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THE PARTIAL REGULARITY FOR MINIMIZERS OF SPLITTING TYPE VARIATIONAL INTEGRALS UNDER GENERAL GROWTH CONDITIONS II. THE NONAUTONOMOUS CASE

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We consider splitting type variational problems with general growth conditions and prove the partial regularity (and the full regularity in 2D) of minimizers in the case of x-dependence. The results obtained generalize the results of Bildhauer and Fuchs concerning such problems with power growth conditions. Bibliography: 17 titles.

1. Introduction

The partial regularity and full regularity in 2D of minimizers of splitting type variational problems with general growth conditions were proved by the author in [**1**]. These results generalize the corresponding results established by Bildhauer and Fuchs [**3, 4**] in the case of power growth conditions. In this paper, we extend the statements from [**1**] to the case of x-dependence. Note that the autonomous case was treated in the recent paper by the author [**5**].

The study of the regularity of minimizers $u : \Omega \to \mathbb{R}^N$ of the energy functionals

$$I[u, \Omega] := \int\limits_{\Omega} F(\nabla u)\, dx, \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$ and $F : \mathbb{R}^{nN} \to [0, \infty)$ satisfies an anisotropic growth condition

$$C_1|Z|^p - c_1 \leqslant F(Z) \leqslant C_2|Z|^q + c_2, \qquad Z \in \mathbb{R}^{nN} \tag{1.2}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \geqslant 0$ and exponents $1 < p \leqslant q < \infty$, was pushed by Marcellini [**6, 7**]. The research of Esposito, Leonetti, and Mingione [**8**] shows that the statements do not stay true if one allows an additional x-dependence and considers minimizers of functionals

$$J[u, \Omega] := \int\limits_{\Omega} F(\cdot, \nabla u)\, dx, \tag{1.3}$$

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for $F : \Omega \times \mathbb{R}^{nN} \to [0, \infty)$. This is not only a technical extension of the autonomous situation, and additional assumptions are often necessary.

As is known, in the autonomous case, we cannot expect the regularity of minimizers of the functional (1.1) if p and q are too far apart (cf. counterexamples in [**9, 10**]). To get better results, additional assumptions are necessary. Therefore, Bildhauer, Fuchs, and Zhong considered decomposable integrands

$$F(Z) = f(\widetilde{Z}) + g(Z_n)$$

where $Z = (Z_1, \ldots, Z_n)$, $Z_i \in \mathbb{R}^N$, and $\widetilde{Z} = (Z_1, \ldots, Z_{n-1})$. Under power growth conditions on the C^2-functions f and g, they get a very general theory in the case $p \geqslant 2$ (cf. [**3, 4, 11**]). In [**1, 5**], we generalized these statements to the case

$$f(\widetilde{Z}) = a(|\widetilde{Z}|), \quad g(Z_n) = b(|Z_n|)$$

where a and b are N-functions. Thereby the main assumptions were formulated as follows:

$$\frac{h'(t)}{t} \approx h''(t),$$

$$h \text{ has superquadratic growth,}$$

where h denotes a or b. The following results were established in [**1, 5**] on the basis of higher integrability theorems in [**12**]:

- the full $C^{1,\alpha}$-regularity in the case $n = 2$,
- the partial $C^{1,\alpha}$-regularity in the general vector case provided that

$$b(t) \leqslant ct^{\omega} a(t), \quad a(t) \geqslant \vartheta t^{\frac{\omega}{2}(n-2)} \quad \text{for } \omega \leqslant 2 \text{ and large } t, \tag{1.4}$$

- the full $C^{1,\alpha}$-regularity in the case $N = 1$ provided that

$$b(t) \leqslant ct^2 a(t), \quad a(t) \leqslant ct^2 b(t) \quad \text{for } t \gg 1.$$

Comparing with the power growth situation, one can see that the above conditions are natural generalizations to the case of N-functions (except for the case $N = 1$, cf. [**3, 4, 11**]).

From now on, we consider minimizers of the functionals

$$\mathcal{I}[w] := \int\limits_{\Omega} \Big[a(\cdot, |\widetilde{\nabla} w|) + b(\cdot, |\partial_n w|) \Big] \, dx, \tag{1.5}$$

where a and b are of class $C^2(\overline{\Omega} \times [0, \infty), [0, \infty))$ and possess the following properties (h denotes a or b):

$$\begin{gathered} h(x, \cdot) \text{ is strictly increasing and convex,} \\ \lim_{t \to 0} \frac{h(x,t)}{t} = 0 \text{ and } \lim_{t \to \infty} \frac{h(x,t)}{t} = \infty \end{gathered} \tag{A1}$$

for all $x \in \overline{\Omega}$. Furthermore, we assume that for all $t \geqslant 0$

$$\widehat{\varepsilon} \, \frac{h^{'}(x,t)}{t} \leqslant h^{''}(x,t) \leqslant \widehat{h} \, \frac{h^{'}(x,t)}{t} \tag{A2}$$

uniformly with respect to $x \in \overline{\Omega}$, where $\widehat{\varepsilon}, \widehat{h} > 0$ are constants. Assume that

$$a(x,t) \leqslant c_1 b(x,t) \quad \text{for all } x \in \overline{\Omega} \text{ and large } t, \tag{A3}$$

where $c_1 > 0$. To have the superquadratic growth condition, we assume that

$$\frac{h^{'}(x,t)}{t} \geqslant h_0 > 0 \quad \forall t \geqslant 0 \tag{A4}$$

for all $x \in \overline{\Omega}$. To handle with terms involving derivatives in the spatial variable, we require

$$|\partial_\gamma h^{'}(x,t)| \leqslant c_2 h^{'}(x,t) \quad \forall\ (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+ \tag{A5}$$

for all $\gamma \in \{1, \dots, n\}$, where $c_2 \geqslant 0$ is a constant.

Remark 1.1. 1. Assumptions (A1)–(A4) can be regarded as generalizations of the corresponding conditions in [**1, 5**] to the case of x-dependence. So, it is possible to show that the (p,q)-growth condition is satisfied in the same way as in (1.2) for the function F.

2. A simple example is given by the function

$$F(x,Z) := \alpha(x) a(|\widetilde{Z}|) + \beta(x) b(|Z_n|), \quad (x,Z) \in \overline{\Omega} \times \mathbb{R}^{nN},$$

where the functions a and b of class $C^2([0,\infty),[0,\infty))$ satisfy the autonomous assumptions from [**1, 5**] and strictly positive functions α, β belong to the class $C^1(\overline{\Omega})$.

At the first step, we establish results on higher integrability.

Theorem 1.2 (higher integrability)**.** *Let Assumptions* (A1)–(A5) *hold, and let* $u \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional* (1.5). *Then the following assertions hold*:

(a) $b(\cdot, |\partial_n u|)|\partial_n u|^2$ *belongs to the space* $L^1_{\text{loc}}(\Omega)$,

(b) *if*

$$b(x,t) \leqslant c t^\omega a(x,t) \quad \textit{for large } t \textit{ and } \omega \leqslant 2, \tag{A6}$$

then $a(\cdot, |\widetilde{\nabla} u|)|\widetilde{\nabla} u|^2$ *belongs to the space* $L^1_{\text{loc}}(\Omega)$. *Furthermore,* $u \in W^{2,2}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

Remark 1.3. 1. The main point of the proof of Theorem 1.2 is a regularization procedure: if we work with an ordinary regularization (cf. [**3**], for example), we do not have the convergence $u_\delta \to u$, where u_δ is a minimizer of the regularized problem, because of the x-dependence. Note that the same problem was mentioned in [**13, 14**]. The approach of [**14**], which is based on a regularization from below with $h_M \leqslant h$, where h denotes a or b, does not resolve the problem because it is impossible to get a uniform variant of Assumption (A2) for the function h_M. Therefore, we modify the regularization described in [**15**].

