})$:

$$CV_n \leqslant \lambda_n(\mathbf{B}^{(0)}_{V,\mathbb{Z}^2}) \leqslant C'V_n.$$

2. *If, in addition, the numbers V_n form a moderately varying sequence, then*

$$\lambda_n(\mathbf{B}^0_{V,\mathbb{Z}^2}) \sim V_n.$$

4 Operators on Metric Graphs

Let Γ be a metric graph with edge lengths l_e. As in [2], we associate with Γ the combinatorial graph $G = G(\Gamma)$ with the same sets $\mathscr{V}, \mathscr{E}$ and the same connection relations. We suppose the general conditions on G formulated at the beginning of Section 2 to be fulfilled. To any edge e of $G(\Gamma)$ we assign the weight $g_e = l_e^{-1}$. The basic Hilbert space is now $L^2(\Gamma)$ with respect to the measure induced by the Lebesgue measure on the edges.

The Sobolev space $\mathscr{H}^1(\Gamma)$ consists of all continuous functions φ on Γ such that $\varphi \in H^1(e)$ on each edge and

$$\int_\Gamma (|\varphi'|^2 + |\varphi|^2)dx < \infty.$$

The operator $-\Delta$ on Γ is defined via its quadratic form

$$\mathbf{a}_\Gamma[\varphi] = \int_\Gamma (|\varphi'|^2 dx$$

with the form-domain $\mathscr{H}^1(\Gamma)$. The Laplacian Δ acts as φ'' on each edge, and the functions from $\operatorname{Dom}\Delta$ satisfy the natural (Kirchhoff) conditions at each vertex.

We list our assumptions about the graph Γ.

Assumption 4.1. One of the following assertions holds.

(i) The edge lengths l_e of the graph Γ are bounded from above, the corresponding combinatorial graph $G(\Gamma)$ has global dimension $D > 2$ and contains a sparse subset, say Y.

(ii) The graph $G(\Gamma)$ is $\mathbb{Z}^2$.

To define the Birman–Schwinger operator, we consider two cases of Assumption 4.1 separately.

In case (i), by [2, Theorem 4.1], the graph Γ has the same global dimension D. Therefore, an inequality of the form (2.3) with $\mathbf{a}_\Gamma$ replacing $\mathbf{a}_G$ holds for all functions in $\mathscr{H}^1(\Gamma)$ with compact support. Consequently, the space $\mathscr{H}(\Gamma)$, the closure of the set of compactly supported functions in the metric $\mathbf{a}_\Gamma$, is a space of functions (cf. [2] for details).

In case (ii), i.e., $G(\Gamma) = \mathbb{Z}^2$, one should consider the subspace $\mathscr{H}_0^1(\Gamma)$ in $\mathscr{H}^1(\Gamma)$ consisting of functions vanishing at the vertex $\mathbf{0}$. A Hardy type inequality, derived easily from (3.3), implies that the space $\mathscr{H}_0(\Gamma)$, the closure of the set of compactly supported functions in $\mathscr{H}_0^1(\Gamma)$ in the metric $\mathbf{a}_\Gamma$, is a space of functions as well.

Further on, to unify notation, we suppress the subscript 0 when dealing with $\mathscr{H}_0(\Gamma)$. So, $\mathscr{H}(\Gamma)$ means $\mathscr{H}_0(\Gamma)$ for $G = \mathbb{Z}^2$. Moreover, we fix a sparse subset $Y \subset \mathbb{Z}^2$.

An analogue of the quadratic form (2.4) is given by

$$\mathbf{b}_{V,\Gamma}[\varphi] = \int\limits_\Gamma V|\varphi|^2 dx, \tag{4.1}$$

where $V \geqslant 0$ is a function in $L^1(\Gamma)$. We denote by $\mathbf{B}_{V,\Gamma}$ the operator in $\mathscr{H}(\Gamma)$ generated by the quadratic form (4.1).

We construct potentials V on Γ whose support is a vicinity (in Γ) of the set Y. Using such potentials, we are able to prove the following statements that can be considered as analogues of Theorems 2.1 and 2.3.

