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FREE VIBRATIONS OF PIEZOCERAMIC HOLLOW CYLINDERS WITH RADIAL POLARIZATION

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We consider an axisymmetric problem of free longitudinal vibrations of hollow piezoelectric cylinders for some types of boundary conditions on the end faces. The piezoceramic material is polarized in the radial direction. The side faces of a cylinder are covered with short-circuited thin electrodes. The method of solution of the problem is based on the combination of the spline collocation method along the longitudinal coordinate and the step-by-step search method along the radial coordinate. We present results of a numerical analysis of a cylinder of PZT 4 ceramic in a wide range of changes in the geometric parameters of the cylinder.

The solution of dynamic problems for thick-walled elements as spatial problems of the theory of elasticity is connected with substantial difficulties caused by the complexity of the system of initial partial differential equations and the necessity to satisfy boundary conditions on surfaces that bound a body. These difficulties increase significantly under conditions of coupling of fields and the anisotropy of piezoelectric materials [1, 2, 8, 9].

It should be noted that, in the literature, only individual works devoted to the investigation of the problem of vibrations of piezoceramic cylinders of finite length performed within the framework of the three-dimensional theory of elasticity are known [10–12, 14, 15].

The method of solution based on the combination of the spline collocation method and step-by-step search method for investigating the stress-strain state and analyzing the spectrum of natural frequencies of vibrations of elastic bodies was used in [3–5, 7]. In [6], this approach was used for investigating free axisymmetric vibrations of hollow piezoceramic cylinders with the polarization of the piezoceramic in the axial direction.

The aim of the present work is to investigate natural axisymmetric vibrations of piezoceramic cylinders of finite length of piezoceramic polarized in the radial direction. In this case, the lateral surfaces of a cylinder are free from external actions and are covered with short-circuited thin electrodes. On the end faces of the cylinder, rigid fixing is considered.

The formulated problem is described by the a system of coupled equations, which consists of

– axisymmetric longitudinal motion equations, which, in the cylindrical coordinate system (r, θ, z), have the form

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial \sigma_{rz}}{\partial z} + \rho\omega^2 \tilde{u}_r - 0 ,$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\sigma_{rz} + \frac{\partial \sigma_{zz}}{\partial z} + \rho\omega^2 \tilde{u}_z = 0 ; \qquad (1)$$

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– equations of electrostatics

$$\frac{\partial D_r}{\partial r} + \frac{1}{r}D_r + \frac{\partial D_z}{\partial z} = 0, \qquad E_r = -\frac{\partial \varphi}{\partial r}, \qquad E_z = -\frac{\partial \varphi}{\partial z}; \tag{2}$$

– geometric relations

$$\varepsilon_{rr} = \frac{\partial \tilde{u}_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\tilde{u}_r \qquad \varepsilon_{zz} = \frac{\partial \tilde{u}_z}{\partial u_z}, \qquad \varepsilon_{rz} = \frac{\partial \tilde{u}_z}{\partial r} + \frac{\partial \tilde{u}_r}{\partial z}. \tag{3}$$

Here σ_{ij} are the components of the stress tensor, ρ is the density of the material, ω is the circular frequency, $\tilde{u}_i$ are the components of the vector of displacements $(u(r,\theta,z,t) = \tilde{u}(r,\theta,z)e^{i\omega t}$, D_i are the components of the electric induction vector, E_i are the components of the electric field vector, φ is the electrostatic potential, and ε_{ij} are the components of the strain tensor.

Physical relations for a piezoceramic material polarized in the radial direction have the form

$$\sigma_{rr} = c_{33}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - e_{33}E_z,$$

$$\sigma_{\theta\theta} = c_{13}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{12}\varepsilon_{zz} - e_{31}E_z,$$

$$\sigma_{zz} = c_{13}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{11}\varepsilon_{zz} - e_{31}E_z,$$

$$\sigma_{rz} = 2c_{55}\varepsilon_{rz} - e_{15}E_r,$$

$$D_r = e_{33}\varepsilon_{rr} + e_{13}\varepsilon_{\theta\theta} + e_{13}\varepsilon_{zz} + \varepsilon_{33}E_z,$$

$$D_z = 2e_{15}\varepsilon_{rz} + \varepsilon_{11}E_z. \tag{4}$$

Here c_{ij} are the components of the tensor of the modulus of elasticity, e_{ij} are the components of the tensor of piezomodulus, and ε_{ij} are the components of the dielectric constant tensor of the material.