2. In the nonautonomous situation, the superquadratic growth condition is already required for proving the higher integrability, unlike the autonomous case (cf. [**12**]).

3. In comparison with [**12**], we need Assumption (A6) to get higher integrability. The reason is that the condition

$$b(x,t) \leqslant ct^2 a(x,t^2) \quad \text{for large } t$$

required in [**12**] is not extended to the regularized functions a_M and b_M.

As in the proof in [**1, 5**], further assumptions are required in the general vector case

$$\frac{h'(x,t)}{t} \leqslant h''(x,t) \quad \text{for } t \geqslant 0 \text{ if } \omega < 1, \tag{A7}$$

where $x \in \overline{\Omega}$ is arbitrary and $h = a$ or $h = b$, and

$$a(x,t) \geqslant \vartheta t^{\frac{\omega}{2}(n-2)} \text{ for large } t \tag{A8}$$

for $\vartheta > 0$, where ω is defined in (A6).

Theorem 1.4 (partial $C^{1,\alpha}$-regularity)**.**

(a) *Let Assumptions* (A1)–(A6) *with* $\omega < 2$, (A7), *and* (A8) *hold. Suppose that for all* $B \Subset \Omega$

$$\operatorname{argmin}_{y\in B} a(y,t) \textit{ is independent of } t \tag{A9}$$

and

$$a(x,t) \leqslant \theta_1 \, t^{\theta_2|x-y|} a(y,t) \textit{ for all } t \gg 1 \textit{ and all } x,y \in B, \tag{A10}$$

where $\theta_1 > 0$ *and* $\theta_2 \geqslant 0$ *are constants. Then for any local minimizer* $u \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$ *of the functional* (1.5) *there exists an open subset* Ω_0 *of* Ω *such that* $\mathcal{L}^n(\Omega_0 - \Omega) = 0$ *and* $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ *for all* $\alpha < 1$.

(b) *If* $n = 2$, *then* $\Omega_0 = \Omega$ *without assuming* (A3), (A6)–(A10), *and the condition* $u \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$.

(c) *If Assumptions* (A1), (A2), *and* (A4)–(A6) *hold with* $\omega \leqslant 2$ *and* $N = 1$, *then any local minimizer* $u \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}(\Omega)$ *of the functional* (1.5) *belongs to the space* $C^{1,\alpha}(\Omega)$ *for all* $\alpha < 1$ *provided that*

$$a(x,t) \leqslant ct^2 b(x,t) \quad \textit{for large } t \tag{A11}$$

uniformly with respect to $x \in \overline{\Omega}$.

Remark 1.5. 1. The results of [**1, 5**] concerning partial regularity are extended to the nonautonomous case with the only restriction that we should assume that $b(x,t) \leqslant ct^\omega a(x,t)$ for ω really smaller than 2. The reason is that we cannot prove a uniform variant of the inequality

$$b(t) \leqslant ct^\omega a(x, t^\omega)$$

for our regularization (cf. Section 2).

2. The results are extended completely in the case $n = 2$ or $N = 1$.

3. As was mentioned in [**5**], we can remove the assumption $u \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$ if $n = 2$.

Remark 1.6. 1. From (A9) we get the existence of $y^* \in B$ such that

$$a(y^*, t) \leqslant a(y, t) \text{ for all } (y, t) \in B \times [0, \infty).$$

This is necessary to prove the continuous growth condition in iteration of blow up. The corresponding examples of density (cf. [**14**]), show that (A9) and (A10) are natural conditions in the case of x-dependence.

4. Sharp conditions for the regularity of minimizers of nonautonomous anisotropic variational integrals are indicated in [**8**], where a condition of the form (A9) was used (cf. (74)). Thus, we can proceed that this assumption is necessary for regularity.

5. We cannot consider minimizers of the functional

$$\int\limits_{\Omega} \Big[(1 + |\widetilde{\nabla} w|^2)^{\frac{p(x)}{2}} + (1 + |\partial_n w|^2)^{\frac{p(x)}{2}}\Big] \, dx$$

for $p, q \in W^{1,\infty}_{\text{loc}}(\Omega, [2, \infty))$ since the functions

$$a(x,t) := \big(1 + t^2\big)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \big(1 + t^2\big)^{\frac{q(x)}{2}} - 1$$

do not satisfy Assumption (A5).

2. Auxiliaries and Higher Integrability

First of all, we define a regularization. Let

$$h_M(x,t) := \int\limits_0^t s g_M(x,s)\, ds, \quad t \geqslant 0,$$

where $M \gg 1$, h denotes a or b, and

$$g_M(x,t) := g(x,0) + \int\limits_0^t \eta(s) g'(x,s)\, ds,$$

$$g(x,t) := \frac{h'(x,t)}{t}.$$

Here, $\eta \in C^1([0,\infty))$ is a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta' \leqslant 0$, $|\eta'| \leqslant c/M$, $\eta \equiv 1$ on $[0, 3M/2]$, and $\eta \equiv 0$ on $[2M, \infty)$.

Lemma 2.1. *The sequence* (h_M) *satisfies the following conditions*:

(1) $h_M \in C^2(\overline{\Omega} \times [0,\infty))$, $h_M(x,t) = h(x,t)$ *for all* $t \leqslant 3M/2$, *and*

$$\lim_{M\to\infty} h_M(x,t) = h(x,t) \quad \forall\ (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+,$$

(2) $h_M \leqslant h$, $g_M \leqslant g$, *and* $h''_M \leqslant c(M)$ *on* $\overline{\Omega} \times \mathbb{R}_0^+$ *by Assumption* (A2),

(3) *the same assertion holds for* h_M *by Assumption* (A1),

(4) *by Assumption* (A2),

$$\overline{\varepsilon}\,\frac{h_M^{'}(x,t)}{t} \leqslant h_M^{''}(x,t) \leqslant \overline{h}\,\frac{h_M^{'}(x,t)}{t}$$

uniformly with respect to M,

(5) *Assumption* (A3) *is uniformly extended to* a_M *and* b_M :

$$a_M(x,t) \leqslant \overline{c}_1 b_M(x,t) \quad \textit{for all } x \in \overline{\Omega} \textit{ and large } t,$$

(6) *By Assumption* (A4), *the same inequality holds for* h_M *uniformly with respect to* M :

$$\frac{h_M^{'}(x,t)}{t} \geqslant \overline{h}_0 > 0 \quad \textit{for all } t \geqslant 0$$

provided that, in addition, Assumption (A2) *is satisfied,*

(7) *Assumption* (A5) *is extended to* h_M *uniformly with respect to* M :

$$|\partial_\gamma h_M^{'}(x,t)| \leqslant \overline{c}_2 h_M^{'}(x,t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+ \textit{ and all } \gamma \in \{1,\ldots,n\},$$

(8) *if* $b(x,t) \leqslant ct^\omega a(x,t^\omega)$ *for large* t, *then the same inequality holds for* a_M *and* b_M *uniformly with respect to* M.