Theorem 4.2. *Let a graph Γ satisfy Assumption 4.1. Let a sequence $p_n \searrow 0$ be such that $n^{1/2}p_n \to 0$. Then there exists a function $V \in L^1(\Gamma)$, $V \geqslant 0$, such that*

$$Cp_n \leqslant \lambda_n(\mathbf{B}_{V,\Gamma}) \leqslant C' p_n \quad \forall n \in \mathbb{N},$$

where the constants $C_0, C \in (0,\infty)$ do not depend on the sequence $\{p_n\}$.

Theorem 4.3. *Suppose, in addition, that the sequence p_n is moderately varying. Then the function $V \in L^1(\Gamma)$, $V \geqslant 0$, can be chosen in such a way that*

$$\lambda_n(\mathbf{B}_{V,\Gamma}) \sim p_n.$$

Below, we restrict ourselves to the proof of Theorem 4.3. The proof of Theorem 4.2 is much simpler and can be easily reconstructed in a similar way. As in [2, Section 4.3], for analyzing the operators $\mathbf{B}_{V,\Gamma}$ we use the orthogonal (in $\mathscr{H}(\Gamma)$) decomposition

$$\mathscr{H}(\Gamma) = \mathscr{H}_{\mathrm{pl}} \oplus \mathscr{H}_{\mathscr{D}}, \tag{4.2}$$

where the subspace $\mathscr{H}_{\mathrm{pl}}$ is formed by functions $\varphi \in \mathscr{H}(\Gamma)$ that are linear on each edge e and $\mathscr{H}_{\mathscr{D}}$ is formed by functions vanishing at each vertex v. We denote by $\mathbf{B}_{V,\Gamma;\mathrm{pl}}$ and $\mathbf{B}_{V,\Gamma;\mathscr{D}}$ the operators in these subspaces generated by the quadratic forms $\mathbf{b}_{V,\Gamma} \restriction \mathscr{H}_{\mathrm{pl}}$ and $\mathbf{b}_{V,\Gamma} \restriction \mathscr{H}_{\mathscr{D}}$ respectively. Given a function $\varphi \in \mathscr{H}(\Gamma)$, we denote by φ_{pl} and $\varphi_{\mathscr{D}}$ its components in the decomposition (4.2).

Proof of Theorem 4.3. Let $\widehat{V}$ be a sparse potential on the combinatorial graph $G(\Gamma)$ supported on a sparse set $Y \subset \mathscr{V}$ such that $\widehat{V}(v_n) = p_n \searrow 0$, $v_n \in Y$. Along with the operator

$\mathbf{B}_{\widehat{V},G(\Gamma)}$, let us consider the operator $\widehat{\mathbf{B}}_{\widehat{V},\Gamma}$ in $\mathscr{H}(\Gamma)$, associated with the quadratic form

$$\widehat{\mathbf{b}}_{\widehat{V},\Gamma}[\varphi] = \sum_n p_n |\varphi(v_n)|^2.$$

This quadratic form vanishes on $\mathscr{H}_{\mathscr{D}}$, and

$$\widehat{\mathbf{b}}_{\widehat{V},\Gamma}[\varphi] = \mathbf{b}_{\widehat{V},G(\Gamma)}[\varphi \restriction \mathscr{V}]. \tag{4.3}$$

It follows that the nonzero spectra of the operators $\widehat{\mathbf{B}}_{\widehat{V},\Gamma}$ and $\mathbf{B}_{\widehat{V},G(\Gamma)}$ coincide. So, if we allow such point–supported potentials, then all the results of Subsection 2.3 carry over to the metric graphs.

Now, we show how to construct a "genuine" potential $V \in L^1(\Gamma)$ possessing the desired properties. For the sake of simplicity, we suppose that $Y = Y_{\widehat{V}}$ contains no neighboring vertices. This can be always achieved by a thinning down the original sparse set. Choose a sequence $\varepsilon_n > 0$ such that $\varepsilon_n < \min\limits_{e \ni v_n} l_e$. Further assumptions about the behavior of ε_n will be imposed later. For the calculations below it is convenient to denote by $\widehat{\mathscr{E}}$ the set of all edges such that one of its ends lies in Y.

