Journal of Mathematical Sciences, Vol. 162, No. 1, 2009

# ON THE DETERMINATION OF STRESS CONCENTRATION IN A STRETCHED PLATE WITH TWO HOLES

**V. V. Panasyuk and M. P. Savruk**

UDC 539.3

We present a short survey of studies of the elastic interaction of two holes in a stretched plate. Special attention is paid to the study of the concentration of stresses on the contours of closely positioned holes. For two identical elliptic holes, numerical results are obtained by the method of singular integral equations. With the help of the limit transition, we determined the stress intensity factors at the vertices of semi-infinite parabolic notches. A comparison of the numerical data with known analytic solutions for two circular holes and collinear cracks is performed.

The study of the concentration of stresses near holes in structural elements was begun by G. Kirsch as far back as in the year 1898. In Lviv, such investigations were developed by G. M. Savin for 1945–1952. The scientific achievements of the Lviv school of mechanicians were presented and generalized in monographs [10, 11]. These monographs give mainly the solution of the problems concerning the concentration of stresses near a single curvilinear hole with the use of the Kolosov–Muskhelishvili method [2], by constructing the functions of conformal mappings in the form of series. The plane problem of the concentration of stresses near two identical circular holes in the infinite elastic isotropic plane was considered in [24] (see also [10]). The solution of the problem was given in the form of Fourier series using of bipolar coordinates. For two unequal circular holes, such a solution was first constructed by Ya. S. Pidstryhach [6, 9] (see also [7]). His candidate-degree thesis [8], carried out under the guidance of Academician G. M. Savin, was devoted to the investigation of this problem. At that time, this was a significant achievement in this branch of science.

In the present paper, we consider the problem of the interaction of two elliptic holes in the elastic isotropic plane, using the method of singular integral equations. Main attention is focused on the study of the concentration of stresses in the case where the distance between the contours of holes is small. In this case, the maximum stresses on the contours of holes become unboundedly large. In this connection, one meets significant difficulties in the execution of calculations while studying the concentration of stresses near closely positioned holes. The asymptotic behavior of stresses upon approach of the holes to each other was studied by means of analytic methods only in the case of two identical circular holes and collinear cracks. Modern computers and new numerical methods of solution of integral equations allow one to numerically determine the order of singularity of maximum stresses and to find the numerical coefficient of this singularity for holes with various configurations. Knowledge of the singularity of stresses is important for the development of direct numerical methods of solution of similar problems. Such studies can be useful for the derivation of numerical solutions of new problems using the limit transition. Exactly in this way, we obtain the stress concentration factors at the tips of semi-infinite parabolic notches in what follows.

### Two Circular Holes

Consider the solution of the problem of the stress concentration near two unequal circular holes constructed by Ya. S. Pidstryhach [6, 9]. Let the elastic isotropic plate be weakened by two unequal circular holes $L_1$ and

Karpenko Physicomechanical Institute, Ukrainian Academy of Science, Lviv, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 51, No. 2, pp. 112–123, April–June, 2008. Original article submitted April 10, 2008.

 1072-3374/09/1621–0132 

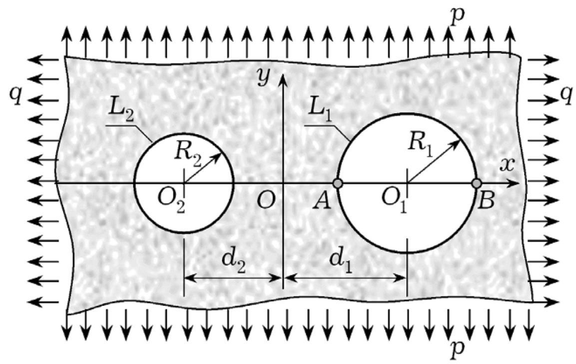


**Fig. 1**

$L_2$ with radii $R_1$ and $R_2$. The centers of the holes lie on the $Ox$ axis at the points $(-d_2,0)$ and $(0,d_1)$ (Fig. 1). We introduce the bipolar coordinates $\xi,\eta$ by means of the relations

$$x = \frac{\tilde{a}\sinh\eta}{\cosh\eta - \cos\xi}, \qquad y = \frac{\tilde{a}\sin\xi}{\cosh\eta - \cos\xi}, \tag{1}$$

where $\tilde{a}$ is a constant parameter with dimension of length.

Consider the coordinate line $\eta = \alpha$. By eliminating the parameter $\xi$ from equalities (1), we find that the curve $\eta = \alpha$ is a circle

$$(x - \tilde{a}\coth\alpha)^2 + y^2 = \tilde{a}^2 \frac{1}{\sinh^2\alpha}$$

with the center at the point $(\tilde{a}\coth\alpha, 0)$ on the $Ox$ axis and with the radius

$$R = \tilde{a}\frac{1}{|\sinh\alpha|}. \tag{2}$$

Thus, for the proper choice of $\alpha_i$, the coordinate lines $\eta = \tilde{\alpha}_i$, $i = 1,2$, coincide with the contours of the holes $L_i$, $i = 1,2$. The values of $\tilde{\alpha}_i$ are given by the relations

$$\sinh\tilde{\alpha}_1 = \frac{M}{2m\lambda_0}, \quad \sinh\tilde{\alpha}_2 = -\frac{M}{2\lambda_0}, \quad \lambda_0 = \frac{d_1 + d_2}{R_1 + R_2}, \quad m_0 = \frac{R_1}{R_2},$$

$$M = \sqrt{(m_0^2 + 1)(\lambda_0^2 - 1)^2 + 2m_0(\lambda_0^4 - 1)}.$$

We can determine now the parameter $\tilde{a}$ from equality (2), by setting $R = R_1$, $\eta = \alpha_1$.

The biharmonic equation for the stress function $U$ in the bipolar coordinates (1) is reduced to the linear differential equation with constant coefficients for the function $gU$ [19]

$$\left(\frac{\partial^4}{\partial\eta^4}+2\frac{\partial^4}{\partial\eta^2\partial\xi^2}+\frac{\partial^4}{\partial\xi^4}-2\frac{\partial^2}{\partial\eta^2}+2\frac{\partial^2}{\partial\xi^2}+1\right)(gU)=0\,, \tag{3}$$

where $g=(\cosh\eta-\cos\xi)/\tilde{a}$.