Proof. By the definition of h_M, we get assertion (1) and the first two statements of assertion (2). To prove the remaining assertions, we need the equalities

$$\frac{h_M'(x,t)}{t} = g_M(x,t) = \eta(t)\frac{h'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} h'(x,s)\,ds \tag{2.1}$$

for $(x,t) \in \overline{\Omega} \times \mathbb{R}_0^+$. By the definition of g, we have

$$g(x,0) = h''(x,0).$$

Therefore,

$$g_M(x,t) = h''(x,0) + \int\limits_0^t \eta(s)\left\{\frac{h''(x,s)}{s} - \frac{h'(x,s)}{s^2}\right\} ds = \eta(t)\frac{h'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} h'(x,s)\,ds.$$

We have

$$h_M''(x,t) = g_M(x,t) + t g_M'(x,t)$$

and, consequently,

$$t g_M'(x,t) = t\eta(t) g'(x,t) = \eta(t)\left[h''(x,t) - \frac{h'(x,t)}{t}\right].$$

By (2.1) and (A2), for $\overline{\varepsilon} := \min\{1,\widehat{\varepsilon}\}$ we have

$$h_M''(x,t) = \eta(t)h''(x,t) + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} h^{'}(x,s)\,ds$$

$$\geqslant \overline{\varepsilon}\left[\eta(t)\frac{h'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta^{'}(s)}{s}\right\} h'(x,s)\,ds\right] = \overline{\varepsilon} g_M(x,t) = \overline{\varepsilon}\frac{h_M'(x,t)}{t}.$$

By (A2) and (2.1), for $\overline{h} := \max\left\{1, \widehat{h}\right\}$ we have

$$h''_M(x,t) = \frac{h'_M(x,t)}{t} + \eta(t)\left[h''(t) - \frac{h'(x,t)}{t}\right]$$
$$\leqslant \frac{h'_M(x,t)}{t} + \left[\widehat{h} - 1\right]\eta(t)\frac{h'(x,t)}{t} \leqslant \overline{h}\,\frac{h'_M(x,t)}{t},$$

which proves assertion (4). Now,

$$h''_M(x,t) \leqslant cg_M(x,t) \leqslant cg(x,0) + c\int\limits_0^{2M} |g'(x,s)|\,ds \leqslant c(M).$$

Since $h_M(x,0) = 0$, we have

$$\lim_{t\to 0}\frac{h_M(x,t)}{t} = h'_M(x,0) = 0.$$

Furthermore,

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t sg_M(x,s)\,ds = \lim_{t\to\infty} tg_M(x,t) = \infty$$

because

$$\lim_{t\to\infty} g_M(x,t) = \int\limits_{3M/2}^{2M}\{-\eta'(s)\}\,g(x,s)\,ds > 0,$$

which follows from (2.1) and the monotonicity of h. By Assumptions (A3) and (A2),

$$a'(x,t) \leqslant cb'(x,t) \quad \text{for } t \geqslant t_0.$$

Thus, by (2.1), for $t \geqslant t_0$ we have

$$\frac{a'_M(x,t)}{t} = \eta(t)\frac{a'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\,ds$$
$$\leqslant c\left[\eta(t)\frac{b'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} b'(x,s)\,ds\right] = c\,\frac{b'_M(x,t)}{t}$$

provided that $3M/2 \geqslant t_0$.

(6) From Assumptions (A1) and (A4) we find that for $t \leqslant 3M/2$

$$h''_M(x,t) \geqslant \overline{h}_0.$$

In the case $3M/2 < t < 2M$,

$$h''_M(x,t) \geqslant \overline{\varepsilon} g_M(x,t) \geqslant \overline{\varepsilon}\left[h_0\eta(t) + h_0\int\limits_{3M/2}^{t}\{-\eta'(s)\}\,ds\right] = h_0\overline{\varepsilon},$$

and for $t > 2M$ we get

$$h''_M(x,t) \geqslant \overline{\varepsilon} h_0 \int\limits_{3M/2}^{2M} \big\{-\eta'(s)\big\}\, ds = h_0 \overline{\varepsilon}.$$

The proof of the estimate for $\partial_\gamma h_M$ can be found in [**15**, p. 14]. To prove the last assertion, based on Assumptions (A6) and (A2), we find

$$b'(x,t) \leqslant c t^\omega a'(x,t) \quad \text{for } t \geqslant t_0.$$

By (2.1), this relation remains valid for $t \geqslant t_0$ provided that $3M/2 \geqslant t_0$ (note that $\eta^{'}(t) = 0$ for $t \leqslant 3M/2$):

$$\begin{aligned} \frac{b'_M(x,t)}{t} &= \eta(t)\frac{b'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} b'(x,s)\, ds \\ &\leqslant c t^\omega \left[\eta(t)\frac{a'(x,t)}{t} + \int\limits_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\, ds\right] = c t^\omega \frac{a'_M(x,t)}{t} \quad \text{for all } t \geqslant t_0. \end{aligned}$$

The lemma is proved. □

Remark 2.2. 1. By [**13**, Lemma A.1], from Assumptions (A1) and (A2) it follows that

$$h(x,2t) \leqslant 2^{\widehat{h}+1} h(x,t) \quad \forall\ t \geqslant 0. \tag{2.2}$$

Thus, by Lemma 2.1, (3) and (4), we obtain the uniform Δ_2-condition on h_M. Based on the same quotation, we deduce

$$h'(x,2t) \leqslant 2^{\widehat{h}} h'(x,t) \quad \forall t \geqslant 0,$$

which is extended to h_M uniformly.

2. By the monotonicity of h' and Assumptions (A1), (A2), for $\mu := 2^{\widehat{h}+1}$ we have

$$\mu^{-1} t h'(x,t) \leqslant h(x,t) \leqslant t h'(x,t) \quad \forall\ t \geqslant 0,$$

which is extended to h_M uniformly.

Now, we define u_M as a unique minimizer of the functional

$$\mathcal{T}_M[w] := \int\limits_B F_M(\cdot, \nabla w)\, dx := \int\limits_B \left[a_M(\cdot, |\widetilde{\nabla} w|) + b_M(\cdot, |\partial_n w|)\right]\, dx$$

in $u + W^{1,2}_0(B, \mathbb{R}^N)$, where $B := B_R(x_0) \Subset \Omega$ is arbitrary. The following assertion concerns some properties of the regularization u_M.

Lemma 2.3. *Let Assumptions* (A1)–(A5) *be satisfied. Then*

(1) u_M *belongs to the space* $W^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^N)$,

(2) $a_M(\cdot, |\nabla \widetilde{u}_M|)|\widetilde{\nabla} u_M|^2$ *and* $b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2$ *belong to the space* $L^1_{\mathrm{loc}}(B)$,

(3) *if* $n=2$ *or* $N=1$, *then* $u_M \in W^{1,\infty}_{\mathrm{loc}}(B,\mathbb{R}^N)$,

(4) *for* $\gamma \in \{1,\dots,n\}$ $\partial_\gamma u_M$ *is a solution to the equation*

$$\int\limits_B D_P^2 F_M(\cdot,\nabla u_M)(\nabla w,\nabla\varphi)\,dx + \int\limits_B \partial_\gamma D_P F_M(\cdot,\nabla u_M) : \nabla\varphi\,dx = 0$$

for all $\varphi \in W^{1,2}_0(B,\mathbb{R}^N)$ *such that* $\mathrm{spt}(\varphi) \Subset B$,

(5) u_M *belongs to* $W^{1,2}(B,\mathbb{R}^N)$, *is uniformly bounded, and*

$$\sup_M \int\limits_B F_M(\cdot,\nabla u_M)\,dx < \infty,$$

(6) *if* $u \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, *then* $\sup\limits_M \|u_M\|_\infty < \infty$.

