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MODULES OVER FORMAL MATRIX RINGS

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Abstract. This work contains some new and known results on modules over formal matrix rings. The main results are presented with proofs.

In ring theory, various matrix rings play an important role. Above all, we mean formal matrix rings. Formal matrix rings generalize the notion of matrix rings of order n over a given ring. An important class of formal matrix rings consists of *Morita context* rings (e.g., see [41], [34, Sec. 18C], or Sec. 8 of the present paper). The class of formal matrix rings contains an appreciable subclass of triangular matrix rings. These rings often appear in the representation theory of Artinian algebras (e.g., see [4]); they provide examples of rings with asymmetrical properties (e.g., see [25, 51]). One section of the book [15] is devoted to rings of triangular matrices.

Every ring with nontrivial idempotents is isomorphic to some formal matrix ring. The endomorphism ring of a decomposable module also is a formal matrix ring. Therefore, studies of formal matrix rings are appropriate. These studies are quite useful for solving some problems on endomorphism rings of Abelian groups.

The study of modules over formal matrix rings is also of certain interest. Such modules consist of "column vectors" or "row vectors." Examples of such modules are columns or rows of the matrix ring of order n over some ring.

In Secs. 1 and 2 of this paper, we present general properties of formal matrix rings and modules over them. In Secs. 3 and 4, we study various submodules of modules over formal matrix rings. In Secs. 5 and 7, we study injective, flat, projective, and hereditary modules over such rings. In Sec. 8, we briefly recall familiar results on equivalences of module categories. In so doing, the presentation uses a certain formal matrix ring. Section 9 contains some applications to endomorphism rings of Abelian groups.

All rings are assumed to be associative and with nonzero identity element; modules are assumed to be unitary. Unless otherwise stated, modules are assumed to be left modules. We write homomorphisms from the left on the arguments. Except for Sec. 9, the composition of mappings $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ is denoted by $\alpha\beta$. Thus, $(\alpha\beta)(x) = \beta\big(\alpha(x)\big)$ for all $x \in X$. (In Sec. 9, we assume that $(\alpha\beta)(x) = \alpha\big(\beta(x)\big)$.) For a ring T, the category of all left T-modules is denoted by T-mod. We often use familiar methods of transformations of homomorphism groups and tensor products into modules, and also natural isomorphisms related to these objects.

1. Constructions and Properties of Formal Matrix Rings

We define a *formal matrix ring*, which is also called a *generalized matrix ring*. Let R and S be two rings, M be an R-S-bimodule, and let N be an S-R-bimodule. We denote by K the set of all matrices of the form

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix},$$

where $r \in R$, $s \in S$, $m \in M$, and $n \in N$. The set K is an Abelian group with respect to the matrix addition. To turn K into a ring, we have to know how to calculate the "product" $mn \in R$ and the "product" $nm \in S$. We can correctly do this as follows. We assume that there are bimodule homomorphisms

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$\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. We set $mn = \varphi(m \otimes n)$ and $nm = \psi(n \otimes m)$ for all $m \in M$ and $n \in N$. Now we can multiply matrices in K similarly to ordinary matrices:

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}\begin{pmatrix} r_1 & m_1 \\ n_1 & s_1 \end{pmatrix} = \begin{pmatrix} rr_1 + mn_1 & rm_1 + ms_1 \\ nr_1 + sn_1 & nm_1 + ss_1 \end{pmatrix}, \quad r, r_1 \in R, \quad s, s_1 \in S, \quad m, m_1 \in M, \quad n, n_1 \in N.$$

We note that rm_1, ms_1, nr_1, and sn_1 are the corresponding module products. We also assume that for all $m, m' \in M$ and $n, n' \in N$, the associativity relations $(mn)m' = m(nm')$ and $(nm)n' = n(mn')$ hold. Then K is a ring with respect to the considered operations of addition and multiplication. For checking ring axioms, we also have to consider main properties of tensor products and the property that φ and ψ are bimodule homomorphisms. The converse is also true: if K is a ring, then the considered associativity relations hold. The ring K is called the *formal matrix ring*; it is also denoted by

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

When it is necessary to emphasize that the ring K is constructed with the use of the homomorphisms φ and ψ, we write $K(\varphi, \psi)$. If $N = 0$ or $M = 0$, then K is a ring of triangular matrices. For defining the ring, we do not need the homomorphisms φ and ψ.

The images I and J of the homomorphisms φ and ψ are ideals of the rings R and S, respectively; these ideals are called the *trace ideals* of the ring K. We say that K is a ring *with zero trace ideals* provided that $\varphi = 0 = \psi$, i.e., $I = 0 = J$. Formal triangular matrix rings are rings with zero trace ideals. We denote by MN (NM) the set of all finite sums of elements of the form mn (respectively, nm). The relations

$$I = MN, \quad J = NM, \quad IM = MJ, \quad NI = JN$$

hold. What is the correct formulation of the problem of the study of formal matrix rings?

It is natural to assume that the study of the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is the determination of interrelations between properties of this ring and properties of the rings R and S, the bimodules M and N, and the homomorphisms φ and ψ.

For convenience and brevity, we identify matrices with the corresponding elements. For example, we identify the matrix

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$$

with the element $r \in R$ and so on. Similar rules are used for sets of matrices. For example, the set of matrices

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$$

is presented in the form (X, Y) (or in the form X if $Y = 0$). Similar rules are used for matrices with zero upper row.

If $M = 0 = N$, then $K = R \times S$ is the direct product of rings. Basically, we assume that the rings $R \times S$ are formal matrix rings. We note that sometimes the rings $R \times S$ are not considered as formal matrix rings (in such a case, the class of formal matrix rings is not necessarily closed with respect to factor rings and it does not contain commutative rings).

Let T be some ring. In T, we preserve the previous addition and define a new multiplication $\circ$ by the relation $x \circ y = yx$, $x, y \in T$. As a result, we obtain a new ring T°, which is called the ring *opposite* to T. It is directly verified that the ring opposite to

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is isomorphic to the formal matrix ring

$$\begin{pmatrix} R^\circ & N \\ M & S^\circ \end{pmatrix},$$

where N is considered as an R°-S°-bimodule, and M is considered as an S°-R°-bimodule. In addition, we note that

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} \cong \begin{pmatrix} S & N \\ M & R \end{pmatrix}.$$

If V is a right T-module, then the relation $tv = vt$, $t \in T$, $v \in V$, defines on V the structure of a left T°-module, and conversely.

Let K be some ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

of formal matrices. We consider the structure of ideals and factor rings of K. We use our representation of matrices for presenting the relation

$$K = \begin{pmatrix} eKe & eK(1-e) \\ (1-e)Ke & (1-e)K(1-e) \end{pmatrix}.$$

With the use of such a method, the action of the corresponding homomorphisms φ and ψ coincides with the multiplication in the ring K. If L is some ideal of the ring K, then it is directly verified that L coincides with the set of matrices

$$\begin{pmatrix} eLe & eL(1-e) \\ (1-e)Le & (1-e)L(1-e) \end{pmatrix},$$

where eLe and $(1-e)L(1-e)$ are ideals of the rings R and S, respectively, and $eL(1-e)$ and $(1-e)Le$ are subbimodules in M and N, respectively. The subgroups placed in one of the four positions in L coincide with the sets of the corresponding components of all elements of L.

We form the matrix group

$$\bar{K} = \begin{pmatrix} eKe/eLe & eK(1-e)/eL(1-e) \\ (1-e)Ke/(1-e)Le & (1-e)K(1-e)/(1-e)L(1-e) \end{pmatrix}.$$

In fact, we have the formal matrix ring $\bar{K}$ considered in the above general sense. In $\bar{K}$, multiplication of matrices is induced by the multiplication in K. It is directly verified that the mapping

$$K/L \to \bar{K}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix} + L \to \begin{pmatrix} \bar{r} & \bar{m} \\ \bar{n} & \bar{s} \end{pmatrix},$$

is a ring isomorphism, where the bar denotes the corresponding residue class.

If an abstract ring T contains a nonzero idempotent $e \neq 1$, then T is canonically isomorphic to the formal matrix ring

$$\begin{pmatrix} eTe & eT(1-e) \\ (1-e)Te & (1-e)T(1-e) \end{pmatrix}$$

under the correspondence

$$x \to \begin{pmatrix} exe & ex(1-e) \\ (1-e)xe & (1-e)x(1-e) \end{pmatrix}, \quad x \in T.$$

The class of all formal matrix rings coincides with the class of endomorphism rings over various rings. Let $G = A \oplus B$ be a right module over some ring T. The endomorphism ring of G is canonically isomorphic to the matrix ring

$$\begin{pmatrix} \mathrm{End}_T(A) & \mathrm{Hom}_T(B, A) \\ \mathrm{Hom}_T(A, B) & \mathrm{End}_T(B) \end{pmatrix}$$

with ordinary operations of addition and multiplication of matrices (if it is necessary, the product is the composition of homomorphisms). It is clear that we are dealing with a formal matrix ring. The corresponding bimodule homomorphisms are defined as compositions.

In addition to rings of triangular matrices, we can consider one more interesting type of formal matrix rings. These rings are matrix rings with entries in a given ring R. Let R be some ring, and let K_1 be the "ordinary" ring of 2×2 matrices over R. Two coinciding R-R-bimodule isomorphisms

$$\omega\colon R\otimes_R R\to R,\quad x\otimes y\to xy.$$

correspond to the ring K_1. Starting from other bimodule homomorphisms $R\otimes_R R\to R$, we can obtain rings of 2×2 matrices over R, considered as formal matrix rings, which are not isomorphic to K_1.

We note that every R-R-bimodule endomorphism α of the bimodule R coincides with multiplication of the ring R by some central element. Indeed, there exist elements $s,t\in R$ such that $\alpha(x)=sx$ and $\alpha(x)=xt$ for all $x\in R$. For $x=1$, we obtain $s=t$. Therefore, $sx=xs$, i.e., s is a central element. Further, we take some R-R-bimodule homomorphism $\varphi\colon R\otimes_R R\to R$. We have $\varphi=\alpha\omega$ for some bimodule endomorphism α of the ring R. We take the element $s\in Z(R)$ such that $\alpha(x)=sx$, $x\in R$ ($Z(R)$ denotes the center of the ring R). Then we obtain $\varphi(x\otimes y)=sxy$, $x,y\in R$.

Now we assume that

$$K(\varphi,\psi)=\begin{pmatrix} R & R\\ R & R\end{pmatrix}$$

is some matrix ring with R-R-bimodule homomorphisms $\varphi,\psi\colon R\otimes_R R\to R$ satisfying the following two associativity laws:

$$\varphi(x\otimes y)z=x\psi(y\otimes z),\quad \psi(x\otimes y)z=x\varphi(y\otimes z).$$

Let we have elements $s,t\in Z(R)$ such that

$$\varphi(x\otimes y)=sxy,\quad \psi(x\otimes y)=txy,\quad x,y\in R.$$

For $x=y=z=1$, we obtain

$$s=\varphi(1\otimes 1)1=1\psi(1\otimes 1)=t,\quad \varphi=\psi.$$

Thus, in the ring $K(\varphi,\psi)$, matrices are multiplied according to the relation

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} e & f\\ g & h\end{pmatrix}=\begin{pmatrix} ae+sbg & af+bh\\ ce+dg & scf+dh\end{pmatrix}. \tag{$*$}$$

We denote this ring $K(\varphi,\psi)$ by K_s. The element s is called the *multiplier* of the ring K_s. The converse is also true. Namely, every central element s of the ring R defines a ring of 2×2 matrices over R in which multiplication satisfies the relation $(*)$. Consequently, this is the ring K_s. For $s=0$, we obtain the trivial ring K_s with $\varphi=\psi=0$. For $s=1$, we obtain the "ordinary" ring.

We return to arbitrary formal matrix rings. Such a ring $K(\varphi,\psi)$ is defined with the use of two bimodule homomorphisms φ and ψ. In general case, the choice of another pair of homomorphisms leads to a different ring. We obtain the problem of classification of formal matrix rings according to the corresponding pairs of bimodule homomorphisms. More precisely, the following *isomorphism problem* arises. Let $K(\varphi,\psi)$ and $K(\varphi_1,\psi_1)$ be two formal matrix rings

$$\begin{pmatrix} R & M\\ N & S\end{pmatrix}$$

with corresponding bimodule homomorphisms φ, ψ and φ_1, ψ_1. Determine the interrelations between the homomorphisms φ, ψ and φ_1, ψ_1 that are equivalent to the existence of an isomorphism $K(\varphi,\psi)\cong K(\varphi_1,\psi_1)$. In general case, this problem seems to be quite difficult. We consider the problem for matrix rings with entries in a given ring R.

The isomorphism problem for matrix rings K_s has the following form. For any two central elements s, t of the ring R, determine conditions under which the rings K_s and K_t are isomorphic to each other.

The center of the ring T is denoted by $Z(T)$. The following lemma is verified by direct calculation.

Lemma 1.1.

(1) *The center of the formal matrix ring*

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

consist of all diagonal matrices

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix},$$

where $r \in Z(R)$, $s \in Z(S)$, $rm = ms$, $nr = sn$ *for all* $m \in M$ *and* $n \in N$.

(2) *Let* $s \in Z(R)$. *Then*

$$Z(K_s) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \,\middle|\, r \in Z(R) \right\}.$$

In this part of the work, the isomorphism problem is studied; in this part, we do not assume that ring direct products $R \times S$ *are formal matrix rings.*

Lemma 1.2. *If a ring* R *is not a formal matrix ring, then* $K_0 \ncong K_s$ *for any nonzero central element* t.

Proof. We recall that K_0 is the ring K_s with $s = 0$. We assume that $K_0 \cong K_t$ for some nonzero $t \in Z(R)$. We fix a ring isomorphism $f\colon K_0 \to K_t$. Let I be the ideal

$$\begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix}$$

of the ring K_0 and let J be the ideal $f(I)$ of the ring K_t. It is directly verified that the ideal J consists of matrices

$$\begin{pmatrix} X & A \\ B & Y \end{pmatrix},$$

where X, Y, A, and B are some ideals of the ring R (they are interrelated with the use of the element t).

We show that A and B are proper ideals. We assume that $A = R$. Then

$$tR \subseteq X, \quad tR \subseteq Y, \quad t \in X, \quad t \in Y, \quad \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \in J.$$

By Lemma 1.1, we obtain

$$\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \in Z(K_t), \quad f^{-1}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \in I \cap Z(K_0) = 0.$$

This implies that the relation $t = 0$ is impossible. Therefore, $A \neq R$. Similarly, we have $B \neq R$.

The isomorphism f induces an isomorphism between the factor rings K_0/I and K_t/J. The first ring is isomorphic to $R \times R$, and the second ring is isomorphic to the formal matrix ring

$$\begin{pmatrix} R/X & R/A \\ R/B & R/Y \end{pmatrix}.$$

Consequently, $R \times R$ is a formal matrix ring. Therefore, R is a formal matrix ring. This is a contradiction. □

Lemma 1.3. *Let* R *be an arbitrary ring,* α *be an automorphism of* A, *and let* s *and* v *be central elements of the ring* R, *where* v *is invertible. Then*

$$K_s \cong K_{vs} \cong K_{\alpha(s)} \cong K_{v\alpha(s)}.$$

Proof. The isomorphism $K_s \cong K_{vs}$ satisfies the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ v^{-1}c & d \end{pmatrix}.$$

The isomorphism $K_s \cong K_{\alpha(s)}$ satisfies the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{pmatrix}.$$

The isomorphism $K_s \cong K_{v\alpha(s)}$ follows from the two considered isomorphisms. □

Corollary 1.4. *If R is a simple ring, then $K_s \cong K_t$ for any two nonzero central elements s and t.*

Proof. Since the center of a simple ring is a field, the assertion follows from Lemma 1.3. □

Theorem 1.5. *Let R be a commutative ring and let s and t be two elements of R such that at least one of them is not a zero-divisor. The rings K_s and K_t are isomorphic to each other if and only if there exist an invertible element $v \in R$ and an automorphism α of the ring R with $t = v\alpha(s)$.*

Proof. The sufficiency of the condition is contained in Lemma 1.3. Now we assume that there is an isomorphism $f\colon K_s \to K_t$. By Lemma 1.2, we can assume that $s \neq 0$ and t is not a zero-divisor. The isomorphism f induces an isomorphism of the centers $\alpha\colon Z(K_s) \to Z(K_t)$. It follows from Lemma 1.1 and agreements on identifying matrices with elements that we can consider α as an automorphism of the ring R. We consider the ideal

$$sK_s = \begin{pmatrix} sR & sR \\ sR & sR \end{pmatrix}$$

of the ring K_s. Its image under the action of f is an ideal of the ring K_t, which coincides with

$$f(s)K_t = \alpha(s)K_t = \begin{pmatrix} \alpha(s)R & \alpha(s)R \\ \alpha(s)R & \alpha(s)R \end{pmatrix}.$$

The isomorphism f also induces an isomorphism of the factor rings $\bar{f}\colon K_s/sK_s \to K_t/\alpha(s)K_t$. The ring K_s/sK_s is a trivial matrix ring (i.e., with multiplier 0), and the ring $K_t/\alpha(s)K_t$ is a matrix ring with multiplier $\bar{t} = t + \alpha(s)R$ over the ring $R/\alpha(s)R$. The rings R/sR and $R/\alpha(s)R$ are isomorphic to each other. Therefore, it follows from Lemma 1.2 that $\bar{t} = 0$ or $t \in \alpha(s)R$. Thus, $t = \alpha(s)x$, $x \in R$. By considering the converse isomorphism f^{-1}, we similarly obtain that $s = \alpha^{-1}(t)y$, $y \in R$. Then

$$t = \alpha(s)x = t\alpha(y)x, \quad t(1 - \alpha(y)x) = 0, \quad x\alpha(y) = 1,$$

since t is not a zero-divisor. Therefore, the element x is invertible. We have $t = v\alpha(s)$, where v is an invertible element and α is an automorphism of the ring R. □

Corollary 1.6. *Let R be either a commutative domain or a commutative local ring, and let $s, t \in R$. There is an isomorphism $K_s \cong K_t$ if and only if there exist an invertible element $v \in R$ and an automorphism α of the ring R such that $t = v\alpha(s)$.*

Proof. If R is a commutative domain, then the assertion directly follows from Theorem 1.5. If R is a commutative local ring, then we repeat the argument from the proof of Theorem 1.5 until we obtain the relation $t(1 - \alpha(y)x) = 0$. Further, it follows from Lemma 1.2 that we can assume that $t \neq 0$. Then the element $1 - \alpha(y)x$ is not invertible. Since the ring R is local, the element $\alpha(y)x$ is invertible. Therefore, the element x is invertible. □

We describe the Jacobson radical of a formal matrix ring. The Jacobson radical of some ring T is denoted by $J(T)$. We use the properties of the radical presented below. A right ideal L of the ring T is contained in $J(T)$ if and only if the element $1 - x$ is right invertible for any element $x \in L$. A similar assertion holds for left ideals.

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be an arbitrary formal matrix ring. We define four subbimodules in the bimodules M and N. We set

$$A_{\mathrm{l}}(M) = \{m \in M \mid Nm \subseteq J(S)\}, \quad A_{\mathrm{r}}(M) = \{m \in M \mid mN \subseteq J(R)\},$$
$$A_{\mathrm{l}}(N) = \{n \in N \mid Mn \subseteq J(R)\}, \quad A_{\mathrm{r}}(N) = \{n \in N \mid nM \subseteq J(S)\}.$$

Further, we set

$$A(M) = A_{\mathrm{l}}(M) \cap A_{\mathrm{r}}(M), \quad A(N) = A_{\mathrm{l}}(N) \cap A_{\mathrm{r}}(N).$$

Now we consider the following sets of matrices:

$$A_{\mathrm{l}}(K) = \begin{pmatrix} J(R) & A_{\mathrm{l}}(M) \\ A_{\mathrm{l}}(N) & J(S) \end{pmatrix}, \quad A_{\mathrm{r}}(K) = \begin{pmatrix} J(R) & A_{\mathrm{r}}(M) \\ A_{\mathrm{r}}(N) & J(S) \end{pmatrix},$$
$$A(K) = \begin{pmatrix} J(R) & A(M) \\ A(N) & J(S) \end{pmatrix}.$$

It is directly verified that we have obtained left, right, and two-sided ideals of the ring K, respectively.

Theorem 1.7. *There are relations*

$$J(K) = A_{\mathrm{l}}(K) = A_{\mathrm{r}}(K) = A(K).$$

Proof. It is sufficient to prove the inclusions

$$J(K) \subseteq A(K), \quad A_{\mathrm{l}}(K) \subseteq J(K), \quad A_{\mathrm{r}}(K) \subseteq J(K).$$

We have

$$J(K) = \begin{pmatrix} X & B \\ C & Y \end{pmatrix}.$$

We have the relations

$$X = eJ(K)e = J(eKe) = J(R), \quad \text{where} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, we obtain that $Y = J(S)$. Further, we have $B \subseteq A(M)$ and $C \subseteq A(N)$. The inclusion $J(K) \subseteq A(K)$ has been proved.

In $A_{\mathrm{r}}(K)$, we take an arbitrary matrix

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}$$

and the identity matrix E. The matrices

$$E - \begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}, \quad E - \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}$$

are right invertible in K. The matrices

$$\begin{pmatrix} x & xm \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ yn & y \end{pmatrix},$$

are the right inverse matrices, where x and y are right inverse elements for $1 - r$ and $1 - s$, respectively. Consequently, the matrices

$$\begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix}$$

are contained in $J(K)$. Therefore, $A_{\mathrm{r}}(K) \subseteq J(K)$. Similarly, $A_{\mathrm{l}}(K) \subseteq J(K)$. □

The prime radical of the ring K has a similar structure.

