VARIATION ON A THEME OF CAFFARELLI AND VASSEUR

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Recently, using DiGiorgi-type techniques, Caffarelli and Vasseur have shown that a certain class of weak solutions to the drift diffusion equation with initial data in L^2 gain Hölder continuity, provided that the BMO norm of the drift velocity is bounded uniformly in time. We show a related result: a uniform bound on the BMO norm of a smooth velocity implies a uniform bound on the C^β norm of the solution for some $\beta > 0$. We apply elementary tools involving the control of Hölder norms by using test functions. In particular, our approach offers a third proof of the global regularity for the critical surface quasigeostrophic (SQG) *equation in addition to the two proofs obtained earlier. Bibliography: 6 titles.*

1. Introduction

In preprint [1], Caffarelli and Vasseur proved that certain weak solutions of the drift diffusion equation with $(-\Delta)^{1/2}$ dissipation gain Hölder regularity, provided that the velocity u is uniformly bounded in the BMO norm. The proof uses DiGiorgi-type iterative techniques. The goal of this paper is twofold. First, we wanted to provide additional intuition for the Caffarelli–Vasseur theorem by presenting an elementary proof of a related result. Second, we think that, perhaps, the method of this paper may be useful in other situations.

Throughout this paper, our setting for the space variable will be a d-dimensional torus $\mathbb{T}^d$. Equivalently, we may think of the problem set in $\mathbb{R}^d$ with periodic initial data. With the latter interpretation in mind, we recall the definition of the BMO norm:

$$\|f\|_{\mathrm{BMO}} = \sup_{B\in\mathbb{R}^d} \frac{1}{|B|} \int\limits_B |f(x) - \overline{f}_B|\, dx. \tag{1.1}$$

Here B stands for a ball in $\mathbb{R}^d$, $|B|$ is its volume, and $\overline{f}_B$ represents the mean of the function f over B.

Theorem 1.1. *Assume that $\theta(x,t)$, $u(x,t)$ are $C^\infty(\mathbb{T}^d)$ for all $t \in [0,T]$ and are such that*

$$\theta_t = (u\cdot\nabla)\theta - (-\Delta)^{1/2}\theta \tag{1.2}$$

holds for any $t \geq 0$. Assume that the velocity u is divergence-free and satisfies a uniform bound $\|u(\cdot,t)\|_{\mathrm{BMO}} \leq B$ for $t \in [0,T]$. Then there exists $\beta = \beta(B,d) > 0$ such that

$$\|\theta(x,t)\|_{C^\beta(\mathbb{T}^d)} \leq C(\theta(x,0)) \tag{1.3}$$

for any $t \in [0,T]$.

Remark. In fact, we get the control of Hölder continuity in terms of just the L^1 norm of θ_0 if we are willing to allow time dependence in (1.3). Namely, the following bound is also true:

$$\|\theta(x,t)\|_{C^\beta(\mathbb{T}^d)} \leq C(B, \|\theta(x,0)\|_{L^1}) \min{(1,t)}^{-d-\beta}. \tag{1.4}$$

Thus a uniform bound on the BMO norm of u implies a uniform bound on a certain Hölder norm of θ. The dimension d is arbitrary.

Our result is different from [1]. For one thing, [1] contains a local regularization version, which we do not attempt here. However our proof is simpler and is quite elementary. At the expense of extra technicalities, it can be extended to more general settings.

Theorem 1.1 can be used to give a third proof of the global regularity of the critical surface quasigeostrophic (SQG) equation, which has recently been established in [5] and [1]. We discuss this in Sec. 5. Throughout the paper, we will denote by C different constants dependent on the dimension d only.

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2. Preliminaries

First, we need an elementary tool to characterize Hölder-continuous functions. Define a function $\Omega(x)$ on $\mathbb{T}^d$ by

$$\Omega(x) = \begin{cases} |x|^{1/2}, & |x| < 1/2 \\ \dfrac{1}{\sqrt{2}}, & |x| \geq 1/2 \end{cases} \tag{2.1}$$

(thinking of the $\mathbb{R}^d$ picture, Ω is defined as above on a unit cell and is continued by periodicity). Let $A > 1$ be a parameter to be fixed later.