We assume that, on the contours of the holes $L_i$, $i=1,2$, the normal $\sigma_\eta^{(i)}$ and tangential $\tau_{\xi\eta}^{(i)}$ stresses are given. We present them in the form of trigonometric series

$$\sigma_\eta^{(i)}=\sum_{n=0}^{\infty}\sigma_n^{(i)}\cos n\xi, \qquad \tau_{\xi\eta}^{(i)}=\sum_{n=1}^{\infty}\tau_n^{(i)}\sin n\xi \tag{4}$$

and assume that the stresses are vanishing at the infinity:

$$(gU)_\infty=0\,. \tag{5}$$

The solution of Eq. (3) can be obtained by the method of separation of variables. Satisfying the conditions of uniqueness of displacements, we choose the solution in the form

$$\frac{1}{a}gU=D_1\eta\cosh\eta+K(\cosh\eta-\cos\xi)\ln(\cosh\eta-\cos\xi)+\sum_{n=1}^{\infty}f_n(\eta)\cos n\xi\,,$$

where

$$f_n(\eta)=A_n\cosh(n+1)\eta+B_n\cosh(n-1)\eta+C_n\sinh(n+1)\eta+D_n\sinh(n-1)\eta\,,$$

$$f_1(\eta)=A_1\cosh 2\eta+B_1+C_1\sinh 2\eta-D_1\eta, \qquad n>1\,.$$

The arbitrary constants $A_n$, $B_n$, $C_n$, and $D_n$ should be chosen so that the boundary conditions (4) are satisfied. The constant $K$ is determined from condition (5), which is reduced to the equation

$$\sum_{n=1}^{\infty}(A_n+B_n)=0\,,$$

if we consider that the infinite point of the plane has the coordinates $\xi=\eta=0$.

Using the known formulas [19] for the normal $\sigma_\eta$ and tangential $\tau_{\xi\eta}$ stresses in bipolar coordinates in terms of the function $gU$, we get that the boundary conditions (4) yield a system of linear algebraic equations for the unknown constants. For specific external loads, the required constants $A_n, B_n, C_n, D_n$ can be analytically given in terms of the known coefficients $\sigma_n^{(i)}$ and $\tau_n^{(i)}$. Ya. S. Pidstryhach constructed the solutions in the cases where the edges of holes undergo the action of constant pressures that are different on different contours, as well as in the case of the uniform tension of a plate without any load on the contours of holes.

We note that the analytic solution of this problem in the form of Fourier series can be obtained also by the Muskhelishvili method, by using the conformal mapping of an elastic region on a circular ring. Just in this way, R. A. Haddon [22] constructed the solution of this problem under a uniaxial tension at infinity whose direction forms an arbitrary angle to the line of the centers of holes.

We now consider the case of a biaxial tension of an elastic plane with two identical holes with unloaded contours ( $R_1 = R_2 = R,\ d_1 = d_2 = d,\ \tilde{\alpha}_1 = \tilde{\alpha}_2 = \alpha$ ) due to the forces applied at infinity $\sigma_y^\infty = p$ and $\sigma_x^\infty = q$. By using the bipolar coordinates (1), C.-B. Ling [24] constructed a solution of the problem in the form of Fourier series and obtained the formula for determination of the circular stresses $\sigma_\xi$ on the contours of the holes

$$\sigma_\xi = 2T(\cosh\alpha - \cos\xi)\left[K\sinh\alpha\left(1 + 4\sum_{n=1}^{\infty}\frac{\sinh n\alpha\cos n\alpha}{\sinh 2n\alpha + n\sinh 2\alpha}\right)\right.$$

$$\left.\mp 2\sum_{n=1}^{\infty}\frac{n(n\sinh n\alpha\sinh\alpha - \cosh n\alpha\cosh\alpha)\cos n\xi}{\sinh 2n\alpha + n\sinh 2\alpha}\right], \tag{6}$$

where

$$K = \frac{\dfrac{1}{2} \pm \dfrac{1}{2} \mp 2\sinh^2\alpha\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sinh 2n\alpha + n\sinh 2\alpha}}{\dfrac{1}{2} + \tanh\alpha\sinh^2\alpha - 4\displaystyle\sum_{n=2}^{\infty}\frac{e^{-n\alpha}\sinh n\alpha + n\sinh\alpha(n\sinh\alpha + \cosh n\alpha)}{n(n^2-1)(\sinh 2n\alpha + n\sinh 2\alpha)}}. \tag{7}$$

Here, the upper and lower signs correspond, respectively, to longitudinal tension from the forces $T = q\ (q \neq 0,\ p = 0)$ and to transverse tension from the forces $T = p\ \ (p \neq 0,\ q = 0)$.

The given solution is not valid for the values of the dimensionless parameter $\gamma = \dfrac{d}{R} - 1 < 1$. On the basis of solution (6), Yu. A. Ustinov [17, 18] deduced the approximate formulas for the stress intensity factors $k_A, k_B$ at the points $A$ and $B$ (Fig. 1) on the contours of the holes for small values of the parameter $\gamma$:

$$k_A = \frac{\sigma_A}{p} = 2\left[\frac{2}{\tilde{\varepsilon}^2}\sinh^2\tilde{\varepsilon}(K_1 + \tilde{m}K_2) + 1 - \tilde{m}\right]\cosh\tilde{\varepsilon}\,\frac{\cosh\tilde{\varepsilon} + 1}{\sinh 2\tilde{\varepsilon} + 2\tilde{\varepsilon}}, \qquad 0 < \tilde{\varepsilon} \le 1,$$

$$k_B = \frac{\sigma_B}{p} = 3.87 - 1.49\tilde{\varepsilon} + 1.23\tilde{\varepsilon}^2 - \tilde{m}(0.981 - 0173\tilde{\varepsilon} - 0018\tilde{\varepsilon}^2), \qquad 0 \le \tilde{\varepsilon} \le 0.3, \tag{8}$$

where

$$\tilde{\varepsilon} = \ln\left(1 + \gamma + \sqrt{2\gamma + \gamma^2}\right) \quad \text{and} \quad \tilde{m} = \frac{p}{q}.$$

The constants $K_1$ (transverse tension) and $K_2$ (longitudinal tension) are determined from the equations