Proof. (1), (3), (5). Assertion (1) follows from [**13**, Lemma 2.5], and assertion (3) is proved in [**15**, Theorem 1.1, (ii), (iii)] for $p=q=2$. For assertion (5) we refer to [**14**, Lemma 1.2].

(2). The minimization of $\mathcal{T}_M$ is a variational problem with splitting condition and power growth conditions with $p=q=2$. As was noted in [**12**, Remark 3b], it is possible to extend the approach of [**12**, Theorem 1] to the nonautonomous case. Hence we get $\nabla u_M \in L^4_{\mathrm{loc}}(B,\mathbb{R}^{nN})$. By the quadratic growth of a_M and b_M, we obtain the required assertion.

(4). It is clear that $\partial_\gamma u_M$ is a solution if only $\varphi \in C_0^\infty(B,\mathbb{R}^N)$ are allowed for test functions. But $D_P^2 F_M(\cdot,\nabla u_M)$ are bounded (cf. Lemma 2.1, (2)); moreover, $\partial_\gamma D_P F_M(\cdot,\nabla u_M) \in L^2(B,\mathbb{R}^{nN})$ which follows from Lemma 2.1, (2) and (4), and Assumption (A5). Assertion (4) is obtained by approximation.

(6). The uniform boundedness of u_M is obtained by the maximum principle [**16**]. □

Proof of Theorem 1.2. We set

$$\Gamma_M := 1+|\nabla u_M|^2, \quad \widetilde{\Gamma}_M := 1+|\widetilde{\nabla} u_M|^2, \quad \Gamma_{n,M} := 1+|\partial_n u_M|^2.$$

We want to estimate the integral

$$\int\limits_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\partial_n u_M|^2\,dx$$

independently of M in the same way as in [**12**]. Hence we consider $\eta \in C_0^\infty(B)$ such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_r(x_0)$ for $r<R$ and $|\nabla\eta| \leqslant c/(R-r)$. Integrating by parts and using the uniform bound on u_M (cf. Lemma 2.3), we see that it suffices to consider only the term

$$\int\limits_B \eta^{2k}|\partial_n\left[b_M(\cdot,|\partial_n u_M|)\right]||\partial_n u_M|\,dx. \tag{2.3}$$

Here, one can see

$$T_2 \leqslant c\int\limits_B \eta^{2k}|\partial_n b_M(\cdot,|\partial_n u_M|)||\partial_n u_M|\,dx + c\int\limits_B \eta^{2k} b_M'(\cdot,|\partial_n u_M|)|\partial_n u_M||\partial_n\partial_n u_M|\,dx$$

$$:= cT_2^1 + cT_2^2.$$

By Lemma 2.1, (7),

$$|\partial_n b_M(x,t)| = \left| \int\limits_0^t \partial_n b'_M(x,s)\,ds \right| \leqslant c\, b_M(x,t).$$

By the Young inequality,

$$T_2^1 \leqslant \tau \int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\,dx + c(\tau)\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\,dx.$$

Furthermore, taking into account Remark 2.2, we get

$$T_2^2 \leqslant \tau \int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\,dx + c(\tau)\int\limits_B \eta^{2k} \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|}|\partial_n\partial_n u_M|^2\,dx.$$

Absorbing τ-terms in (2.3), we get

$$\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\,dx \leqslant\, c(r) + c\int\limits_B \eta^{2k} \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|}|\partial_n\partial_n u_M|^2\,dx, \tag{2.4}$$

where $c(r)$ is a constant such that $c(r) \to \infty$ as $r \to R$, but $c(r)$ is independent of M. To estimate the integral on the right-hand side of (2.4), we need a Caccioppoli type inequality, similar to that in [**12**]. It suffices to consider only the term

$$-\int\limits_B \partial_n D_P F_M(\cdot, \nabla u_M) : \nabla\left\{\eta^{2k}\partial_n u_M\right\}\,dx.$$

The first estimate yields the bound

$$c\int\limits_B |a'_M(\cdot, |\widetilde{\nabla} u_M|)||\widetilde{\nabla}\left\{\eta^{2k}\partial_n u_M\right\}|\,dx + c\int\limits_B |b'_M(\cdot, |\partial_n u_M|)||\partial_n\left\{\eta^{2k}\partial_n u_M\right\}|\,dx$$
$$:= c\,[\mathcal{W}_1 + \mathcal{W}_2]$$

in view of Lemma 2.1, (7). We consider the terms separately:

$$\mathcal{W}_1 \leqslant c\int\limits_B \eta^{2k-1} a'_M(\cdot, |\widetilde{\nabla} u_M|)|\nabla\eta||\partial_n u_M|\,dx + c\int\limits_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|)|\partial_n\widetilde{\nabla} u_M|\,dx$$
$$:= c\left[\mathcal{W}_1^1 + \mathcal{W}_1^2\right].$$

By the Young inequality,

$$\mathcal{W}_1^2 \leqslant\, \tau\int\limits_B \eta^{2k}\frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\partial_n\widetilde{\nabla} u_M|^2\,dx + c(\tau)\int\limits_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|\,dx,$$

which can be treated in the same way as in [**13**, Section 3]. For an upper bound for $\mathcal{W}_1^1$ we can take

$$\int\limits_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|\,dx + \int\limits_B \eta^{2k-2}|\nabla\eta|^2\frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\partial_n u_M|^2\,dx.$$

The second integral can be estimated in the same way as in [**3**, Section 3] because all the assumptions on a and b are extended uniformly to a_M and b_M. Taking into account Remark 2.2 and Lemma 2.3, (5), we can estimate the first integral independently of M. Thus,

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2 \, dx \leqslant c(r). \tag{2.5}$$

Now, we want to estimate the integral

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2 \, dx. \tag{2.6}$$

As above, after integrating by parts, the only difference with the calculations of [**13**] is the integral

$$\int_B u_M \eta^{2k} \partial_\gamma \left[a_M(\cdot, |\widetilde{\nabla} u_M|) \right] \partial_\gamma u_M \, dx.$$

We have

$$\begin{aligned} U_2 &\leqslant c \int_B \eta^{2k} |\partial_\gamma a_M(\cdot, |\widetilde{\nabla} u_M|)||\widetilde{\nabla} u_M| \, dx + c \int_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M||\partial_\gamma \partial_\gamma u_M| \, dx \\ &:= c\, U_2^1 + c\, U_2^2. \end{aligned}$$

Using Remark 2.2 and Lemma 2.1, (7), we find

$$U_2^1 \leqslant \tau \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2 \, dx + c(\tau) \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \, dx$$

and

$$U_2^2 \leqslant \tau \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2 \, dx + c(\tau) \int_B \eta^{2k} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_\gamma \widetilde{\nabla} u_M|^2 \, dx.$$