Definition 2.1. *We say that a C^∞ function φ defined on $\mathbb{T}^d$ belongs to $\mathcal{U}_r(\mathbb{T}^d)$ if*

$$\|\varphi(x)\|_{L^\infty} \leq \frac{A}{r^d}, \tag{2.2}$$

$$\int\limits_{\mathbb{T}^d} \varphi(x)\, dx = 0, \tag{2.3}$$

$$\|\varphi(x)\|_{L^1} \leq 1, \tag{2.4}$$

$$\int\limits_{\mathbb{T}^d} |\varphi(x)|\Omega(x - x_0)\, dx \leq r^{1/2} \quad \textit{for some} \quad x_0 \in \mathbb{T}^d. \tag{2.5}$$

Observe that the classes $\mathcal{U}_r$ are invariant under shift. We will write $f(x) \in a\mathcal{U}_r(\mathbb{T}^d)$ if $f(x)/a \in \mathcal{U}_r(\mathbb{T}^d)$. The choice of the exponent $1/2$ in (2.1) and (2.5) is arbitrary and can be replaced by any positive number less than 1 with an appropriate adjustment of the range of β in Lemma 2.2 below.

The classes $\mathcal{U}_r$ can be used to characterize Hölder spaces as follows. We denote

$$\|f\|_{C^\beta(\mathbb{T}^d)} = \sup_{x,y \in \mathbb{T}^d} \frac{|f(x) - f(y)|}{|x - y|^\beta}, \tag{2.6}$$

omitting the $\|f\|_{L^\infty}$ term commonly included on the right-hand side. The seminorm (2.6) is sufficient for our purposes since θ remains bounded automatically, and moreover we could without loss of generality restrict consideration to a mean zero θ, invariant under evolution, for which (2.6) is equivalent to the usual Hölder norm.

Lemma 2.2. *A bounded function $\theta(x)$ lies in $C^\beta(\mathbb{T}^d)$, $0 < \beta \leq 1/2$, if and only if there exists a constant C such that for every $0 < r \leq 1$,*

$$\left| \int\limits_{\mathbb{T}^d} \theta(x)\varphi(x)\, dx \right| \leq C r^\beta \tag{2.7}$$

for all $\varphi \in \mathcal{U}_r$. Moreover,

$$\|\theta\|_{C^\beta(\mathbb{T}^d)} \leq C(\beta) \sup_{\varphi \in \mathcal{U}_r,\, 0 < r \leq 1} r^{-\beta} \left| \int\limits_{\mathbb{T}^d} \theta(x)\varphi(x)\, dx \right|. \tag{2.8}$$

Remark. The lemma holds for each fixed A in (2.2). It will be clear from the proof that the constant C in (2.8) does not depend on A, provided that A was chosen sufficiently large.

Proof. Assume first that $\theta \in C^\beta$. Consider any $\varphi \in \mathcal{U}_r$, and observe that

$$\int\limits_{\mathbb{T}^d} \theta(x)\varphi(x)\, dx = \int\limits_{\mathbb{T}^d} (\theta(x) - \theta(x_0))\varphi(x)\, dx \leq C \int\limits_{\mathbb{T}^d} |x - x_0|^\beta |\varphi(x)|\, dx.$$

Using the Hölder inequality, we get

$$\begin{aligned} \int\limits_{\mathbb{T}^d} |x - x_0|^\beta |\varphi(x)|\, dx &\leq \Big(\int\limits_{\mathbb{T}^d} |\varphi(x)|\, dx \Big)^{1-2\beta} \Big(\int\limits_{\mathbb{T}^d} |x - x_0|^{1/2} |\varphi(x)|\, dx \Big)^{2\beta} \\ &\leq C \Big(\int\limits_{\mathbb{T}^d} |\varphi(x)|\, dx \Big)^{1-2\beta} \Big(\int\limits_{\mathbb{T}^d} \Omega(x - x_0) |\varphi(x)|\, dx \Big)^{2\beta}. \end{aligned} \tag{2.9}$$

By (2.4) and (2.5), the right-hand side of (2.9) does not exceed r^β.

Conversely, consider a periodization θ_p of θ in $\mathbb{R}^d$. Recall a well-known characterization of Hölder continuous functions in $\mathbb{R}^d$ (e.g., see [6]):

$$\theta \in C^\beta \;\Leftrightarrow\; \|\theta\|_{L^\infty} \le Q, \quad \|\Delta_j(\theta)\|_\infty \le Q2^{-\beta j} \quad \text{for any} \quad j. \tag{2.10}$$

Moreover, if the right-hand side of (2.10) is satisfied, then $\|\theta\|_{C^\beta} \le CQ$.

