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ON ELASTIC WAVES IN A WEDGE

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The existence of waves propagating along the edge of an elastic wedge has been established by many authors by physically rigorous arguments on the base of numerical computations. A mathematically rigorous proof for a wedge with aperture angle less than $\pi/2$ was presented by I. Kamotskii. We supplement the I. Kamotskii result and prove the existence of fundamental modes for some range of aperture angles greater than $\pi/2$. Bibliography: 7 titles.

1. Introduction

The existence of waves traveling along the edge of an isotropic elastic wedge (wedge waves) has been established by many authors (for example, see [1–4]) on the "physical" level of rigor with the help of numerical calculations. These waves represent the fundamental type of oscillations of a solid, along with volume and surface waves. As a rule, their velocities are less than the velocities of the Rayleigh wave on a plane surface [5]. Localized near an edge and possessing no dispersion, such waves realize waveguide propagation of oscillations.

In paper [1], with the help of the method of finite elements, Lagasse approximately determined the phase velocity as a function of the angle of a wedge for two lower antisymmetric modes. An independent investigation has been conducted by Maradudin with collaborators [2], who considered an unbounded wedge and, expanding the displacement field in Laguerre polynomials, found a numerical solution of the problem. The latter authors have succeeded in computing the phase velocities of the symmetric mode and several antisymmetric modes, depending on the angle of the wedge. Together with the fundamental (of lower order) antisymmetric mode, the symmetric mode has been studied in paper [3]. Representing the displacements as a sum of two solutions of equations of elasticity theory, each of which satisfies the boundary condition precisely only on one side of the wedge and approximately on the other side, Tiersten and Rubin obtained a numerical solution with the help of the projection method. A rigorous proof of the existence of a localized waveguide mode for a wedge with aperture angle less than $\frac{\pi}{2}$ was given by I. V. Kamotskii in [6].

The idea of [6] is the reduction of the problem to studying the spectrum of a self-adjoint operator in the Hilbert space $L_2(\Omega)$, where Ω is an angle in $\mathbb{R}^2$. Having proved that the essential spectrum of this operator coincides with the ray $[c_R^2k^2, +\infty)$, where c_R is the velocity of the Rayleigh wave and k is the length of the wave vector, I. V. Kamotskii then constructs a test function on which the Rayleigh quotient is less than $c_R^2k^2$ and thus the operator has at least one point of the discrete spectrum, to which corresponds a localized wedge wave in the initial problem.

Developing the method used in [6], we offer a more complicated ansatz for test functions. As a result, we prove the existence of localized wedge waves for some interval of values of angles greater than $\pi/2$. Moreover, our ansatz enables one to separate the cases of existence of symmetric and antisymmetric modes. For some (negative) values of the Poisson coefficient $\sigma = \frac{\lambda}{2(\lambda+\mu)}$, the existence of waves of both types is proved.

2. Statement of the problem. A result due to I. Kamotskii

Let

$$\Omega = \{(x_1, x_2) : r > 0, \phi \in (0, \varphi)\}$$

be an angle on the plane $\mathbb{R}^2$, (r, ϕ) be polar coordinates.

Assume that in the volume that is occupied by a wedge $K = \Omega \times \mathbb{R}$ (see Fig. 1), the equations of motion of an isotropic solid are fulfilled. If the dependence of the displacement on time is harmonic $U(x)e^{-i\omega t}$, then the equations for U have the form

$$LU \equiv L(\partial_1, \partial_2, \partial_3)U := -\partial_m \sigma_{nm} = \rho\omega^2 U_n, \quad n = 1, 2, 3. \tag{1}$$

Here ρ is the density of matter (in the sequel, without loss of generality, we set $\rho = 1$), $U = (U_1, U_2, U_3)$ is the displacement vector, and the σ_{nm} are components of the stress tensor

$$\sigma_{nm} = \lambda \partial_l U_l \delta_{nm} + \mu(\partial_m U_n + \partial_n U_m),$$

where λ and μ are the Lamé coefficients and δ_{nm} is the Kronecker symbol. Henceforth summation over repeating indices is implied.

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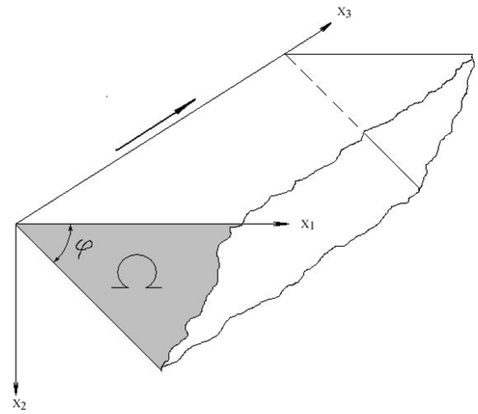

FIG. 1

On the surface ∂K of the wedge, conditions of the absence of stresses are satisfied:

$$NU \equiv N(\partial_1, \partial_2, \partial_3)U := \sigma_{nm}\nu_m = 0, \quad n = 1, 2, 3. \tag{2}$$

Here, (ν_1, ν_2, ν_3) is the vector of the external normal.