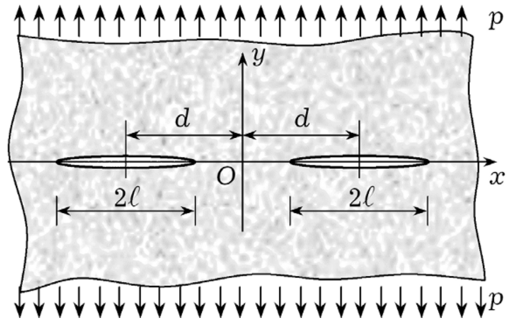


**Fig. 2**

$$K_s\left(0.716-0.0393\tilde{\varepsilon}^2+\frac{2\sinh^4\tilde{\varepsilon}}{\tilde{\varepsilon}^2(\sinh 2\tilde{\varepsilon}+2\tilde{\varepsilon})}\right)$$

$$=\frac{1}{2}+(-1)^s\frac{1}{2}+(-1)^s\frac{\sinh^2\tilde{\varepsilon}}{\sinh 2\tilde{\varepsilon}+2\tilde{\varepsilon}}-(-1)^s\frac{\sinh^2\tilde{\varepsilon}}{\tilde{\varepsilon}^2}(0.769-0.179\tilde{\varepsilon}^2+0.341\tilde{\varepsilon}^4),\quad s=1,2\,. \tag{9}$$

This implies that $k_A\to 2.22\gamma^{-1/2}$ ($q=0$) and $k_A\to 1.97\gamma^{-1/2}$ ($q=p$) as $\gamma\to 0$. In formulas (7) and (9), we corrected some mistakes in [18].

We note that the method of bipolar coordinates allows one not only to consider the limiting case where the parameter $\gamma\to 0$, but also to obtain solutions of problems in the case where the contours of two circular holes intersect each other, i.e., to consider a curvilinear hole with nonsmooth edge formed by two arcs with the same [5, 10, 11, 19, 25] or different [19] radii. In particular, these solutions obtained in a closed form (in terms of a Fourier integral) yield the stress intensity factor at the corner tip of a symmetric lune-shaped hole [1].

## Two Collinear Cracks with the Same Length [4]

Let the plate stretched at infinity be weakened by two collinear cracks of the same length $2\ell$ (Fig. 2). The exact solution of this problem is available. The formulas for the stress intensity factors at the near ($K_{\mathrm{I}}^-$) and remote ($K_{\mathrm{I}}^+$) tips of the cracks take the form

$$K_{\mathrm{I}}^{\pm}=\pm\frac{p\sqrt{\pi\ell}}{\lambda\sqrt{1\pm\lambda}}\left[1\pm\lambda-\frac{E(\lambda)}{K(\lambda)}\right], \tag{10}$$

where $K(\lambda), E(\lambda)$ are complete elliptic integrals of the first and second kind, respectively; $\lambda=\ell/d$; $2d$ is the distance between the centers of the cracks.

Using the expansion of the ratio $E(\lambda)/K(\lambda)$ for small values of the parameter $\lambda$,

$$\frac{E(\lambda)}{K(\lambda)}=1-\frac{1}{2}\lambda^2-\frac{1}{16}\lambda^4+O(\lambda^6)\,,$$

and its asymptotic behavior as $\lambda\to 1$,

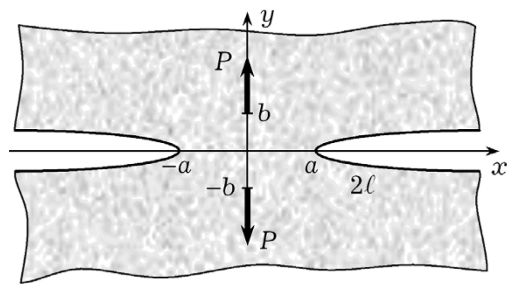


**Fig. 3**

$$\frac{E(\lambda)}{K(\lambda)} \approx -\frac{1}{\ln\sqrt{1-\lambda}},$$

we can write a closed approximate formula for this ratio,

$$\frac{E(\lambda)}{K(\lambda)} = \frac{1-(1-\lambda)\ln\sqrt{1-\lambda}}{1-\ln\sqrt{1-\lambda}},$$

which agrees with the exact expression (10) as $\lambda \to 0$ and $\lambda \to 1$.

The approximate formulas for the stress intensity factors (10) take the form

$$K_{\mathrm{I}}^{+} = \frac{p\sqrt{\pi\ell}\,(1-2\ln\sqrt{1-\lambda})}{\sqrt{1+\lambda}\,(1-\ln\sqrt{1-\lambda})},$$

$$K_{\mathrm{I}}^{-} = \frac{p\sqrt{\pi\ell}}{\sqrt{1-\lambda}\,(1-\ln\sqrt{1-\lambda})}. \qquad (11)$$

The relative errors of these formulas do not exceed 0.3% for $K_{\mathrm{I}}^{+}$ and 1.1% for $K_{\mathrm{I}}^{-}$ in the whole range of variation of the parameter $\lambda$.

Relations (11) yield the following asymptotics as $\lambda \to 1$:

$$K_{\mathrm{I}}^{+} \to p\sqrt{2\pi\ell}, \qquad K_{\mathrm{I}}^{-} \to -\frac{p\sqrt{\pi\ell}}{\sqrt{1-\lambda}\,\ln\sqrt{1-\lambda}}. \qquad (12)$$

Let the elastic plane contain two semi-infinite cracks, whose tips are positioned at the distance $2a$. Consider the symmetric tension of the plane by concentrated forces $P$ applied at the points $(0, b)$ and $(0, -b)$ on the $Oy$ axis (Fig. 3).

The stress intensity factor at the crack tip

$$K_{\mathrm{I}} = \frac{Pb}{\sqrt{\pi a}\sqrt{a^2+b^2}}\left[1-\frac{2a^2}{(1+\kappa)(a^2+b^2)}\right], \qquad (13)$$

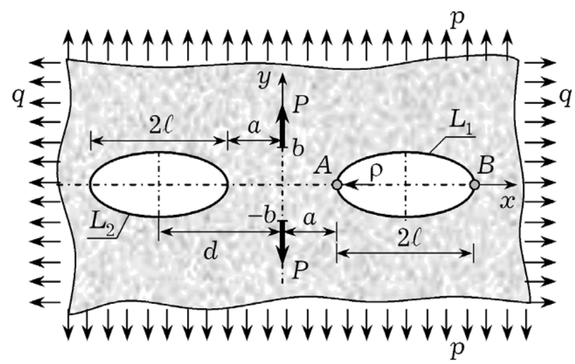


**Fig. 4**

where $\kappa = 3 - 4\nu$ for a plane strain, and $\kappa = (3-\nu)/(1+\nu)$ for a plane stress; and $\nu$ is the Poisson's ratio.