Here the Δ_j are the Littlewood–Paley projections

$$\Delta_j(\theta) = \theta * \Psi_{2^{-j}},$$

where $\Psi_t(x) = t^{-d}\Psi(x/t)$, and $\widehat{\Psi}(\xi) = \eta(\xi) - \eta(2\xi)$ with $\eta \in C_0^\infty$, $0 \le \eta(\xi) \le 1$, $\eta(\xi) = 1$ if $|\xi| \le 1$, and $\eta(\xi) = 0$, $|\xi| \ge 2$. Observe that Ψ belongs the Schwartz class $\mathcal{S}$, $\int_{\mathbb{R}^d} \Psi\, dx = 0$, $\int_{\mathbb{R}^d} |\Psi(x)|\, dx \le C$, $\int_{\mathbb{R}^d} |x|^{1/2}|\Psi(x)|\, dx \le C$, and $\|\Psi\|_{L^\infty} \le C$. We define

$$\widetilde{\Psi}_j(x) = c \sum_{n \in \mathbb{Z}^d} \Psi_{2^{-j}}(x+n);$$

then $\widetilde{\Psi}_j(x) \in \mathcal{U}_{2^{-j}}(\mathbb{T}^d)$ if c is sufficiently small (independently of j). Moreover,

$$\int_{\mathbb{T}^d} \theta(x)\widetilde{\Psi}_j(x-y)\, dx = c\int_{\mathbb{R}^d} \theta_p(x)\Psi_{2^{-j}}(x-y)\, dx. \tag{2.11}$$

By assumption, the left-hand side in (2.11) does not exceed $Q2^{-j\beta}$. Thus, by the criterion (2.10), θ_p is C^β and so is θ. The remark after (2.10) implies that (2.8) is true. □

The proof of Theorem 1.1, which we will outline at the beginning of the next section, relies on the transfer of the evolution to a test function. Here is an elementary lemma that allows us to do that. Let $\varphi^t(x,s)$ be the solution of

$$\varphi^t_s = -(u(x,t-s)\cdot\nabla)\varphi^t - (-\Delta)^{1/2}\varphi^t, \quad \varphi^t(x,0) = \varphi(x). \tag{2.12}$$

Lemma 2.3. *Let $\theta_0, \varphi \in C^\infty(\mathbb{T}^d)$, and let $\theta(x,t)$ be the solution of* (1.2) *with $\theta(x,0) = \theta_0(x)$. Then we have*

$$\int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\, dx = \int_{\mathbb{T}^d} \theta_0(x)\varphi^t(x,t)\, dx.$$

Proof. We claim that for $0 \le s \le t$, the expression

$$\int_{\mathbb{T}^d} \theta(x,t-s)\varphi^t(x,s)\, dx \tag{2.13}$$

remains constant. A direct computation, using (2.12), (1.2) and the fact that u is divergence-free, shows that the s-derivative of (2.13) is zero. Substituting $s=0$ and $s=t$ into (2.13) proves the lemma. □

3. The proof of the main result

We outline our plan for the proof of Theorem 1.1. Conceptually, the proof is quite simple: we integrate the solution against a test function from $\mathcal{U}_r$, transfer the evolution to the test function, and prove estimates on the test function evolution.

The key to the proof of Theorem 1.1 is the following result.

Theorem 3.1. *Let $v(x,s) \in C^\infty(\mathbb{T}^d), 0 \le s \le T$, be a divergence-free d-dimensional vector field, and let $\psi(x,s)$ solve*

$$\psi_s = -(v\cdot\nabla)\psi - (-\Delta)^{1/2}\psi, \quad \psi(x,0) = \psi(x). \tag{3.1}$$

Assume that

$$\max_{s\in[0,T]} \|v(\cdot,s)\|_{\mathrm{BMO}} \le B.$$

Then the constant $A = A(B,d)$ in (2.2) can be chosen in such a way that the following is true.

Suppose $\psi \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$. Then there exist constants δ and $K > 0$, which depend only on B and the dimension d, such that

$$\psi(x,s) \in \left(\frac{r}{r+Ks}\right)^{\delta/K} \mathcal{U}_{r+Ks}(\mathbb{T}^d) \tag{3.2}$$

if $r + Ks \le 1$ and $\psi(x,s) \in r^{\delta/K}\mathcal{U}_1(\mathbb{T}^d)$ otherwise.

We assume that Theorem 3.1 is true and prove Theorem 1.1.