Multiplying the left-hand side of (1) by $\overline{U_n}$ (the bar means complex conjugation) and integrating by parts, we obtain the identity

$$(LU, U)_K + (NU, U)_{\partial K} = a_K(U, U) \equiv \int_K a(\partial_1, \partial_2, \partial_3; U, U)\, dx, \tag{3}$$

where

$$a(\partial_1, \partial_2, \partial_3; U, U) = \sigma_{nm}\overline{\partial_m U_n}.$$

We shall seek the solution of problem (1), (2) in the form

$$U(x_1, x_2, x_3) = u(x_1, x_2)e^{ikx_3},$$

where k is the wave number. Then for the vector $u(x_1, x_2)$ we get the following boundary-value problem in the domain Ω:

$$\begin{aligned} \mathcal{L}u &\equiv L(\partial_1, \partial_2, ik)u = \omega^2 u \quad \text{in} \quad \Omega, \\ \mathcal{N}u &\equiv N(\partial_1, \partial_2, ik)u = 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{4}$$

The following identity is derived similarly to (3):

$$(\mathcal{L}u, u)_\Omega + (\mathcal{N}u, u)_{\partial\Omega} = a_\Omega(ik; u, u) \equiv \int_\Omega a(\partial_1, \partial_2, ik; u, u)\, dx_1 dx_2. \tag{5}$$

The symmetric quadratic form on the right-hand side in (5) is positive and closed in the Hilbert space $(H^1(\Omega))^3$. For this reason (for example, see [7, Chap. 10]), to the boundary-value problem (4) corresponds a self-adjoint operator $\mathcal{A}(ik)$ in the space $L_2(\Omega)$. As is known, the greatest lower bound of the spectrum of the operator $\mathcal{A}(ik)$ is equal to $\inf\limits_{(H^1(\Omega))^3} \Phi(u)$, where

$$\Phi(u) := \frac{a_\Omega(ik; u, u)}{(u, u)_\Omega}$$

is the Rayleigh quotient.

If $u(x_1, x_2) \in L_2(\Omega)$ is a nontrivial eigenfunction of the operator $\mathcal{A}(ik)$, $k > 0$, then the function $U(x, t) = u(x_1, x_2)e^{i(kx_3 - \omega t)}$, $\omega > 0$, will be called a *waveguide mode.* Since there is a plane of symmetry in the problem, the operator is reduced by the decomposition of the space $(H^1(\Omega))^3$ into the subspaces of antisymmetric ($\mathcal{H}_a$) and symmetric ($\mathcal{H}_s$) displacements, which makes it possible to exist both antisymmetric (bending) and symmetric modes.

As was mentioned in the Introduction, in paper [6] it is shown that the essential spectrum of the operator $\mathcal{A}(ik)$ coincides with the ray $[c_R^2k^2, +\infty)$, where c_R is the velocity of the Rayleigh wave. For this reason, below the frequency $\omega_R^2 = c_R^2k^2$, only the discrete spectrum may occur, the presence of which corresponds to the existence of localized waveguide modes of the wedge.

To prove the existence of a discrete spectrum, it is sufficient to present a function $u^{test} \in (H^1(\Omega))^3$ such that

$$\Phi(u^{test}) < c_R^2k^2. \tag{6}$$

In [6], as a test function it was taken $u_n^{test}(x_1, x_2) = e^{-\frac{x_1}{n}}u^R(x_2)$, n is a large parameter, where

$$U^R(x_1, x_2, x_3, t) = e^{-i\omega t + ikx_3}u^R(x_2) \tag{7}$$

is the Rayleigh wave traveling in the direction (0,0,1) and leaving the plane $x_2 = 0$ free of stresses. It turns out that

$$\Phi(u_n^{test}) = c_R^2k^2 - cn^{-1} + O(n^{-2}), \quad n \to \infty,$$

where $c > 0$ for $\varphi < \frac{\pi}{2}$. Taking n large enough, we obtain (6), which proves the existence of a waveguide mode for $\varphi < \frac{\pi}{2}$ and for all values of σ.

3. A generalization of I. Kamotskii's result

Together with (8), we consider the Rayleigh wave

$$V^R(x_1, x_2, x_3, t) = e^{-i\omega t + ikx_3}u^R(\sin(\varphi)x_1 - \cos(\varphi)x_2), \tag{8}$$

traveling in the direction (0,0,1) and leaving the other side $\sin(\varphi)x_1 - \cos(\varphi)x_2 = 0$ of the wedge free of stresses.

As a test function, we take a linear combination of "truncated" profiles of the Rayleigh waves (7) and (8):

$$v_n^{test}(x_1, x_2) = \alpha e^{-\frac{x_1}{n}}u^R(x_2) + \beta e^{-\frac{\cos(\varphi)x_1 + \sin(\varphi)x_2}{n}}u^R(\sin(\varphi)x_1 - \cos(\varphi)x_2).$$

It is obvious that

$$\Phi(v_n^{test}) = \frac{\mathfrak{A}_n(\alpha, \beta)}{\mathfrak{B}_n(\alpha, \beta)},$$

where $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are quadratic forms on $\mathbb{R}^2$. For this reason, the inequality $\Phi(v_n^{test}) < c_R^2k^2$ for some α and β is equivalent to the presence of a negative eigenvalue of the form $\mathfrak{A}_n - c_R^2k^2\mathfrak{B}_n$.