Approaching the parameter $b$ to infinity, we obtain the solution of the problem

$$K_{\mathrm{I}} = \frac{P}{\sqrt{\pi a}},$$

where $P$ is the principal vector of a tensile load given at infinity.

## Two Elliptic Holes

Consider the biaxial tension of the infinite isotropic plate with two identical elliptic holes $L_k$, $k = 1, 2$, whose contours are unloaded, by forces $p$ and $q$. We refer each contour $L_k$ to the local coordinate system $x_k O_k y_k$. In the basic Cartesian coordinate system $xOy$, the points $O_k$ are defined by the complex-valued coordinates $z_1^0 = d$ and $z_2^0 = -d$, and the $O_k x_k$ axes form the angles $\alpha_1 = 0$ and $\alpha_2 = \pi$ with the $Ox$ axis (Fig. 4, $P = 0$).

We solved the first basic problem of the plane elasticity theory for an infinite body with smooth curvilinear holes, whose contours do not intersect each other, by using the method of singular integral equations [12]. The complex-valued potentials of stresses of such a problem take the form

$$\Phi(z) = \Phi_0(z) + \frac{1}{2\pi}\sum_{k=1}^{2}\int_{L_k}\frac{g_k'(t_k)\exp(i\alpha_k)}{T_k - z}dt_k, \qquad T_k = t_k\exp(i\alpha_k) + z_k^0,$$

$$\Psi(z) = \Psi_0(z) + \frac{1}{2\pi i}\sum_{k=1}^{2}\frac{M_k}{(z - z_k^0)^2} + \frac{1}{2\pi}\sum_{k=1}^{2}\int_{L_k}\left[\frac{\overline{g_k'(t_k)}\exp(-i\alpha_k)}{T_k - z}\overline{dt_k} - \frac{\overline{T_k}g_k'(t_k)\exp(i\alpha_k)}{(T_k - z)^2}dt_k\right],$$

where $g_k'(t_k)$ is the required continuous function on the contours of holes; the potentials $\Phi_0(z)$ and $\Psi_0(z)$ that determine the basic stress-strain state in the continuous plane without holes are

$$\Phi_0(z) = p + q/4, \qquad \Psi_0(z) = p - q/2,$$

and

$$M_k = i \int_{L_k} [\overline{T_k}\, g_k'(t_k) \exp(i\alpha_k)\, dt_k - T_k\, \overline{g_k'(t_k)} \exp(-i\alpha_k)\, \overline{dt_k}\,] .$$

The system of two singular integral equations of the problem takes the form [12]

$$\sum_{k=1}^{2} \int_{L_k} \left[ \mathcal{K}_{nk}(t_k, t_n')\, g_k'(t_k)\, dt_k + \mathcal{L}_{nk}(t_k, t_n')\, \overline{t_n'(t_k)}\, \overline{dt_k} \right]$$

$$- \frac{1}{2i} \sum_{k=1}^{2} \frac{M_k \exp(-2i\alpha_n)}{(\overline{T_n'} - \overline{z_k^0})^2} \frac{\overline{dt_n'}}{dt_n'} + a_n \frac{ds_n'}{dt_n'} = \pi p_n(t_n'), \quad n = 1,2 , \tag{14}$$

where $s_n'$ is the arc abscissa of a point $t_n' \in L_n$; $T_n' = t_n' \exp(i\alpha_n) + z_n^0$;

$$\mathcal{K}_{nk}(t_k, t_n') = \frac{\exp(i\alpha_k)}{2} \left[ \frac{1}{T_k - T_n'} + \frac{\exp(-2i\alpha_k)}{\overline{T_k} - \overline{T_n'}} \frac{\overline{dt_n'}}{dt_n'} \right],$$

$$\mathcal{L}_{nk}(t_k, t_n') = \frac{\exp(-i\alpha_k)}{2} \left[ \frac{1}{\overline{T_k} - \overline{T_n'}} - \frac{T_k - T_n'}{(\overline{T_k} - \overline{T_n'})^2} \frac{\overline{dt_n'}}{dt_n'} \exp(-2i\alpha_k) \right],$$

$$p_n(t_n') = -\left\{ \Phi_0(T_n') + \overline{\Phi_0(T_n')} + \frac{\overline{dt_n'}}{dt_n'} [T_n'\, \overline{\Phi_0'(T_n')} + \overline{\Psi_0(T_n')}] \right\} .$$

To the left-hand side of the integral equations (14), we added zero functionals

$$a_n = \int_{L_n} g_n'(t_n)\, dt_n , \qquad n = 1,2 ,$$

which ensure the unconditional solvability and uniqueness of their solution.

Choosing the local coordinate systems to be symmetric by the origin and taking the similar symmetry of the elastic region and the external load into account, we obtain

$$g_1'(t) = g_2'(t) = g'(t), \qquad t \in L_1 = L .$$

The last conditions allow us to reduce the complex potentials of stresses to the form

$$\Phi(z) = \Phi_0(z) + \frac{1}{\pi} \int_L \frac{T g'(t)}{T^2 - z^2} dt , \qquad T = t + d ,$$

$$\Psi(z)=\Psi_0(z)+\frac{1}{\pi}\int_L\left\{\frac{T}{T^2-z^2}\overline{g'(t)dt}-\frac{\bar{T}(T^2+z^2)}{(T^2-z^2)^2}g'(t)dt\right\}$$

$$+\frac{z^2+d^2}{\pi(z^2-d^2)}\left\{\int_L[\bar{t}\,g'(t)dt-t\overline{g'(t)dt}]-id\int_L[g'(t)dt+\overline{g'(t)dt}]\right\}. \qquad (15)$$

In this case, the system of two integral equations (14) is reduced to one singular integral equation

$$\int_L\{\mathcal{K}(t,t')g'(t)dt+\mathcal{L}(t,t')\overline{g'(t)}\overline{dt}\}=\pi p(t'), \qquad t'\in L, \qquad (16)$$

where

$$\mathcal{K}(t,t')=\frac{T}{T^2-T'^2}+\left[\frac{\bar{T}}{\bar{T}^2-\bar{T}'^2}-\frac{\bar{T}(\bar{T}'^2+a^2)}{(\bar{T}'^2-a^2)^2}\right]\frac{\overline{dt'}}{dt'}+\frac{ds'}{dt'}, \qquad T'=t'+d,$$

$$\mathcal{L}(t,t')=\frac{\bar{T}}{\bar{T}^2-\bar{T}'^2}+\left[\frac{2T'\bar{T}\bar{T}'-T(\bar{T}^2+\bar{T}'^2)}{(\bar{T}^2-\bar{T}'^2)^2}+\frac{T(\bar{T}'^2+a^2)}{(\bar{T}'^2-a^2)^2}\right]\frac{\overline{dt'}}{dt'}.$$

Here, $s'$ is the arc abscissa of the point $t'\in L$.