In the remaining part of this section, we present some remarks on formal matrix rings of order $n \geq 2$. The considered case $n = 2$ is sufficient for understanding how we can correctly define such rings.

Let $R_1, \dots, R_n$ be rings and let M_{ij} be R_i-R_j-bimodules such that $M_{ii} = R_i$, $i, j = 1, \dots, n$. We assume that for any $i, j, k = 1, \dots, n$ such that $i \neq k$ and $k \neq j$, an R_i-R_j-bimodule homomorphism

$$\varphi_{ikj} \colon M_{ik} \otimes_{R_k} M_{kj} \to M_{ij}$$

is given. For subscripts $i = k$ and $k = j$, we assume that φ_{iij} and φ_{ijj} are the canonical isomorphisms

$$R_i \otimes_{R_i} M_{ij} \to M_{ij}, \quad M_{ij} \otimes_{R_j} R_j \to M_{ij}.$$

We write ab instead of $\varphi_{ikj}(a \otimes b)$. In this notation, we also assume that $(ab)c = a(bc)$ for all elements $a \in M_{ik}$, $b \in M_{kj}$, $c \in M_{jl}$, and subscripts i, j, k, l.

We denote by K the set of all $n \times n$ matrices (a_{ij}) of order n with entries in the bimodules M_{ij}. With respect to standard matrix operations of addition and multiplication, K is a ring.

We say that K is the *formal matrix ring of order n*.

In order to better understand the structure of formal matrix rings, we recall the following interrelations between them, idempotents, and endomorphism rings. A ring T is the ring of all matrices of order n if and only if T has a complete orthogonal system consisting of n nonzero idempotents and if and only if the ring T is isomorphic to the endomorphism ring of some module that is decomposable into a direct sum of n nonzero summands.

In concrete studies, formal matrix rings of any order n appear. In the general theory, matrix rings of order 2 are ordinarily studied. There is no loss of generality, since the case $n > 2$ can be reduced in some sense to the case of matrices of order 2. Namely, a formal matrix ring of order $n > 2$ is isomorphic to some formal matrix ring of order k for every $k = 2, \dots, n-1$. This assertion becomes clear if we consider the representation of matrix rings with the use of idempotents or endomorphism rings. It is sufficient to "enlarge" idempotents or direct summands in a way. This assertion can also be directly proved with the use of the matrix method. For example, we take $k = 2$. We introduce the following notation for sets of matrices. We set $R = R_1$, $M = (M_{12}, \dots, M_{1n})$,

$$N = \begin{pmatrix} M_{21} \\ \vdots \\ M_{n1} \end{pmatrix}, \quad S = \begin{pmatrix} R_2 & M_{23} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n2} & M_{n3} & \dots & R_n \end{pmatrix}.$$

Here S is a formal matrix ring of order $n-1$, M is an R-S-bimodule, N is an S-R-bimodule, and module multiplications are defined as products of rows and columns by matrices. With the use of φ_{ikj} defining multiplication in K, we can define bimodule homomorphisms $M \otimes_S N \to R$ and $N \otimes_R M \to S$ such that two familiar associativity laws hold. As a result,

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is turned into the formal matrix ring of order 2. In addition,

$$K \cong \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

The isomorphism is obtained by decomposing any matrix into four blocks.

We temporarily consider rings of upper triangular matrices of order 3 (we consider them again in Sec. 2). Such a ring Γ can be represented in the form

$$\begin{pmatrix} R & M & L \\ 0 & S & N \\ 0 & 0 & T \end{pmatrix},$$

where R, S, and T are rings, M is an R-S-bimodule, L is an R-T-bimodule, and N is an S-T-bimodule. Among bimodule homomorphisms, only $M \otimes_S N \to L$ remains nonzero (besides situations where one of the factors is R, S, or T). We have two ways to turn Γ into the ring of triangular matrices of order 2. The first method is

$$\begin{pmatrix} r & m & l \\ 0 & s & n \\ 0 & 0 & t \end{pmatrix} \hookrightarrow \begin{pmatrix} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} & \begin{pmatrix} l \\ n \end{pmatrix} \\ (0 \;\; 0) & (t) \end{pmatrix}.$$

In this case, $\left(\begin{smallmatrix} L \\ N \end{smallmatrix}\right)$ is an $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$-$T$-bimodule. In the second case, $\left(\begin{smallmatrix} M & L \end{smallmatrix}\right)$ is an R-$\left(\begin{smallmatrix} S & N \\ 0 & T \end{smallmatrix}\right)$-bimodule.

There are various rings of triangular matrices. Fossum, Griffith, and Reiten [12] study so-called *trivial extensions* of rings defined as follows. If R is a ring and M is a R-bimodule, then we denote by T the direct sum of Abelian groups R and M, $T = \{(r, m) \mid r \in R,\ m \in M\}$. The group T is a ring with multiplication defined by the relation $(r, m)(r_1, m_1) = (rr_1, rm_1 + mr_1)$. This ring is the trivial extension considered above.

Now we consider the ring of triangular matrices

$$\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$$

and the subring

$$\Gamma = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \,\middle|\, r \in R,\ m \in M \right\}$$

in the ring. The rings T and Γ are isomorphic to each other under the correspondence

$$(r, m) \to \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}.$$

Thus, trivial extensions consist of triangular matrices.

Every ring of formal triangular matrices is a trivial extension. Indeed, M can be considered as an $(R \times S)$-$(R \times S)$-bimodule if we assume that $(r, s)m = rm$ and $m(r, s) = ms$. Then we take the trivial extension

$$T = \big\{ \big((r, s), m\big) \mid r \in R,\ s \in S,\ m \in M \big\}$$

of the ring $R \times S$. The correspondence

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \to \big((r, s), m\big)$$

defines an isomorphism of the rings K and T. However, there is a class of rings of triangular matrices containing trivial extensions. Let $f \colon R \to S$ be a ring homomorphism. In the ring

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix},$$

all matrices of the form

$$\begin{pmatrix} r & m \\ 0 & f(r) \end{pmatrix}$$

form a subring.

The paper [21] contains many results on rings of formal triangular matrices. Palmer [45] and Palmer and Roos [46] calculate homological dimensions of semitrivial and trivial extensions of rings.

There are many problems and directions for subsequent studies of the formal matrix ring

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

For example, we can look for conditions under which the ring K is coherent, hereditary, regular, self-injective, and so on. It is interesting to solve some problems related to matrices with special properties. It is known that the ring of all matrices of order $n > 1$ over a given ring satisfies the following property: every matrix is the sum of k invertible matrices for $k = 3$ and $k = 4$. For $k = 2$, a similar property is not necessarily true. These questions are interesting for matrices in K. It is also interesting to know when every element of the ring K is the sum of an idempotent and an invertible element. Rings with such a property are called *clean* rings. If K is a ring and any element of K is the sum of an idempotent and an invertible element that commute with each other, then K is called a *strongly clean* ring.

2. Preliminary Properties of Modules over Formal Matrix Rings

We study the structure of modules over the formal matrix ring

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

We can construct these modules starting from R-modules and S-modules. Let X be an R-module and let Y be an S-module. We assume that we have R-module homomorphisms $f\colon M \otimes_S Y \to X$ and S-module homomorphisms $g\colon N \otimes_R X \to Y$ such that

$$m(nx) = (mn)x, \quad n(my) = (nm)y, \quad m \in M, \quad n \in N, \quad x \in X, \quad y \in Y,$$

where we assume that nx coincides with $g(n \otimes x)$ and my coincides with $f(m \otimes y)$. The group of column vectors

$$\begin{pmatrix} X \\ Y \end{pmatrix}$$

is turned into a K-module if we take the product of a matrix and a column,

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx + my \\ nx + sy \end{pmatrix}$$

as the module multiplication. The homomorphisms f and g are called the *homomorphisms of the module multiplication.*

Every K-module has the form of the module of columns. More precisely, it can be obtained by the indicated method. Let V be a K-module and let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then eV is an R-module, $(1-e)V$ is an S-module, and $\begin{pmatrix} eV \\ (1-e)V \end{pmatrix}$ is a K-module. This becomes clear if we recall remarks from Sec. 1 about representations of matrices and sets of matrices. For example, we can assume that K has the form

$$\begin{pmatrix} eKe & eK(1-e) \\ (1-e)Ke & (1-e)K(1-e) \end{pmatrix},$$

Then module multiplications in the considered modules naturally hold, and homomorphisms of the module multiplication

$$M \otimes_S (1-e)V \to eV, \quad N \otimes_R eV \to (1-e)V$$

are the "restrictions" of the canonical isomorphism $K \otimes_K V \to V$ to the corresponding submodules. The correspondence of elements

$$v \to \begin{pmatrix} ev \\ (1-e)v \end{pmatrix}, \quad v \in V,$$

is an isomorphism between K-modules V and

$$\begin{pmatrix} eV \\ (1-e)V \end{pmatrix}.$$

In particular, the left K-module K has the form

$$\begin{pmatrix} (R, M) \\ (N, S) \end{pmatrix}$$

with homomorphisms of the module multiplication

$$m \otimes (n, s) \to (mn, ms), \quad n \otimes (r, m) \to (nr, nm).$$

A similar argument is true for right K-modules. Every right K-module has the form of the row vector module (X, Y), where X is a right R-module and Y is a right S-module. With this K-module, we associate module homomorphisms $Y \otimes_S N \to X$ and $X \otimes_R M \to Y$ satisfying the corresponding associativity relations. The module multiplication is the product of a row and a matrix. All properties of left K-modules have right-side analogues, which can be also proved by a formal transfer to right modules over the opposite ring K° (see Sec. 1). It is noted in Sec. 1 that the ring K° also is a formal matrix ring.

We call especial attention to the following remark: the left K-module of the form $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ and the elements of this module are also represented below as rows.

For (right or left) K-modules, we agree on the representation form of matrices; we also use this form for the ring K itself. For example, we write X instead of $\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)$ or $(X, 0)$, and we write x instead of $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ or $(x, 0)$, and so on.

Let (X, Y) be some (left) K-module. We set $MY = \operatorname{Im} f$ and $NX = \operatorname{Im} g$. It is clear that MY and NX are the sets of all finite sums of elements of the form my and nx, respectively. There are inclusions $IX \subseteq MY$ and $JY \subseteq NX$. When considering subgroups in M, N, Y, and X, we also use notations that are similar to MN, MY, and NX.

Instead of the module multiplication homomorphisms

$$f\colon M \otimes_S Y \to X, \quad g\colon N \otimes_R X \to Y,$$

it is more convenient sometimes to use the following S-homomorphism f' and the R-homomorphism g':

$$f'\colon Y \to \operatorname{Hom}_R(M, X), \quad f'(y)(m) = f(m \otimes y) = my, \quad y \in Y, \quad m \in M,$$
$$g'\colon X \to \operatorname{Hom}_S(N, Y), \quad g'(x)(n) = g(n \otimes x) = nx, \quad x \in X, \quad n \in N.$$

The homomorphisms f' and g' correspond to the homomorphisms f and g under natural isomorphisms of Abelian groups

$$\operatorname{Hom}_R(M \otimes_S Y, X) \cong \operatorname{Hom}_S\big(Y, \operatorname{Hom}_R(M, X)\big), \quad \operatorname{Hom}_S(N \otimes_R X, Y) \cong \operatorname{Hom}_R\big(X, \operatorname{Hom}_S(N, Y)\big),$$

respectively. When defining K-modules, we can start from the homomorphisms f' and g', which is absolutely equivalent to the original method. Therefore, we can also call f' and g' the *homomorphisms of the module multiplication*.

K-module homomorphisms can be represented by pairs consisting of R-module homomorphisms and S-module homomorphisms. Let (X, Y) and (X_1, Y_1) be two K-modules. We assume that $\alpha\colon X \to X_1$ and $\beta\colon Y \to Y_1$ is an R-homomorphism and an S-homomorphism, respectively, and $\alpha(my) = m\beta(y)$, $\beta(nx) = n\alpha(x)$ for all $m \in M$, $n \in N$, $x \in X$, $y \in Y$. Then the mapping

$$(X, Y) \to (X_1, Y_1), \quad (x, y) \to \big(\alpha(x), \beta(y)\big),$$

is a K-homomorphism. Now let $\Phi\colon (X, Y) \to (X_1, Y_1)$ be some K-homomorphism. It follows from the relations

$$\Phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(x, y)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Phi(x, y), \quad \Phi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(x, y)\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Phi(x, y)$$

that Φ acts as $\Phi(x, y) = \big(\alpha(x), \beta(y)\big)$, where α and β are the mappings $X \to X_1$ and $Y \to Y_1$, respectively. It is directly verified that α is an R-homomorphism, β is an S-homomorphism, and $\alpha(my) = m\beta(y)$, $\beta(nx) = n\alpha(x)$ for all elements from these relations. Thus, there are reasons to consider K-module homomorphisms as pairs (α, β).

The presented material on the structure of K-modules can be represented in the category-theoretical form. We have proved that the category of K-modules is equivalent to some category of "fours." We define the category $\mathcal{A}(K)$. The objects of $\mathcal{A}(K)$ are expressions (X, Y, f, g), where X is an R-module, Y is an S-module, $f\colon M \otimes_S Y \to X$ is an R-homomorphism, $g\colon N \otimes_R X \to Y$ is an S-homomorphism, and the following diagrams are commutative:

$$\begin{array}{ccccc}
M \otimes_S N \otimes_R X & \xrightarrow{1\otimes g} & M \otimes_S Y & \xrightarrow{f} & X \\
\Big\downarrow{\scriptstyle \varphi\otimes 1} & & & & \Big\downarrow{\scriptstyle 1_X}, \\
R \otimes_R X & & \xrightarrow{\mu} & & X
\end{array}$$

$$\begin{array}{ccccc}
N \otimes_R M \otimes_S Y & \xrightarrow{1\otimes f} & N \otimes_R X & \xrightarrow{g} & Y \\
\Big\downarrow{\scriptstyle \psi\otimes 1} & & & & \Big\downarrow{\scriptstyle 1_Y}, \\
S \otimes_S Y & & \xrightarrow{\nu} & & Y
\end{array} \qquad (1)$$

where μ and ν are the canonical homomorphisms. Morphisms $(X, Y, f, g) \to (X_1, Y_1, f_1, g_1)$ are pairs (α, β), where $\alpha \in \mathrm{Hom}_R(X, X_1)$ and $\beta \in \mathrm{Hom}_S(Y, Y_1)$ such that the following diagrams are commutative:

$$\begin{array}{ccc}
M \otimes_S Y & \xrightarrow{f} & X \\
\Big\downarrow{\scriptstyle 1\otimes\beta} & & \Big\downarrow{\scriptstyle \alpha} \\
M \otimes_S Y_1 & \xrightarrow{f_1} & X_1
\end{array}, \qquad
\begin{array}{ccc}
N \otimes_R X & \xrightarrow{g} & Y \\
\Big\downarrow{\scriptstyle 1\otimes\alpha} & & \Big\downarrow{\scriptstyle \beta} \\
N \otimes_R X_1 & \xrightarrow{g_1} & Y_1
\end{array}. \qquad (2)$$

Theorem 2.1 (Palmer [45], Green [16])**.** *The categories K-mod and $\mathcal{A}(K)$ are equivalent.*

Proof. We define a functor $F\colon K\text{-mod} \to \mathcal{A}(K)$. For the K-module V, we take $(eV, (1-e)V, f, g)$ as $F(V)$, where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

eV, and $(1-e)V$ are the corresponding modules from the beginning of the section, $f\colon M \otimes_S (1-e)V \to eV$ and $g\colon N \otimes_R eV \to (1-e)V$ are the "restrictions" of the canonical isomorphism $K \otimes_K V \to V$. Diagrams (1) are commutative. Let $\Phi\colon V \to W$ be a K-module homomorphism. Then $F(\Phi)\colon F(V) \to F(W)$ is the pair (α, β), where α is the restriction of Φ to eV and β is the restriction of Φ to $(1-e)V$. We note that $\Phi(eV) = e\Phi(eV) \subseteq eW$ and similarly for $(1-e)V$. It is directly verified that diagrams (2) are commutative.

Now we define a functor $G\colon \mathcal{A}(K) \to K\text{-mod}$. Let $(X, Y, f, g) \in \mathcal{A}(K)$. We assume that $G(X, Y, f, g)$ is the group of row vectors $\{(x, y) \mid X \in X,\ y \in Y\}$. The action K on $G(X, Y, f, g)$ is defined by the relation

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} (x, y) = (rx + f(m \otimes y), g(n \otimes x) + sy).$$

As a result, we obtain a K-module $G(X, Y, f, g)$. If $(\alpha, \beta)\colon (X, Y, f, g) \to (X_1, Y_1, f_1, g_1)$ is a morphism in $\mathcal{A}(K)$, then we define $G(\alpha, \beta)$ by the relation

$$G(\alpha, \beta) = \big(\alpha(x), \beta(y)\big), \quad x \in X, \quad y \in Y.$$

It is directly verified that $G(\alpha, \beta)$ is a K-homomorphism.

It remains to verify that the functors F and G define an equivalence between the categories K-mod and $\mathcal{A}(K)$. Namely, the composition GF is naturally equivalent to the identity functor of the category K-mod, and the composition FG is naturally equivalent to the identity functor of the category $\mathcal{A}(K)$. We define natural transformations σ and τ of functors as follows. If V is some K-module, then $GF(V)$ is the K-module $(eV, (1-e)V)$ according to our definitions. The mapping $\sigma_V\colon GF(V) \to V$, $\sigma_V(ev, (1-e)v) = v$,

$v \in V$, is a K-isomorphism. In addition, if $\varphi\colon V \to W$ is a K-module homomorphism, then $GF(\varphi)\sigma_W = \sigma_V\varphi$. Consequently, σ is a natural equivalence.

Now we consider an object $A = (X, Y, f, g)$ of the category $\mathcal{A}(K)$. $FG(A)$ practically coincides with A, and we can take the identity morphism of the object A as the morphism $\tau_A\colon FG(A) \to A$. If A' is one more object of the category $\mathcal{A}(K)$ and $\psi\colon A \to A'$ is a morphism, then it is directly verified that the diagram of objects in $\mathcal{A}(K)$

$$\begin{array}{ccc} FG(A) & \xrightarrow{FG(\psi)} & FG(A') \\ {\scriptstyle\tau_A}\downarrow & & \downarrow{\scriptstyle\tau_{A'}} \\ A & \xrightarrow{\psi} & A' \end{array}$$

is commutative. Consequently, τ is a natural equivalence and the theorem has been proved. □

There are simple but quite useful constructions of K-modules based on the tensor product and the group Hom. Let X be an R-module. The group of row vectors $(X, N \otimes_R X)$ is a K-module (we consider $N \otimes_R X$ as the canonical S-module) such that the homomorphisms of the module multiplication are the homomorphism

$$M \otimes_S (N \otimes_R X) \to X, \quad m(n \otimes x) \to (mn)x,$$

and the identity automorphism $N \otimes_R X \to N \otimes_R X$. Thus, $m(n \otimes x) = (mn)x$ and $nx = n \otimes x$ in our notation. Starting from the S-module Y, the K-module $(M \otimes_S Y, Y)$ is similarly defined. We use the notation $T(X) = N \otimes_R X$ and $T(Y) = M \otimes_S Y$. The modules $\big(X, T(X)\big)$ and $(T(Y), Y)$ have some specific property.

Lemma 2.2. *Let X be an R-module, (A, B) be a K-module, and let $\alpha\colon X \to A$ be an R-homomorphism. Then there exists a unique S-homomorphism $\beta\colon T(X) \to B$ such that $(\alpha, \beta)\colon \big(X, T(X)\big) \to (A, B)$ is a K-module homomorphism.*

A similar assertion holds for any S-module Y, an S-homomorphism $Y \to B$, and the K-modules (A, B), $(T(Y), Y)$.

Proof. The mapping $N \times X \to B$, $(n, x) \to n\alpha(x)$, $n \in N$, $x \in X$ is S-balanced. Consequently, there exists an S-homomorphism $\beta\colon T(X) \to B$ acting on generators as $\beta(n \otimes x) = n\alpha(x)$. The pair (α, β) defines a K-homomorphism, since $\alpha\big(m(n \otimes x)\big) = m\beta(n \otimes x)$ and $\beta(nx) = n\alpha(x)$ for all $m \in M$, $n \in N$, $x \in X$.

The uniqueness of β is meant in the sense indicated below. If $(\alpha, \gamma)\colon \big(X, T(X)\big) \to (A, B)$ is some K-homomorphism, then $\gamma = \beta$. Indeed, it follows from the definition of the module $\big(X, T(X)\big)$ that

$$\gamma(n \otimes x) = \gamma(nx) = n\alpha(x) = \beta(n \otimes x), \quad \gamma = \beta.$$

For the modules (A, B) and $(T(Y), Y)$, the assertion is similarly verified. □

Now we consider the group of row vectors $\big(X, \mathrm{Hom}_R(M, X)\big)$, where the group $\mathrm{Hom}_R(M, X)$ is considered as an S-module as usual. In fact, we have a K-module with the homomorphisms of the module multiplication

$$M \otimes_S \mathrm{Hom}_R(M, X) \to X, \quad m \otimes \alpha \to \alpha(m),$$
$$N \otimes_R X \to \mathrm{Hom}_R(M, X), \quad n \otimes x \to \beta,$$

where

$$\beta(m) = (mn)x, \quad m \in M, \quad n \in N, \quad x \in X, \quad \alpha \in \mathrm{Hom}_R(M, X).$$

Corresponding to the agreements on matrix representations, we have the relations $m\alpha = \alpha(m)$ and $(nx)(m) = (mn)x$. Similarly, the S-module Y provides the K-module $(\mathrm{Hom}_S(N, Y), Y)$ with module multiplications

$$n\gamma = \gamma(n), \quad (my)(n) = (nm)y, \quad n \in N, \quad m \in M, \quad y \in Y, \quad \gamma \in \mathrm{Hom}_S(N, Y).$$

We write $H(X)$ and $H(Y)$ instead of $\mathrm{Hom}_R(M,X)$ and $\mathrm{Hom}_S(N,Y)$, respectively. The modules $\big(X,H(X)\big)$ and $(H(Y),Y)$ have one important property related to homomorphisms.