We set the following boundary conditions on the surfaces of the cylinder:

– the lateral surfaces $r = R_0 \pm h$ are free from external forces $\sigma_{rr} = \sigma_{rz} = 0$ and covered by thin electrode that are short-circuited $\varphi = 0$;

– the end faces of the cylinder $z = \pm L/2$ are rigidly restrained $\tilde{u}_r = 0$, $\tilde{u}_z = 0$ and free from electrodes $D_z = 0$.

Here R_0 is the radius of the middle surface of the cylinder, h is the half-thickness of the cylinder, and L is the length of the cylinder.

Substituting relations (3) and (4) into Eqs. (1) and (2) and resolving them for $\frac{\partial^2\varphi}{\partial r^2}$, $\frac{\partial^2\tilde{u}_r}{\partial r^2}$, $\frac{\partial^2\tilde{u}_z}{\partial r^2}$, we get

$$\frac{\partial^2\varphi}{\partial r^2} = -\frac{\Delta_3}{\Delta}\frac{\partial^2\varphi}{\partial z^2} - \left(\frac{\tilde{e}_{13}\tilde{e}_{33}}{\Delta} - 1\right)\frac{1}{r}\frac{\partial\varphi}{\partial r} + \left(\frac{\tilde{c}_{11}}{r^2} - \Omega^2\right)\frac{\tilde{e}_{33}}{\Delta}u_r$$

$$+ \frac{\Delta_4}{\Delta}\frac{\partial^2\tilde{u}_r}{\partial z^2} - \frac{\tilde{c}_{33}\tilde{e}_{13}}{\Delta}\frac{1}{r}\frac{\partial\tilde{u}_r}{\partial r} + \frac{\Delta_1}{\Delta}\frac{1}{r}\frac{\partial\tilde{u}_z}{\partial z} + \frac{\Delta_2}{\Delta}\frac{\partial^2\tilde{u}_z}{\partial r\partial z},$$

$$\frac{\partial^2\tilde{u}_r}{\partial r^2} = -\frac{\Delta_7}{\Delta}\frac{\partial^2\varphi}{\partial z^2} - \frac{\tilde{e}_{15}\tilde{e}_{13}}{\Delta}\frac{1}{r}\frac{\partial\varphi}{\partial r} + \left(\frac{\tilde{c}_{11}}{r^2} - \Omega^2\right)\frac{\tilde{\varepsilon}_{33}}{\Delta}\tilde{u}_r$$

$$- \frac{\Delta_8}{\Delta}\frac{\partial^2\tilde{u}_r}{\partial z^2} - \left(\frac{\tilde{e}_{13}\tilde{e}_{33}}{\Delta} + 1\right)\frac{1}{r}\frac{\partial\tilde{u}_r}{\partial r} + \frac{\Delta_5}{\Delta}\frac{1}{r}\frac{\partial\tilde{u}_z}{\partial z} - \frac{\Delta_6}{\Delta}\frac{\partial^2\tilde{u}_z}{\partial r\partial z},$$

$$\frac{\partial^2\tilde{u}_z}{\partial r^2} = -\frac{\tilde{e}_{15}}{\tilde{c}_{55}}\frac{1}{r}\frac{\partial\varphi}{\partial z} - \frac{\tilde{e}_{13} - \tilde{e}_{15}}{\tilde{c}_{55}}\frac{\partial^2\varphi}{\partial r\partial z} - \frac{\tilde{c}_{12} + \tilde{c}_{55}}{\tilde{c}_{55}}\frac{1}{r}\frac{\partial\tilde{u}_r}{\partial z}$$

$$- \frac{\tilde{c}_{13} + \tilde{c}_{55}}{\tilde{c}_{55}}\frac{\partial^2\tilde{u}_r}{\partial r\partial z} - \frac{\Omega^2}{\tilde{c}_{55}}\tilde{u}_z - \frac{\tilde{c}_{11}}{\tilde{c}_{55}}\frac{\partial^2\tilde{u}_z}{\partial z^2} - \frac{1}{r}\frac{\partial\tilde{u}_z}{\partial r}. \quad (5)$$