We write the parametric equation of the elliptic contour of a hole in the form

$$t=\ell\omega(\xi)=\ell(\cos\xi+im\sin\xi), \qquad 0\le\xi\le 2\pi, \qquad (17)$$

where $m=c/\ell$; and $\ell$ and $c$ are the major and minor semiaxes of the ellipse.

Making the substitution

$$t=\ell\omega(\xi), \quad 0\le\xi\le 2\pi, \qquad t'=\ell\omega(\eta), \quad 0\le\eta\le 2\pi,$$

we write the integral equation (16) in the canonical form

$$\frac{1}{\pi}\int_0^{2\pi}[M(\xi,\eta)u(\xi)+N(\xi,\eta)\overline{u(\xi)}]d\xi=p(\eta), \qquad 0\le\eta\le 2\pi,$$

where we introduced the notation

$$M(\xi\eta)=\ell\tilde{\mathcal{K}}(\omega(\xi),\omega(\eta)), \qquad N(\xi\eta)=\ell\tilde{\mathcal{L}}(\omega(\xi),\omega(\eta)),$$

$$u(\xi)=g'(\omega(\xi))\omega'(\xi), \qquad p(\eta)=p(\omega(\eta)).$$

The required $2\pi$-periodic continuous function $u(\xi)$ has quasisingularities at the tips of the contour $L$ on the major axis of the ellipse for small values of the parameter $\varepsilon$, which complicates the derivation of sufficiently exact numerical solutions in this case. In recent years, in order to enhance the exactness of solutions of equations of such a type, various nonlinear transformations of the integration variable are used. This causes the accumulation of quadrature nodes and collocation nodes in the vicinity of the contour tip. Such transformations lead, in essence, to a change of the parametric equations of a curvilinear contour that allow one to obtain more precise numerical solutions. We note that the parametric equation of the elliptic contour in the form (17) is not efficient from this viewpoint due to the accumulation of quadrature nodes on sections of the contour with minimum curvature.

In this case, we performed the change of variables [23]

$$\xi = G(\tau) = \tau - \frac{1}{2}\sin 2\tau, \quad 0 \le \tau \le 2\pi, \qquad \eta = G(\theta), \quad 0 \le \theta \le 2\pi .$$

The function $G(\tau)$ uniquely maps the interval $\tau \in [0, 2\pi]$ on the interval $\xi \in [0, 2\pi]$. As a result, we obtain the integral equation

$$\frac{1}{\pi}\int_0^{2\pi}\left[M(\xi,\eta)u^*(\tau) + N(\xi,\eta)\overline{u^*(\tau)}\right]G'(\tau)\,d\tau = p^*(\theta), \quad 0 \le \theta \le 2\pi ,$$

where

$$u^*(\tau) = u(G(\tau)) \,, p^*(\tau) = p(G(\theta)) .$$

A discrete analog of the integral equation is the system of $2n$ linear algebraic equations

$$\frac{1}{\pi}\sum_{k=1}^{2n}\left[M(\xi_k,\eta_m)u^*(\tau_k) + N(\xi_k,\eta_m)\overline{u^*(\tau_k)}\right]G'(\tau_k) = p^*(\theta_m) \,, \qquad m = 1,\ldots,2n \,, \tag{18}$$

where

$$\xi_k = G(\tau_k), \qquad \tau_k = \frac{\pi(2k-1)}{2n}, \qquad k = 1,\ldots,2n \,,$$

$$\eta_m = G(\theta_m), \quad \theta_m = \frac{2\pi(m-1)}{2n}, \qquad m = 1,\ldots,2n \,.$$

The order of system (18) can be decreased twice with regard for the symmetry of the problem relative to the $Ox$ axis.

After the solution of the system of algebraic equations by formulas (18), we can determine the complex potentials of stresses that allow one to find the stress-strain state in the whole elastic region. In the case where the contours of holes are unloaded, the normal stresses on the contour of a hole can be expressed directly through the function $u^*(\tau)$ [16]:

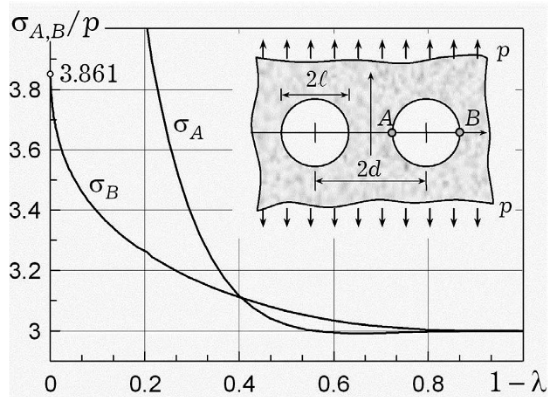


**Fig. 5**

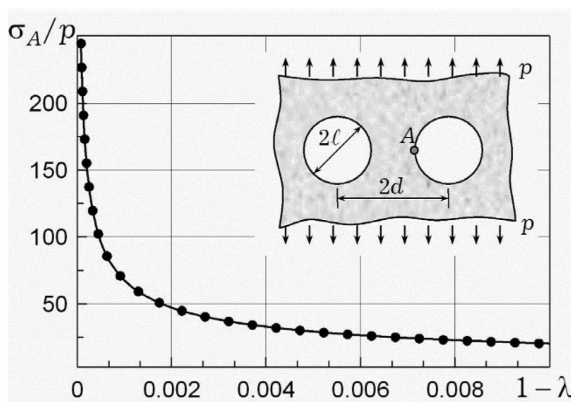


**Fig. 6**

$$\sigma_s(t) = 4\,\mathrm{Re}[\Phi_0(t) + \Phi^+(t)] = 4\,\mathrm{Re}\,\Phi_0(t) - 4\,\mathrm{Im}\, g'(t)$$

$$= 4\,\mathrm{Re}\,\Phi_0(t) - 4\,\mathrm{Im}\left[\frac{u(\xi)}{\omega'(\xi)}\right] = 4\,\mathrm{Re}\,\Phi_0(t) - 4\,\mathrm{Im}\left[\frac{u^*(\tau)}{\omega'(\xi)}\right], \quad t \in L\,.$$