Lemma 2.3. *Let X be an R-module, (A,B) be a K-module, and let $\alpha\colon A\to X$ be an R-homomorphism. We define the mapping $\beta\colon B\to H(X)$ by the relation $\beta(b)(m)=\alpha(mb)$, $b\in B$, $m\in M$. Then β is an S-homomorphism and (α,β) is a K-homomorphism $(A,B)\to\big(X,H(X)\big)$. Such a homomorphism β is unique.*

A similar assertion holds for any S-module Y, an S-homomorphism $B\to Y$, and K-modules (A,B) and $(H(Y),Y)$.

Proof. The mapping β is a homomorphism of Abelian groups. In addition, for arbitrary $s\in S$, $b\in B$, and $m\in M$, we have

$$\beta(sb)(m)=\alpha\big(m(sb)\big),\quad \big(s\beta(b)\big)(m)=\beta(b)(ms)=\alpha\big((ms)b\big).$$

Since $m(sb)=(ms)b$, we have $\beta(sb)=s\beta(b)$. Therefore, β is an S-homomorphism.

The pair (α,β) is a K-homomorphism provided that the relations

$$\alpha(mb)=m\beta(b),\quad \beta(na)=n\alpha(a),\quad m\in M,\quad n\in N,\quad a\in A,\quad b\in B,$$

hold. These relations follow from the relation $m\beta(b)=\beta(b)(m)=\alpha(mb)$. In addition,

$$\beta(na)(m)=\alpha\big(m(na)\big)=\alpha\big((mn)a\big)=mn\alpha(a)=\big(n\alpha(a)\big)(m),\quad \beta(na)=n\alpha(a).$$

We assume that $(\alpha,\gamma)\colon(A,B)\to\big(X,H(X)\big)$ is a K-homomorphism. It follows from the definition of the module $\big(X,H(X)\big)$ that $\gamma(b)(m)=m\gamma(b)$, $b\in B$, $m\in M$. On the other hand, $m\gamma(b)=\alpha(mb)$ and $\beta(b)(m)=\alpha(mb)$, whence $\gamma(b)(m)=\beta(b)(m)$ and $\gamma=\beta$.

A similar proof holds for the modules (A,B) and $(T(Y),Y)$. ☐

Corollary 2.4. *For any R-module X, there are canonical ring isomorphisms*

$$\mathrm{End}_K\big(X,T(X)\big)\cong\mathrm{End}_R(X)\cong\mathrm{End}_K\big(X,H(X)\big).$$

A similar assertion holds for endomorphism rings of the modules Y, $(T(Y),Y)$, and $(H(Y),Y)$.

Remarks. For any module (X,Y), we have the following four homomorphisms:

$$(1,g)\colon\big(X,T(X)\big)\to(X,Y),\quad (f,1)\colon(T(Y),Y)\to(X,Y),$$
$$(1,f')\colon(X,Y)\to\big(X,H(X)\big),\quad (g',1)\colon(X,Y)\to(H(Y),Y).$$

It follows from Lemmas 2.2 and 2.3 that f, g and f', g' are only possible homomorphisms provided the second mapping is the identity mapping. These homomorphisms will be quite useful in what follows.

We can consider the presented constructions from the category-theoretical viewpoint. For any formal matrix ring K, there exists a ring homomorphism

$$R\times S\to K,\quad (r,s)\to\begin{pmatrix} r & 0\\ 0 & s\end{pmatrix}.$$

Consequently, every K-module can be considered as an $(R\times S)$-module. This provides the "forgetting" functor $E\colon K\text{-mod}\to(R\times S)\text{-mod}$. We note that if K is a formal matrix ring with zero trace ideals, then the "diagonal" mapping

$$K\to R\times S,\quad \begin{pmatrix} r & m\\ n & s\end{pmatrix}\to(r,s)$$

is a homomorphism; consequently, any $(R\times S)$-module is a K-module in this case. Now we define two functors T and H from $(R\times S)$-mod in K-mod. With the $(R\times S)$-module (X,Y), the functors T and H associate the K-modules $\big(X,T(X)\big)\oplus(T(Y),Y)$ and $\big(X,H(X)\big)\oplus(H(Y),Y)$, respectively. If $(\alpha,\beta)\colon(X,Y)\to(X_1,Y_1)$ is an $(R\times S)$-module homomorphism, then $T(\alpha,\beta)$ and $H(\alpha,\beta)$ are obviously defined induced homomorphisms $T(X,Y)\to T(X_1,Y_1)$ and $H(X,Y)\to H(X_1,Y_1)$, respectively. In fact, T is the functor $K\otimes_{R\times S}(-)$, where $T(X,Y)=K\otimes_{R\times S}(X,Y)$, and H is the functor $\mathrm{Hom}_{R\times S}(K,-)$, where

$H(X,Y) = \mathrm{Hom}_{R\times S}\big(K,(X,Y)\big)$. The functors T and H transfer $(R\times S)$-module homomorphisms into the corresponding induced K-module homomorphisms. The functor T (the functor H) is left conjugated (respectively, right conjugated) to the functor E. These conjugacy situations appeared in Lemmas 2.2 and 2.3.

The functors T and H are also related to each other in some way. We mean that there exists a natural transformation $\theta\colon T\to H$. The corresponding natural homomorphism $\theta(X,Y)\colon T(X,Y)\to H(X,Y)$ is the sum of homomorphisms $(1,h)+(h',1)$, where $h\colon T(X)\to H(X)$ maps from $n\otimes x$ into the homomorphism $m\to(mn)x$. We similarly define $(h',1)\colon (T(Y),Y)\to(H(Y),Y)$.

Now we study the form of submodules and factor modules of K-modules. This is not difficult to do, since we know the structure of K-modules themselves. Let $V=(X,Y)$ be a module over the ring K. A subset $W\subseteq V$ is a submodule of the module V if and only if there exist a submodule A of the R-module X and a submodule B of the S-module Y such that $W=(A,B)$, $MB\subseteq A$, and $NA\subseteq B$. The following situation is an important partial case. For two submodules A and B of the modules X and Y, respectively, the sets (A,NA) and (MB,B) are necessarily submodules in (X,Y). If the ring K has zero trace ideals (i.e., $I=0=J$), then we obtain the submodules $(MB,0)$ and $(0,NA)$. The interrelations between the modules (A,NA), (MB,B) and the modules $\big(A,T(A)\big)$, $(T(B),B)$ and $\big(A,H(A)\big)$, $(H(B),B)$ are clear from the remarks after Corollary 2.4.

The group of row vectors $(X/A,Y/B)$ is a K-module. The homomorphisms of the module multiplication

$$M\otimes_S Y/B\to X/A,\quad N\otimes_R X/A\to Y/B$$

are induced by the homomorphisms of the module multiplication of the module (X,Y). Namely, $m\bar{y}=\overline{my}$, where $\bar{y}=y+B$, $\overline{my}=my+A$, and we have a similar situation for the second homomorphism. The factor module V/W can be identified with the module $(X/A,Y/B)$. More precisely, the correspondence $(x,y)+W\to(x+A,y+B)$ is an isomorphism between these modules.

In operating with modules over a ring of formal triangular matrices, some particularities appear. It is not difficult to determine them by considering the previous material with $N=0$. We only consider some details. Let (X,Y) be a K-module. Since $g=0$, there is only f among two homomorphisms of the module multiplication f and g. Two associativity relations hold automatically. An important particularity of the "triangle case" is that for any R-module X, we have the K-module $(X,0)$. A K-module homomorphism $(X,Y)\to(X_1,Y_1)$ is a pair (α,β) consisting of an R-homomorphism $\alpha\colon X\to X_1$ and an S-homomorphism $\beta\colon Y\to Y_1$ satisfying the relation $\alpha(my)=m\beta(y)$ for all $m\in M$ and $y\in Y$. The category $\mathcal{A}(K)$ from Theorem 2.1 is turned into the category of "triples" of the form (X,Y,f). Diagrams (1) are always commutative, and only the first diagram remains in (2). If X is an R-module and Y is an S-module, then K-modules $\big(X,T(X)\big)$ and $(H(Y),Y)$ from Lemmas 2.2 and 2.3 have the form $(X,0)$ and $(0,Y)$, respectively.

Modules over a formal matrix ring of order $n>2$ (these rings are defined at the end of Sec. 1) have a structure that is similar to the structure of modules in the case $n=2$. Such a module is the module of column vectors of height n, and the module multiplication satisfies the multiplication rule "a matrix $\times$ a column." It is not necessary to present details, since all details are clear from the considered case $n=2$. We consider more carefully modules over the ring of formal triangular matrices

$$\Gamma=\begin{pmatrix} R & M & L\\ 0 & S & N\\ 0 & 0 & T\end{pmatrix}$$

of order 3 (this ring is considered in the end of Sec. 1). Let $\varphi\colon M\otimes_S N\to L$ be the R-T-bimodule homomorphism from the definition of the ring Γ. As earlier, we write mn instead of $\varphi(m\otimes n)$. We assume that there are an R-module X, an S-module Y, a T-module Z, R-module homomorphisms $f\colon M\otimes_S Y\to X$ and $h\colon L\otimes_T Z\to X$, an S-module homomorphism $g\colon N\otimes_T Z\to Y$, and $m(nz)=(mn)z$ for all $m\in M$, $n\in N$, and $z\in Z$. We also preserve the previous notation. Then (X,Y,Z) is a Γ-module with module multiplication of the form "a matrix $\times$ a column." Any Γ-module can be obtained by this method.

Γ-module homomorphisms act coordinatewise. A homomorphism $(X, Y, Z) \to (X_1, Y_1, Z_1)$ is a triple (α, β, γ), where $\alpha\colon X \to X_1$, $\beta\colon Y \to Y_1$, and $\gamma\colon Z \to Z_1$ is a homomorphisms of corresponding modules. In addition, it is necessary that the relations

$$\alpha(my) = m\beta(y), \quad \alpha(lz) = l\gamma(z), \quad \beta(nz) = n\gamma(z)$$

hold for all values m, n, l, y, z.

Similarly to the category $\mathcal{A}(K)$, we define the category $\mathcal{A}(\Gamma)$. The objects of $\mathcal{A}(\Gamma)$ are expressions (X, Y, Z, f, g, h), and morphisms are triples (α, β, γ) satisfying commutative diagrams that are similar to diagrams (2). The categories Γ-mod and $\mathcal{A}(\Gamma)$ are equivalent [16]. This can be proved by a direct proof, which is similar to the proof of Theorem 2.1. We can also use Theorem 2.1 two times. We present the details of the second assertion.

The ring Γ is naturally isomorphic to two rings Δ and Λ of triangular matrices of order 2 (see Sec. 1). In turn, the Γ-module (X, Y, Z) with homomorphisms f, g, and h can be considered as a Δ-module $\big((X, Y), Z\big)$ of row vectors of length 2 consisting of the blocks (x, y) and z. Here (X, Y) is the $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$-module obtained with the use of the homomorphism $f\colon M \otimes_S Y \to X$, and the homomorphism

$$\begin{pmatrix} L \\ N \end{pmatrix} \otimes_T Z \to (X, Y)$$

is $h + g$. The module multiplication of the Γ-module (X, Y, Z) induces a module multiplication of the Δ-module $\big((X, Y), Z\big)$ acting on blocks. Similarly, the Γ-module (X, Y, Z) is turned into the Λ-module $\big(X, (Y, Z)\big)$ such that (Y, Z) is the $\left(\begin{smallmatrix} S & N \\ 0 & T \end{smallmatrix}\right)$-module obtained with the use of the homomorphism $g\colon N \otimes_T Z \to Y$, and the homomorphism

$$(M, L) \otimes_{\left(\begin{smallmatrix} S & N \\ 0 & T \end{smallmatrix}\right)} (Y, Z) \to X$$

is $f + h$ (it is necessary to consider the relation $m(nz) = (mn)z$). The Γ-module (X, Y, Z) and the Δ-module $\big((X, Y), Z\big)$ are practically identical. When using more precise argument, we can say that the categories Γ-mod and Δ-mod are equivalent. Now it is clear how we can twice apply Theorem 2.1 to obtain the equivalence of the categories Γ-mod and $\mathcal{A}(\Gamma)$.

There are some other interesting important results related to the following topic: the reduction of the study of the module over an arbitrary ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

to the study of modules over some triangular matrix ring. We present some quite general simple theorem related to this topic.

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring with bimodule homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. We fix some ideal L of the ring R containing the trace ideal I (for example, $L = I$ or $L = R$). Then there exists a ring of triangular matrices

$$\begin{pmatrix} R & M & L \\ 0 & S & N \\ 0 & 0 & R \end{pmatrix}$$

of order 3. Such rings were considered above. We take φ as the bimodule homomorphism $M \otimes_S N \to L$; this is correct, since $\operatorname{Im}\varphi = I \subseteq L$.

Let $V = (X, Y)$ be some K-module with homomorphisms of the module multiplication $f\colon M \otimes_S Y \to X$ and $g\colon N \otimes_R X \to Y$. We can construct the Γ-module $W = (X, Y, X)$ with homomorphisms of the module multiplication $f\colon M \otimes_S Y \to X$, $h\colon L \otimes_R X \to X$, and $g\colon N \otimes_R X \to Y$, where h is the canonical homomorphism $l \otimes x \to lx$, $l \in L$, $x \in X$. The relation $m(nz) = (mn)z$ is true, since it is turned into the relation $m(nx) = (mn)x$, which follows from the existence of the K-module (X, Y).

We determine the interrelations between K-module homomorphisms and Γ-module homomorphisms. Let $(\alpha, \beta)\colon (X, Y) \to (X_1, Y_1)$ be a K-module homomorphism. We assert that $(\alpha, \beta, \alpha)\colon (X, Y, X) \to (X_1, Y_1, X_1)$ is a Γ-module homomorphism. Indeed, we have three required relations

$$\alpha(my) = m\beta(y), \quad \beta(nx) = n\alpha(x), \quad \alpha(lx) = l\alpha(x).$$

Conversely, we assume that $(\alpha, \beta, \gamma)\colon (X, Y, X) \to (X_1, Y_1, X_1)$ is a Γ-module homomorphism, and we assume that $L = R$. Then

$$r\alpha(x) = \alpha(rx) = r\gamma(x) \text{ for all } r \in R, \quad x \in X.$$

This implies that $\gamma = \alpha$. Thus, every homomorphism $(X, Y, X) \to (X_1, Y_1, X_1)$ is a "triple" (α, β, α).

We formulate the obtained interrelations between K-modules and Γ-modules in the category-theoretical form. We take the ring

$$\begin{pmatrix} R & M & R \\ 0 & S & N \\ 0 & 0 & R \end{pmatrix}$$

as Γ. We define the covariant functor $F\colon K\text{-mod} \to \Gamma\text{-mod}$. The functor F transfers a K-module $V = (X, Y)$ into the Γ-module $F(V) = (X, Y, X)$. The functor F transfers a K-module homomorphism (α, β) into the Γ-module homomorphism (α, β, α).

Theorem 2.5. *The functor F is a full embedding from the category* K-mod *into the category* Γ-mod.

Proof. We mean that F defines an equivalence between K-mod and the complete subcategory in the category Γ-mod consisting of modules of the form (X, Y, X). For the proof of this assertion, all required properties are presented in the text before Theorem 2.5. ☐

Corollary 2.6. *Let V be a K-module.*

(1) *The endomorphism rings of the K-module V and the Γ-module $F(V)$ are isomorphic to each other.*
(2) *The K-module V is indecomposable if and only if the Γ-module $F(V)$ is indecomposable.*

Hirano [26] explicitly considers representations of the Γ-modules, where Γ is an arbitrary ring of triangular matrices

$$\begin{pmatrix} R & M & L \\ 0 & S & N \\ 0 & 0 & T \end{pmatrix},$$

as a module over two rings of triangular matrices of order 2 indicated in Sec. 1. The paper of Green [16] contains an interesting study of the problem considered in Theorem 2.5. In particular, there are constructed several functors from the category of K-modules into the category of modules over various rings of triangular matrices of order 3. The author tries to select Γ such that both rings K and Γ are rings of finite (infinite) representation type.

The mapping $f\colon V \to W$ of modules over the ring T is said to be *T-homogeneous* if $f(tv) = tf(v)$ for all $t \in T$ and $v \in V$. A module V is said to be *endoformal* if every T-homogeneous mapping $V \to V$ is an endomorphism; this means that any T-homogeneous mapping is additive. Maxson [39] have proved that all modules over the matrix ring of order $n > 1$ are endoformal. The case of modules over formal matrix rings is not clear.

3. Small and Essential Submodules

This section and the next section are of illustrative character. The presented results are not difficult. We present some methods of studying modules over formal matrix rings.

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring and let (X, Y) be a K-module. Some submodules of the module (X, Y) play very important role in various problems. The submodules MY and NX, defined in Sec. 2, are example of such submodules. There are two additional such submodules $L(X)$ and $L(Y)$ defined by the relations

$$L(X) = \{x \in X \mid nx = 0 \text{ for every } n \in N\},$$
$$L(Y) = \{y \in Y \mid my = 0 \text{ for every } m \in M\}.$$

If the ring K has zero trace ideals, then $MY \subseteq L(X)$ and $NX \subseteq L(Y)$.

In this section, we assume that K is a ring with zero trace ideals, i.e., the trace ideals I and J of the ring K are equal to zero. In this case, $mn = 0 = nm$ for all $m \in M$ and $n \in N$.

In addition to small and essential submodules, we describe finitely generated, hollow, and uniform K-modules. Here it is convenient to give a remark similar to the remark in Sec. 1 on the character of studies in K-modules. When we consider the description of some K-module (X, Y), it is reasonable to look for this description in terms of the R-module X, the S-module Y, and the actions of the bimodules M and N on Y and X, respectively.

Proposition 3.1. *The K-module (X, Y) is finitely generated if and only if the R-module X/MY and the S-module Y/NX are finitely generated.*

Proof. Let (X, Y) be a finitely generated K-module with finite generator system $(x_1, y_1), \ldots, (x_k, y_k)$. We take an arbitrary element $x \in X$. We have

$$(x, 0) = t_1(x_1, y_1) + \cdots + t_k(x_k, y_k),$$

where

$$t_i = \begin{pmatrix} r_i & m_i \\ n_i & s_i \end{pmatrix}, \quad i = 1, \ldots, k.$$

Then

$$x = \sum_{i=1}^{k}(r_i x_i + m_i y_i), \quad x + MY = r_1(x_1 + MY) + \cdots + r_k(x_k + MY).$$

Consequently, $\{x_i + MY\}_{i=1}^{k}$ and $\{y_i + NX\}_{i=1}^{k}$ are generator systems for X/MY and Y/NX, respectively.

Now we assume that the modules X/MY and Y/NX are finitely generated, and $\{x_i + MY\}_{i=1}^{k}$ and $\{y_i + NX\}_{i=1}^{l}$ are their generator systems. We assert that

$$\{(x_i, 0), (0, y_j) \mid i = 1, \ldots, k, \ j = 1, \ldots, l\}$$

is a generator system of the K-module (X, Y). It is sufficient to verify that all elements of the form $(x, 0)$ and $(0, y)$ are linear combinations of elements of the presented system. For $(x, 0)$, it is verified as follows (the case $(0, y)$ is similarly considered). We have

$$x = r_1 x_1 + \cdots + r_k x_k + m_1 b_1 + \cdots + m_i b_i,$$

where $r \in R$, $m \in M$, and $b \in Y$. In turn, each of the elements $b_1, \ldots, b_i$ is equal to the sum of the form

$$s_1 y_1 + \cdots + s_l y_l + n_1 a_1 + \cdots + n_j a_j,$$

where $s \in S$, $n \in N$, $a \in X$. We substitute these sums into the expression for x and consider the relation $mn = 0$. We obtain that

$$x = r_1 x_1 + \cdots + r_k x_k + c_1 y_1 + \cdots + c_l y_l$$

for some $c_1, \ldots, c_l \in M$. Now it is clear how to represent $(x, 0)$ in the required form. We only note that

$$\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} (0, y_1) = (c_1 y_1, 0). \qquad \square$$

We denote by (σ, τ) the canonical homomorphism $(X, Y) \to (X/MY, Y/NX)$. We recall that (MY, NX) is a submodule in (X, Y), and the factor module $(X, Y)/(MY, NX)$ can be identified with the module $(X/MY, Y/NX)$, as we agreed in Sec. 2.

A submodule A of some module V is said to be *small* if $B = V$ for any submodule B in V with $A + B = V$.