Proof of Theorem 1.1. Let $\beta = \delta/K$. Since $\theta(x,0)$ is smooth, we have

$$\left| \int_{\mathbb{T}^d} \theta(x,0)\varphi(x)\,dx \right| \le C(\theta(x,0)) r^{\beta} \tag{3.3}$$

for all $\varphi(x) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$. But, by Lemma 2.3,

$$\int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx = \int_{\mathbb{T}^d} \theta(x,0)\varphi^t(x,t)\,dx.$$

By Theorem 3.1, $\varphi^t(x,t)$ belongs to $\left(\frac{r}{r+Kt}\right)^{\beta} \mathcal{U}_{r+Kt}(\mathbb{T}^d)$ if $r + Kt \le 1$, and to $r^{\beta}\mathcal{U}_1(\mathbb{T}^d)$ otherwise. Then (3.3) implies that

$$\left| \int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx \right| \le C(\theta(x,0)) r^{\beta} \tag{3.4}$$

for all $\varphi(x) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$. $\square$

Observe that $C(\theta(x,0))$ will depend only on the L^1 norm of $\theta(x,0)$ if we are willing to allow time dependence in (3.4):

$$\begin{aligned}\left| \int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx \right| &= \left| \int_{\mathbb{T}^d} \theta(x,0)\varphi^t(x,t)\,dx \right| \le \|\theta(x,0)\|_{L^1} \|\varphi^t(x,t)\|_{L^\infty} \\ &\le A(B,d) r^{\beta} \min(1, r+Kt)^{-d-\beta} \le C(B,d,\|\theta(x,0)\|_{L^1}) \min(1,t)^{-d-\beta} r^{\beta}.\end{aligned}$$

This proves bound (1.4) in the Remark after Theorem 1.1.

Thus it remains to prove Theorem 3.1.

4. The evolution of the test function

The proof of Theorem 3.1 is based on the following lemma, which looks at what happens over small time increments.

Lemma 4.1. *Under assumptions of Theorem 3.1, we can choose $A = A(B,d)$ in such a way that the following is true. There exist positive δ, K, and γ (dependent only on B and d) such that for all $0 \le s \le \gamma r$, if $\psi(x,0) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$, then*

$$\psi(x,s) \in \left(1 - \frac{\delta s}{r}\right) \mathcal{U}_{r+Ks}(\mathbb{T}^d). \tag{4.1}$$

Estimate (4.1) is valid as long as $r + Ks \le 1$; otherwise the solution just remains in $\mathcal{U}_1$.

Proof. We need to check four conditions. First, the equation for ψ preserves the mean zero property, so that $\int_{\mathbb{T}^d} \psi(x,s)\,dx = 0$ for all s.

Next, consider *the L^∞ norm.* We set $M(s) = \|\psi(\cdot,s)\|_{L^\infty}$. Consider any point x_0 where the maximum or minimum value is achieved. Without any loss of generality, we may assume that $x_0 = 0$ and $\psi(0,s) = M(s)$. Then

$$\partial_s \psi(0,s) = -(-\Delta)^{1/2}\psi(0,s) = C \sum_{n \in Z^d} \int_{\mathbb{T}^d} \frac{\psi(y,s) - M(s)}{|y+n|^{d+1}}\,dy. \tag{4.2}$$

Here we used a well-known formula for the fractional Laplacian (e.g., see [3]). Since $\|\psi(\cdot,s)\|_{L^1(\mathbb{T}^d)} \le 1$ (see the argument below on the L^1 norm monotonicity), it is clear that the contribution to the right-hand side of (4.2) from the central period cell is maximal when $\psi(y)$ is the characteristic function of a ball of radius $cM(s)^{-1/d}$ centered at the origin. This yields the estimate

$$\partial_s\psi(0,s) \le -C \int\limits_{cM(s)^{-1/d}}^{r^{-1}} M(s)|y|^{-d-1}\,dy \le -C_1 M(s)^{\frac{d+1}{d}} + C_2 r M(s) \le -CM(s)^{\frac{d+1}{d}}. \tag{4.3}$$

The argument is valid for all sufficiently large $M(s)$, which is the only situation we need to consider if A was chosen large enough. The same bound holds for any point x_0 where $M(s)$ is attained and, by continuity, in some neighborhoods of such points. Thus, we have (4.3) in an open set U. Owing to the smoothness of ψ, away from U we have

$$\max_{x\notin U} |\psi(x,\tau)| < M(\tau)$$

for every τ during some period of time after s. Thus we obtain

$$\frac{d}{ds}M(s) \le -CM^{\frac{d+1}{d}}(s), \quad M(0) \le Ar^{-d}. \tag{4.4}$$