We obtained the numerical results for the stresses $\sigma_s(t)$ at the vertexes of the holes $A$ and $B$ ($\sigma_A$ and $\sigma_B$) as functions of the relative distance between the holes $1-\lambda$ ($\lambda = \ell / d$) for various values of the parameter $m$, i.e., for various ratios of the minor and major semiaxes of the ellipse. In Fig. 5, we give such dependences for $m = 1$ (circular holes) under transverse tension. In the limiting case where the distance between the holes tends to zero, the relative stress $\sigma_B / p = 3.861$, which is in good agreement with both the result of C.-B. Ling [24] ($\sigma_B / p = 3.869$) and solution (8), and the stress $\sigma_A$ tends to infinity.

For small distances between the holes (Fig. 6), the numerical results are well approximated by the curve $\sigma_A / p = -2.008 + 2.199 / \sqrt{1-\lambda}$ (continuous line) constructed by the method of least squares. Then, by extrapolation, we obtain the coefficient of the singularity of maximum stresses on the contour of a circular hole $\sigma_A \sqrt{1-\lambda} / p = 2.199$ which is close to 2.22, which follows from solution (8).

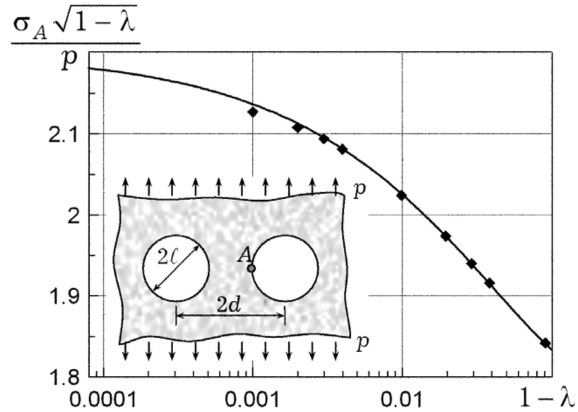


**Fig. 7**

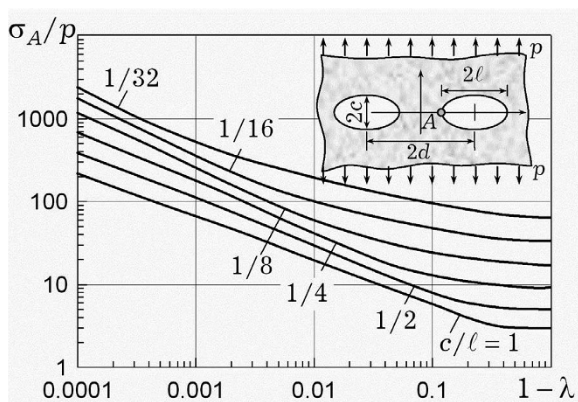


**Fig. 8**

The behavior of maximum stresses on the contours of circular holes on their approach to each other was studied also numerically using of integral equations [21]. The results from [21] (squares) and those obtained by us (continuous curve) are compared in Fig. 7. In [21], the value of the coefficient of singularity of maximum stresses on the contour of a circular hole obtained by extrapolation was somewhat less: $\sigma_A \sqrt{1-\lambda}\, / \, p = 2.13$.

Knowing the coefficient of the singularity of stresses, we can construct the approximating formulas for the stresses $\sigma_A$ which are valid in the whole range of variation of the relative distance between the holes. In the case of uniaxial tension under consideration, we obtained the approximating formula

$$\frac{\sigma_A}{p} = \begin{cases} 3 - \lambda^2 (0.916 - 1.931\lambda), & 0 \le \lambda < 0.7, \\ 2.199 / \sqrt{1-\lambda} - 1.949 + 2.438\sqrt{1-\lambda}, & 0.7 \le \lambda \le 1, \end{cases}$$

whose relative error does not exceed 1% for all values of the parameter $\lambda$, $0 \le \lambda \le 1$.

The contour stresses at the vertex $A$ of the elliptic hole versus the relative distance between the holes are given in Fig. 8 for the uniaxial tension for various ratios of the semiaxes of the ellipse, $c / \ell$. The approximation of numerical data with the help of the method of least squares showed that the elliptic holes have the same singularity of stresses $\sigma_A$ upon approaching each other (the square-root type) as that in the case of circular

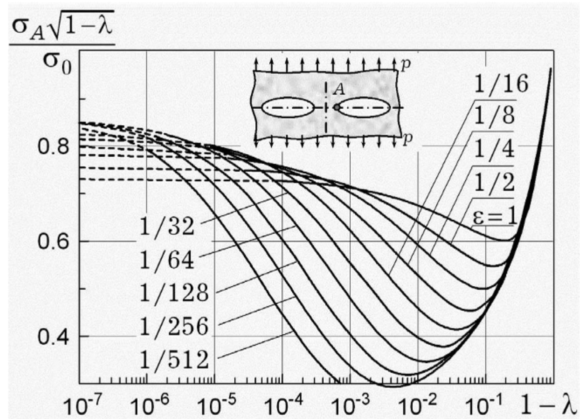


**Fig. 9**

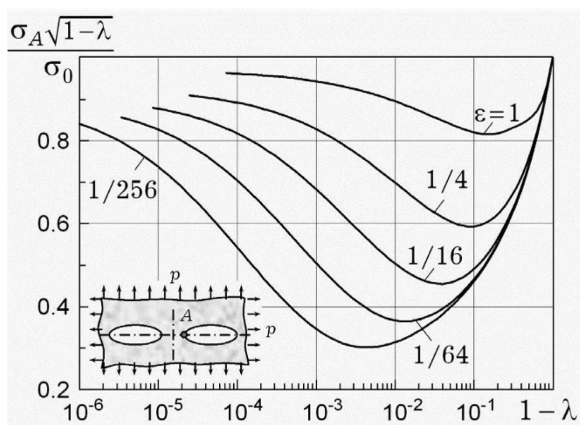


**Fig. 10**

holes. In order to determine the coefficients of this singularity, we present the dependence of the function $\sigma_A\sqrt{1-\lambda}/p$ on the parameter $1-\lambda$ for various values of $\varepsilon=\rho/\ell=c^2/\ell^2$ ($\rho$ is the curvature radius at the vertex $A$) in Fig. 9 (uniaxial tension) and Fig. 10 (uniform tension). These results were obtained numerically (continuous lines) and with the help of the extrapolation by functions of the type $\sigma_A\sqrt{1-\lambda}/p=a_1+a_2\sqrt{1-\lambda}$ for small values of the relative distances between the holes (dashed lines). Here, $\sigma_0=p(1-q/p+2/\sqrt{\varepsilon})$ is the stress at the vertex $A$ for a single elliptic hole ($\lambda=0$).