We absorb the first term in (2.6) and use a Caccioppoli type inequality like in [**12**] for the second term. Then it remains to consider

$$\int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \left\{ \eta^{2k} \partial_\gamma u_M \right\} \, dx.$$

By Lemma 2.1, (7), we obtain the upper bound

$$\begin{aligned} &c \int_B a'_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} \left\{ \eta^{2k} \partial_\gamma u_M \right\}| \, dx + c \int_B b'_M(\cdot, |\partial_n u_M|)|\partial_n \left\{ \eta^{2k} \partial_\gamma u_M \right\}| \, dx \\ &:= c \left[\mathcal{U}_1 + \mathcal{U}_2 \right]. \end{aligned}$$

Consequently,

$$\mathcal{U}_1 \leqslant c \int\limits_B \eta^{2k-1} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\nabla \eta| |\partial_\gamma u_M| \, dx + c \int\limits_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\partial_\gamma \widetilde{\nabla} u_M| \, dx$$
$$:= c \left[\mathcal{U}_1^1 + \mathcal{U}_1^2\right].$$

By the Young inequality, we obtain the inequality

$$\mathcal{U}_1^2 \leqslant \tau \int\limits_B \eta^{2k} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_\gamma \widetilde{\nabla} u_M|^2 \, dx + c(\tau) \int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \, dx$$

which can be treated in a standard way. Furthermore,

$$\mathcal{U}_2 \leqslant \int\limits_B \eta^{2k-1} |\nabla \eta| b'_M(\cdot, |\partial_n u_M|) |\widetilde{\nabla} u_M| \, dx + \int\limits_B \eta^{2k} b'_M(\cdot, |\partial_n u_M|) |\partial_\gamma \partial_n u_M| \, dx.$$

From the Young inequality and Remark 2.2 we obtain the upper bound for the second integral:

$$\tau \int\limits_B \eta^{2k} \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\partial_\gamma \partial_n u_M|^2 \, dx + c(\tau) \int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \, dx$$

which is uncritical. For the first one we see

$$\int\limits_B \eta^{2k-1} |\nabla \eta| b'_M(\cdot, |\partial_n u_M|) |\widetilde{\nabla} u_M| \, dx$$
$$\leqslant \int\limits_B \eta^{2k-2} |\nabla \eta|^2 \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\widetilde{\nabla} u_M|^2 \, dx + \int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \, dx$$

which can be bound in the same way as in [**12**, Section 3]. Therefore, we need the inequality

$$b_M(x,t) \leqslant c t^2 a_M(x, t^2).$$

However, we have the stronger inequality

$$b_M(x,t) \leqslant c t^2 a_M(x, t).$$

Thus, we get

$$\int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M|^2 \, dx \leqslant c(r). \tag{2.7}$$

By Lemma 2.1, (1), (6),

$$\int\limits_B \eta^2 |\nabla^2 u_M|^2 \, dx \leqslant \int\limits_B D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) \, dx.$$

Using a Caccioppoli type inequality as in [**12**], we can obtain an estimate independent of M (note that the right-hand side of the inequality is bounded in the rest of the proof). Thus, we obtain the uniform boundedness of u_M in $W^{2,2}_{\text{loc}}(B,\mathbb{R}^N)$ (cf. Lemma 2.3, (5)). By arguments similar to those at the end of Section 2 in [**14**], we find

$$\begin{aligned} &u_M \rightharpoonup u \ \text{ in } W^{2,2}_{\text{loc}}(B,\mathbb{R}^N),\\ &\nabla u_M \to \nabla u \ \text{ in } L^2_{\text{loc}}(B,\mathbb{R}^{nN}),\\ &\nabla u_M \to \nabla u \text{ a.e.} \end{aligned} \tag{2.8}$$

as $M\to\infty$. This implies $u\in W^{2,2}_{\text{loc}}(\Omega,\mathbb{R}^N)$. By the Fatou lemma, (2.5), and (2.7), we obtain the assertion of Theorem 1.2. $\square$

3. Partial $C^{1,\alpha}$-Regularity

We define the excess function

$$E(x,r) := \fint\limits_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\,dy + \fint\limits_{B_r(x)} \overline{a}(\cdot, |\nabla u - (\nabla u)_{x,r}|)\,dy,$$

where

$$\overline{a}(x,t) := a(x,t)t^{\omega}$$

and $\omega\in(0,2)$ is taken from Assumption (A6). If $r\leqslant R_0$, where $\chi+\theta_2R_0\leqslant 2$, then Theorem 1.2 and Assumption (A10) guarantee the existence of $E(x,r)$. To show this we estimate

$$\begin{aligned}\fint\limits_{B_r(x)} \overline{a}(\cdot,|\nabla u-(\nabla u)_{x,r}|)\,dy &\leqslant \fint\limits_{B_r(x)}\fint\limits_{B_r(x)} \overline{a}(y,|\nabla u(y)-\nabla u(z)|)\,dydz\\ &\leqslant c\fint\limits_{B_r(x)}\fint\limits_{B_r(x)} \overline{a}(y,|\nabla u(y)|)\,dydz + c\fint\limits_{B_r(x)}\fint\limits_{B_r(x)} \overline{a}(y,|\nabla u(z)|)\,dydz.\end{aligned}$$

For the second term we use (A10) and without loss of generality assume that $|\nabla u(z)|\geqslant 1$:

$$\overline{a}(y,|\nabla u(z)|)\leqslant c|\nabla u(z)|^{\theta_2|y-z|}|\nabla u(z)|^{\omega}a(z,|\nabla u(z)|)\leqslant |\nabla u(z)|^2 a(z,|\nabla u(z)|).$$

Considering the integration over the set where $[|\partial_n u|\leqslant|\widetilde{\nabla} u|]$ and its complement, we prove the existence of the excess.

Lemma 3.1. *Let Assumptions* (A1)–(A10) *hold for $\omega<2$, and let $L>0$ be fixed. Then there is $C^*(L)$ such that for every $\tau\in(0,1/4)$ there exists $\varepsilon=\varepsilon(\tau,L)>0$ possessing the following property: if*

$$\begin{aligned}&|(\nabla u)_{x,r}|\leqslant L,\\ &E(x,r)+r^{\gamma^*}\leqslant\varepsilon\end{aligned}\tag{3.1}$$

for a ball $B_r(x)\Subset\Omega$, then

$$E(x,\tau r)\leqslant C^*\tau^2[E(x,r)+r^{\gamma^*}]\tag{3.2}$$

where $\gamma^*\in(0,2)$ is arbitrary.