Proposition 3.2 (Yardykov)**.** *Let (X, Y) be a K-module. The submodule (A, B) is small in (X, Y) if and only if σA is a small submodule in X/MY and τB is a small submodule in Y/NX.*

Proof. We assume that (A, B) is a small submodule in (X, Y). Let us be given the relation $\sigma A + C/MY = X/MY$ for some submodule C in X. Since $\sigma A = (A + MY)/MY$, we obtain

$$A + C = X, \quad (A, B) + (C, Y) = (X, Y), \quad (C, Y) = (X, Y), \quad C = X.$$

Therefore, σA is a small submodule in X/MY. It can be similarly proved that τB is a small submodule in Y/NX.

Now we assume that σA is a small submodule in X/MY and τB is a small submodule in Y/NX. Let $(A, B) + (C, D) = (X, Y)$ for some submodule (C, D) in (X, Y). Then

$$A + C = X, \quad B + D = Y, \quad \sigma A + (C + MY)/MY = X/MY, \quad \tau B + (D + NX)/NX = Y/NX.$$

Since σA is a small submodule in X/MY and τB is a small submodule in Y/NX, we have that $C + MY = X$ and $D + NX = Y$. We multiply the last relation by N and M, respectively. We obtain $NC = NX$ and $MD = MY$. Now we obtain $MY = MD \subseteq C$ and $NX = NC \subseteq D$. Therefore, $C = X$, $D = Y$, and (A, B) is a small submodule in (X, Y). $\square$

A module M is said to be *hollow* if M is not equal to zero and all submodules of M are small in M.

Corollary 3.3. *A nonzero module (X, Y) is hollow if and only if either X/MY is a hollow module and $Y = NX$, or Y/NX is a hollow module and $X = MY$.*

Proof. First, we note that the relations $X = MY$ and $Y = NX$ do not simultaneously hold.

We assume that (X, Y) is a hollow module. We consider two possible cases for the module (X, NX).

(1) $(X, NX) = (X, Y)$. Then $Y = NX$ and $X \neq MY$. We take some submodule A/MY in X/MY, where $A \neq X$. By assumption, (A, NA) is a small submodule in (X, Y). It follows from Proposition 3.2 that A/MY is a small submodule in X/MY. Therefore, X/MY is a hollow submodule.

(2) $(X, NX) \neq (X, Y)$. It follows from Proposition 3.2 that X/MY is a small submodule in X/MY, whence $X = MY$. Then $(MY, Y) = (X, Y)$. Similar to (1), we obtain that Y/NX is a hollow submodule.

Now we assume that X/MY is a hollow module and $Y = NX$. (The case where Y/NX is a hollow module and $X = MY$, is considered similarly.) We take some proper submodule (A, B) in (X, Y). It is clear that $A \neq X$. By assumption, σA is a small submodule in X/MY, and it is obvious that τB is a small submodule in Y/NX. By Proposition 3.2, (A, B) is a small submodule in (X, Y). Therefore, (X, Y) is a hollow module. $\square$

A module is said to be *local* if it is finitely generated and has exactly one maximal submodule. It is easy to verify that local modules coincide with finitely generated hollow modules. By Proposition 3.1 and Corollary 3.3, we obtain following result.

Corollary 3.4. *The module (X, Y) is local if and only if either X/MY is a local module and $Y = NX$ or Y/NX is a local module and $X = MY$.*

A submodule A of the module V is said to be *essential* (or *large*) if A has nonzero intersection with every nonzero submodule of the module V. In this case, V is called an *essential extension* of the module A.

Proposition 3.5 (Yardykov)**.** *The submodule (A, B) of the module (X, Y) is essential if and only if $A \cap L(X)$ is an essential submodule in $L(X)$ and $B \cap L(Y)$ is an essential submodule in $L(Y)$.*

Proof. Let (A, B) be an essential submodule in (X, Y). We verify that $A \cap L(X)$ is an essential submodule in $L(X)$. For a nonzero element $x \in L(X)$, there exists a matrix

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix},$$

such that

$$0 \neq \begin{pmatrix} r & m \\ n & s \end{pmatrix} (x, 0) \in (A, B) \quad \text{or} \quad (rx, nx) \in (A, B).$$

In addition, $nx = 0$, since $x \in L(X)$. Then $rx \neq 0$, $rx \in A \cap L(X)$ and $A \cap L(X)$ is an essential submodule in $L(X)$. Similarly, $B \cap L(Y)$ is an essential submodule in $L(Y)$.

Now we assume that $A \cap L(X)$ is an essential submodule in $L(X)$, $B \cap L(Y)$ is an essential submodule in $L(Y)$, and (x, y) is a nonzero element in (X, Y). We prove that $K(x, y) \cap (A, B) \neq 0$. If $y \neq 0$, then we can assume that $x = 0$. We consider the case where $y \notin L(Y)$. Then $my \neq 0$ for some $m \in M$. Since $my \in L(X)$ and $A \cap L(X)$ is an essential submodule in $L(X)$, we have that $0 \neq rmy \in A \cap L(X)$ for some $r \in R$. Then

$$\begin{pmatrix} 0 & rm \\ 0 & 0 \end{pmatrix} (0, y) = (rmy, 0) \in (A, B).$$

If $y \in L(Y)$, then $0 \neq sy \in B \cap L(Y)$, where $s \in S$, since $B \cap L(Y)$ is an essential submodule in $L(Y)$. Then $s(0, y) = (0, sy) \in (A, B)$. The case $y = 0$ is considered similarly. ☐

A module is said to be *uniform* if the intersection of any two its nonzero submodules is not equal to zero.

Corollary 3.6. *The module (X, Y) is uniform if and only if either $L(X) = 0$ and Y is uniform, or $L(Y) = 0$ and X is uniform.*

Proof. Let (X, Y) be a uniform module. The intersection of the submodules $(L(X), 0)$ and $\big(0, L(Y)\big)$ is equal to zero. Therefore, at least one of the modules $L(X)$, $L(Y)$ is equal to zero. For example, if $L(X) = 0$, then $L(Y) = Y$, since $MY \subseteq L(X) = 0$. We take an arbitrary nonzero submodule B in Y. Since (MB, B) is an essential submodule in (X, Y), it follows from Proposition 3.5 that $B \cap L(Y)$ is an essential submodule in $L(Y)$. Therefore, Y is an essential extension of the module B and the module Y is uniform. If $L(Y) = 0$, then we use a similar argument.

For the proof of the converse assertion, we assume that $L(X) = 0$ and Y is uniform. As earlier, $L(Y) = Y$. Let (A, B) be an arbitrary nonzero submodule in (X, Y). Then $B \neq 0$, since otherwise $A \neq 0$ and $A \subseteq L(X)$, which is impossible. It follows from Proposition 3.5 that (A, B) is an essential submodule in (X, Y). Therefore, the module (X, Y) is uniform. ☐

Remark. We can apply all results of this section to modules over the ring

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

of triangular matrices (see [22]). We only need to do the corresponding changes, since the relations $NX = 0$ and $L(X) = X$ always hold for the module (X, Y) over such a ring. For example,

> (X, Y) is a hollow module if and only if either X is a hollow module and $Y = 0$ or Y is a hollow module and $X = MY$;
> (X, Y) is a uniform module if and only if either $X = 0$ and Y is a uniform module, or $L(Y) = 0$ and X is a uniform module.

4. The Socle and the Radical

In this section, K is an arbitrary formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

First, we describe simple K-modules. Then we use this description and determine the structure of minimal and maximal submodules, the socle, and the radical.

A nonzero module is said to be *simple* if it does not have nontrivial submodules.

Proposition 4.1 (Yardykov [53]). *A module (X, Y) is simple if and only if either X and Y are simple modules, $X = MY$, and $Y = NX$, or X is a simple module and $Y = 0$, or $X = 0$ and Y is a simple module.*

Before proving Proposition 4.1, we present the following remark. Since inclusions $IX \subseteq MY$ and $JY \subseteq NX$ are always true by Sec. 2, it follows from the relations $X = IX$ and $Y = JY$ that $X = MY$ and $Y = NX$. The converse is also true. Similarly, $IX, JY \neq 0$ if and only if $MY, NX \neq 0$. Therefore, in Proposition 4.1, we can also write $X = IX$, $Y = JY$ or $IX, JY \neq 0$, or $MY, NX \neq 0$.

Proof of Proposition 4.1. Let (X, Y) be a simple module and let $X, Y \neq 0$. For any nonzero submodules A in X and B in Y, we have that (A, NA) and (MB, B) are submodules in (X, Y). Therefore, $A = X$, $B = Y$, and X, Y are simple modules. In particular, $X = MY$ and $Y = NX$. If one of the modules X, Y is equal to zero, then it is clear that the second module is necessarily simple.

Now we assume that X and Y are simple modules and $X = MY$, $Y = NX$. A nontrivial submodule in (X, Y) can be only of the form $(X, 0)$ or $(0, Y)$. This is impossible by the relations $X = MY$ and $Y = NX$. If one of the modules X, Y is simple and the second module is equal to zero, then it is clear that (X, Y) is a simple module. $\square$

Corollary 4.2. *Let (X, Y) be a K-module.*

1. *If $L(X) = 0 = L(Y)$, then the module (X, Y) is simple if and only if X and Y are simple modules.*
2. *If K is a ring with zero trace ideals, then the module (X, Y) is simple if and only if either the module X is simple and $Y = 0$ or $X = 0$ and the module Y is simple.* $\square$

A nonzero (proper) submodule of the module V is said to be *minimal* (respectively, *maximal*) if it is a minimal (respectively, maximal) element in the lattice of all submodules of the module V.

Corollary 4.3. *Let (A, B) be a submodule of the K-module (X, Y).*

1. *(A, B) is minimal if and only if either A and B are minimal, $A = MB$, and $B = NA$, or A is minimal and $B = 0$ (then $NA = 0$), or $A = 0$ (then $MB = 0$) and B is minimal.*
2. *(A, B) is maximal if and only if either A and B are maximal, $MY \not\subseteq A$, and $NX \not\subseteq B$ (this is equivalent to the property that $IX \not\subseteq A$ and $JY \not\subseteq B$), or A is maximal and $B = Y$ (then $MY \subseteq A$), or $A = X$ (then $NX \subseteq B$) and B is maximal.*

Proof. Assertion (1) directly follows from Proposition 4.1, since every minimal submodule is a simple module. In relation to assertion (2), we note that (A, B) is a maximal submodule if and only if $(X, Y)/(A, B) = (X/A, Y/B)$ is a simple module. We can again use Proposition 4.1. If the first possibility of this proposition holds, then $X/A = M(Y/B)$ and $Y/B = N(X/A)$. Since $M(Y/B) = (MY + A)/A$, we have that $X = MY + A$ and $MY \not\subseteq A$ by the maximality of A. In addition, $NX \not\subseteq B$. It can be similarly proved that $IX \not\subseteq A$ and $JY \not\subseteq B$; this also follows from the remark after Proposition 4.1. $\square$

The sum of all minimal submodules of the module V is called the *socle* V; it is denoted by $\operatorname{Soc} V$. If V does not have minimal submodules, then $\operatorname{Soc} V = 0$ by definition.

Corollary 4.4. *Let (X, Y) be some module. The socle of (X, Y) is equal to $\big(\operatorname{Soc} L(X), \operatorname{Soc} L(Y)\big) + \sum(A, NA)$, where the summation is over all minimal submodules A in X such that $IA \neq 0$ and NA is* *a minimal submodule in B. The last summand is also equal to $\sum(MB,B)$, where the summation is over all minimal submodules B in Y such that $JB \neq 0$ and MB is a minimal submodule in A; this summand is also equal to $\sum(A,B)$, where the summation is over all above A and B.*

Proof. Since there are three types of minimal submodules, we obtain three sums of corresponding minimal submodules, and we have

$$\operatorname{Soc}(X,Y) = \sum(A,0) + \sum(0,B) + \sum(A,B).$$

The first summand is $\operatorname{Soc} L(X)$, the second summand is $\operatorname{Soc} L(Y)$, and the third summand coincides with each of the three sums indicated in the corollary. □

Corollary 4.5. *Let (X,Y) be a K-module.*

(1) *If $L(X) = 0 = L(Y)$, then* $\operatorname{Soc}(X,Y) = (\operatorname{Soc} X, \operatorname{Soc} Y)$.

(2) *If K is a ring with zero trace ideals, then* $\operatorname{Soc}(X,Y) = \big(\operatorname{Soc} L(X), \operatorname{Soc} L(Y)\big)$, *and (X,Y) is an essential extension* $\operatorname{Soc}(X,Y)$ *if and only if X is an essential extension* $\operatorname{Soc} L(X)$ *and Y is an essential extension* $\operatorname{Soc} L(Y)$. □

The intersection of all maximal submodules of the module V is called the *radical* of the module M; it is denoted by $\operatorname{Rad} V$. If V does not have maximal submodules, then $\operatorname{Rad} V = V$ by definition.

The canonical homomorphism $(X,Y) \to (X/MY, Y/NX)$ is denoted by (σ,τ) below.

Corollary 4.6. *The radical of the module (X,Y) coincides with*

$$\big(\sigma^{-1}(\operatorname{Rad} X/MY), \tau^{-1}(\operatorname{Rad} Y/NX)\big) \cap \Big(\bigcap(A,B)\Big),$$

where the intersection is over all submodules (A,B) in (X,Y) such that A and B are maximal submodules and $MY \not\subseteq A$, $NX \not\subseteq B$.

Proof. We know that there are three forms of maximal submodules in (X,Y). Therefore, we can consider three groups of corresponding maximal submodules. We have

$$\operatorname{Rad}(X,Y) = \Big(\bigcap(A,Y)\Big) \cap \Big(\bigcap(X,B)\Big) \cap \Big(\bigcap(A,B)\Big),$$

$$\bigcap(A,Y) = (\sigma^{-1}(\operatorname{Rad} X/MY), Y), \quad \bigcap(X,B) = \big(X, \tau^{-1}(\operatorname{Rad} Y/NX)\big).$$ □

Corollary 4.7. *Let (X,Y) be a K-module.*

(1) *If $NX = Y$ and $MY = X$, then* $\operatorname{Rad}(X,Y) = (\operatorname{Rad} X, \operatorname{Rad} Y)$.

(2) *If K is a ring with zero trace ideals, then*

$$\operatorname{Rad}(X,Y) = \big(\sigma^{-1}(\operatorname{Rad} X/MY), \tau^{-1}(\operatorname{Rad} Y/NX)\big),$$

and $\operatorname{Rad}(X,Y)$ *is a small submodule in (X,Y) if and only if* $\operatorname{Rad} X/MY$ *is a small submodule in X/MY and* $\operatorname{Rad} Y/NX$ *is a small submodule in Y/NX.*

Proof. (1) It follows from Corollary 4.6 that $\operatorname{Rad}(X,Y) = \bigcap(A,B)$, where (A,B) runs all submodules in (X,Y) such that A and B are maximal submodules. It is not obvious that this intersection coincides with $(\operatorname{Rad} X, \operatorname{Rad} Y)$. Since $\bigcap(A,B) = \Big(\bigcap A, \bigcap B\Big)$, it is sufficient to prove that for every maximal submodule A in X, there exists a maximal submodule B in Y with the property that (A,B) is a submodule in (X,Y) (then (A,B) is maximal), and to prove a similar assertion for every maximal submodule B in Y.

Let A be some maximal submodule in X. The subset (A,Y) does not form a submodule (since $MY = X$). Therefore, the set of all submodules of the form (A,D) is inductive and nonempty (since it contains the submodule (A,NA)). By the Zorn lemma, this set contains a maximal element (A,B). This submodule is a maximal submodule in (X,Y) (we consider the relation $NX = Y$). By Corollary 4.3, B is a maximal submodule in Y, which is required. For the maximal submodule B in Y, we can present a similar argument.

(2) If (A,B) is a submodule in (X,Y) such that A and B are maximal and $MY \not\subseteq A$, $NX \not\subseteq B$, then $IX \not\subseteq A$ and $JY \not\subseteq B$. Since $I = 0 = J$, the intersection $\bigcap(A,B)$ from Corollary 4.6 is equal to zero. For the radical, the smallness criterion follows from Proposition 3.2. $\square$

Remark. All results of this section can be applied to modules over the ring

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

of triangular matrices (see also the end of the previous section). The corresponding results are obtained in [22]. For example,

$$\operatorname{Soc}(X,Y) = \big(\operatorname{Soc} X, \operatorname{Soc} L(Y)\big), \quad \operatorname{Rad}(X,Y) = (\sigma^{-1}(\operatorname{Rad} X/MY), \operatorname{Rad} Y).$$

5. Injective Modules and Injective Hulls

We determine the structure of injective modules over a formal matrix ring. In this section, K is an arbitrary formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

In this section and the next section, we usually deal with a K-module denoted by (A,B).

Let X be an R-module and let Y be an S-module. Considered before Lemma 2.3, the K-modules $\big(X, \operatorname{Hom}_R(M,X)\big)$ and $(\operatorname{Hom}_S(N,Y), Y)$ play a very important role. We agreed to denote these modules $\big(X, H(X)\big)$ and $(H(Y),Y)$, respectively. We systematically use Lemma 2.3. For example, the following result is practically a corollary of this lemma.

Proposition 5.1. $\big(X,H(X)\big)$ *is an injective K-module if and only if X is an injective R-module. A similar assertion holds for the S-module Y and the K-module $(H(Y),Y)$.*

Proof. Let X be an injective R-module. We assume that there is a K-module monomorphism $(i,j)\colon (A,B) \to (C,D)$ and a K-module homomorphism $(\alpha,\beta)\colon (A,B) \to \big(X,H(X)\big)$. Since the module X is injective, there exists a homomorphism $\gamma\colon C \to X$ with $i\gamma = \alpha$. By Lemma 2.3, there exists a homomorphism $\delta\colon D \to H(X)$ such that $(\gamma,\delta)\colon (C,D) \to \big(X,H(X)\big)$ is a K-module homomorphism. By Lemma 2.3, $j\delta = \beta$. Consequently, $(i,j)(\gamma,\delta) = (\alpha,\beta)$ and the module $\big(X,H(X)\big)$ is injective.

Conversely, let $\big(X,H(X)\big)$ be an injective module. We assume that $i\colon A \to C$ is a monomorphism and $\alpha\colon A \to X$ is an R-module monomorphism. It follows from Lemma 2.3 that there exist K-module homomorphisms

$$(i,j)\colon \big(A,H(A)\big) \to \big(C,H(C)\big), \quad (\alpha,\beta)\colon \big(A,H(A)\big) \to \big(X,H(X)\big).$$

We assert that j is a monomorphism. Indeed, if $j(\eta) = 0$, where $\eta \in H(A)$, then $j(\eta)(m) = i(m\eta) = i\big(\eta(m)\big) = 0$ for any $m \in M$, whence $\eta = 0$. Therefore, (i,j) also is a monomorphism. Since the module $\big(X,H(X)\big)$ is injective, there exists a homomorphism $(\gamma,\delta)\colon \big(C,H(C)\big) \to \big(X,H(X)\big)$ such that $(i,j)(\gamma,\delta) = (\alpha,\beta)$. Consequently, $i\gamma = \alpha$ and the module X is injective.

In the case of the module $(H(Y),Y)$, we use a similar argument. $\square$

Remark. It follows from Proposition 5.1 that the K-module $(X,0)$ is injective if and only if X is an injective R-module and $\operatorname{Hom}_R(M,X) = 0$. A similar result holds for the module $(0,Y)$.

Proposition 5.1 can be reformulated as follows. Let H be the functor defined in the remarks after Corollary 2.4. We will see soon that any injective K-module is isomorphic to the module $H(X,Y)$ for some injective modules X and Y.

Let (A,B) be some K-module and let f' and g' be the homomorphisms defined at the beginning of Sec. 2. These homomorphisms play an important role. They correspond to the homomorphisms f and g of the module multiplication under the isomorphisms indicated there. It is more convenient to write f and g instead of f' and g', respectively. Thus, f is an S-homomorphism $B \to \operatorname{Hom}_R(M,A)$, where

$f(b)(m) = mb$, $b \in B$, $m \in M$, and g is an R-homomorphism $A \to \mathrm{Hom}_S(N, B)$, where $g(a)(n) = na$, $a \in A$, $n \in N$. We have exact sequences

$$0 \to L(A) \to A \xrightarrow{g} \mathrm{Hom}_S(N, B),$$

$$0 \to L(B) \to B \xrightarrow{f} \mathrm{Hom}_R(M, A)$$

of R-modules and S-modules, respectively, where $L(A)$ and $L(B)$ are the submodules defined in Sec. 3. The mappings

$$(1, f)\colon (A, B) \to \big(A, H(A)\big), \quad (g, 1)\colon (A, B) \to (H(B), B)$$

are K-homomorphisms (see Lemma 2.3 and remarks after Corollary 2.4); they are often used. In addition, if X is an R-module, $\big(X, H(X)\big)$ is a K-module, and I is one of the trace ideals of the ring K, then $L(X) = \{x \in X \mid Ix = 0\}$, $L\big(H(X)\big) = 0$.

We pass to the description of injective K-modules. Here we consider two important cases. Later, we will prove that the general case can be reduced to these two cases.

Theorem 5.2. *Let (A, B) be a module with $L(A) = 0 = L(B)$. The module (A, B) is injective if and only if A and B are injective modules. In addition, f and g are isomorphisms.*

Proof. Let (A, B) be an injective module. It follows from the assumption that f and g are monomorphisms. Therefore, $(1, f)$ is a monomorphism. Since the module (A, B) is injective, the image $\mathrm{Im}(1, f)$ is a direct summand in $\big(A, H(A)\big)$. The complement summand has the form $(0, Z)$, where Z is a some direct summand in $H(A)$. Consequently, $MZ = 0$ and $Z \subseteq L\big(H(A)\big)$. As was noted above, $L\big(H(A)\big) = 0$ and $Z = 0$. Thus, $(1, f)$ is an isomorphism. Therefore, $\big(A, H(A)\big)$ is an injective module and f is an isomorphism. By Proposition 5.1, the module A is injective. It can be similarly proved that g is an isomorphism and the module B is injective.