Define the potential V on Γ as follows:

$$V(y) = \begin{cases} \dfrac{p_n}{\varepsilon_n \deg v_n} & \text{if } \operatorname{dist}(y, v_n) < \varepsilon_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

We have

$$\mathbf{b}_{V,\Gamma}[\varphi] = \mathbf{b}_{V,\Gamma}[\varphi_{\mathrm{pl}}] + \mathbf{b}_{V,\Gamma}[\varphi_{\mathscr{D}}] + 2\operatorname{Re}\mathbf{b}_{V,\Gamma}[\varphi_{\mathrm{pl}}, \varphi_{\mathscr{D}}]. \tag{4.5}$$

We inspect each term separately.

Any edge $e \in \widehat{\mathscr{E}}$ can be written as $e = (v_n, v_n')$, where $v_n \in Y$ and $v_n' \notin Y$. The vertex v_n' that corresponds to the vertex v_n is not unique, so that here we have some slight abuse of notation, however what is important is that v_n' is determined uniquely by e. We identify any such edge with the interval $(0, l_e)$, so for $y \in e$ we have

$$\varphi_{\mathrm{pl}}(y) = l_e^{-1}\left(\varphi(v_n)(l_e - y) + \varphi(v_n')y\right).$$

From here we derive

$$\mathbf{b}_{V,\Gamma}[\varphi_{\mathrm{pl}}] = \sum_{e \in \widehat{\mathscr{E}}} \frac{p_n}{l_e^2 \varepsilon_n \deg\, v_n} \int_0^{\varepsilon_n} |\varphi(v_n)(l_e - y) + \varphi(v_n')y|^2 dy.$$

In each summand, the term that will be shown to be dominant is

$$I_e[\varphi] = \frac{p_n}{\varepsilon_n \deg\, v_n} \int_0^{\varepsilon_n} |\varphi(v_n)|^2 dy = \frac{p_n}{\deg\, v_n} |\varphi(v_n)|^2.$$

So, by (4.3),

$$\sum_{e \in \widehat{\mathscr{E}}} I_e[\varphi] = \mathbf{b}_{\widehat{V},G(\Gamma)}[\varphi_{\mathrm{pl}}].$$

To estimate the difference $\mathbf{b}_{V,\Gamma}[\varphi_{\rm pl}] - \mathbf{b}_{\widehat{V},G(\Gamma)}[\varphi_{\rm pl}]$, we use the elementary inequality

$$\big||A|^2 - |B|^2\big| \leqslant |A-B|(|A|+|B|)$$

and obtain

$$\begin{aligned}\big||\varphi(v_n)(l_e-y)+\varphi(v_n')y|^2 - |\varphi(v_n)|^2 l_e^2\big| &\leqslant y|\varphi(v_n)-\varphi(v_n')|\left((2l_e-y)|\varphi(v_n)|+y|\varphi(v_n')|\right)\\ &\leqslant 2l_e y(|\varphi(v_n)|^2+|\varphi(v_n')|^2).\end{aligned}$$

Now, the summation over all edges $e \in \widehat{\mathscr{E}}$ gives

$$\big|\mathbf{b}_{V,\Gamma}[\varphi_{\rm pl}] - \mathbf{b}_{\widehat{V},G(\Gamma)}[\varphi_{\rm pl}]\big| \leqslant \sum_{e=(v_n,v_n')\in\widehat{\mathscr{E}}} R(e,\varepsilon_n;\varphi\restriction\mathscr{V}), \tag{4.6}$$

where

$$R(e,\varepsilon_n) = \frac{p_n\varepsilon_n}{l_e \deg\, v_n}(|\varphi(v_n)|^2+|\varphi(v_n')|^2).$$