An immediate calculation shows that

$$\mathfrak{A}_n(\alpha, \beta) - c_R^2k^2\mathfrak{B}_n(\alpha, \beta) = \mathfrak{M}(\alpha, \beta) + O(n^{-1}), \quad n \to \infty,$$

where $\mathfrak{M}$ is a quadratic form the matrix of which $\begin{bmatrix} \mathfrak{m}_1 & \mathfrak{m}_2 \\ \mathfrak{m}_2 & \mathfrak{m}_1 \end{bmatrix}$ does not depend on n and has a complicated but closed-form expression.

It is obvious that the form $\mathfrak{M}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. It is also easy to see that if to a negative eigenvalue corresponds the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then for n large enough the inequality $\Phi(v_n^{test}) < c_R^2k^2$ is achieved on an antisymmetric function, which yields $\inf_{\mathcal{H}_a} \Phi(u) < c_R^2k^2$ and thus proves the existence of an antisymmetric mode. Similarly, the presence of a negative eigenvalue with the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ proves the existence of a symmetric mode.

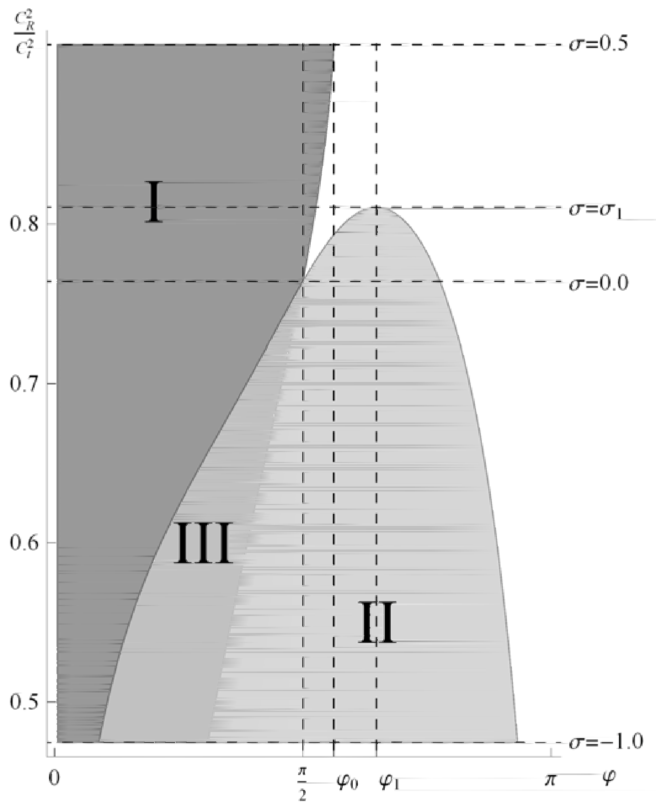

FIG. 2

With the help of elementary facts of linear algebra, we obtain the following sufficient conditions of the existence of waveguide modes[1]

I	$\lvert\mathfrak{m}_1\rvert < \mathfrak{m}_2$	antisymmetric mode
II	$\lvert\mathfrak{m}_1\rvert < -\mathfrak{m}_2$	symmetric mode
III	$\lvert\mathfrak{m}_2\rvert < -\mathfrak{m}_1$	both antisymmetric and symmetric modes

A further complication of the ansatz with the use of more that two Rayleigh waves does not extend the domains of existence of an antisymmetric mode. The range of the values of parameters for which a symmetric mode exists can be somewhat extended by adding in the test function to the Rayleigh waves propagating along both sides of the wedge a third wave traveling along the plane $\cos(\varphi/2)x_1 + \sin(\varphi/2)x_2 = 0$.

The results of computations are shown in Fig. 2. In the figure it is seen that a fundamental antisymmetric mode may propagate as localized in a wedge with aperture angle $\varphi < \varphi_0 \approx 101.25°$, which agrees well with the results in [2]. The existence of a symmetric mode is proved, in addition to unreal (negative) values of σ, only for $\sigma < \sigma_1 \approx 0.139$ for some intervals of angles greater than $\pi/2$ (to the value $\sigma_1 \approx 0.139$ corresponds $\varphi_1 \approx 116.65°$), which agrees with the results in [3]. It is curious that in a rectangular wedge ($\varphi = \pi/2$), for $\sigma > 0$ an antisymmetric mode may propagate and for $\sigma < 0$ a symmetric mode may travel.

It should be mentioned the case where the quadratic form $\mathfrak{m}$ is identical zero for $\lambda = 0$ and $\varphi = \frac{\pi}{2}$. In this case, as in [6], we failed to prove the existence of any mode, and we believe that it merely does not exist.

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[1]The boundary between zones I and III (II and III) belongs to zone I (to II, respectively).

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