Now we assume that modules A and B are injective. We again use the homomorphisms $(1, f)$ and $(g, 1)$. We consider the K-module $\big(\mathrm{Hom}_S\big(N, H(A)\big), H(A)\big)$ and a K-homomorphism

$$(h, 1)\colon \big(A, H(A)\big) \to \big(\mathrm{Hom}_S\big(N, H(A)\big), H(A)\big)$$

described in Lemma 2.3. With the use of a direct calculation, it is verified that $h = gf_*$, where $f_*\colon \mathrm{Hom}_S(N, B) \to \mathrm{Hom}_S\big(N, H(A)\big)$ is the homomorphism induced by the homomorphism f. Since f_* and g are monomorphisms, h is a monomorphism. By Proposition 5.1, the module $\big(A, H(A)\big)$ is injective. By repeating the argument from the first part of the proof, we can verify that h is an isomorphism. Consequently, the monomorphism f_* is an epimorphism. Therefore, f_* and g are isomorphisms. We can similarly prove that f is an isomorphism. Thus, $(1, f)\colon (A, B) \to \big(A, H(A)\big)$ is an isomorphism and the module (A, B) is injective, which is required. □

In any module (A, B), the subsets $(L(A), 0)$, $\big(0, L(B)\big)$, and $\big(L(A), L(B)\big)$ are submodules. In studies of injective K-modules, we have one important situation, where $\big(L(A), L(B)\big)$ is an essential submodule in (A, B). (This is necessarily true if the trace ideals of the ring K are equal to zero.)

We recall several notions of ring theory. Let V be a module over some ring and let G and Z be two submodules in V. The submodule G is said to be *closed* (in V) if G does not have proper essential extensions in V. The submodule G is called a *closure* of the submodule Z (in V) if $Z \subseteq G$, G is an essential extension of the module Z and G is closed in V. In V, every submodule has at least one closure, which is not always unique. The symbol $\bar{Z}$ denotes some closure of the submodule Z.

Theorem 5.3. *Let (A, B) be a module such that $\big(L(A), L(B)\big)$ is an essential submodule in (A, B). The module (A, B) is injective if and only if there exist closures $\overline{L(A)}$ and $\overline{L(B)}$ such that they are injective,*

$$L(A) \cap M\overline{L(B)} = 0, \quad N\overline{L(A)} \cap L(B) = 0,$$

$$\mathrm{Hom}_R(M, \overline{L(A)}) \subseteq \mathrm{Im}\, f, \quad \mathrm{Hom}_S(N, \overline{L(B)}) \subseteq \mathrm{Im}\, g.$$

Proof. We assume that the module (A, B) is injective. There exist closures (A_1, B_2) and (A_2, B_1) of the submodules $(L(A), 0)$ and $\big(0, L(B)\big)$, respectively such that $(A_1, B_2) \cap (A_2, B_1) = 0$. Any closed submodule of an injective module is injective. Consequently, $(A_1, B_2) \oplus (A_2, B_1)$ is an essential injective submodule in (A, B); therefore, $(A, B) = (A_1, B_2) \oplus (A_2, B_1)$. It is directly verified that A_1 is an essential extension of the module $L(A)$ and B_1 is an essential extension of the module $L(B)$. The direct summands A_1 and B_1 are closed submodules. Therefore, $A_1 = \overline{L(A)}$ and $B_1 = \overline{L(B)}$. By Lemma 2.3, we have the homomorphism $(1, f)\colon (A_1, B_2) \to \big(A_1, H(A_1)\big)$; more precisely, we have to take the restriction of f to B_2 instead of f. Since $B_2 \cap L(B) = 0$, we have that $(1, f)$ is a monomorphism. Since the module (A_1, B_2) is injective, we repeat the argument from the proof of Theorem 5.2 and obtain that $(1, f)$ is an isomorphism. Consequently, the module $\big(A_1, H(A_1)\big)$ is injective. Then the module A_1 is injective by Proposition 5.1. In addition, $\mathrm{Hom}_R(M, A_1) \subseteq \mathrm{Im}\, f$. Finally, it follows from inclusions $L(A) \subseteq A_1$ and $MB_1 \subseteq A_2$ that $L(A_1) \cap MB_1 = 0$. The remaining assertions can be similarly proved; in particular, $(g, 1)\colon (A_2, B_1) \to (H(B_1), B_1)$ is an isomorphism.

Now we assume that there exist closures $\overline{L(A)}$ and $\overline{L(B)}$ such that they are injective, $L(A) \cap M\overline{L(B)} = 0$, $N\overline{L(A)} \cap L(B) = 0$, $\mathrm{Hom}_R(M, \overline{L(A)}) \subseteq \mathrm{Im}\, f$, and $\mathrm{Hom}_S(N, \overline{L(B)}) \subseteq \mathrm{Im}\, g$. We set $A_1 = \overline{L(A)}$ and $B_1 = \overline{L(B)}$. We consider also the submodules (A_1, NA_1) and (MB_1, B_1). We note that the intersection of them is equal to zero. We take also modules $\big(A_1, H(A_1)\big)$ and $(H(B_1), B_1)$, which are injective by Proposition 5.1. We also consider homomorphisms

$$(1, f)\colon (A_1, NA_1) \to \big(A_1, H(A_1)\big), \quad (g, 1)\colon (MB_1, B_1) \to (H(B_1), B_1),$$

where f and g denote the restrictions of the homomorphisms to the corresponding submodules. In fact, we have monomorphisms, since $\mathrm{Ker}\, f = L(B) \cap NA_1 = 0$ (similar relations hold for $\mathrm{Ker}\, g$). The sum of mappings $(1, f) + (g, 1)$ is extended to a monomorphism $(A, B) \to \big(A_1, H(A_1)\big) \oplus (H(B_1), B_1)$. We identify (A, B) with the image of this monomorphism. We note that the role of the submodule $L(A)$ in the above sum coincides with its role in (A, B).

We have direct decompositions

$$A = A_1 \oplus A_2, \quad B = B_1 \oplus B_2, \quad \text{where} \quad A_2 = A \cap H(B_1), \quad B_2 = B \cap A_1.$$

It is clear that there exist K-modules (A_1, B_2) and (A_2, B_1) and a direct decomposition $(A, B) = (A_1, B_2) \oplus (A_2, B_1)$. It is convenient to return to the original module (A, B) and assume that this decomposition is a decomposition of this module. We take the monomorphism $(1, f)\colon (A_1, B_2) \to \big(A_1, H(A_1)\big)$. Let $\alpha \in H(A_1)$. Since $H(A_1) \subseteq \mathrm{Im}\, f$, we have that $\alpha = f(b)$ for some $b \in B$. We have $b = c + d$, where $c \in B_2$, $d \in B_1$. For every $m \in M$, we have

$$\alpha(m) = f(b)(m) = mb = mc + md, \quad mc, mb \in A_1, \quad md \in A_2.$$

Therefore, $md = 0$ and $mb = mc$. Then we obtain

$$\alpha(m) = mc = f(c)(m), \quad \alpha = f(c).$$

We have proved that $(1, f)$ is an isomorphism. Thus, $(A, B) \cong \big(A_1, H(A_1)\big) \oplus (H(B_1), B_1)$ and the module (A, B) is injective. $\square$

Corollary 5.4. *Let the conditions and notation of Theorem* 5.3 *hold. Then there is the canonical isomorphism*

$$(A, B) \cong \big(A_1, H(A_1)\big) \oplus (H(B_1), B_1).$$

Remark. Let K be an arbitrary formal matrix ring and let (A, B) be any K-module. Then (A, B) is an essential extension of the module $(L(A) + MB, L(B) + NA)$, and (A, B) is an essential extension of the module $\big(L(A), L(B)\big)$ in the case, where K is a ring with zero trace ideals. Indeed, let $(a, b) \in (A, B)$ and $a \neq 0$. If $na = 0$ for all $n \in N$, then $a \in L(A)$, and if $na \neq 0$ for some $n \in N$, then

$$\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} (a, b) = (0, na) \in (0, NA).$$

The case $b \neq 0$ is considered similarly. If K is a ring with zero trace ideals, then $MB \subseteq L(A)$, $NA \subseteq L(B)$, $L(A) + MB = L(A)$, $L(B) + NA = L(B)$.

Corollary 5.5. *Let (A, B) be a K-module.*

(1) *If K is a ring with zero trace ideals, then the module (A, B) is injective if and only if the modules $L(A)$ and $L(B)$ are injective,* $\operatorname{Hom}_R\big(M, L(A)\big) \subseteq \operatorname{Im} f$, $\operatorname{Hom}_S\big(N, L(B)\big) \subseteq \operatorname{Im} g$.

(2) *If*

$$K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is a ring of triangular matrices, then the module (A, B) is injective if and only if the modules A and $L(B)$ are injective and $f\colon B \to \operatorname{Hom}_R(M, A)$ is an epimorphism.

Proof. (1) Since K is a ring with zero trace ideals, any module $\big(X, H(X)\big)$ satisfies the relations $L(X) = X$ and $L\big(H(X)\big) = 0$. Modules of the form $(H(Y), Y)$ satisfy similar relations.

We assume that the module (A, B) is injective. We identify (A, B) with the isomorphic image from Corollary 5.4. Then $L(A) = \overline{L(A)}$ and $L(B) = \overline{L(B)}$. Therefore, the modules $L(A)$ and $L(B)$ are injective. The remaining part of (1) follows from Theorem 5.3.

(2) The assertion follows from (1) and the relation $L(A) = A$. □

Theorem 5.6. *An arbitrary K-module (A, B) is injective if and only if some closure of the submodule $\big(L(A), L(B)\big)$ is injective, and there exist closures $\overline{L(A)}$ and $\overline{L(B)}$ such that the factor modules $A/\Big(\overline{L(A)} + g^{-1}H\big(\overline{L(B)}\big)\Big)$ and $B/\Big(\overline{L(B)} + f^{-1}H\big(\overline{L(A)}\big)\Big)$ are injective.*

Proof. We assume that the module (A, B) is injective. All closed submodules of injective modules are injective. There exists a direct decomposition $(A, B) = (G, H) \oplus (C, D)$, where the first summand is some closure of the submodule $\big(L(A), L(B)\big)$. We can apply Theorem 5.3 to the module (G, H). Consequently, there exist injective closures A_1 and B_1 of the modules $L(A)$ and $L(B)$, respectively. There exists a direct decomposition $(G, H) = (A_1, B_2) \oplus (A_2, B_1)$ for some submodules A_2 and B_2. In addition, it follows from the proof of Theorem 5.3 that the mappings

$$(1, f)\colon (A_1, B_2) \to \big(A_1, H(A_1)\big), \quad (g, 1)\colon (A_2, B_1) \to (H(B_1), B_1)$$

are isomorphisms. The module (C, D) is injective and $L(C) = 0 = L(D)$. By Theorem 5.2, the modules C and D are injective. It remains to note that

$$C \cong A/(A_1 \oplus A_2) = A/\big(A_1 + g^{-1}H(B_1)\big).$$

Similar relations are also true for another factor module.

Let the conditions of the theorem hold. Consequently, $(A, B) = (G, H) \oplus (C, D)$ for some closure (G, H) of the submodule $\big(L(A), L(B)\big)$ and some module (C, D) with $L(C) = 0 = L(D)$. We can apply Theorem 5.3 to the module (G, H). As above, there exists a direct decomposition $(G, H) = (A_1, B_2) \oplus (A_2, B_1)$ and the mappings $(1, f)$ and $(g, 1)$ are isomorphisms. By repeating the above argument, we obtain that the modules C and D are injective. By Theorem 5.2, the module (C, D) is injective. Therefore, the module (A, B) is injective. □

From the theorems proved above, we obtain the following conclusion.

Remark. Every injective module (A, B) has a direct decomposition

$$(A, B) = (A_1, B_2) \oplus (A_2, B_1) \oplus (C, D),$$
$$A_1 = \overline{L(A)}, \quad B_1 = \overline{L(B)}, \quad L(C) = 0 = L(D),$$

and the canonical mappings

$$B_2 \to \operatorname{Hom}_R(M, A_1), \quad A_2 \to \operatorname{Hom}_S(N, B_1), \quad D \to \operatorname{Hom}_R(M, C), \quad C \to \operatorname{Hom}_S(N, D)$$

are isomorphisms.

Corollary 5.7 (Müller [43]). *The module (A,B) is injective if and only if there exist an injective R-module X and an injective S-module Y such that $(A,B) \cong \big(X, H(X)\big) \oplus \big(Y, H(Y)\big)$.*

Proof. Corollary 5.7 follows from the previous remark, Theorem 5.2 and Theorem 5.3. □

We can use the obtained information about injective modules for describing injective hulls. An injective hull of some module V is denoted by $\hat{V}$.

Lemma 5.8. *Let V be a module over some ring, C_1 and C_2 be closed submodules in V such that $C_1 \cap C_2 = 0$, and let $C_1 \oplus C_2$ be an essential submodule in V. Then*

$$\hat{V} = \hat{C}_1 \oplus \hat{C}_2, \quad \textit{where} \quad \hat{C}_1 \cong \widehat{V/C_1}, \quad \hat{C}_2 \cong \widehat{V/C_2}.$$

Proof. Since $\hat{V} \cong \hat{C}_1 \oplus \hat{C}_2$, we have the relations $C_2 \cong (C_1 \oplus C_2)/C_1 \subseteq V/C_1$, and V/C_1 is an essential extension of the module $(C_1 \oplus C_2)/C_1$. Indeed, let B be a submodule such that $C_1 \subseteq B$ and $C_1 \neq B$. Since C_1 is a closed submodule in V, we have $B \cap C_2 \neq 0$. Consequently, $B/C_1 \cap (C_1 \oplus C_2)/C_1 \neq 0$. Therefore, the module C_2 is isomorphic to an essential submodule in V/C_1, whence $\hat{C}_2 \cong \widehat{V/C_1}$. The second isomorphism can be similarly proved. □

Corollary 5.9 (Müller [43]). *Let (A,B) be a K-module.*

(1) *If $L(A) = 0 = L(B)$, then there exists a K-module $(\hat{A}, \hat{B})$ and this module is an injective hull of the module (A,B). In addition, there are the canonical isomorphisms $\hat{A} \cong \mathrm{Hom}_S(N, \hat{B})$ and $\hat{B} \cong \mathrm{Hom}_R(M, \hat{A})$.*

(2) *If (A,B) is an essential extension of the module $\big(L(A), L(B)\big)$, then the module*

$$\Big(\widehat{L(A)}, H\big(\widehat{L(A)}\big)\Big) \oplus \big(H\big(\widehat{L(B)}\big), \widehat{L(B)}\big)$$

is an injective of the module (A,B).

(3) *An injective hull of the module (A,B) has the form $U \oplus V$, where U is an injective hull of the module $\big(L(A), L(B)\big)$ and there exists a closure W of the submodule $\big(L(A), L(B)\big)$ such that V is an injective hull of the factor module $(A,B)/W$. The module $\big(L(A), L(B)\big)$ satisfies the conditions of (2), and the module $(A,B)/W$ satisfies the conditions of (1).*

Proof. (1) We consider the K-module $\big(\hat{A}, H(\hat{A})\big)$. By Proposition 5.1, this module is injective. Since $H(A) \subseteq H(\hat{A})$, we can consider the homomorphism $(1,f)\colon (A,B) \to \big(\hat{A}, H(\hat{A})\big)$. Since $L(B) = 0$, we have that $(1,f)$ is a monomorphism. Its image is an essential submodule in $\big(\hat{A}, H(\hat{A})\big)$. Otherwise, $\big(\hat{A}, H(\hat{A})\big)$ contains some injective hull of the image of the form $(\hat{A}, Y)$ for some proper submodule Y. Then $H(\hat{A}) = Y \oplus Z$, where $Z \neq 0$ and $MZ = 0$. This contradicts the relation $L\big(H(\hat{A})\big) = 0$; this argument is similar to the argument from the beginning of the proof of Theorem 5.2. Thus, $\big(\hat{A}, H(\hat{A})\big)$ is the injective hull of the module (A,B). Similarly, the module $(H(\hat{B}), \hat{B})$ is also the injective hull of the module (A,B). We identify the module (A,B) with its isomorphic image in these two hulls. Then the identity mapping of the module (A,B) is extended to an isomorphism $(\alpha,\beta)\colon \big(\hat{A}, H(\hat{A})\big) \to (H(\hat{B}), \hat{B})$. In addition, α is an extension of the monomorphism $g\colon A \to \mathrm{Hom}_S(N,B)$, and β is an extension of the converse isomorphism for $f\colon B \to \mathrm{Im}\, f$. This is assumed when we speak of the canonical character of the endomorphisms indicated in the corollary.

(2) An injective hull of the module (A,B) coincides with the injective hull of the module $\big(L(A), L(B)\big)$, which is equal to $(L(A), 0) \oplus \big(0, L(B)\big)$. It follows from the proof of Theorem 5.3 that the injective hull of the module $(L(A), 0) \oplus \big(0, L(B)\big)$ coincides with the sum indicated in the corollary.

(3) The assertion is verified with the use of Lemma 5.8. □

6. Flat Modules

In this section, we describe flat modules over a formal matrix ring K with zero trace ideals. We use right modules and the right-side analogues of assertions and constructions obtained and defined before. We often use the trace ideals I and J of the ring K (see Sec. 1). We also write the following remark. The isomorphisms considered in the text are canonical; this means that their actions satisfy certain rules, which can be easily indicated.

In the study of flat modules, any information about tensor products is useful. Tensor products of K-modules can be obtained with the use of tensor products of R-modules and tensor products of S-modules. Let $U = (C, D)$ be a right K-module and let $V = (A, B)$ be a left K-module.

Proposition 6.1. *There is an isomorphism of Abelian groups*

$$U \otimes_K V \cong (C \otimes_R A \oplus D \otimes_S B)/H,$$

where the subgroup H is generated by all elements of the form

$$c \otimes mb - cm \otimes b, \quad d \otimes na - dn \otimes a,$$

where

$$c \in C, \quad d \in D, \quad a \in A, \quad b \in B, \quad m \in M, \quad n \in N.$$

Proof. The group $C \otimes_R A \oplus D \otimes_S B$ is isomorphic to the group $U \otimes_{R\times S} V$ under the correspondence of generators $c \otimes a + d \otimes b \to (c, d) \otimes (a, b)$. We set $G_1 = U \otimes_{R\times S} V$ and $G_2 = U \otimes_K V$. We use the definition of the tensor product as the factor group of a free group. Let F be a free Abelian group with the basis consisting of all expressions

$$\big((c, d), (a, b)\big), \quad c \in C, \quad d \in D, \quad a \in A, \quad b \in B.$$

Then $G_1 = F/H_1$ and $G_2 = F/H_2$, where H_1 and H_2 are the subgroups generated by elements of the familiar form. We indicate the difference between these subgroups. The generator system of the group H_1 contains all elements of the form

$$\big((c, d), (ra, sb)\big) - \big((cr, ds), (a, b)\big), \quad r \in R, \quad s \in S,$$

the generator system of the group H_2 contains all elements of the form

$$\big((c, d), k(a, b)\big) - \big((c, d)k, (a, b)\big), \quad k = \begin{pmatrix} r & m \\ n & s \end{pmatrix} \in K,$$

and all remaining generators of these two groups coincide. Therefore, $H_1 \subseteq H_2$. We have the relations

$$G_2 = F/H_2 \cong (F/H_1)/(H_2/H_1) = G_1/H, \quad \text{where} \quad H = H_2/H_1.$$

The factor group H_2/H_1 is generated by the images of all generators of the group H_2, i.e., by elements of the form

$$\big((c, d) \otimes (mb, na)\big) - \big((dn, cm) \otimes (a, b)\big).$$

With the use of the isomorphism considered above, we obtain that $G_2 \cong (C \otimes_R A \oplus D \otimes_S B)/H$, where the subgroup H is not renamed and H is generated by all elements of the form $cm \otimes b + d \otimes na - dn \otimes a - cm \otimes b$; therefore, H is generated by all elements indicated above. □

Remarks. We present some general remarks and notation related to the K-module (A, B). We denote by L the ideal

$$\begin{pmatrix} I & IM \\ JN & J \end{pmatrix}$$

of the ring K and we set $\bar{K} = K/L$. We can identify the factor ring $\bar{K}$ with the matrix ring

$$\begin{pmatrix} R/I & M/IM \\ N/JN & S/J \end{pmatrix}.$$

The last ring is denoted by

$$\begin{pmatrix} \bar{R} & \bar{M} \\ \bar{N} & \bar{S} \end{pmatrix}.$$

Since $\bar{M}\bar{N} = 0 = \bar{N}\bar{M}$, we have that $\bar{K}$ is a ring with zero trace ideals. The module (A, B) has the submodules (IA, JB) and (MB, NA), and $(IA, JB) \subseteq (MB, NA)$. As was noted above, the factor modules $(A, B)/(IA, JB)$ and $(A, B)/(MB, NA)$ can be identified with the modules $(A/IA, B/JB)$ and $(A/MB, B/NA)$, respectively. Since $L(A, B) = (IA, JB)$, we have that $(A/IA, B/JB)$ and $(A/MB, B/NA)$ is $\bar{K}$-modules.