Proof. We extend the ideas of [**1, 5**]. For $z \in B_1 := B_1(0)$ we set

$$u_m(z) := \frac{1}{\lambda_m r_m}\big(u(x_m + r_m z) - a_m - r_m A_m z\big),$$
$$a_m := (u)_{x_m,r_m},$$
$$A_m := (\nabla u)_{x_m,r_m};$$

$(f)_{x,r}$ denotes the mean value of a function f over the ball $B_r(x)$. For $\lambda_m^2 := E(x_m, r_m) + r_m^{\gamma^*}$ from (3.1) we deduce that

$$|A_m| \leqslant L, \ \fint_{B_1} |\nabla u_m|^2\, dz + \lambda_m^{-2} \fint_{B_1} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m|) dz + \lambda_m^{-2} r_m^{\gamma^*} = 1. \tag{3.3}$$

Scaling, we can write (3.2) in the form

$$\fint_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^2\, dz + \lambda_m^{-2} \fint_{B_\tau} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m - (\nabla u_m)_{0,\tau}|) dz > C_* \tau^2. \tag{3.4}$$

Using (3.3) and passing to subsequences, if necessary, we have

$$A_m \to: A, \ \ u_m \rightharpoonup: \overline{u} \quad \text{in } W^{1,2}(B_1, \mathbb{R}^N), \ \ (\overline{u})_{0,1} = 0, \ \ (\nabla \overline{u})_{0,1} = 0, \tag{3.5}$$

$$\lambda_m \nabla u_m \to 0 \quad \text{in } L^2(B_1, \mathbb{R}^{nN}) \ \text{ and a.e. on } B_1. \tag{3.6}$$

Analyzing the proof in [**1, 5**], we see that it is not difficult to verify the limit equation. Thus, to complete the proof of Lemma 3.1, we need to show that

$$\nabla u_m \to \nabla \overline{u} \text{ in } L^2_{\text{loc}}(B), \tag{3.7}$$

$$\lim_{m\to\infty} \lambda_m^{-2} \fint_{B_r} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m - (\nabla u_m)_{0,r}|) dz = 0 \quad \forall\, r < 1. \tag{3.8}$$

If we want to establish a Caccioppoli type inequality as in [**1, 5**], we need to estimate, in addition, the integral

$$\int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \left\{\eta^2 \left[\partial_\gamma u_M - P\right]\right\} dx,$$

where $P \in \mathbb{R}^{nN}$ is arbitrary. By Lemma 2.1, (7), we find

$$T_M^1 := \int_B a_M'(\cdot, |\widetilde{\nabla} u_M|) |\partial_\gamma \widetilde{\nabla} u_M| \eta^2\, dx,$$

$$T_M^2 := \int_B a_M'(\cdot, |\widetilde{\nabla} u_M|) |\nabla u_M - P| \eta |\nabla \eta|\, dx,$$

$$T_M^3 := \int\limits_B b_M'(\cdot, |\partial_n u_M|)|\partial_\gamma \partial_n u_M|\eta^2\, dx,$$

$$T_M^4 := \int\limits_B b_M'(\cdot, |\partial_n u_M|)|\nabla u_M - P|\eta|\nabla\eta|\, dx.$$

Using the Young inequality and taking into account Remark 2.2, we find

$$T_M^1 \leqslant \tau \int\limits_B \frac{a_M'(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\partial_\gamma \widetilde{\nabla} u_M|^2 \eta^2\, dx + c(\tau)\int\limits_B a_M(\cdot, |\widetilde{\nabla} u_M|)\eta^2\, dx$$

for $\tau > 0$. For T_M^2 the same arguments lead to the upper bound

$$c(\eta)\int\limits_B a_M'(\cdot, |\widetilde{\nabla} u_M|)\, dx + c(\eta)\int\limits_B a_M(\cdot, |\widetilde{\nabla} u_M|)\, dx + c(\eta)\int\limits_{B\cap \operatorname{spt}\eta} \frac{a_M'(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\partial_n u_M|^2\, dx.$$

Similarly,

$$T_M^3 \leqslant \tau \int\limits_B \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|}|\partial_\gamma \partial_n u_M|^2 \eta^2\, dx + c(\tau)\int\limits_B b_M(\cdot, |\partial_n u_M|)\eta^2,$$

$$T_M^4 \leqslant c(\eta)\int\limits_B b_M'(\cdot, |\partial_n u_M|)\, dx + c(\eta)\int\limits_B b_M(\cdot, |\partial_n u_M|)\, dx + c(\eta)\int\limits_{B\cap \operatorname{spt}\eta} \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|}|\widetilde{\nabla} u_M|^2\, dx.$$

After absorption of τ-integrals we need to justify that we can interchange integrals in the remaining terms as $M \to \infty$. We follow arguments of [**1, 5**]. For arbitrary $\varkappa > 0$ we choose a subset $S \subset B$ such that $\nabla u_M \to \nabla \overline{u}$ uniformly on S and $\mathcal{L}^n(B - S) \leqslant \varkappa$ (here, we need (3.6) and the Egorov theorem). Arguing in the same way as in [**1, 5**], we see that the integrals over $B - S$ are less than $c\varkappa^\mu$. Furthermore, we need to establish the convergence almost everywhere, based on

$$\widetilde{\psi}_M := \int\limits_0^{|\widetilde{\nabla} u_M|} \sqrt{\frac{a_M'(x,t)}{t}}\, dt,$$

$$\psi_M^{(n)} := \int\limits_0^{|\partial_n u_M|} \sqrt{\frac{b_M'(x,t)}{t}}\, dt$$

against $\widetilde{\psi}$ and $\psi^{(n)}$ (with a suitable definition). By the arguments at the end of Section 2, $\nabla u_M \to \nabla u$ a.e. Thus, we need to establish the convergence almost everywhere of

$$\widetilde{\chi}_M(x,s) := \int\limits_0^s \sqrt{\frac{a_M'(x,t)}{t}}\, dt,$$

$$\chi_M^{(n)}(x,s) := \int\limits_0^s \sqrt{\frac{b_M'(x,t)}{t}}\, dt.$$

By Lemma 2.1, (2), this is true in view of the Lebesgue dominated convergence theorem. In addition,

$$\widetilde{\psi}_{M,x} := \int_0^{|\widetilde{\nabla} u_M|} \nabla_x \sqrt{\frac{a'_M(x,t)}{t}}\, dt,$$

$$\psi^{(n)}_{M,x} := \int_0^{|\partial_n u_M|} \nabla_x \sqrt{\frac{b'_M(x,t)}{t}}\, dt.$$

But, by Lemma 2.1, (7), these terms can be bounded by $\widetilde{\psi}_M$ and $\psi^{(n)}_M$ which can be estimated in the same way as in [**1, 5**]. In the limit version of the essential Caccioppoli type inequality, it is necessary to add

$$T^1 := \int_B a(\cdot, |\widetilde{\nabla} u|)\eta^2\, dx,$$

$$T^2 := \int_B a'(\cdot, |\widetilde{\nabla} u|)|\nabla u - P|\eta|\nabla\eta|\, dx,$$

$$T^3 := \int_B b(\cdot, |\partial_n u|)\eta^2\, dx,$$

$$T^4 := \int_B b'(\cdot, |\partial_n u|)|\nabla u - P|\eta|\nabla\eta|\, dx$$

on the right-hand side. To prove (3.7), we scale and set

$$T^1_m := \frac{r_m^2}{\lambda_m^2}\int_{B_1} a(x_m + r_m z, |\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)\eta^2\, dz,$$

$$T^2_m := \frac{r_m^2}{\lambda_m^2}\int_{B_1} a'(x_m + r_m z, |\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)|\lambda_m \nabla u_m|\eta\frac{|\nabla\eta|}{r_m}\, dz,$$

$$T^3_m := \frac{r_m^2}{\lambda_m^2}\int_{B_1} b(x_m + r_m z, |A^{(n)}_m + \lambda_m \partial_n u_m|)\eta^2\, dz,$$

$$T^4_m := \frac{r_m^2}{\lambda_m^2}\int_{B_1} b'(x_m + r_m z, |A^{(n)}_m + \lambda_m \partial_n u_m|)|\lambda_m \nabla u_m|\eta\frac{|\nabla\eta|}{r_m}\, dz.$$