Choosing ε_n decaying fast enough, we can grant an arbitrarily fast decay of the eigenvalues of the operator corresponding to the quadratic form on the right-hand side of (4.6). This yields

$$\lambda_n(\mathbf{B}_{V,\Gamma;{\rm pl}}) \sim p_n.$$

Now we go over to the quadratic form $\mathbf{b}_V[\varphi_{\mathscr{D}}]$. The associated operator $\mathbf{B}_{V,\Gamma;\mathscr{D}}$ can be identified with the direct orthogonal sum of operators $\mathbf{B}_{V,e}$, $e \in \widehat{\mathscr{E}}$, acting in the spaces $H^{1,0}(e)$, and the Rayleigh quotient for each operator $\mathbf{B}_{V,e}$ is

$$\frac{\displaystyle\int_e V(y)|u(y)|^2dy}{\displaystyle\int_e |u'(y)|^2dy}, \qquad u \in H^{1,0}(e),$$

where the weight function $V(y)$ is defined by (4.4). For the eigenvalue estimates of such operators we use the following assertion.

Proposition 4.4. *Let $V(y) \geqslant 0$ be a monotone function on an interval $e=(0,a)$, $a \leqslant \infty$, such that*

$$\int_e \sqrt{V(y)}dy < \infty.$$

Then for any $\lambda > 0$

$$n(\lambda,\mathbf{B}_{V,e}) \leqslant \frac{2}{\pi}\lambda^{-1/2}\int_e \sqrt{V(y)}dy.$$

Proof. For $a=\infty$, due to the Birman–Schwinger principle, this is an equivalent reformulation of the well-known Calogero estimate (cf. [12] and also [13, Theorem XIII.9(b))]). The case $a<\infty$ follows from here by the standard variational argument. □

Applying Proposition 4.4 to each edge $e = (v_n, v'_n) \in \widetilde{\mathscr{E}}$, we obtain

$$n(\lambda, \mathbf{B}_{V,\Gamma;\mathscr{D}}) = \sum_{e \in \widehat{\mathscr{E}}} n(\lambda, \mathbf{B}_{V,e;\mathscr{D}}) \leqslant \frac{2}{\pi} \lambda^{-1/2} \sum_{e \in \widehat{\mathscr{E}}} \left(\frac{p_n \varepsilon_n}{deg\, v_n} \right)^{1/2}. \tag{4.7}$$

If the series on the right converges, then

$$n(\lambda, \mathbf{B}_{V,\Gamma;\mathscr{D}}) = O(\lambda^{-1/2})$$

(or, equivalently, $\lambda_j(\mathbf{B}_{V,\Gamma;\mathscr{D}}) \leqslant Cj^{-2}$) and hence the operator $\mathbf{B}_{V,\Gamma;\mathscr{D}}$ does not contribute to the asymptotics of any greater order.

It remains to inspect the last term in (4.5). It is equal to

$$\Omega[\varphi] = 2\,\mathrm{Re} \sum_{e=(v_n,v'_n)\in\widehat{\mathscr{E}}} \frac{p_n}{l_e^2 \varepsilon_n \deg v_n} \int_0^{\varepsilon_n} (\varphi(v_n)(l_e - y) + \varphi(v'_n) y) \overline{\varphi_{\mathscr{D}}(y)} dy.$$

The main contribution to this sum is given by the similar expression, say $\Omega_0[\varphi]$, with the first factor in the integrand replaced by $l_e \varphi(v_n)$. For the spectral estimates we choose one more sequence γ_n vanishing as $n \to \infty$.

Each term in $\Omega_0[\varphi]$ does not exceed

$$\frac{p_n}{l_e \varepsilon_n \deg\, v_n} |\varphi(v_n)| \int_0^{\varepsilon_n} |\varphi_{\mathscr{D}}(y)| dy \leqslant \gamma_n p_n |\varphi(v_n)|^2 + \frac{p_n}{4\gamma_n (\deg v_n)^2} \varepsilon_n \int_0^{\varepsilon_n} |\varphi_{\mathscr{D}}(y)|^2 dy.$$

Here, the first terms on the right-hand side correspond to the quadratic form that, by Theorem 2.1, generates an operator whose eigenvalues behave as $o(p_n)$ due to the factor γ_n. The second terms are quadratic forms of the same type as the ones considered above for the operator $\mathbf{B}_{V,\Gamma;\mathscr{D}}$ with the same weight function. Applying the estimate (4.7) (and again choosing ε_n decaying fast enough), we see that the corresponding estimate is of the Weyl type. So, these terms do not affect the spectral asymptotics of any order greater than $O(\lambda^{-1/2})$.