This is valid for all times while $M(s)$ remains sufficiently large. Solving (4.4), we get the estimate

$$M(s) \le \frac{M(0)}{(1+CM(0)^{1/d}s)^d} \le Ar^{-d}(1 - CA^{1/d}r^{-1}s)$$

for all sufficiently small s. This implies that

$$\|\psi(\cdot,s)\|_{L^\infty} \le Ar^{-d}(1 - CA^{1/d}r^{-1}s) \tag{4.5}$$

for all sufficiently small $s \le \gamma(A,d)r$. Note that γ is independent of ψ or v other than through the value of A, which will be chosen below depending on the value of B only. Estimate (4.5) agrees with the properties of the $(1-\frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class, provided that

$$\delta + dK \le CA^{1/d}. \tag{4.6}$$

Next, we consider *the concentration condition*

$$\int\limits_{\mathbb{T}^d} \Omega(x-x_0)|\psi(x)|\,dx \le r^{1/2}.$$

Consider $x(s) \in \mathbb{T}^d$ satisfying

$$x'(s) = \overline{v}_{B_r(x(s))} \equiv \frac{1}{|B_r|} \int\limits_{B_r(x(s))} v(y,s)\,dy, \quad x(0) = x_0. \tag{4.7}$$

Here $B_r(x)$ stands for the ball of radius r centered at x, and $|B_r|$ is its volume. We will estimate $\int\limits_{\mathbb{T}^d} \Omega(x - x(s))|\psi(x,s)|\,dx$. We write $\psi(x) = \psi_+(x) - \psi_-(x)$, where $\psi_\pm(x) \ge 0$ and have disjoint support. We denote by $\psi_\pm(x,s)$ the solutions of (3.1) with $\psi_\pm(x,0) = \psi_\pm(x)$. Then, by the linearity and maximum principle,

$$|\psi(x,s)| = |\psi_+(x,s) - \psi_-(x,s)| \le \psi_+(x,s) + \psi_-(x,s),$$

and thus

$$\int\limits_{\mathbb{T}^d} \Omega(x-x(s))|\psi(x,s)|\,dx \le \int\limits_{\mathbb{T}^d} \Omega(x-x(s))\psi_+(x,s)\,dx + \int\limits_{\mathbb{T}^d} \Omega(x-x(s))\psi_-(x,s)\,dx. \tag{4.8}$$

We estimate the first integral on the right-hand side of (4.8); the second one can be handled in the same way. We have

$$\begin{aligned}
\Big|\partial_s \int_{\mathbb{T}^d} \Omega(x-x(s))\psi_+\,dx\Big| &= \Big|\int_{\mathbb{T}^d} \Big(\Omega(x-x(s))\Big((-v\cdot\nabla)\psi_+ - (-\Delta)^{1/2}\psi_+\Big) - \nabla(\Omega(x-x(s)))\cdot x'(s)\psi_+\Big)\,dx\Big| \\
&= \Big|\int_{\mathbb{T}^d} \nabla(\Omega(x-x(s)))\cdot\big(v-\overline{v}_{B_r(x(s))}\big)\,\psi_+\,dx - \int_{\mathbb{T}^d}(-\Delta)^{1/2}\Omega(x-x(s))\psi_+\,dx\Big| \\
&\le C\Big(\int_{\mathbb{T}^d}|x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}||\psi_+|\,dx + \int_{\mathbb{T}^d}|x-x(s)|^{-1/2}|\psi_+|\,dx\Big).
\end{aligned} \tag{4.9}$$

We used the divergence-free condition on v and (4.7) at the second step, and estimated

$$|\nabla\Omega(x-x_0)| \le C|x-x_0|^{-1/2}, \quad |(-\Delta)^{1/2}\Omega(x-x_0)| \le C|x-x_0|^{-1/2}.$$

Consider the two integrals in (4.9). Since $\|\psi_+\|_{L^1} \le 1/2$ and $\|\psi_+\|_{L^\infty} \le Ar^{-d}$, the integral $\int_{\mathbb{T}^d} |x-x(s)|^{-1/2}|\psi_+|\,dx$ is maximal when ψ_+ is a characteristic function of a ball centered at $x(s)$ of radius $crA^{-1/d}$. This gives an upper bound of $Cr^{-1/2}A^{1/2d}$ for this integral. To estimate the first integral in (4.9), we split $\mathbb{T}^d = \cup_{k=0}^N E_k$, where

$$E_k = \{x:\ r2^{k-1} < |x-x(s)| \le r2^k\}\cap\mathbb{T}^d, \quad k>0, \quad E_0 = B_r(x(s)).$$

Recall (e.g., see [6]) that for any BMO function f, any ball B, and any $1\le p<\infty$,

$$\|f-\overline{f}_B\|_{L^p(B)} \le c_p|B|^{1/p}\|f\|_{\mathrm{BMO}}. \tag{4.10}$$