To bound these expressions uniformly with respect to M, we separate into the sets $[|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m| \leqslant K]$ and $[|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m| > K]$ and use the uniform boundedness of $\lambda_m^{-2} r_m^2$. By (3.3),

$$T^1_m \leqslant c(K) + c(K)\int_{B_1} \overline{a}(x_m + r_m z, \lambda_m\, |\nabla u_m|)dz \leqslant c(K).$$

By Assumption (A6), we obtain the same estimate for T_m^3. Taking into account Remark 2.2, we deduce

$$\begin{aligned}
T_m^2 &\leqslant c(\eta, K) \int\limits_{B_1} |\nabla u_m| \, dz + c(\eta, K) \int\limits_{B_1 \cap [\ldots > K]} a(x_m + r_m z, \lambda_m |\nabla u_m|) dz \\
&\leqslant c(\eta, K) + c(\eta, K) \int\limits_{B_1} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m|) dz \\
&\leqslant c(\eta, K),
\end{aligned}$$

where the L^2-bound for ∇u_m and (3.3) were used. We estimate T_m^4 in a similar way with the help of Assumption A6). Proving (3.8), we define

$$\widetilde{\psi}_m := \frac{1}{\lambda_m} \int\limits_{|\widetilde{A}_m|}^{|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|} \sqrt{\frac{a'(x,t)}{t}} dt,$$

$$\psi_m^{(n)} := \frac{1}{\lambda_m} \int\limits_{|A_m^{(n)}|}^{|A_m^{(n)} + \lambda_m \partial_n u_m|} \sqrt{\frac{b'(x,t)}{t}} dt.$$

Following the argument of [**1, 5**], we again get uniform $W^{1,2}_{\mathrm{loc}}$-bounds (additionally to the terms $T_m^1, \ldots, T_m^4$) and can complete the proof of the blow up lemma just in the same way as in [**1, 5**]. Now, we can iterate this lemma as in [**17**], for example. The only difference is connected with the inequality

$$E(x_0, r) \leqslant c(\tau) E(x_0, \tau^k R), \quad \tau^{k+1} R \leqslant r \leqslant \tau^k R.$$

However, by Assumption (A9),

$$E(x_0, r) \leqslant c(\tau) E(x_0, \tau^k R) + c(\tau) r. \tag{3.9}$$

By the convexity and Δ_2-condition on $\overline{a}$,

$$\begin{aligned}
\fint\limits_{B_r(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{r,x_0}|) \, dy \leqslant c &\fint\limits_{B_r(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R, x_0}|) \, dy \\
+c &\fint\limits_{B_r(x_0)} \overline{a}(y, |(\nabla u)_{\tau^k R, x_0} - (\nabla u)_{r,x_0}|) \, dy.
\end{aligned}$$

It is obvious that the first integral is estimated by

$$c(\tau) \fint\limits_{B_{\tau^k R}(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R, x_0}|) \, dy.$$

For the second integral we introduce

$$y^* := \operatorname{argmin}_{B_r(x_0)} a(y, t) \tag{3.10}$$

which is independent from t by Assumption (A9). Then we get the bound

$$\fint_{B_r(x_0)} \left|\overline{a}(y, |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|) - \overline{a}(y^*, |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|)\right| dy$$

$$+ \fint_{B_r(x_0)} \overline{a}(y^*, |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|)\, dy.$$

The first term can be estimated by

$$\sup_{t\in[0,1]} \left|\nabla_x \overline{a}\left(y + t(y^* - y), |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|\right)\right| |y^* - y| \leqslant c(\tau) r.$$

By Assumption (A5), we have the inequality

$$|(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}| \leqslant \fint_{B_r(x_0)} |\nabla u - (\nabla u)_{\tau^k R,x_0}|\, dz$$

$$\leqslant c(\tau) \fint_{B_{\tau^k R}(x_0)} |\nabla u - (\nabla u)_{\tau^k R,x_0}|\, dz \leqslant c(\tau) \left[E(x_0, \tau^k R) + 1\right] \leqslant c(\tau)$$

because

$$E(x_0, \tau^k R) \leqslant \varepsilon$$

which follows by iterating the blow up lemma (cf. [**17**]). By the Jensen inequality and Assumption (A9),

$$\fint_{B_r(x_0)} \overline{a}(y^*, |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|)\, dy \leqslant \fint_{B_r(x_0)} \overline{a}(y^*, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy$$

$$\leqslant \fint_{B_r(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy c(\tau) \fint_{B_{\tau^k R}(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy$$

with an appropriate choice of y^*. Hence we obtain (3.9). □

Proof of Theorem 1.4, (b). As was noted in [**5**], a 2D-result can be deduced from the proof of [**15**]. □

4. Regularity Results for $N = 1$

Let $N = 1$.

Lemma 4.1. *For all $t < \infty$ and $B_\rho \Subset B$*

$$\sup_M \|\nabla u_M\|_{L^t(B_\rho)} < \infty.$$

We want to estimate the integral

$$\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx, \tag{4.1}$$

where $\eta \in C_0^\infty(B)$ is a cut-off function such that $\eta \equiv 1$ on $B_r(x_0)$ for $\rho < R$ and $0 \leqslant \eta \leqslant 1$. Following [**12**], we integrate by parts and, using uniform local bounds for u_M (cf. Lemma 2.3, (6)), find

$$\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx \leqslant c(\eta)\left[1 + I_1 + I_2 + I_3 + I_4\right], \tag{4.2}$$

where

$$I_1 := \int\limits_{\operatorname{spt}(\eta)} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx,$$

$$I_2 := \int\limits_{\operatorname{spt}(\eta)} \frac{a_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|^2}\left[b_M(x,\cdot)^{-1}\left(\frac{a_M(\cdot, |\widetilde{\nabla} u_M|)}{\tau|\widetilde{\nabla} u_M|^2}\right)\right]^{\alpha+2} dx,$$

$$I_3 := \int\limits_B \eta^{2k}|\partial_n b_M(\cdot, |\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx,$$

$$I_4 := \int\limits_B |\partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla[\partial_n u_M \eta^2 \Gamma_{n,M}^{\frac{\alpha}{2}}]|\, dx.$$

Note that, comparing with [**12**], we have additionally I_3 and I_4 because of x-dependence. Since

$$a_M(x,t) \leqslant ct^2 b_M(x,t) \quad \text{for large } t$$

(cf. Lemma 2.1, (6)), I_2 can be bound by

$$c(\tau)\left[1 + \int\limits_{\operatorname{spt}(\eta)} a_M(\cdot, |\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\, dx\right],$$

where we used the uniform Δ_2-condition on b_M^{-1}. This fact follows from the uniform variant of Assumption (A2). From Lemma 1.1 (7) and the Young inequality we find

$$\begin{aligned} I_2 \leqslant{}& c\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx \\ &+ \tau\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx + c(\tau)\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx. \end{aligned}$$