The estimates for the quadratic form $\Omega[\varphi] - \Omega_0[\varphi]$ are much easier. We have

$$|\Omega[\varphi] - \Omega_0[\varphi]| \leqslant 2 \sum_{e=(v_n,v'_n)\in\widehat{\mathscr{E}}} \frac{p_n |\varphi(v'_n) - \varphi(v_n)|}{l_e^2 \deg\, v_n} \int_0^{\varepsilon_n} |\varphi_{\mathscr{D}}(y)| dy$$

Each term can be estimated through

$$\frac{p_n^2 (|\varphi(v'_n)|^2 + |\varphi(v_n)|^2) \varepsilon_n}{l_e^4 (\deg v_n)^2} + \int_0^{\varepsilon_n} |\varphi_{\mathscr{D}}(y)|^2 dy.$$

The first terms give a quadratic form whose eigenvalues decay as fast as we wish provided that the sequence ε_n is chosen in an appropriate way. The same is true for the second terms, after we apply the estimate (4.7). □

References

1. G. Rozenblum and M. Solomyak, "Counting Schrödinger boundstates: semiclassics and beyond," In: *Sobolev Spaces in Mathematics. II. Applications in Analysis and Partial Differential Equations*, pp. 329–354, International Mathematical Series **9**, Springer and Tamara Rozhkovskaya Publisher (2009).

2. G. Rozenblum and M. Solomyak, "On the spectral estimates for the Schrödinger type operators: the case of small local dimension" [in Russian], *Funk. Anal. Pril.* **44**, No. 4, 21–33 (2010); English transl.: *Funct. Anal. Appl.* **44**, No. 4, 259–269 (2010).

3. G. Rozenblum and M. Solomyak, "On the spectral estimates for the Schrödinger operator on $\mathbb{Z}^d$, $d \geqslant 3$" [in Russian], *Probl. Mat. Anal.* **41**, 107-126 (2009); English transl.: *J. Math. Sci., New York*, **159**, No. 2, 241–263 (2009).

4. D. Pearson, "Singular continuous measures in scattering theory," *Commun. Math. Phys.* **60**, 13–36 (1976).

5. S. Molchanov and B. Vainberg, " Spectrum of multidimensional Schrödinger operators with sparse potentials," In: *Analytical and Computational Methods in Scattering and Applied Mathematics (Newark, DE, 1998)*, pp. 231-254, Chapman and Hall/CRC Res. Notes Math. **417**, Boca Raton, FL (2000).

6. S. Molchanov and B. Vainberg, "Scattering on the system of the sparse bumps: multidimensional case," *Appl. Anal.* **71**, No. 1-4, 167-185 (1999).

7. N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Univ. Press, Cambridge (1992).

8. M. Sh. Birman and M. Solomyak, *Spectral Theory of Selfadjoint Operators in Hilbert Space* [in Russian], 2nd ed. Lan', S. Petersburg etc. (2010); English transl.: D. Reidel Publishing Co., Dordrecht (1987).

9. I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space* [in Russian], Nauka, Moscow (1965); English transl.: Am. Math. Soc., Providence, RI (1969).

10. W. H. McCrea and F. J. W. Whipple, "Random paths in two and three dimensions," *Proc. Royal Soc. Edinburgh* **60**, 281–298 (1940).

11. F. Spitzer, *Principles of Random Walk*, Springer (2001).

12. F. Calogero, "Upper and lower limits for the number of bound states in a given central potential," *Commun. Math. Phys* **1**, 80–88 (1965).

13. M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press (1978).

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