Here, we introduce the notation

$$\Delta = e_{33}{}^2 + c_{33}\varepsilon_{33}, \qquad \Delta_1 = (c_{12} - c_{13})e_{33} + c_{33}e_{13},$$

$$\Delta_2 = c_{33}(e_{15} + e_{33}) - (c_{13} + c_{55})e_{33}, \qquad \Delta_3 = c_{33}\varepsilon_{11} + e_{15}e_{33},$$

$$\Delta_4 = c_{33}e_{15} - c_{55}e_{33}, \qquad \Delta_5 = (c_{12} - c_{13})\varepsilon_{33} - e_{13}e_{33},$$

$$\Delta_6 = (c_{13} + c_{55})\varepsilon_{33} + (e_{15} + e_{33})e_{33}, \qquad \Delta_7 = e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33},$$

$$\Delta_8 = e_{15}e_{33} + c_{55}\varepsilon_{33}, \quad (6)$$

and dimensionless quantities

$$\Omega = \omega h\sqrt{\frac{\rho}{\lambda}}, \qquad \tilde{c}_{ij} = \frac{c_{ij}}{\lambda}, \qquad \tilde{e}_{ij} = \frac{e_{ij}}{\sqrt{\varepsilon_0\lambda}}, \qquad \tilde{\varepsilon}_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}. \quad (7)$$

Here, ω is the circular frequency, $\lambda = 10^{10}$ Pa, and ε_0 is the dielectric constant of vacuum.

We seek the functions $\varphi(r,z)$, $\tilde{u}_r(r,z)$, and $\tilde{u}_z(r,z)$ in the form

$$\varphi(r,z) = \sum_{i=0}^{N} v_i(x)\varphi_{2i}(z), \qquad \tilde{u}_r(r,z) = \sum_{i=0}^{N} w_i(x)\varphi_{2i}(z),$$

$$\tilde{u}_z(r,z) = \sum_{i=0}^{N} u_i(x)\varphi_{1i}(z), \tag{8}$$

where

$$x = \frac{r - R_0}{h},$$

$u_i(x)$, $v_i(x)$, and $w_i(x)$ are the desired functions of the variable x, and $\varphi_{ji}(z)$, $j = 1, 2$, $i = 0, 1, \ldots, N$, are linear combinations of B-splines on a homogeneous mesh $\Delta: -\frac{L}{2} = z_0 < z_1 < \ldots < z_n = \frac{L}{2}$, which take into account the boundary conditions on the end faces of the cylinder for $z = -\frac{L}{2}$ and $z = \frac{L}{2}$. Note that derivatives of the components of the solution vector not higher than derivatives of the second order enter into system (5). Consequently, we can restrict ourselves to the approximation by spline functions of the third order.

Introducing the notation

$$\Phi_j = \left[\varphi_{ji}(\xi_k)\right], \qquad k, i = 0, \ldots, N, \quad j = 1, 2,$$

$$\bar{u} = \left[u_0, u_1, \ldots, u_N\right]^\top, \qquad \bar{v} = \left[v_0, v_1, \ldots, v_N\right]^\top, \qquad \bar{w} = \left[w_0, w_1, \ldots, w_N\right]^\top,$$

$$\bar{a}_{k\ell} = \left[a_{k\ell}(x, \xi_0, \Omega^2),\ a_{k\ell}(x, \xi_1, \Omega^2), \ldots,\ a_{k\ell}(x, \xi_N, \Omega^2)\right]^\top,$$

$$(k, \ell) \in \left\{(k, \ell) \mid k, \ell = 1, \ldots, 6\right\}, \tag{9}$$

we transform system (5) into the system of $6(N+1)$ linear differential equations for the functions $\bar{u}$, $\bar{\bar{u}}$, $\bar{v}$, $\bar{\bar{v}}$, $\bar{w}$, $\bar{\bar{w}}$:

$$\frac{d\bar{u}}{dx} = \bar{\bar{u}}, \qquad \frac{d\bar{v}}{dx} = \bar{\bar{v}}, \qquad \frac{d\bar{w}}{dx} = \bar{\bar{w}},$$