Now we consider the ideal

$$L_1 = \begin{pmatrix} I & M \\ N & J \end{pmatrix}$$

of the ring K. There exists an isomorphism

$$K/L_1 \cong R/I \times S/J = \bar{R} \times \bar{S}.$$

Further, it follows from the relation $L_1(A, B) = (MB, NA)$ that $(A/MB, B/NA)$ is $(\bar{R} \times \bar{S})$-module.

With the use of the above notation, we formulate the following result.

Corollary 6.2. *Let (A, B) be a flat K-module.*

(1) *$(A/IA, B/JB)$ is a flat $\bar{K}$-module and $M/IM \otimes_S B/NA \cong MB/IA$, $N/JN \otimes_R A/MB \cong NA/JB$.*
(2) *A/MB is a flat $\bar{R}$-module and B/NA is a flat $\bar{S}$-module.*
(3) *If $I = 0$, then $M \otimes_S B/NA \cong MB$ and A/MB is a flat R-module. In addition, if $N = 0$, then we have $M \otimes_S B \cong MB$, and B is a flat S-module.*

Proof. (1) We have the relation $(A/IA, B/JB) = (A, B)/L(A, B)$. It is known that $(A, B)/L(A, B)$ is a flat module. For example, this follows from the Chase's criterion [11, Proposition 11.33].

We set $\bar{A} = A/IA$ and $\bar{B} = B/JB$. Since $(\bar{A}, \bar{B})$ is a flat module, there exists an isomorphism

$$(0, \bar{M}) \otimes_{\bar{K}} (\bar{A}, \bar{B}) \cong (0, \bar{M})(\bar{A}, \bar{B}) = \bar{M}\bar{B},$$

where $(0, \bar{M})$ is a right ideal of the ring $\bar{K}$. By Proposition 6.1, the tensor product from the left part is isomorphic to the factor group $(\bar{M} \otimes_{\bar{S}} \bar{B})/\bar{H}$, and the subgroup $\bar{H}$ is generated by elements of the form $\bar{m} \otimes \bar{n}\bar{a}$ for all $\bar{m} \in \bar{M}$, $\bar{n} \in \bar{N}$, and $\bar{a} \in \bar{A}$ (we consider that $\bar{M}\bar{N} = 0$). The group $\bar{H}$ is the image of the induced mapping $\bar{M} \otimes_{\bar{S}} \bar{N}\bar{A} \to \bar{M} \otimes_{\bar{S}} \bar{B}$. Consequently, we obtain the isomorphism $(0, \bar{M}) \otimes_K (\bar{A}, \bar{B}) \cong \bar{M} \otimes_{\bar{S}} \bar{B}/\bar{N}\bar{A}$. Thus, we obtain the isomorphism

$$\bar{M} \otimes_{\bar{S}} \bar{B}/\bar{N}\bar{A} \cong \bar{M}\bar{B}, \quad \bar{m} \otimes (\bar{b} + \bar{N}\bar{A}) \to \bar{m}\bar{b}.$$

With the use more detailed representation, this isomorphism has the form $M/IM \otimes_S B/NA \cong MB/IA$. The second isomorphism can be similarly proved.

(2) Similar to (1), it can be proved that $(A/MB, B/NA)$ is a flat $(\bar{R} \times \bar{S})$-module.

(3) The assertion directly follows from (1) and (2). □

Remarks. The right-side version of the construction of the K-modules $\big(X, H(X)\big)$ and $(H(Y), Y)$ from Sec. 2 has the following form. If Z is a right R-module, then the group of row vectors $\big(Z, \mathrm{Hom}_R(N, Z)\big)$ is a right K-module. The homomorphisms of the module multiplication are defined similarly to the case of left modules. Similarly, the right S-module Z leads to the right K-module $(\mathrm{Hom}_S(M, Z), Z)$.

To K-modules, we will apply one standard method of passing from left modules to right modules. Let (X, Y) be a K-module and let G be an arbitrary Abelian group. The group of additive homomorphisms $\mathrm{Hom}\big((X, Y), G\big)$ is a right K-module with module multiplication defined by the relation

$$(\eta k)(x, y) = \eta\big(k(x, y)\big), \quad \eta \in \mathrm{Hom}\big((X, Y), G\big), \quad k \in K, \quad x \in X, \quad y \in Y.$$

Similarly, the group $\mathrm{Hom}(X, G)$ ($\mathrm{Hom}(Y, G)$) is a right R-module (respectively, S-module). We can consider the group of row vectors $\big(\mathrm{Hom}(X, G), \mathrm{Hom}(Y, G)\big)$ as a right K-module. The module multiplications are defined by the relations

$$(\alpha m)y = \alpha(my), \quad (\beta n)x = \beta(nx),$$

where

$$\alpha \in \mathrm{Hom}(X, G), \quad \beta \in \mathrm{Hom}(Y, G), \quad m \in M, \quad n \in N, \quad x \in X, \quad y \in Y.$$

There exists the canonical K-module isomorphism

$$\mathrm{Hom}\big((X, Y), G\big) \to \big(\mathrm{Hom}(X, G), \mathrm{Hom}(Y, G)\big),$$
$$\mathrm{Hom}\big((X, Y), G\big) \ni \eta \to (\eta|_X, \eta|_Y) \in \big(\mathrm{Hom}(X, G), \mathrm{Hom}(Y, G)\big).$$

With the use of this isomorphism, we identify K-modules $\mathrm{Hom}\big((X, Y), G\big) \to \big(\mathrm{Hom}(X, G), \mathrm{Hom}(Y, G)\big)$.

If V is a module over some ring T, then the right T-module $\mathrm{Hom}(V, \mathbb{Q}/\mathbb{Z})$ is called the *character module* of the module V; it is denoted by V^*. The character module of the K-module (X, Y) is (X^*, Y^*). It is well known that the T-module V is flat if and only if the character module V^* is injective. Since we have obtained the description of injective K-modules, we can conditionally assume that there is the description of flat K-modules. It is quite another matter that it is difficult to formulate this description in terms of the original module (X, Y). However, sometimes this is possible. For example, it is easy to obtain the following result (cf. Proposition 5.1).

Proposition 6.3. *The K-module $\big(X, T(X)\big)$ is flat if and only if X is a flat R-module. A similar assertion holds for S-module Y and the K-module $(T(Y), Y)$. Thus, the functor T from Sec.* 2 *preserve flat modules.*

Proof. We agree to identify the character module of the module $\big(X, T(X)\big)$ with the right K-module $(X^*, T(X)^*)$. There are natural isomorphisms of right S-modules

$$T(X)^* \cong \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R\big(N, \mathrm{Hom}(X, \mathbb{Q}/\mathbb{Z})\big) = \mathrm{Hom}_R(N, X^*).$$

Thus, the character module of the K-module $\big(X, T(X)\big)$ coincides with $\big(X^*, \mathrm{Hom}_R(N, X^*)\big)$. By the right-side analogue of Proposition 5.1, the module $\big(X^*, \mathrm{Hom}_R(N, X^*)\big)$ is injective if and only if the module X^* is injective. The last property is equivalent to the property that the module X is flat.

The case of the module Y is considered similarly. □

Remark. It directly follows from Proposition 6.3 that the module $(A, 0)$ ($(0, B)$) is flat if and only if A is a flat module and $N \otimes_R A = 0$ (respectively, B is a flat module and $M \otimes_S B = 0$). In addition, Theorem 5.2 is practically equivalent to the following result.

Corollary 6.4. *Let (A, B) be a K-module and let $NA = B$, $MB = A$. Then the module (A, B) is flat if and only if A and B are flat modules.*

Proof. The character module of the module (A, B) coincides with (A^*, B^*). Since $(\eta m)b = \eta(mb)$ for all $\eta \in A^*$, $m \in M$, and $b \in B$, we have that

$$L(A^*) = \mathrm{Hom}(A/MB, \mathbb{Q}/\mathbb{Z}) = (A/MB)^*,$$

where we identify the module $\mathrm{Hom}(A/MB, \mathbb{Q}/\mathbb{Z})$ with the set of all $\eta \colon A \to \mathbb{Q}/\mathbb{Z}$ with $\eta(MB) = 0$. In addition, $L(B^*) = (B/NB)^*$. Therefore, it follows from the assumption that we have $L(A^*) = 0 = L(B^*)$. The module (A^*, B^*) satisfies the conditions of the right-side version of Theorem 5.2. Consequently, the module (A^*, B^*) is injective if and only if the modules A^* and B^* are injective. The last property is equivalent to the property that A and B are flat modules. □

Under the additional condition that the trace ideals of the ring K are equal to zero, we obtain a complete result.

Theorem 6.5. *If K is a ring with zero trace ideals, then the K-module (A, B) is flat if and only if A/MB is a flat R-module, B/NA is a flat S-module, and there are isomorphisms*

$$M \otimes_S B/NA \cong MB, \quad N \otimes_R A/MB \cong NA.$$

Proof. If (A, B) is a flat K-module, then by Corollary 6.2(3), A/MB is a flat R-module, B/NA is a flat S-module, $M \otimes_S B/NA \cong MB$, and $N \otimes_R A/MB \cong NA$.

We prove the sufficiency of the conditions. There exist K-modules $(MB, 0)$ and (A, NA). We consider the factor module $(A, NA)/(MB, 0)$, which coincides with the module $(A/MB, NA)$. Then $M(NA) = 0$ and $n(a + MB) = na$ for all $n \in N$ and $a \in A$. By assumption, the S-modules $N \otimes_R A/MB$ and NA are isomorphic to each other. This isomorphism is induced by the correspondence $n \otimes (a + MB) \to na$ between generators. By Lemma 2.2, the identity mapping of the module A/MB induces a K-module homomorphism $\big(A/MB, T(A/MB)\big) \to (A/MB, NA)$, which coincides with the above isomorphism on the second place. Now it follows from Proposition 6.3 and the conditions of the theorem that $(A/MB, NA)$ is a flat module. It can be similarly proved that $(MB, B/NA)$ is a flat module. Therefore, the character modules $(A/MB, NA)^*$ and $(MB, B/NA)^*$ are injective. Now we prove that the direct sum of these two modules is isomorphic to the module $(A, B)^*$. Thus, we have to verify that there exists a right K-module isomorphism

$$\big((A/MB)^* \oplus (MB)^*, (NA)^* \oplus (B/NA)^*\big) \cong (A^*, B^*). \tag{1}$$

We will use the following relations from the proof of Corollary 6.4:

$$(A/MB)^* M = 0 = (B/NA)^* N. \tag{2}$$

Since the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective, there exists an exact sequence of right R-modules

$$0 \to (A/MB)^* \to A^* \xrightarrow{\pi} (MB)^*. \tag{3}$$

As was noted earlier, we identify the module $(A/MB)^*$ with its image in A^*, which consists of all homomorphisms $A \to \mathbb{Q}/\mathbb{Z}$ annihilating MB. In addition, we note that the mapping π associates with an arbitrary homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ the restriction of it to MB. The R-module $(A/MB)^*$ is injective, since A/MB is a flat module. Consequently, there exists a direct decomposition $A^* = (A/MB)^* \oplus V$ with summand V that is isomorphic to the module MB^*. To obtain the required isomorphism (1), we have to particularly choose a module V and an isomorphism between $(MB)^*$ and V. Since the sequence (3) splits, there exists a monomorphism $\varepsilon\colon (MB)^* \to A^*$ such that $\varepsilon\pi$ is the identity mapping. We take $V = \operatorname{Im}\varepsilon$; we take ε as the isomorphism between $(MB)^*$ and V. As a result, we obtain the isomorphism of right K-modules

$$\Phi\colon (A/MB)^* \oplus (MB)^* \to A^*,$$

which acts identically on $(A/MB)^*$, and $\Phi(\alpha)|_{MB} = \alpha$ for all $\alpha \in (MB)^*$. We also can obtain an isomorphism $\Psi\colon (NA)^* \oplus (B/NA)^* \to B^*$ with similar properties.

We prove that the pair (Φ, Ψ) defines the isomorphism (1). It is sufficient to verify the fulfillment of the right-side analogues of two relations indicated in Sec. 2. Namely, we verify that

$$\Phi(\beta n) = \Psi(\beta)n, \quad \Psi(\alpha m) = \Phi(\alpha)m$$

for all

$$\beta \in (NA)^* \oplus (B/NA)^*, \quad \alpha \in (A/MB)^* \oplus (MB)^*, \quad n \in N, \quad m \in M.$$

In checking, we consider relations (2) and the choice of isomorphisms Φ and Ψ. We take concrete β and n and obtain $\beta = \gamma + \delta$, where $\gamma \in (NA)^*$ and $\delta \in (B/NA)^*$. Since $\delta n = 0$, we have that $\beta n = \gamma n \in (A/MB)^*$. For any element $a \in A$, it follows from the definition of the mapping Φ that $\Phi(\beta n) = \Phi(\gamma n)a$. On the other hand, it follows from the definition of the mapping Ψ that

$$(\Psi(\delta)n)a = \big(\Psi(\delta)\big)(na) = 0, \quad (\Psi(\beta)n)a = (\Psi(\gamma)n)a = \big(\Psi(\gamma)\big)(na).$$

Since $\gamma(na) = (\gamma n)a$, we have $\Phi(\beta n) = \Psi(\beta)n$. The second relation is similarly verified. Thus, the module $(A, B)^*$ is injective. Therefore, (A, B) is a flat module. $\square$

Corollary 6.6 ([12]). *The module* (A, B) *over the ring*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is flat if and only if A/MB *and* B *is a flat modules and* $M \otimes_S B \cong MB$.

7. Projective and Hereditary Modules and Rings

Over a formal matrix ring K with zero trace ideals, projective modules and hereditary modules admit a satisfactory description. Then we apply this description to determine conditions, under which the ring K is hereditary.

As earlier, I and J denote the trace ideals of the ring K. In this section, we usually denote K-modules by (P, Q).

Our first result can be proved with the use of the arguments similar to the arguments from the proof of Proposition 5.1. We need only refer to Lemma 2.2 instead of Lemma 2.3.

Proposition 7.1. *If* X *is a projective* R*-module and* Y *is a projective* S*-module, then* $\big(X, T(X)\big)$ *and* $(T(Y), Y)$ *are projective* K*-modules. The converse is also true.*

It follows from Proposition 7.1 that the K-module $(X, 0)$ is projective if and only if X is a projective R-module and $N \otimes_R X = 0$. We similarly obtain that the assertion holds for the K-module $(0, Y)$.

In Sec. 6, we defined ideals L and L_1 of the ring K.

Corollary 7.2. *Let* (P, Q) *be a projective* K*-module.*

(1) $(P/IP, Q/JQ)$ *is a projective* K/L*-module.*
(2) P/MQ *is a projective* R/I*-module and* Q/NP *is a projective* S/J*-module.*

Proof. If V is a projective module over some ring T and A is an ideal in T, then V/AV is a projective T/A-module. With the use of this property for $T = K$, $A = L$ and $T = K$, $A = L_1$, it is not difficult to prove assertions (1) and (2). □

Theorem 7.3. *Let* K *be a ring with zero trace ideals and let* (P, Q) *be a* K*-module. Then the following conditions are equivalent.*

(1) (P, Q) *is a projective module.*
(2) P/MQ *is a projective* R*-module,* Q/NP *is a projective* S*-module,* $M \otimes_S Q/NP \cong MQ$ *and* $N \otimes_R P/MQ \cong NP$.
(3) *There exist a projective* R*-module* X *and a projective* S*-module* Y *such that* $(P, Q) = (X, NP) \oplus (MQ, Y)$, $M \otimes_S Y \cong MQ$, *and* $N \otimes_R X \cong NP$.
(4) *There exist a projective* R*-module* X *and a projective* S*-module* Y *such that* $(P, Q) \cong \big(X, T(X)\big) \oplus (T(Y), Y)$.

Proof. (1) $\Longrightarrow$ (2) By Corollary 7.2, P/MQ is a projective R-module and Q/NP is a projective S-module. By Corollary 6.2, $M \otimes_S Q/NP \cong MQ$ and $N \otimes_R P/MQ \cong NP$.

(2) $\Longrightarrow$ (3) First, we present one remark. Since $Y \subseteq Q$, we have the induced homomorphism $M \otimes_S Y \to M \otimes_S Q$. The composition of this homomorphism with the homomorphism of the module multiplication $M \otimes_S Q \to MQ$ provides the homomorphism $M \otimes_S Y \to MQ$. In (3), it is assumed that this homomorphism is an isomorphism. We have $P = X \oplus MQ$ and $Q = NP \oplus Y$, where $X \cong P/MQ$ and $Y \cong Q/NP$. Since (X, NP) and (MQ, Y) are K-modules, we obtain the relations

$$(P, Q) = (X \oplus MQ, NP \oplus Y) = (X, NP) \oplus (MQ, Y).$$

(3) $\Longrightarrow$ (4) The assertion follows from Lemma 2.2 and the property that

$$\big(X, T(X)\big) \cong (X, NP), \quad (T(Y), Y) \cong (MQ, Y).$$

The implication (4) $\Longrightarrow$ (1) follows from Proposition 7.1. □

For the ring of triangular matrices, we obtain the following corollary; in the corollary, the equivalence (1) $\Longleftrightarrow$ (2) has been proved in [22].

Corollary 7.4. *Let*

$$K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

and let (P, Q) *be a* K*-module. Then the following conditions are equivalent.*

(1) (P, Q) *is a projective module.*
(2) P/MQ *is a projective* R*-module,* Q *is a projective* S*-module, and* $M \otimes_S Q \cong MQ$*.*
(3) Q *is a projective* S*-module,* $M \otimes_S Q \cong MQ$*, and there exists a projective* R*-module* X *with* $(P, Q) = (X, 0) \oplus (MQ, Q)$*.*
(4) Q *is a projective* S*-module and there exists a projective* R*-module* X *with* $(P, Q) \cong (X, 0) \oplus (T(Q), Q)$*.*

A module is said to be *hereditary* if all its submodules are projective. We begin our study of hereditary modules with the following result.

Proposition 7.5. *If* (P, Q) *is a hereditary module, then* P *and* Q *are hereditary modules.*

Proof. As usual, we prove the assertion for one of the modules P and Q, since for the second module, the assertion can be proved similarly. We take an arbitrary submodule A in P. By assumption, (A, NP) is a projective submodule in (P, Q). Let $\{a_t \mid t \in T\}$ be some generator system of the R-module A. Then $\{(a_t, 0) \mid t \in T\}$ is a generator system of the K-module (A, NA). By the dual basis lemma [11, Lemma 3.23], there exist K-module homomorphisms $F_t\colon (A, NA) \to K$, $t \in T$, such that every element $v \in (A, NA)$ is equal to $\sum\limits_{t \in T} F_t(v)(a_t, 0)$, where almost all elements $F_t(v)$ are equal to zero. We denote by h the additive homomorphism

$$K \to R, \quad \begin{pmatrix} r & * \\ * & * \end{pmatrix} \to r$$

(hereafter, $*$ denote elements that are inessential for us). For every $t \in T$, there exists an additive homomorphism $f_t\colon A \to R$ such that $f_t(a) = (F_t h)(a, 0)$, $a \in A$. We verify that $f_t(ra) = r f_t(a)$ for all $r \in R$ and $a \in A$. We have

$$f_t(ra) = h\big(F_t(ra, 0)\big) = h\big(rF_t(a, 0)\big) = h\left(r\begin{pmatrix} c & * \\ * & * \end{pmatrix}\right) = h\begin{pmatrix} rc & * \\ * & * \end{pmatrix} = rc,$$

$$rf_t(a) = r\Big(h\big(F_t(a, 0)\big)\Big) = rh\begin{pmatrix} c & * \\ * & * \end{pmatrix} = rc.$$

Consequently, all f_t are R-module homomorphisms. For every element $a \in A$, we have

$$(a, 0) = \sum F_t(a, 0)(a_t, 0) = \sum \begin{pmatrix} r_t & * \\ * & * \end{pmatrix}(a_t, 0) = \sum (r_t a_t, *), \quad a = \sum r_t a_t,$$

where the subscripts $t \in T$ are omitted, $r_t \in R$, and almost all r_t are equal to zero. Thus, $a = \sum f_t(a) a_t$. By the dual basis lemma, the R-module A is projective. Consequently, P is a hereditary module. ☐

The use of Theorem 7.3 leads to the description of hereditary K-modules.

Theorem 7.6. *Let* K *be a ring with zero trace ideals and let* (P, Q) *be a* K*-module. The module* (P, Q) *is hereditary if and only if the following conditions hold.*

(1) P *and* Q *are hereditary modules.*
(2) *For any submodule* B *in* Q*, the module* P/MB *is projective and* $M \otimes_S B \cong MB$*.*
(3) *For any submodule* A *in* P*, the module* Q/NA *is projective and* $N \otimes_R A \cong NA$*.*

Proof. We assume that (P, Q) is a hereditary module. By Proposition 7.5, P and Q are hereditary modules. We take some submodule B in Q and the K-module (MB, B). The module (MB, B) is projective, since it is a submodule of the hereditary module (P, Q). It follows from Theorem 7.3 that $M \otimes_S B \cong MB$. Now we take the submodule $(P, B + NP)$ of the module (P, Q). By Theorem 7.3, the module P/MB is projective. A similar argument is also true for any submodule A in P.