Numerical values of the coefficients of the singularity of stresses $\sigma_A$ are given in Table 1. We note that, as the ratio of the semiaxes of the ellipse decreases, this coefficient increases. Such an increase can be explained by the change of a type of the singularity from the power to the power-logarithmic one if an elliptic hole is transformed in a crack ($\varepsilon\to0$), which follows from asymptotics (12).

The relative contour stresses $\sigma_B/\sigma_0$ at the vertex $B$ of the elliptic hole versus the relative distance between the holes are shown in Fig. 11 (uniaxial tension) and Fig. 12 (biaxial tension).

Upon approach of the holes to each other ($\lambda\to1$), the stresses $\sigma_B$ tend to a finite value close to the result obtained by the formula for the equivalent ellipse [5], i.e., for an elliptic hole with the same curvature radius $\rho$ at the vertex $A$ and with the twice larger major semiaxis $2\ell$:

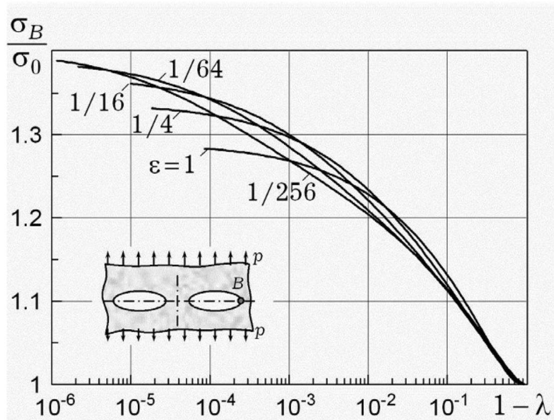


**Fig. 11**

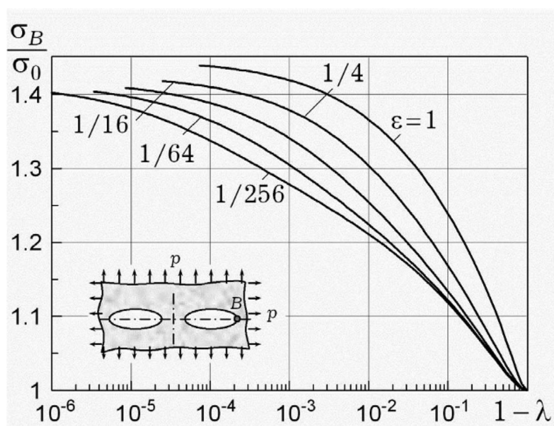


**Fig. 12**

$$\sigma_e = p(1 - q/p + 2\sqrt{2/\varepsilon}).$$

The values calculated by this formula are given in Table 1 to the right from the dashed vertical line.

### Two Semi-Infinite Parabolic Notches

Consider the unbounded elastic plane weakened by two symmetric semi-infinite parabolic notches with the symmetry axes on the $Ox$ axis and the distance $2a$ between the vertexes. By the definition in [20], the boundary-value problem of the plane elasticity theory for such a region is referred to singular problems, whose solutions cannot be obtained, if a constant load is set at infinity. Therefore, we assume that the stresses are absent at infinity, and the plate is stretched by the concentrated forces $P$ applied at the points $(0,b)$ and $(0,-b)$ on the $Oy$ axis. We get the solution of this problem by passage to the limit starting from the corresponding problem of two elliptic holes (Fig. 4, $p = q = 0$), where the distance between the near vertexes is constant and is equal to $2a$, and the major axis of the ellipse $\ell$ increases infinitely. In this case, the problem of two elliptic holes should be solved for an external load that is defined by the complex-valued potentials

**Table 1**

| $c/\ell$ | $\sqrt{1-\lambda}\,\sigma_A/p$ | | $\sigma_B/p$ | | | |
|---|---|---|---|---|---|---|
| | $q=0$ | $q=p$ | $q=0$ | | $q=p$ | |
| 1 | 2.20 | 1.94 | 3.86 | 3.83 | 2.89 | 2.83 |
| 1/2 | 3.93 | 3.70 | 6.66 | 6.66 | 5.70 | 5.66 |
| 1/4 | 7.44 | 7.23 | 12.26 | 12.32 | 11.34 | 11.32 |
| 1/8 | 14.5 | 14.3 | 23.53 | 23.63 | 22.6 | 22.63 |
| 1/16 | 28.6 | 28.3 | 45.87 | 46.25 | 45.2 | 45.25 |

**Table 2**

| $b/a$ | Approximating formulas |
|---|---|
| 1 | $\sigma_A/\sigma_n = 2.216 - 0.493(\rho/\ell)^{0.203}$ |
| 2 | $\sigma_A/\sigma_n = 2.021 - 0.897(\rho/\ell)^{0.212}$ |
| 5/2 | $\sigma_A/\sigma_n = 1.982 - 1.063(\rho/\ell)^{0.217}$ |
| 10/3 | $\sigma_A/\sigma_n = 1.935 - 1.307(\rho/\ell)^{0.237}$ |
| 5 | $\sigma_A/\sigma_n = 1.899 - 1.671(\rho/\ell)^{0.254}$ |

$$\Phi_0(z) = \frac{bP}{\pi(1+\kappa)}\frac{1}{z^2+b^2},$$

$$\Psi_0(z) = \frac{\kappa bP}{\pi(1+\kappa)}\frac{1}{z^2+b^2} + \frac{bP}{\pi(1+\kappa)}\frac{z^2-b^2}{(z^2+b^2)^2}.$$