$$\frac{d\bar{\bar{u}}}{dx} = \Phi_2^{-1}\left[\left(\bar{a}_{11}\Phi_2 + \bar{a}_{12}\Phi_2''\right)\bar{u} + \bar{a}_{13}\Phi_2\bar{\bar{u}} + \bar{a}_{14}\Phi_1'\bar{v} + \bar{a}_{15}\Phi_1'\bar{\bar{v}} + \bar{a}_{16}\Phi_1'\bar{w} + \bar{a}_{17}\Phi_1'\bar{\bar{w}}\right],$$

$$\frac{d\bar{\bar{v}}}{dx} = \Phi_1^{-1}\left[\bar{a}_{21}\Phi_2'\bar{u} + \bar{a}_{22}\Phi_2'\bar{\bar{u}} + \bar{a}_{23}\Phi_1''\bar{v} + \bar{a}_{24}\Phi_1\bar{\bar{v}} + \left(\bar{a}_{25}\Phi_1 + \bar{a}_{26}\Phi_1''\right)\bar{w} + \bar{a}_{27}\Phi_1'\bar{\bar{w}}\right],$$

$$\frac{d\bar{\bar{w}}}{dx} = \Phi_1^{-1}\left[\bar{a}_{31}\Phi_2'\bar{u} + \bar{a}_{32}\Phi_2'\bar{\bar{u}} + \bar{a}_{33}\Phi_1''\bar{v} + \bar{a}_{34}\Phi_1\bar{\bar{v}} + \left(\bar{a}_{35}\Phi_1 + \bar{a}_{36}\Phi_1''\right)\bar{w} + \bar{a}_{37}\Phi_1\bar{\bar{w}}\right]. \tag{10}$$

Here,

$$\bar{a}_{11} = -\frac{\Omega^2}{\tilde{c}_{55}}, \qquad \bar{a}_{12} = -\frac{\tilde{c}_{11}}{\tilde{c}_{55}}, \qquad \bar{a}_{13} = -\frac{1}{x}, \qquad \bar{a}_{14} = -\frac{\tilde{e}_{15}}{\tilde{c}_{55}}\frac{1}{x},$$

$$\bar{a}_{15} = -\frac{\tilde{e}_{13}+\tilde{e}_{15}}{\tilde{c}_{55}}\frac{1}{x}, \qquad \bar{a}_{16} = -\frac{\tilde{c}_{12}+\tilde{c}_{55}}{\tilde{c}_{55}}\frac{1}{x}, \qquad \bar{a}_{17} = -\frac{\tilde{c}_{13}+\tilde{c}_{55}}{\tilde{c}_{55}},$$

$$\bar{a}_{21} = \frac{\Delta_1}{\Delta}\frac{1}{x}, \qquad \bar{a}_{22} = \frac{\Delta_2}{\Delta}, \qquad \bar{a}_{23} = -\frac{\Delta_3}{\Delta}, \qquad \bar{a}_{24} = \left(\frac{\tilde{e}_{13}\tilde{e}_{33}}{\Delta} - 1\right)\frac{1}{x},$$

$$\bar{a}_{25} = \left(\frac{\tilde{c}_{11}}{x^2} - \Omega^2\right)\frac{\tilde{e}_{33}}{\Delta}, \qquad \bar{a}_{26} = \frac{\Delta_4}{\Delta}, \qquad \bar{a}_{27} = \frac{\tilde{c}_{33}\tilde{e}_{13}}{\Delta}\frac{1}{x}, \qquad \bar{a}_{31} = \frac{\Delta_5}{\Delta}\frac{1}{x},$$

$$\bar{a}_{32} = -\frac{\Delta_6}{\Delta}, \qquad \tilde{a}_{33} = -\frac{\Delta_7}{\Delta}, \qquad \bar{a}_{34} = \frac{\tilde{e}_{15}\tilde{\varepsilon}_{33}}{\Delta}\frac{1}{x},$$

$$\bar{a}_{35} = \left(\frac{\tilde{c}_{11}}{x^2} - \Omega^2\right)\frac{\tilde{\varepsilon}_{33}}{\Delta}, \qquad \bar{a}_{36} = -\frac{\Delta_8}{\Delta}, \qquad \bar{a}_{37} = -\left(\frac{\tilde{e}_{13}\tilde{e}_{33}}{\Delta} + 1\right)\frac{1}{x}.$$