We absorb the first term on the right-hand side of (4.1). Furthermore, we write

$$
\begin{aligned}
I_4 &\leqslant \int\limits_B |\eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_n \nabla u_M \Gamma_{n,M}^{\frac{\alpha}{2}}|\,dx \\
&+2k \int\limits_B |\eta^{2k-1} \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \eta \partial_n u_M \Gamma_{n,M}^{\frac{\alpha}{2}}|\,dx \\
&+\alpha \int\limits_B |\eta^{2k} \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_n \nabla u_M \Gamma_{n,M}^{\frac{\alpha-2}{2}} \partial_n u_M^2|\,dx \\
&:= I_4^1 + I_4^2 + I_4^3.
\end{aligned}
$$

Taking into account the splitting type structure and Lemma 2.1 (7), we deduce

$$
I_4^1 \leqslant c \int\limits_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\partial_n \widetilde{\nabla} u_M| \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + c \int\limits_B \eta^{2k} b'_M(\cdot, |\partial_n u_M|) |\partial_n \partial_n u_M| \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx.
$$

Taking into account Remark 2.2, we obtain the following upper bound for the first integral:

$$
\tau \int\limits_B \eta^{2k} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_n \widetilde{\nabla} u_M|^2 \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + c(\tau) \int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx.
$$

For the second integral we use the same arguments. We can absorb τ-terms in a Caccioppoli type inequality (cf. [**12**, Section 5]). Similarly,

$$
I_4^2 \leqslant c \int\limits_B \eta^{2k-2} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\nabla \eta| \Gamma_{n,M}^{\frac{\alpha+1}{2}}\,dx + c \int\limits_B \eta^{2k-1} b'_M(\cdot, |\partial_n u_M|) |\nabla \eta| \Gamma_{n,M}^{\frac{\alpha+1}{2}}\,dx.
$$

By Remark 2.2, the first term is estimated by

$$
\int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + \int\limits_B \eta^{2k-2} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\nabla \eta|^2 \Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx.
$$

The second term exactly corresponds to the term $\mathcal{S}_3$ in [**12**, Section 3], and the estimation of this term leads us to I_2. The second integral in the estimate for I_4^2 is bounded by

$$
c(\eta)\,[1 + I_1]
$$

(cf. Lemma 2.3, (5)). Combining all the above estimates and taking into account Remark 2.2, we finally obtain the inequality

$$
\begin{aligned}
&\int\limits_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx \\
&\qquad \leqslant c(\eta) \Bigg[1 + \int\limits_{\mathrm{spt}(\eta)} b_M(\cdot, |\partial_n u_M|) \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + \int\limits_{\mathrm{spt}(\eta)} a_M(\cdot, |\widetilde{\nabla} u_M|) \widetilde{\Gamma}_M^{\frac{\alpha}{2}}\,dx \Bigg] \\
&\qquad + c \int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \Gamma_{n,M}^{\frac{\alpha}{2}}\,dx. \qquad (4.3)
\end{aligned}
$$

Now, we separate the integrand in the last term $=: \mathcal{I}$. Therefore, for $\tau > 0$ we define an N-function $\mathcal{K}_\tau$ by the formula (we can ignore the case $\alpha = 0$)

$$\mathcal{K}_\tau(x,t) := \tau t^{\frac{\alpha+2}{\alpha}} b_M(x, t^{\frac{1}{\alpha}}). \tag{4.4}$$

The conjugate function $\mathcal{K}_\tau^*$ satisfies the inequality

$$\mathcal{K}_\tau^*(x,s) \leqslant s\widehat{b}_M(x,\cdot)^{-1}\left(\frac{s}{\tau}\right),$$

where

$$\widehat{b}_M(x,t) := t^{\frac{2}{\alpha}} b_M(x, t^{\frac{1}{\alpha}}).$$

By Lemma 2.1, (8), for $t \geqslant 1$ we have

$$\frac{a_M(x,t)}{\tau} \leqslant \frac{ct^2 b_M(x,t)}{\tau} = \frac{c\widehat{b}_M(x,t^\alpha)}{\tau}. \tag{4.5}$$

It is obvious that

$$\widehat{b}_M(x,t) = \lambda_M(x, t^{\frac{1}{\alpha}}) \text{ for } \lambda_M(x,t) := t^2 b_M(x,t)$$

and

$$\widehat{b}_M(x,\cdot)^{-1}(t) = \left[\lambda_M(x,\cdot)^{-1}(t)\right]^\alpha.$$

Using Lemma 2.1, (4), we obtain the uniform Δ_2-condition on $\lambda_M(x,\cdot)^{-1}$ and thereby on $\widehat{b}_M(x,\cdot)^{-1}$. Thus, from (4.5) it follows that

$$\widehat{b}_M(x,\cdot)^{-1}\left(\frac{a_M(x,t)}{\tau}\right) \leqslant c(\tau) t^\alpha.$$

By the Young inequality for N-functions, we get

$$\begin{aligned}\mathcal{I} &\leqslant c\left[1 + \int_B \eta^{2k}\mathcal{K}_\tau(|\partial_n u_M|^\alpha)\,dx + \int_B \eta^{2k}\mathcal{K}_\tau^*(a_M(\cdot, |\widetilde{\nabla} u_M|))\,dx\right] \\ &\leqslant c\left[1 + \tau\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^{\alpha+2}\,dx + c(\tau)\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^\alpha\,dx\right].\end{aligned}$$

Inserting this expression into (4.3) and absorbing τ-terms, we find

$$\begin{aligned}&\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx \\ &\leqslant c(\eta)\left[1 + \int_{\mathrm{spt}(\eta)} b_M(\cdot, |\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + \int_{\mathrm{spt}(\eta)} a_M(\cdot, |\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\,dx\right].\end{aligned} \tag{4.6}$$

Note that the relation between a_M and b_M is symmetric and they have the same properties. Therefore, using the same arguments, we can show that

$$\begin{aligned}
&\int\limits_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \widetilde{\Gamma}_M^{\frac{\beta+2}{2}}\, dx \\
&\leqslant c(\eta)\left[1 + \int\limits_B b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\beta}{2}}\, dx + \int\limits_B a_M(\cdot, |\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\beta}{2}}\, dx\right].
\end{aligned} \tag{4.7}$$

Now, we iterate (4.6) and (4.7) with the induction base $\alpha = 0$. Using Lemma 2.3, (5), we arrive at the assertion of Lemma 4.1.

Now, to obtain assertion (c) of Theorem 1.4, we need to show that

$$\sup_M \|\nabla u_M\|_{L^\infty(B_\rho)} < \infty, \tag{4.8}$$

where $B_\rho \Subset B$. Note that

$$\lambda |X|^2 \leqslant D_P^2 F_M(x, Z)(X, X) \leqslant \Lambda(1 + |Z|^2)^{\frac{q-2}{2}} |X|^2$$

and

$$|\partial_\gamma D_P F_M(Z)| \leqslant c(1 + |Z|^2)^{\frac{q-1}{2}}$$

for all $Z, X \in \mathbb{R}^{nN}$, $x \in \overline{\Omega}$, $\gamma \in \{1, \dots, n\}$ uniformly with respect to M. Using this growth estimates and Lemma 4.1, we can obtain (4.8) by the same arguments as in [**14**, Lemma 5.4].

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