Now we assume that the conditions of the theorem hold. Let (A, B) be some submodule in (P, Q). We have the relations

$$P = X \oplus MB, \quad M \otimes_S B \cong MB, \quad Q = Y \oplus NA, \quad N \otimes_R A \cong NA,$$

where X and Y are some projective modules. Since $MB \subseteq A$ and $NB \subseteq A$, there exist decompositions

$$A = (A \cap X) \oplus MB, \quad B = (B \cap Y) \oplus NA,$$

$$(A, B) = (A \cap X, NA) \oplus \big(MB, (B \cap Y)\big).$$

We verify that the last decomposition satisfies the conditions of Theorem 7.3(3). Indeed, the modules $A \cap X$ and $B \cap Y$ are projective. Further, we have

$$MB \cong M \otimes_S B \cong M \otimes_S (B \cap Y) \oplus M \otimes_S NA.$$

However, $M \otimes_S NA \cong MNA = 0$. Therefore, $M \otimes_S (B \cap Y) \cong MB$. Similarly, $N \otimes_S (A \cap X) \cong NA$. By Theorem 7.3, the module (A, B) is projective. Therefore, (P, Q) is a hereditary module. □

Corollary 7.7. *The module (P, Q) over the ring of triangular matrices*

$$K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is hereditary if and only if P and Q are hereditary modules, $M \otimes_S B \cong MB$, and for any submodule B in Q, the module P/MB is projective.

Remark. For right K-modules, there are assertions that are similar to the results in Secs. 3–6 (for example, the analogues of Theorems 7.3 and 7.6 hold). Some details are presented at the beginning of Sec. 2.

A ring T is said to be *left* (*right*) *hereditary* if T is a hereditary left (respectively, right) T-module, i.e., every left (right) ideal of the ring T is a projective left (right) T-module.

We apply Theorem 7.6 to the ring K.

Corollary 7.8. *The formal matrix ring K with zero trace ideals is left hereditary if and only if the following conditions hold.*

(1) *The rings R and S are left hereditary.*
(2) *M is a flat S-module, N is a flat R-module, and $M \otimes_S N = 0 = N \otimes_R M$.*
(3) *M/ML is a projective R-module for any left ideal L of the ring S.*
(4) *N/NL is a projective S-module for any left ideal L of the ring R.*

Proof. We note that the ring K is left hereditary if and only if the left K-modules (R, N) and (M, S) are hereditary.

Let (R, N) be a hereditary K-module. By Theorem 7.6, the ring R is left hereditary. In addition, for any left ideal L of the ring R, we have that the S-module N/NL is projective and the canonical homomorphism $N \otimes_R L \to NL$ is an isomorphism. The last property is equivalent to the property that N is a flat R-module. Finally, $M \otimes_S N = 0 = N \otimes_R M$. With the use of the hereditary K-module (M, S), the remaining conditions are similarly verified.

We assume that conditions (1)–(4) hold. It follows from Theorem 7.6 that the left K-modules (R, N) and (M, S) are hereditary. We only present some remarks. Since N is a flat S-module, for S-submodule B in N, we have $M \otimes_S B \subseteq M \otimes_S N = 0$. Since $MB = 0$, we have $M \otimes_S B = MB$. In addition, $N \otimes_R L \cong NL$ for any left ideal L of the ring R, since N is a flat R-module. It can be similarly proved that (M, S) is a hereditary module. □

We present three corollaries for the ring of triangular matrices. In Sec. 9, the first corollary is used in studies of Abelian groups with hereditary endomorphism rings.

Corollary 7.9 (Goodearl [15])**.** *The ring*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is left hereditary if and only if the rings R and S are left hereditary, M is a flat S-module, and M/ML is a projective R-module for any left ideal L of the ring S.

Corollary 7.10. *If R and S are Artinian semiprimitive rings, then the ring*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is left and right hereditary for every R-S-bimodule M.

Corollary 7.11. *The ring*

$$\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$$

is left (*or right*) *hereditary if and only if R is an Artinian semiprimitive ring.*

Proof. First, we consider the case of left hereditary rings. If R is an Artinian semiprimitive ring, then by Corollary 7.10, the ring

$$\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$$

is left hereditary.

We assume that the ring

$$\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$$

is left hereditary. By Corollary 7.9, for any left ideal L of the ring R, the module $R/RL = R/L$ is projective. Therefore, L is a direct summand of the module ${}_RR$. Then R is an Artinian semiprimitive ring.

In the case of right hereditary rings, the proof uses the passage to the opposite ring

$$\begin{pmatrix} R^\circ & 0 \\ R^\circ & R^\circ \end{pmatrix}$$

(such rings are mentioned in Sec. 1). □

The following well-known property follows from Corollary 7.11: for any division ring D, the ring

$$\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$$

is hereditary. The following result can be considered as a generalization of this property.

Corollary 7.12. *Let D and F be two division rings and let*

$$K = \begin{pmatrix} D & V \\ W & F \end{pmatrix}$$

be a formal matrix ring. The ring K is left (*or right*) *hereditary if and only if $D \cong F$, V and W are one-dimensional D-spaces and F-spaces, and either K is not a ring with zero trace ideals or K is the ring of* (*upper or lower*) *triangular matrices.*

Proof. Let the ring K be left or right hereditary. Since D and F are division rings, the following three cases are only possible for the trace ideals I and J of the ring K: (1) $I = 0$; (2) $I = D$ and $J = 0$; (3) $J = F$. By considering the products VWV and WVW, it is easy to verify that either $I = D$ and $J = F$, or $I = 0 = J$. In the first case, we obtain that V and W are one-dimensional D-spaces and F-spaces and $D \cong \mathrm{End}_F(V)$ by Lemma 8.3. For $I = 0 = J$, it follows from Corollary 7.8 that $V \otimes_F W = 0$; therefore, $V = 0$ or $W = 0$.

Conversely, if the formal matrix ring

$$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$$

is not a ring with zero trace ideals, then by Corollary 1.4, this ring is isomorphic to the "ordinary" ring of 2×2 matrices over D. The case of the ring of triangular matrices is contained in Corollaries 7.10 and 7.11. ☐

A ring T is said to be *left perfect* if any left T-module has the projective hull. A ring T is left perfect if and only if every flat left T-module is projective.

Corollary 7.13. *If the ring*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is left perfect, then the rings R and S are left perfect. For rings with zero trace ideals, the converse is true.

Proof. Let the ring K be left perfect. We take an arbitrary flat R-module X. Then $\big(X, T(X)\big)$ is a flat module. Since K is left perfect, the module $\big(X, T(X)\big)$ is projective. By Propositions 6.3 and 7.1, the module X is projective. Therefore, the ring R is left perfect. We can similarly prove that the ring S is left perfect.

Conversely, we assume that the rings R and S are left perfect. By Theorems 6.5 and 7.3, any flat left K-module is projective. Therefore, K is left perfect. ☐

Remark. In general case, it is not known whether the assertion, which is converse to Corollary 7.13, is true.

8. Equivalences between Categories R-mod, S-mod, and K-mod

It is natural to separately consider formal matrix rings

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

with $I = R$ and $J = S$. This case is opposite to the case $I = 0 = J$, where K is a ring with zero trace ideals. Such rings appear in studies of equivalences between the categories of R-modules and S-modules. In this field, the main results are the so-called Morita theorems. Sections 21 and 22 of [2] contain a detailed presentation of various topics related to equivalences between module categories. Here we determine the role of the ring K in studies of equivalences of categories. After this, it is clear that the structure of modules over the ring K with $I = R$ and $J = S$ does not depend on the bimodules M and N, and it is determined by the structure of corresponding R-modules and S-modules.

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring, and $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$ be bimodule homomorphisms defined in Sec. 1. The images I and J of these homomorphisms are called the *trace ideals* of the ring K.

We recall that two pairs of module multiplication homomorphisms

$$f\colon M\otimes_S B\to A,\quad g\colon N\otimes_R A\to B,$$
$$f'\colon B\to \operatorname{Hom}_R(M,A),\quad g'\colon A\to \operatorname{Hom}_S(N,B)$$

are related to any K-module (A,B); see the beginning of Sec. 2.

Lemma 8.1. *If $I=R$ and $J=S$, then f, g, f', and g' are isomorphisms.*

Proof. We have

$$1=m_1n_1+\cdots+m_kn_k,\quad \text{where}\quad m_i\in M,\quad n_i\in N,\quad i=1,\ldots,k.$$

Since $A=IA\subseteq MB$, we have that f is surjective. We assume that $f(x_1\otimes b_1+\cdots+x_l\otimes b_l)=0$ for some $x_j\in M$, $b_j\in B$, $j=1,\ldots,l$. We have the relations

$$\sum_j x_j\otimes b_j=\sum_{i,j}(m_in_i)(x_j\otimes b_j)=\sum_{i,j}m_i(n_ix_j)\otimes b_j$$
$$=\sum_{i,j}m_i\otimes n_i(x_jb_j)=\sum_i(m_i\otimes n_i)\cdot\sum_j x_jb_j=\sum_i(m_i\otimes n_i)\cdot 0=0.$$

Consequently, f is an isomorphism.

Similarly, it follows from the relation $J=S$ that

$$1=n_1m_1+\cdots+n_km_k,\quad \text{where}\quad n_i\in N,\quad m_i\in M,\quad i=1,\ldots,k$$

(We use the same symbols n_i and m_i; this does not lead to confusion.) Similarly, we obtain that g is an isomorphism.

If $f'(b)=0$, then

$$b=\sum_i(n_im_i)b=\sum_i n_i(m_ib)=0,$$

since $m_ib=0$. Let α be an arbitrary homomorphism $M\to A$. For any $m\in M$, we have

$$\alpha(m)=\alpha\Big(m\sum_i n_im_i\Big)=\alpha\Big(\sum_i(mn_i)m_i\Big)=\sum_i(mn_i)\alpha(m_i)=m\Big(\sum_i n_i\alpha(m_i)\Big).$$

Therefore,

$$\alpha=f'\Big(\sum_i n_i\alpha(m_i)\Big).$$

Thus, f' is an isomorphism. Similarly, we obtain that g' is an isomorphism. □

Corollary 8.2. *Let $I=R$, $J=S$, and let (A,B) be a K-module.*

(1) *$MB=A$, $NA=B$, and there are the canonical K-module isomorphisms $\big(A,T(A)\big)\cong(A,B)$, $(T(B),B)\cong(A,B)$, $(A,B)\cong\big(A,H(A)\big)$, and $(A,B)\cong(H(B),B)$.*

(2) *There are the canonical ring isomorphisms $\operatorname{End}_R A\cong\operatorname{End}_K(A,B)\cong\operatorname{End}_S B$.*

Proof. In (1), the required isomorphisms are $(1,g)$, $(f,1)$, $(1,f')$, and $(g',1)$, respectively; see remarks after Corollary 2.4. Assertion (2) follows from Corollary 2.4. □

We recall several notions from module theory. Let C be some R-S-bimodule. For every element $r\in R$, the mapping $\alpha_r\colon c\to rc$, $c\in C$, is an S-homomorphism; it is called the *homothety* of the R-module C with coefficient r. The ring homomorphism $R\to\operatorname{End}_S C$, $r\to\alpha_r$, is called the *homothety mapping.* There exists one more homothety mapping, namely, $S\to\operatorname{End}_R C$, $s\to\beta_s$, where $\beta_s(c)=cs$, $s\in S$, $c\in C$. For example, let C be an R-module and let $S=\operatorname{End}_R C$. Then C is an R-S-bimodule. Consequently, there is a homothety mapping $R\to\operatorname{End}_S C$. Here $\operatorname{End}_S C$ is the biendomorphism ring of the R-module C.

Hereafter, G^n denotes the direct sum of n isomorphic copies of the module G.

A module G over a ring T is called a *generator* if one of the following equivalent conditions holds.

(1) The sum of the images of all homomorphisms $G \to R$ coincides with R.
(2) For any T-module X, the sum of the images of all homomorphisms $G \to X$ coincides with X.
(3) For any T-module homomorphisms $\alpha\colon G \to X$ and $\beta, \gamma\colon X \to Y$, from the relation $\alpha\beta = \alpha\gamma$ follows the relation $\beta = \gamma$.
(4) There exist a positive integer n and a T-module H such that $G^n \cong T \oplus H$.

Finitely generated projective generators are often called *progenerators*.

We apply Lemma 8.1 to the ring K. First, we note that each of the K-modules (R, N) and (M, S) gives four homomorphisms of the module multiplication. To familiar homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$, we add homomorphisms $\varphi'\colon N \to \mathrm{Hom}_R(M, R)$ and $\psi'\colon M \to \mathrm{Hom}_S(N, S)$, the canonical isomorphisms $N \otimes_R R \to N$, $M \otimes_S S \to M$, and two homothety mappings $R \to \mathrm{End}_S N$, $S \to \mathrm{End}_R M$. We note that $\varphi'(n)(m) = mn$, $n \in N$, $m \in M$; the homomorphism ψ' acts similarly. Among eight homomorphisms of the module multiplication for right K-modules (R, M) and (N, S), there are homomorphisms $N \to \mathrm{Hom}_S(M, S)$, $M \to \mathrm{Hom}_R(N, R)$, and two homothety mappings $R \to \mathrm{End}_S M$, $S \to \mathrm{End}_R N$.

Lemma 8.3. *Let K be a formal matrix ring such that $I = R$ and $J = S$.*

(1) *All of the above 16 bimodule homomorphisms are isomorphisms.*
(2) *Each of the modules ${}_RM$, M_S, ${}_SN$, and N_R is a progenerator.*

Proof. Assertion (1) is a partial case of Lemma 8.1.

Similarly to Lemma 8.1, let

$$1 = m_1n_1 + \cdots + m_kn_k, \quad \text{where} \quad m_i \in M, \quad n_i \in N, \quad i = 1, \ldots, k.$$

We set $\alpha_i = \varphi'(n_i)$, $i = 1, \ldots, k$. We consider the homomorphism

$$\gamma = \alpha_1 + \cdots + \alpha_k\colon M^k \to R.$$

Since

$$\gamma(m_1 + \cdots + m_k) = m_1n_1 + \cdots + m_kn_k = 1,$$

we have that γ is an epimorphism onto the projective module R. Therefore, γ splits, and $M^k \cong R \oplus X$ for some module X. Thus, M is a generator. We can repeat this argument for the remaining three modules. In particular, there is an isomorphism $M^k \cong S \oplus Y$ for some right S-module Y. Now it follows from isomorphisms of left R-modules

$$R^k \cong \mathrm{Hom}_S(M, M)^k \cong \mathrm{Hom}_S(M^k, M) \cong \mathrm{Hom}_S(S \oplus Y, M) \cong \mathrm{Hom}_S(S, M) \oplus \mathrm{Hom}_S(Y, M) \cong M \oplus X$$

that M is a finitely generated projective R-module. We can repeat these argument for remaining modules. □

If there is the formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

then we can consider the so-called *pre-equivalence situation* or the *Morita context* $(R, S, M, N, \varphi, \psi)$, where R and S are rings, ${}_RM_S$ and ${}_SN_R$ are bimodules, $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$ are bimodule homomorphisms, and the associativity laws

$$(mn)m' = m(nm'), \quad (nm)n' = n(mn') \quad \text{for all} \quad m, m' \in M, \quad n, n' \in N$$

hold. There exists an obvious bijective correspondence between formal matrix rings and pre-equivalence situations. Therefore, it is convenient to call the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

a *pre-equivalence situation* or a *Morita context.* If φ and ψ are isomorphisms, then $(R, S, M, N, \varphi, \psi)$ or the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is called an *equivalence situation.*

Remark. We verify that some "standard" pre-equivalence situation can be obtained if we start from an arbitrary module. Let M be a module over some ring R. We denote by S the endomorphism ring of the R-module M. Then M is an R-S-bimodule. Then we set $M^* = \operatorname{Hom}_R(M, R)$. The group M^* is an S-R-bimodule, where

$$(s\alpha)m = \alpha\big(s(m)\big), \quad (\alpha r)m = \alpha(mr), \quad \alpha \in M^*, \quad s \in S, \quad r \in R, \quad m \in M.$$

There exist an R-R-bimodule homomorphism $\varphi\colon M \otimes_S M^* \to R$ and S-S-bimodule homomorphism $\psi\colon M^* \otimes_R M \to S$ defined by the relations

$$\varphi\Big(\sum m_i \otimes \alpha_i\Big) = \sum \alpha_i(m_i), \quad \Big(\psi\Big(\sum \alpha_i \otimes m_i\Big)\Big)(m) = \sum \alpha_i(m)m_i,$$

where $m_i, m \in M$, $\alpha_i \in M^*$. For φ and ψ, two associativity laws hold. Consequently, we have a pre-equivalence situation $(R, S, M, M^*, \varphi, \psi)$ and the corresponding formal matrix ring. This ring satisfies the following properties.

Lemma 8.4.

(1) *The mapping φ (ψ) is surjective if and only if M is an R-generator (respectively, a finitely generated projective R-module).*

(2) *If M is an R-progenerator, then M satisfies the conditions and assertions of Lemma* 8.3.

Proof. (1) The image of the mapping φ is the sum of the images of all homomorphisms from M in R. It follows from the definition of a generator that φ is surjective if and only if M is a generator.

The mapping ψ is surjective if and only if the identity mapping of the module M is contained in the image of ψ, i.e., there exist homomorphisms $\alpha_1, \ldots, \alpha_k\colon M \to R$ and elements $m_1, \ldots, m_k$ such that $m = \sum \alpha_i(m)m_i$ for all $m \in M$. This means that $\{\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_k\}$ is a dual basis of the module M. The last condition is equivalent to the property that M is a finitely generated projective R-module.

(2) Assertion (2) directly follows from (1). □

We formulate some result on equivalences of categories, which is sometimes called the *first Morita theorem.*

Theorem 8.5 (the first Morita theorem)**.** *Let the ring*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be an equivalence situation. In such a case, the categories R-mod*,* S-mod*, and* K-mod *are equivalent to each other* (*the corresponding equivalences are presented in the proof*)*.*

Proof. We define the functor $T_N = N \otimes_R (-)\colon R\text{-mod} \to S\text{-mod}$ by the relation $T_N(X) = N \otimes_R X$ for any R-module X. The functor T_N transfers R-module homomorphisms into induced S-module homomorphisms. The functor T_M is similarly defined.

We prove that the functors T_N and T_M are mutually inverse equivalences between the categories R-mod and S-mod. We have to verify that the composition $T_M T_N$ $(T_N T_M)$ is naturally equivalent to the identity functor of the category R-mod (respectively, S-mod). This follows from the property that for any R-module X, there exist natural isomorphisms

$$(T_M T_N)X \cong (M \otimes_S N) \otimes_R X \cong R \otimes_R X \cong X.$$

Similarly, $(T_N T_M)Y \cong Y$ for an arbitrary S-module Y.

The equivalence of the categories R-mod and S-mod also defines the functors

$$H_M = \mathrm{Hom}_R(M, -), \quad H_N = \mathrm{Hom}_S(N, -), \quad \text{where} \quad H_M(X) = \mathrm{Hom}_R(M, X), \quad H_N(Y) = \mathrm{Hom}_S(N, Y),$$

and homomorphisms are again transferred into the induced homomorphisms. Indeed,

$$(H_N H_M)X = \mathrm{Hom}_S\big(N, \mathrm{Hom}_R(M, X)\big) \cong \mathrm{Hom}_R(M \otimes_S N, X) \cong \mathrm{Hom}_R(R, X) \cong X, \quad (H_M H_N)Y \cong Y.$$

We note that the functors T_N and T_M, H_M and H_N are closely related to the functors T and H defined in Sec. 2. Of course, the functors T_N and H_M are naturally equivalent. The same is true for the functors T_M and H_N. A natural isomorphism between $T_N(X)$ and $H_M(X)$ is the homomorphism h defined after Corollary 2.4,

$$\big(h(n \otimes x)\big)m = (mn)x, \quad n \in N, \quad x \in X, \quad m \in M.$$

By Lemma 8.1, the homomorphism h is an isomorphism, since h is the homomorphism of the module multiplication for the K-module $\big(X, H_M(X)\big)$.

Now we define the functors $(1, T_N)$: R-mod $\to$ K-mod and $(1, 0)$: K-mod $\to$ R-mod. The first functor practically is the restriction of the functor T from Sec. 2. Namely, $(1, T_N)X = \big(X, T_N(X)\big)$ and $(1, 0)(X, Y) = X$ for any R-module X and the K-module (X, Y). Both functors transfer homomorphisms into induced homomorphisms. We have $\big((1, 0)(1, T_N)\big)X = X$ and

$$\big((1, T_N)(1, 0)\big)(X, Y) = (1, T_N)X = \big(X, T_N(X)\big) \cong (X, Y),$$

where we consider the homomorphism of the module multiplication g as the isomorphism between $T_N(X)$ and Y (see Lemma 8.1 and Corollary 8.2). Thus, $(1, T_N)$ and $(1, 0)$ are mutually inverse equivalences of the categories R-mod and K-mod. We similarly define the functors $(1, H_M)$ and $(1, 0)$, which play the same role ($(1, H_M)$ as the restriction of the functor H from Sec. 2). The equivalence of the categories S-mod and K-mod can be proved similarly.

Both equivalences can be obtained by applying the first part of the proof. For this purpose, we take a standard pre-equivalence situation arising with the use of the module $R \oplus M$. We also represent this module in the form (R, M) to emphasize that we are dealing with an R-K-bimodule. Since $\mathrm{End}_R(R \oplus M) \cong K$ and $\mathrm{Hom}_R\big((R, M), R\big) \cong (R, N)$ ((R, N) is K-R-bimodule), we have the corresponding matrix ring

$$\begin{pmatrix} R & (R, M) \\ \begin{pmatrix} R \\ N \end{pmatrix} & K \end{pmatrix}$$

(such a formal matrix ring exists for any ring K). In fact, it follows from Lemma 8.4 that we are dealing with an equivalence. It follows from the above that the functors $T_{(R,N)}$ and $T_{(R,M)}$ define the equivalence between the categories R-mod and K-mod. In essence, $(1, T_N)$ and $(1, 0)$ are these functors. ☐

Under the conditions of Theorem 8.5, we say that an equivalence situation $(R, S, M, N, \varphi, \psi)$ or the corresponding matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

defines the equivalence of the categories R-mod and S-mod.