If the relative curvature radius $\rho/a$ at the notch tip is small, then the approximate value of maximum stresses at this vertex can be determined by the formula [13]

$$\sigma_{\max} = 2K_{\mathrm{I}} / \sqrt{\pi\rho} ,$$

where $K_{\mathrm{I}}$ is the stress intensity factor for two semi-infinite cracks (13). In the case where $\rho / a = 1$, we present the approximating formulas for the relative stresses $\sigma_A / \sigma_n$ ($\sigma_n = P / 2a$) at the vertex $A$ of an elliptic hole that were obtained for small values of the parameter $\rho / \ell$ ($0 \le \rho / \ell \le 10^{-5}$) numerically ($\nu = 0.3$, a plane stressed state) in Table 2. The first terms in these formulas are the approximate values of the stress intensity factors at the vertex of a parabolic notch. On the basis of these data for parabolic notches, we constructed the approximating dependence of the relative stresses $\sigma_A / \sigma_n$:

$$\frac{\sigma_A}{\sigma_n} = \frac{1.87 + 2.89(a/b)^2}{\left[1 + (a/b)^2\right]^{1.1}}, \quad 0.1 \le \frac{a}{b} \le 1.$$

By extrapolating this dependence for $a/b = 0$, we obtain the value $\sigma_A / \sigma_n = 1.87$ in the case where the tensile load is set at infinity, whose principal vector is equal to $P$. The corresponding value of the stress concentration factor for two hyperbolic notches [3] is $\sigma_A / \sigma_n = 1.56$.

We note that the above-presented procedure for determining the solution of the problem for parabolic notches is not efficient because it involves two passages to the limit in the numerical solution. Here, we demonstrated only the possibility of construction of such solutions. For this purpose, it is much more efficient to use, for example, the problem on a periodic system of holes in the elastic plane, which does not require the double passages to the limit. Just in such a way with the use of the common approach to the problem concerning the concentration of stresses near angular acute and rounded notches, the stress intensity factor and the stress concentration factor for two such notches are obtained in [14, 15].

The authors thank Dr. A. Kazberuk for his help in the preparation of this article.

## REFERENCES


1. N. F. Morozov, *Mathematical Problems of the Theory of Cracks* [in Russian], Nauka, Moscow (1984).
2. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Nauka, Moscow (1966); ***English translation:*** Springer (1977).
3. H. Neuber, *Kerbspanungslehre*, Springer, Berlin (1937).
4. V. V. Panasyuk, M. P. Savruk, and A. P. Datsyshin, *Distribution of Stresses near Cracks in Plates and Shells* [in Russian], Naukova Dumka, Kiev (1976).
5. R. E. Peterson, *Stress Concentration Factors*, Wiley, New York (1974).
6. Ya. S. Pidstryhach, "Stresses in a plane weakened by two unequal circular holes," *Dop. Akad. Nauk URSR,* No. 6, 456–460 (1953).
7. Ya. S. Pidstryhach, *Selected Works* [in Ukrainian], Naukova Dumka, Kiev (1995).
8. Ya. S. Pidstryhach, *Stresses near Two Unequal Circular Holes in the Elastic Plane* [in Russian], Author's Abstract of the Candidate Degree Thesis (Phys.-Mat. Sci.), Lviv (1954).
9. Ya. S. Pidstryhach, "Stresses near two unequal holes in the plane field," *Vopr. Mash. Prochn. Mashinostr.,* **4**, Issue 3, 60–71 (1954).
10. G. N. Savin, *Concentration of Stresses near Holes* [in Russian], Gostekhizdat, Moscow–Leningrad (1951).
11. G. N. Savin, *Distribution of Stresses near Holes* [in Russian], Naukova Dumka, Kiev (1968).
12. M. P. Savruk, *Two-Dimensional Problems of Elasticity for Cracked Bodies* [in Russian], Naukova Dumka, Kiev (1981).
13. M. P. Savruk, *Stress Intensity Factors in Cracked Bodies*, in: V. V. Panasyuk (editor), *Fracture Mechanics and Strength of Materials* [in Russian], Vol. 2, Naukova Dumka, Kiev (1988).
14. M. P. Savruk and A. Kazberuk, "A single approach to solution of problems on the stress concentration near acute and rounded angular notches," *Prikl. Mekh.,* **43**, No. 2, 70–87 (2007).

15. M. P. Savruk and A. Kazberuk, "Connection between the stress intensity factors and the stress concentration factors for acute and rounded angular notches," *Fiz.-Khim. Mekh. Mater.,* **42**, No. 6, 17–26 (2006).
16. M. P. Savruk, P. N. Osiv, and I. V. Prokopchuk, *Numerical Analysis in Plane Problems of the Theory of Cracks* [in Russian], Naukova Dumka, Kiev (1989).
17. Yu. A. Ustinov, "Concentration of stresses in a half-plane and a plane with circular holes on tension," *Izv. Akad. Nauk SSSR. Mekh.,* No. 1, 145–148 (1965).
18. Yu. A. Ustinov, "Solution of some problems for doubly connected domains with approached circular boundaries," in: *Stress Concentration* [in Russian], Naukova Dumka, Kiev (1965), pp. 270–275.
19. Ya. S. Uflyand, *Bipolar Coordinates in Elasticity Theory* [in Russian], Gostekhizdat, Moscow (1950).
20. G. P. Cherepanov, *Mechanics of Brittle Fracture* [in Russian], Nauka, Moscow (1974).
21. Z. P. Duan, R. Kienzler, and G. Herrmann, "The integral equation method and its application to defect mechanics," *J. Mech. Phys. Solids,* **34**, No. 6, 539–561 (1986).
22. R. A. Haddon, "Stresses in an infinite plate with two unequal circular holes," *Quart. J. Mech. Appl. Math.,* **20**, 277–291 (1967).
23. P. R. Johnston, "Application of sigmoidal transformations to weakly singular and near-singular boundary element integrals," *Int. J. Numer. Meth. Eng.,* **45**, 1333–1348 (1999).
24. C.-B. Ling, "On the stresses in a plate containing two circular holes," *J. Appl. Phys.,* **19**, No. 1, 77–82 (1948).
25. C.-B. Ling, "The stresses in a plate containing an overlapped circular hole," *J. Appl. Phys.,* **19**, No. 4, 405–411 (1948).