We can write this system in the matrix form

$$\frac{d\bar{\mathcal{R}}}{dx} = A(x,\Omega)\bar{\mathcal{R}}. \tag{11}$$

The nonzero elements of the matrix A are as follows:

$$A_{12} = 1, \qquad A_{21} = \Phi_2^{-1}\left(\bar{a}_{11}\Phi_2 + \bar{a}_{12}\Phi_2''\right), \qquad A_{22} = \Phi_2^{-1}\bar{a}_{13}\Phi_2,$$

$$A_{23} = \Phi_2^{-1}\bar{a}_{14}\Phi_2'', \qquad A_{24} = \Phi_2^{-1}\bar{a}_{15}\Phi_1', \qquad A_{25} = \Phi_2^{-1}\bar{a}_{16}\Phi_1',$$

$$A_{26} = \Phi_2^{-1}\bar{a}_{17}\Phi_1', \qquad A_{34} = 1, \qquad A_{41} = \Phi_1^{-1}\bar{a}_{21}\Phi_2',$$

$$A_{42} = \Phi_1^{-1}\bar{a}_{22}\Phi_2', \qquad A_{43} = \Phi_1^{-1}\bar{a}_{23}\Phi_1'', \qquad A_{44} = \Phi_1^{-1}\bar{a}_{24}\Phi_1,$$

$$A_{45} = \Phi_1^{-1}\left(\bar{a}_{25}\Phi_1 + \bar{a}_{26}\Phi_1''\right), \qquad A_{46} = \Phi_2^{-1}\bar{a}_{27}\Phi_1', \qquad A_{56} = 1,$$

$$A_{61} = \Phi_1^{-1}\bar{a}_{31}\Phi_2', \qquad A_{62} = \Phi_1^{-1}\bar{a}_{32}\Phi_2', \qquad A_{63} = \Phi_1^{-1}\bar{a}_{33}\Phi_1'',$$

$$A_{64} = \Phi_1^{-1}\bar{a}_{34}\Phi_1, \qquad A_{65} = \Phi_1^{-1}\left(\bar{a}_{35}\Phi_1 + \bar{a}_{36}\Phi_1''\right), \qquad A_{26} = \Phi_2^{-1}\bar{a}_{37}\Phi_1.$$

Table 1

Ordinal number of frequency	$N = 24$	$N = 32$	Values of natural frequencies obtained in [13]
1	0.7072	0.6738	0.6737
2	0.9264	0.8951	0.8970
3	1.0513	1.024	1.020
4	1.3522	1.350	1.354
5	1.8775	1.874	1.874
6	1.9135	1.941	1.938

Table 2

Ordinal number of frequency	Natural frequencies of an elastic cylinder $N = 20$	Natural frequencies of a piezoceramic cylinder $N = 24$
1	0.7787	0.7787
2	0.8061	0.8060
3	1.0321	1.0330
4	1.2421	1.2410
5	1.6565	1.6520
6	1.7410	1.7430

The boundary conditions take the form

$$B_1\bar{\mathcal{R}}(-1) = 0, \qquad B_2\bar{\mathcal{R}}(1) = 0 .$$

The results of numerical investigations obtained by the spline approximation method for $N = 24$ and $N = 32$ were compared with results obtained in [13]. We considered PZT 4 piezoceramics as a material of the cylinder. On the end faces, conditions of hinge support were considered. The results of comparison are presented in Table 1.

For the case where, on the end faces of the cylinder, rigid fixing is set, we performed a comparison of natural frequencies with results obtained on the basis of an analogous technique developed for an elastic cylinder and described in [3]. As a material of the cylinder, we considered PZT 4 piezoceramic with piezomodulus equal to zero. The results of the comparison are presented in Table 2.

In Fig. 1, we show dependences of the first five frequencies (i is the ordinal number of the frequency) on the relative length of the cylinder L/h (for $\varepsilon = h/R_0 = 0.25$). As a material of the cylinder, we considered