By Hölder's inequality,

$$\int_{B_r(x(s))} |x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}|\,|\psi_+|\,dx \le \|\,|x-x(s)|^{-1/2}\|_{L^p(B_r(x(s))}\|v-\overline{v}_{B_r(x(s))}\|_{L^z(B_r(x(s)))}\|\psi_+\|_{L^q(B_r(x(s)))},$$

where $p^{-1}+z^{-1}+q^{-1}=1$. Now

$$\|\psi_+\|_{L^q} \le \|\psi_+\|_{L^1}^{1/q}\|\psi_+\|_{L^\infty}^{1-1/q} \le A^{1-\frac{1}{q}}r^{\frac{d}{q}-d}.$$

Using (4.10), we also see that

$$\|v-\overline{v}_{B_r(x(s))}\|_{L^z(B_r(x(s)))} \le C(z,d)r^{\frac{d}{z}}B.$$

Finally, for any $p<2d$,

$$\||x-x(s)|^{-1/2}\|_{L^p(B_r(x(s))} \le C(p,d)r^{\frac{d}{p}-\frac{1}{2}}.$$

Taking z very large, and p very close to $2d$, we conclude that for any $q>\frac{2d}{2d-1}$, we have

$$\int_{B_r(x(s))} |x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}||\psi_+(x)|\,dx \le CBA^{1-\frac{1}{q}}r^{\frac{d}{q}+\frac{d}{z}+\frac{d}{p}-d-1/2} \le C(\sigma,d)BA^{\sigma}r^{-1/2}, \tag{4.11}$$

where σ is any number greater than $\frac{1}{2d}$. Furthermore, for $k>0$,

$$\begin{aligned}
&\int_{E_k} |x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}||\psi_+|\,dx \le C2^{-k/2}r^{-1/2}\int_{E_k}|v-\overline{v}_{B_r(x(s))}||\psi_+(x)|\,dx \\
&\le C2^{-k/2}r^{-1/2}\Big(\int_{B_{r2^k}(x(s))}|v-\overline{v}_{B_{r2^k}(x(s))}||\psi_+(x)|\,dx + \int_{B_{r2^k}(x(s))}|\overline{v}_{B_{r2^k}(x(s))}-\overline{v}_{B_r(x(s))}||\psi_+(x)|\,dx\Big).
\end{aligned} \tag{4.12}$$

Recall that (e.g., see [6])

$$|\overline{v}_{B_{r2^k}(x(s))} - \overline{v}_{B_r(x(s))}| \le Ck\|v\|_{\mathrm{BMO}}.$$

Therefore the last integral in (4.12) does not exceed CkB. The first integral can be estimated by

$$\|v - \overline{v}_{B_{r2^k}}\|_{L^{\frac{q}{q-1}}(B_{r2^k})}\|\psi_+\|_{L^q(B_{r2^k})} \le C(q,d)B2^{k(d-\frac{d}{q})}A^{1-\frac{1}{q}},$$

where q is any number greater than 1. Thus, in particular,

$$\int_{E_k} |x - x(s)|^{-1/2}|v - \overline{v}_{B_1(x(s))}||\psi_+(x)|\,dx \le CB2^{-3k/8}(k2^{-k/8} + A^{1/8d})r^{-1/2} \tag{4.13}$$

if $q = \frac{8d}{8d-1}$. Adding (4.11) and (4.13), we obtain

$$\int_{\mathbb{T}^d} |x - x(s)|^{-1/2}|v - \overline{v}_{B_r(x(s))}||\psi_+(x,s)|\,dx \le CBA^{3/4d}r^{-1/2},$$

provided that A is large enough (the exponent for A may be greater than $\frac{1}{2d}$). Coming back to (4.9) and (4.8), we see that

$$\int_{\mathbb{T}^d} |x - x(s)|^{1/2}|\psi(x,s)|\,dx \le r^{1/2} + Csr^{-1/2}(A^{1/2d} + BA^{3/4d}). \tag{4.14}$$

This is consistent with the $(1 - \frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class if

$$\left(1 - \frac{\delta s}{r}\right)(r + Ks)^{1/2} \ge r^{1/2} + Csr^{-1/2}(A^{1/2d} + BA^{3/4d}).$$

Since γ is chosen sufficiently small, this condition reduces to the condition

$$\frac{K}{2} - \delta > C(A^{1/2d} + BA^{3/4d}). \tag{4.15}$$

Finally, we consider the L^1 *norm.* Recall (e.g., see [3]) that for a C^∞ function $\psi(x)$,

$$(-\Delta)^{1/2}\psi(x) = \lim_{\epsilon\to 0} \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{T}^d\cap|x-y|\ge\epsilon} \frac{\psi(x) - \psi(y)}{|x - y - n|^{d+1}}\,dy. \tag{4.16}$$