The *second Morita theorem* states that all equivalences of two module categories arise from an equivalence situation.

Theorem 8.6 (the second Morita theorem). *Let R and S be two rings such that the categories* R-mod *and* S-mod *are equivalent. Then every equivalence of the categories* R-mod *and* S-mod *is defined by some ring*

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

Proof. We assume that the functors $F\colon R\text{-mod} \to S\text{-mod}$ and $G\colon S\text{-mod} \to R\text{-mod}$ are mutually inverse equivalences. It is sufficient to prove that there exists some equivalence situation $(R, S, M, N, \varphi, \psi)$ or the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

Indeed, by the previous theorem, the functors T_N and T_M (also H_M and H_N) provide an equivalence between R-mod and S-mod. In this case, it is known that the functor F is equivalent to T_N, and the functor G is equivalent to T_M.

We denote by M the R-module $G(S)$. It can be turned into a right S-module such that M is turned into an R-S-bimodule. We do this as follows. For the elements $m \in M$ and $s \in S$, we assume that $ms = \alpha(m)$, where the endomorphism α of the R-module M corresponds to s under the composition of ring isomorphisms $S \cong \operatorname{End}_S S \cong \operatorname{End}_R M$, where the second isomorphism is one of familiar properties of equivalences of categories. We assume that $S = \operatorname{End}_R M$. We consider a standard pre-equivalence situation defined by the bimodule M, and the corresponding ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

where $N = \operatorname{Hom}_R(M, R)$. Under equivalences of categories, generator (finitely generated projective) modules pass to modules with the same property (such properties are said to be *category-theoretical*). Consequently, M is an R-progenerator. Thus,

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is an equivalence situation by Lemma 8.4, which is required. □

Two rings R and S are said to be *equivalent* (in the sense of Morita) or *Morita-equivalent* if the categories R-mod and S-mod are equivalent. The notion of a Morita-equivalence is left-right symmetrical. If R-mod and S-mod are equivalent, then by Theorems 8.6 and 8.5, the ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

is an equivalence situation, and conversely. Then the opposite ring

$$\begin{pmatrix} R^\circ & N \\ M & S^\circ \end{pmatrix}$$

(see Sec. 1) is an equivalence situation, and conversely. Consequently, the categories R°-mod and S°-mod are equivalent. Therefore, the categories mod-R and mod-S are equivalent.

Corollary 8.7. *For two rings R and S, the following conditions are equivalent.*

(1) *The rings R and S are equivalent.*
(2) *There exists an R-progenerator M such that $S \cong \operatorname{End}_R M$.*
(3) *There exists a right R-progenerator N such that $S \cong \operatorname{End}_R N$.*
(4) *There exists an equivalence situation*

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

Proof. The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) follow from Theorem 8.6.

The equivalence (1) $\Longleftrightarrow$ (4) has been proved in Theorems 8.5 and 8.6.

(2) $\Longrightarrow$ (1) We consider a standard pre-equivalence situation (R, S, M, M^*) (see Remark before Lemma 8.4). The rings R and S are equivalent by Lemma 8.4 and Theorem 8.5.

(3) $\Longrightarrow$ (2) The R°-module N is a progenerator, and $S^\circ \cong \operatorname{End}_{R^\circ} N$. It was proved that the rings R° and S° are equivalent. Therefore, the rings R and S are also equivalent. □

Corollary 8.8. *Let R be a ring and let M be an R-module.*

(1) *If M is a progenerator, then the rings R and $\operatorname{End}_R M$ are equivalent.*
(2) *For any positive integer n, the ring R_n of all $n \times n$ matrices is equivalent to the ring R.*

Proof. (1) The assertion follows from Corollary 8.7.

(2) The assertion follows from the property that the ring R_n is isomorphic to the endomorphism ring of the free module R^n, which is a progenerator. $\square$

We return to modules over the formal matrix ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

If the trace ideals I and J of this ring coincide with R and S, respectively (i.e., K is an equivalence situation), then all three categories R-mod, S-mod, and K-mod are equivalent. Similarly, the categories of right modules mod-R, mod-S, and mod-K are also equivalent to each other. We note that the equivalences can be defined with the use of the functors of the tensor product and Hom; their forms are indicated in the proof of Theorem 8.5. These functors preserve all module properties of the category-theoretical type. In particular, these functors preserve flat modules, projective modules, and hereditary modules considered before. Thus we have the following result.

Corollary 8.9. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be an equivalence situation and let (A, B) be a K-module. Then the following conditions are equivalent.

(1) *A is a flat (projective, hereditary) R-module.*
(2) *B is a flat (respectively, projective, hereditary) S-module.*
(3) *(A, B) is a flat (respectively, projective, hereditary) K-module.*

Proof. The assertion follows from Propositions 6.3 and 7.1, and the K-module isomorphisms $\big(A, T(A)\big) \cong (A, B) \cong (T(B), B)$ proved above. $\square$

Corollary 8.10. *Under the conditions of Corollary 8.9, the following conditions are equivalent.*

(1) *The ring K is left (right) hereditary.*
(2) *The ring R is left (right) hereditary.*
(3) *The ring S is left (right) hereditary.*

Corollary 8.11. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be an equivalence situation.

(1) *If (A, B) is a K-module, then the mappings $X \to NX$ and $X \to (X, NX)$ are isomorphisms from the lattice of all submodules of the module A onto the lattice of all submodules of the module B and the lattice of all submodules of the module (A, B), respectively. In the first case, the inverse isomorphism is defined by the rule $Y \to MY$, where Y is an arbitrary submodule in B.*
(2) *The correspondence $L \to NL$ is an isomorphism from the lattice of all left ideals of the ring R onto the lattice of all submodules of the S-module N, and ideals of R correspond to subbimodules of the bimodule N. A similar assertion holds for the ring S and the bimodule M.*
(3) *The assertions for right modules are similar to (1) and (2).*
(4) *The mappings $X \to NXM$ and $Y \to MYN$ are mutually inverse isomorphisms between the lattices of ideals of the rings R and S.*

(5) *The correspondence*

$$X \to \begin{pmatrix} X & XM \\ NX & NXM \end{pmatrix}$$

is an isomorphism from the lattice of ideals of the ring R *onto the lattice of ideals of the ring* K.

Proof. (1) The assertion follows from Corollary 8.2.

(2) The assertion follows from (1) applied to the K-modules (R, N) and (M, S).

(3) The assertion follows from (1) and (2) and the symmetry argument.

(4) The assertion follows from (2).

(5) The assertion follows from (4) applied to the ring

$$\begin{pmatrix} R & (R, M) \\ \begin{pmatrix} R \\ N \end{pmatrix} & K \end{pmatrix}$$

from the proof of Theorem 8.5. □

In the next section, the following two well-known properties are used.

Corollary 8.12. *For a ring* R*, the following conditions are equivalent.*

(1) R *is a left hereditary ring.*

(2) *For any nonzero idempotent* $e \in R$*, the ring* R *is left hereditary.*

(3) *For any positive integer* n*, the ring* R_n *of all* $n \times n$ *matrices over* R *is left* (*right*) *hereditary.*

(4) *There exists a positive integer* n *such that the ring* R_n *of all* $n \times n$ *matrices over* R *is left* (*right*) *hereditary.*

A similar assertion holds for right hereditary rings.

Proof. The implication (1) $\Longrightarrow$ (2) follows from Proposition 7.5 if we identify the ring R with the formal matrix ring (as in Sec. 1).

The implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1) There exists a nonzero idempotent $e \in R_n$ such that $R \cong eR_ne$. Since the implication (1) $\Longrightarrow$ (2) has been proved and the ring R_n is left (right) hereditary, the ring R is left (right) hereditary.

(1) $\Longrightarrow$ (3) The ring R_n is left hereditary if and only if the module of column vectors $(R, \dots, R)$ of length n is a hereditary R_n-module. This module can be considered as the module $\big(R, (R, \dots, R)\big)$ over the formal matrix ring

$$\begin{pmatrix} R & (R, \dots, R) \\ \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} & R_{n-1} \end{pmatrix}$$

of order 2. Then we apply Corollary 8.9 to the last module and obtain that the ring R_n is left hereditary. □

If K is an equivalence situation, then the study of K-modules is almost always reduced to the study of R-modules or S-modules. We consider another extreme case, where the trace ideals of the ring K are equal to zero. Then the study of K-modules can often be reduced to the study of R-modules or S-modules, but some additional difficulties appear. This is confirmed by our studies. The "intermediate" case, where I and J are nontrivial ideals, is quite difficult.

Corollary 8.9 completely describes flat, projective, and hereditary modules in the case where K is an equivalence situation. On the other hand, Theorems 6.5, 7.3, and 7.6 contain satisfactory characterizations of such modules over the ring K with zero trace ideals. The structures of flat, projective, and hereditary modules over an arbitrary formal matrix ring K are not known. The same is true for regular modules. (A module M is said to be *regular* if every cyclic submodule of M is a direct summand in M.) It is also important to know when the ring K is left hereditary, right hereditary, or regular.

Finally, we give the following remark. Let M be some R-module. It is interesting to study the formal matrix ring

$$K = \begin{pmatrix} R & M \\ \operatorname{Hom}_R(M,R) & \operatorname{End}_R M \end{pmatrix}$$

(a standard pre-equivalence situation), and also two subrings of triangular matrices in K.

9. Hereditary Endomorphism Rings of Abelian Groups

In this final section, we use Corollary 7.9 to describe some Abelian groups with hereditary endomorphism rings. The word "group" means "Abelian group." Groups are considered as $\mathbb{Z}$-modules.

In Sec. 9*, the composition of homomorphisms is defined "from the right to the left":* $(\alpha\beta)(x) = \alpha\big(\beta(x)\big)$.

For a group G, we denote by $\operatorname{End} G$ the endomorphism ring of G. Let the group G be equal to the direct sum $A \oplus B$. Then we can identify the ring $\operatorname{End} G$ with the formal matrix ring

$$\begin{pmatrix} \operatorname{End} A & \operatorname{Hom}(B,A) \\ \operatorname{Hom}(A,B) & \operatorname{End} B \end{pmatrix}.$$

If A is a fully invariant subgroup, then $\operatorname{Hom}(A,B) = 0$ and we obtain the ring of triangular matrices. We will often deal with such a situation.

We use the following notation:

- p is some prime integer;
- $\mathbb{Z}(p)$ is a cyclic group of order p;
- $\mathbb{Z}(p^\infty)$ is a quasicyclic p-group;
- $\mathbb{Q}$ is the additive group or field of rational numbers;
- $\hat{\mathbb{Z}}_p$ is the group or ring of p-adic integers.

We have the ring isomorphisms $\operatorname{End} \mathbb{Z}(p^\infty) \cong \hat{\mathbb{Z}}_p$ and $\operatorname{End} \mathbb{Q} \cong \mathbb{Q}$.

We often deal with divisible groups. Any divisible group D can be represented in the form $D = \bigoplus_p D_p \oplus D_0$, where D_p is a divisible p-group, and D_0 is a divisible torsion-free group; D_p is either the zero group or the direct sum of some set of copies of the group $\mathbb{Z}(p^\infty)$, and D_0 is either the zero group or the direct sum of some set of copies of the group $\mathbb{Q}$.

Remark. The monograph of Fuchs [13] contains all required notions, properties, and notations from the theory of Abelian groups.

A group is said to be *elementary* if the order of every its nonzero element cannot be divided by squares of integers. An elementary group is the direct sum of elementary p-groups. An elementary nonzero p-group is the direct sum of the groups $\mathbb{Z}(p)$.

For a group G, the largest p-subgroup in G is called the *p-component* of the group G.

We often use Corollary 8.12. For example, if $G = A \oplus B$ and the ring $\operatorname{End} G$ is left (right) hereditary, then the rings $\operatorname{End} A$ and $\operatorname{End} B$ are left (respectively, right) hereditary. Indeed, if e is the projection of the group G onto A with kernel B, then the ring $\operatorname{End} A$ can be identified with the ring $e \cdot \operatorname{End} G \cdot e$. In addition, we formulate the following result.

Proposition 9.1 ([31, Proposition 35.11])**.**

(1) *If G is a group and the ring* $\operatorname{End} G$ *is left or right hereditary, then G is not an infinite direct sum of nonzero groups.*

(2) *Let A be a reduced group with left or right hereditary ring* $\operatorname{End} A$*. Then every p-component A_p of the group A is an elementary p-group of finite rank and $A = A_p \oplus B_p$ for some group B_p.*

(3) *A reduced torsion group G has the left or right hereditary endomorphism ring if and only if G is an elementary group of finite rank (i.e., G is a finite direct sum of the groups $\mathbb{Z}(p)$ for some p).*

For convenience, we repeat Corollary 7.9 and present the right-side analogue of it.

Proposition 9.2. *Let*
$$K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}.$$

(1) *The ring K is left hereditary if and only if of the rings R and S are left hereditary, M is a flat S-module, and M/ML is a projective R-module for any left ideal L of the ring S.*
(2) *The ring K is right hereditary if and only if the rings R and S are right hereditary, M is a flat R-module, and M/LM is a projective S-module for any right ideal L of the ring S.*

Remark. Let D be some divisible group. In studies of groups with hereditary endomorphism rings, we can assume that D is a group of finite rank; this follows from Proposition 9.1. We represent the group D in the form $D = D_t \oplus D_0$, where D_t is a torsion group and D_0 is a torsion-free group. For $D_t \neq 0$ and $D_0 \neq 0$, the ring $\operatorname{End} D$ is the ring of triangular matrices
$$\begin{pmatrix} \operatorname{End} D_t & \operatorname{Hom}(D_0, D_t) \\ 0 & \operatorname{End} D_0 \end{pmatrix}.$$
For example, if $D = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}$, then
$$\operatorname{End} D = \begin{pmatrix} \hat{\mathbb{Z}}_p & A_p \\ 0 & \mathbb{Q} \end{pmatrix},$$
where $A_p = \operatorname{Hom}\bigl(\mathbb{Q}, \mathbb{Z}(p^\infty)\bigr)$ is the additive group of the field of p-adic numbers. This ring is not left hereditary, since A_p is not a projective $\hat{\mathbb{Z}}_p$-module (see Proposition 9.2(1)). At the same time, this ring is right hereditary, since all conditions of Proposition 9.2(2) hold. The same is true for the endomorphism ring of the group $\mathbb{Z}(p_1^\infty) \oplus \cdots \oplus \mathbb{Z}(p_k^\infty) \oplus \mathbb{Q}$, where $p_1, \ldots, p_k$ are distinct prime integers.

Below, we present Theorem 9.3, which answers the following questions.

(1) When is the endomorphism ring of a divisible group left hereditary?
(2) When is the endomorphism ring of a divisible group right hereditary?

Theorem 9.3. *Let D be a nonzero divisible group of finite rank.*

(1) *The ring* $\operatorname{End} D$ *is left hereditary if and only if either D is a torsion-free group or D is a torsion group.*
(2) *The ring* $\operatorname{End} D$ *is right hereditary.*

Proof. (1) Let the ring $\operatorname{End} D$ be left hereditary. By the above remark, the group D does not contain direct summands of the form $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}$. Therefore, either D is a torsion-free group or D is a torsion group.

Conversely, if D is a torsion group, then $\operatorname{End} D$ is a finite direct product of matrix rings over rings of p-adic integers. If D is a torsion-free group, then $\operatorname{End} D$ is a matrix ring over $\mathbb{Q}$. In both cases, the ring $\operatorname{End} D$ is left and right hereditary by Corollary 8.12.

(2) If D is either a torsion-free group or a torsion group, then the ring $\operatorname{End} D$ is right hereditary (see the proof of (1)). Let D be a mixed group, i.e., let D contain a quasicyclic group and the group $\mathbb{Q}$. We denote by C the group $\mathbb{Z}(p_1^\infty) \oplus \cdots \oplus \mathbb{Z}(p_k^\infty) \oplus \mathbb{Q}$, where p_i are all prime integers such that the group D has a direct summand of the form $\mathbb{Z}(p_i^\infty)$. There exist a positive integer n and a group E such that $C^n \cong D \oplus E$. By Corollary 8.12 and the remark before the theorem, the ring $\operatorname{End} C^n$ is right hereditary. Therefore, the ring $\operatorname{End} D$ is right hereditary. □

Now we begin to solve the following problem. We wish to reduce the study of (left or right) heredity of the ring $\operatorname{End} G$ to the case where G is a reduced group.

Let G be a nonreduced nondivisible group. Then $G = D \oplus A$, where D is a nonzero divisible group and A is a nonzero reduced group. We preserve this notation until the completion of the section. The ring $\operatorname{End} G$ coincides with the ring
$$\begin{pmatrix} \operatorname{End} D & \operatorname{Hom}(A, D) \\ 0 & \operatorname{End} A \end{pmatrix}.$$

Theorem 9.4. *The ring* $\operatorname{End} G$ *is left hereditary if and only if* D *is a torsion-free group of finite rank, the ring* $\operatorname{End} A$ *is left hereditary, and* $\operatorname{Hom}(A, D)$ *is a flat* $(\operatorname{End} A)$*-module.*

Proof. Let the ring $\operatorname{End} G$ be left hereditary. It follows from Theorem 9.3 that the group D cannot be mixed. In addition, D is not a torsion group. Indeed, otherwise D is a finite direct sum of quasicyclic p-groups for some prime p. By Proposition 9.2, $\operatorname{Hom}(A, D)$ is a projective $(\operatorname{End} D)$-module. (We note that $\operatorname{End} D$ is a finite direct product of matrix rings over rings $\hat{\mathbb{Z}}_p$.) The group structure of the group $\operatorname{Hom}(A, D)$ is known (see [13, Theorem 47.1]). This implies that $\operatorname{Hom}(A, D)$ cannot be a projective $(\operatorname{End} D)$-module.

Now we assume that D is a torsion-free group of finite rank, the ring $\operatorname{End} A$ is left hereditary and $\operatorname{Hom}(A, D)$ is a flat $(\operatorname{End} A)$-module. Since $\operatorname{End} D$ is a matrix ring over $\mathbb{Q}$, the ring $\operatorname{End} G$ is left hereditary by Proposition 9.2. □

We preserve the notation defined before Theorem 9.4. We pass to right hereditary rings.

Theorem 9.5. *The ring* $\operatorname{End} G$ *is right hereditary if and only if* $G = D \oplus T$*, where* D *is a divisible group of finite rank,* T *is an elementary group of finite rank, and the groups* D *and* T *do not contain nonzero* p*-components for equal* p.

Proof. Let the ring $\operatorname{End} G$ be right hereditary. It follows from Proposition 9.1 that the rank of the group D is finite. We assume that the group A contains elements of infinite order. By Proposition 9.2, $\operatorname{Hom}(A, D)$ is a projective $(\operatorname{End} A)$-module. Now we note that the additive group of the ring $\operatorname{End} A$ is reduced, since A is a reduced group. Therefore, $\operatorname{Hom}(A, D)$ is a reduced group. On the other hand, if D has a direct summand that is isomorphic to $\mathbb{Q}$, then $\operatorname{Hom}(A, D)$ also has a direct summand that is isomorphic to $\mathbb{Q}$. Similarly, if D contains the group $\mathbb{Z}(p^\infty)$, then $\operatorname{Hom}(A, D)$ is a nonreduced group. Thus, we obtain that A is a torsion group. Now it follows from Proposition 9.1 that the structure of the group A satisfies our theorem. The group $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p)$ cannot be a direct summand of the group G. The reason is that the endomorphism ring of this group is the matrix ring

$$\begin{pmatrix} \hat{\mathbb{Z}}_p & \mathbb{Z}(p) \\ 0 & \mathbb{Z}/p\mathbb{Z} \end{pmatrix}.$$

This ring is not right hereditary, since $\mathbb{Z}/p\mathbb{Z}$ is not a flat $\hat{\mathbb{Z}}_p$-module. It follows from the above that the groups D and T have nonzero p-components only for distinct p.

Under the conditions of the theorem, we obtain that the subgroups D and T are fully invariant in G. Therefore, $\operatorname{End} G = \operatorname{End} D \times \operatorname{End} T$. The ring $\operatorname{End} D$ is right hereditary by Theorem 9.3, and the ring $\operatorname{End} T$ is right hereditary by Proposition 9.1. □

Remarks. Since

$$\operatorname{End}(\mathbb{Q} \oplus \mathbb{Z}) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix},$$

it follows from Theorems 9.4 and 9.5 that this ring is left hereditary, but it is not right hereditary.

Right hereditary endomorphism rings of torsion-free groups are studied in [31]. In connection with Theorem 9.4, it is natural to pose the problem of description of groups A such that $\operatorname{Hom}(A, \mathbb{Q})$ is a flat right $(\operatorname{End} A)$-module. For the $(\operatorname{End} A)$-modules $\operatorname{Hom}\big(A, \mathbb{Z}(p)\big)$ and $\operatorname{Hom}\big(A, \mathbb{Z}(p^\infty)\big)$, it is interesting to know when these modules are simple, Artinian, or Noetherian. The book [31] contains a more detailed introduction to this field (e.g., see Problems 11–13).

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