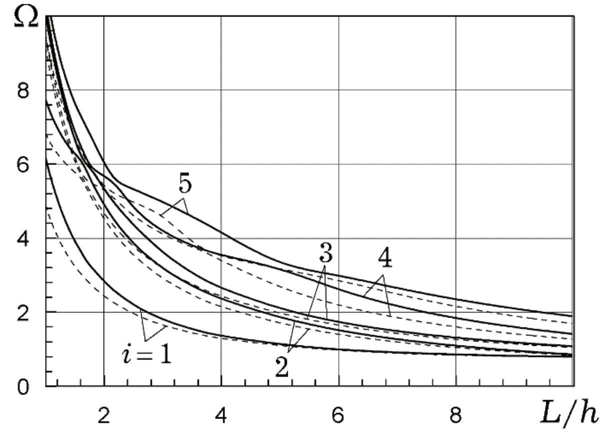

Fig. 1

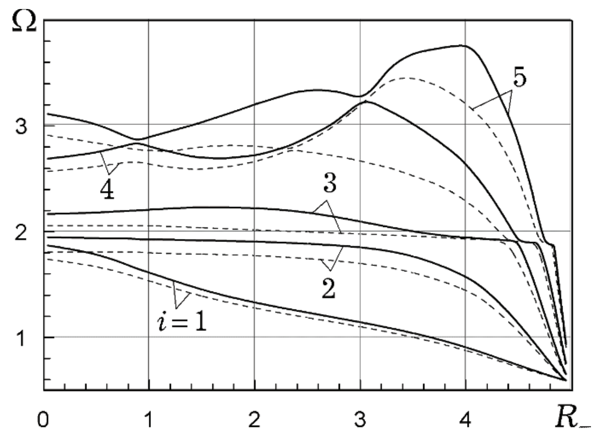

Fig. 2

PZT 4 piezoceramic. We show the values of the natural frequencies with regard to the piezoeffect by solid lines and the values of the natural frequencies without regard to the piezoeffect $\left(e_{ij} = 0\right)$ by dashed lines. In Fig. 1, it is seen that the influence of the piezoeffect leads to the rigidifying of the material, i.e., an increase in the value of the natural frequencies. In this case, in the determination of the first natural frequency, the influence of the piezoeffect can be neglected up to a relative length $L/h = 5$. For the second frequency, a noticeable influence of the piezoeffect is observed for fairly long cylinders $\left(L/h < 8\right)$. For higher frequencies, this influence is pronounced for larger cylinders.

In Fig. 2, we show dependences of the first five natural frequencies on the relative internal diameter of the cylinder R_- for the fixed length of the cylinder $L/h = 5$ and external diameter $R_+ = 5$. The change of the internal diameter is considered in the wide range from 0.05 to 4.95 units, i.e., from a practically solid cylinder to a very thin cylindrical shell. As a material of the cylinder, PZT 4 piezoceramic was again chosen. It follows from the analysis of the shown curves that, with increase in the thickness of the cylinder, the natural frequencies of vibrations increase abruptly. It should be noted that, for the first three frequencies, with increase in the thickness of the cylinder, the natural frequency of vibrations increases. For higher frequencies, after an abrupt increase in the frequency with increasing thickness, it decreases smoothly, which is not observed when the relative length of the cylinder increases (Fig. 1). As the relative length of the cylinder increases, the natural frequency of vibrations always increases. The influence of the piezoeffect for the second natural frequency and higher ones manifests itself even in fairly thin cylinders. Only in the determination of the first natural frequency can it be neglected practically without loss of accuracy in the calculation of the frequency in the segment $R\_ \geq 4$.

Analyzing the demonstrated figures, we can note that only the first and third branches in Fig. 1 and the first two branches in Fig. 2 are relatively simple curves. For the dependences of the higher natural frequencies, the structure of the spectrum is complicated. Segments with small changes in frequencies depending on the geometrical parameters (we call them plateau) with further approach of the values of frequencies (we call these points "points of attraction") are characteristic. Note that these "plateaus" in both Fig. 1 and 2 are located along some characteristic lines. For instance, in Fig. 1, it is the segment $1 \leq \frac{L}{h} \leq 3$ for the second, third, fourth, and fifth frequencies, the segment $4 \leq \frac{L}{h} \leq 6$ is for the fourth and fifth frequencies, and $\frac{L}{h} \approx 5$ is for the fourth and fifth natural frequencies. In Fig. 2, these are the domain $4 \leq R\_ \leq 5$ for the third, fourth, and fifth frequencies, and segments $R\_ \cong 3$ and $R\_ \cong 1$ for the fourth and fifth frequencies. It can be assumed that the described "points of attraction" of the natural frequencies are "unfavorable" for the material because to a small change in the frequency there corresponds a substantial reconstruction of the geometry of vibrations. However, this statement calls for additional investigations.

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