Let S be the set where $\psi(x,s) = 0$, and we define

$$D_\pm = \{x \in \mathbb{T}^d \,\Big|\, \pm\psi(x,s) > 0\}.$$

The sets S and $D_\pm$ depend on s, but we will omit this in the notation to save space. Owing to (3.1) and the incompressibility of v, we have

$$\begin{aligned}\partial_s\|\psi(\cdot,s)\|_{L^1} &= \int_{\mathbb{T}^d\setminus S} \frac{\psi(x,s)}{|\psi(x,s)|}\Big(-v\cdot\nabla\psi(x,s) - (-\Delta)^{1/2}\psi(x,s)\Big)\,dx + \int_S |(-\Delta)^{1/2}\psi(x,s)|\,dx \\ &= -\int_{\mathbb{T}^d\setminus S} \frac{\psi(x,s)}{|\psi(x,s)|}(-\Delta)^{1/2}\psi(x,s)\,dx + \int_S |(-\Delta)^{1/2}\psi(x,s)|\,dx.\end{aligned} \tag{4.17}$$

The integral over S is of course nonzero only if the Lebesgue measure of S is positive. Substituting (4.16) into (4.17) and symmetrizing with respect to x, y, we get

$$\begin{aligned}\partial_s\|\psi(\cdot,s)\|_{L^1} = &-\frac{1}{2}\lim_{\epsilon\to 0} \int_{\substack{((\mathbb{T}^d\setminus S)\times(\mathbb{T}^d\setminus S))\\ \cap|x-y|\ge\epsilon}} \left(\frac{\psi(x,s)}{|\psi(x,s)|} - \frac{\psi(y,s)}{|\psi(y,s)|}\right)\sum_{n\in\mathbb{Z}^d}\frac{\psi(x,s)-\psi(y,s)}{|x-y-n|^{d+1}}\,dxdy \\ &- \int_{\mathbb{T}^d\setminus S}\frac{\psi(x,s)}{|\psi(x,s)|}\left(\sum_{n\in\mathbb{Z}^d}\int_S \frac{\psi(x,s)}{|x-y-n|^{d+1}}\,dy\right)dx + \int_S \left|\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{T}^d\setminus S}\frac{\psi(y,s)}{|x-y-n|^{d+1}}\,dy\right|dx.\end{aligned} \tag{4.18}$$

Note that the expression under the first integral in (4.18) is nonnegative for all x, y, and it is positive if $\psi(x,s)$ and $\psi(y,s)$ have different signs. Also, we note that

$$\int\limits_{S} \Big| \sum_{n\in\mathbb{Z}^d} \int\limits_{\mathbb{T}^d\setminus S} \frac{\psi(y,s)}{|x-y-n|^{d+1}}\, dy \Big|\, dx \le \int\limits_{S} \sum_{n\in\mathbb{Z}^d} \Big| \int\limits_{D_+} \frac{\psi(y,s)}{|x-y-n|^{d+1}}\, dy + \int\limits_{D_-} \frac{\psi(y,s)}{|x-y-n|^{d+1}}\, dy \Big|.$$

Therefore, the combined contribution of the last line in (4.18) over every cell is less than or equal to zero. Leaving only the central cell contributions in (4.18), we get

$$\begin{aligned} \partial_s \|\psi(\cdot,s)\|_{L^1} \le & -\int\limits_{D_+} \psi(x,s) \int\limits_{D_-} \frac{dy}{|x-y|^{d+1}}\, dx + \int\limits_{D_-} \psi(y,s) \int\limits_{D_+} \frac{dx}{|x-y|^{d+1}}\, dy \\ & - \int\limits_{D_+\cup D_-} |\psi(x,s)| \Big(\int\limits_{S} \frac{1}{|x-y|^{d+1}}\, dy \Big) dx + \int\limits_{S} \Big| \int\limits_{D_+} \frac{\psi(y,s)}{|x-y|^{d+1}}\, dy + \int\limits_{D_-} \frac{\psi(y,s)}{|x-y|^{d+1}}\, dy \Big|\, dx. \end{aligned} \tag{4.19}$$

Without loss of generality, we may assume that $1 \ge \|\psi(\cdot,s)\|_{L^1} \ge 9/10$ for every s that we consider, for otherwise the L^1 condition is already satisfied. Also, by (4.14), we may assume that $\int_{\mathbb{T}^d} \Omega(x - x(s))|\psi(x,s)|\, dx \le \frac{11}{10} r^{1/2}$, provided that the time interval $[0, \gamma r]$ that we consider is sufficiently small, with $\gamma = \gamma(A,B)$. These two bounds imply that

$$\int\limits_{\mathbb{T}^d \cap |x-x(s)|\le 400r} |\psi(x,s)|\, dx \ge 4/5.$$

The mean zero condition leads to the condition

$$\pm \int\limits_{D_\pm \cap \{|x-x(s)|\le 400r\}} \psi(x,s)\, dx \ge 3/10. \tag{4.20}$$

We denote $\widetilde{D}_\pm = D_\pm \cap \{|x-x(s)| \le 400r\}$ and $\widetilde{S} = S \cap \{|x-x(s)| \le 400r\}$. Note that if $x \in \widetilde{S}$, then, by (4.20),

$$\pm \int\limits_{D_\pm} \frac{\psi(y,s)}{|x-y|^{d+1}}\, dy \ge \pm \int\limits_{\widetilde{D}_\pm} \frac{\psi(y,s)}{|x-y|^{d+1}}\, dy \ge Cr^{-d-1}.$$

This implies that, owing to the cancellation in the last term of (4.19), we can estimate the last line of (4.19) from above by $-C|\widetilde{S}|r^{-d-1}$. Reducing the integration in the second line of (4.19) to $\widetilde{D}_\pm$, we obtain

$$\partial_s \|\psi(\cdot,s)\|_{L^1} \le -Cr^{-d-1} \Big(|\widetilde{D}_-| \int\limits_{\widetilde{D}_+} \psi(x,s)\, dx + |\widetilde{D}_+| \int\limits_{\widetilde{D}_-} \psi(x,s)\, dx + |\widetilde{S}| \Big) \le -cr^{-1}, \tag{4.21}$$

where c is a fixed positive constant. Here at the last step we used (4.20) and $|\widetilde{D}_+| + |\widetilde{D}_-| + |\widetilde{S}| \ge Cr^d$. Estimate (4.21) is consistent with the $(1 - \frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class if $\delta \le c$.

It remains to note that, if $A = A(B,d)$ is chosen sufficiently large, one can indeed find K and δ so that conditions (4.6), (4.15), and the $\delta \le c$ condition arising from the L^1 norm estimate are all satisfied. It is also clear from the proof that (4.1) then holds for all $s \le \gamma(B,d)r$. The only restriction from the above on the value of r comes from the L^∞ norm condition, which must be consistent with the L^1 and concentration conditions. For convenience, we choose to cap the value of r at 1. □

The proof of Theorem 3.1 is now straightforward.

Proof of Theorem 3.1. From Lemma 4.1 it follows that for any $s > 0$, $\psi(x,s) \in f(s)\mathcal{U}_{r+Ks}(\mathbb{T}^d)$, provided that $f'(s) \ge -\frac{\delta}{r+Ks} f(s)$. Solving this differential equation, we conclude that the factor $f(s) = \left(\frac{r}{r+Ks}\right)^{\delta/K}$ is acceptable. □

5. The critical SQG equation and further discussion

Theorem 1.1 provides an alternative path to the proof of the existence of global regular solutions to the critical surface quasigeostrophic equation

$$\begin{cases} \theta_t = u \cdot \nabla\theta - (-\Delta)^{1/2}\theta, \quad \theta(x,0) = \theta_0(x), \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta), \end{cases}$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a periodic scalar function, and R_1 and R_2 are the usual Riesz transforms in $\mathbb{R}^2$. Indeed, the local existence and uniqueness of a smooth solution starting from H^1 periodic initial data are known (e.g., see [4]). The L^∞ norm of the solution does not increase, by the maximum principle (e.g., see [3]), which implies a uniform bound on the BMO norm of the velocity. Since the local solution is smooth, one can apply Theorem 1.1. This, similarly to [1], implies a uniform bound on some Hölder norm of the solution θ. This improvement over the L^∞ control is sufficient to show the global regularity (see [1] or [2] for slightly different settings, which can be adapted to our case in a standard way).

One can pursue a number of generalizations of Theorem 1.1, reducing, for instance, the assumptions on the smoothness of the solution, the velocity, or initial data. However we chose to present here the case with the most transparent proof, containing the heart of the matter. As follows from the proof, the role of the BMO space is mainly the right scaling: the BMO is the most general function space for which (4.10) is available. The BMO scaling properties are of course also crucial for the proof of [1] to work.

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