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AN INTERACTIVE METHOD OF TACKLING UNCERTAINTY IN INTERVAL MULTIPLE OBJECTIVE LINEAR PROGRAMMING

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Abstract. Mathematical programming models for decision support must explicitly take account of the treatment of the uncertainty associated with the model coefficients along with multiple and conflicting objective functions. Interval programming just assumes that information about the variation range of some (or all) of the coefficients is available. In this paper, we propose an interactive approach for multiple objective linear programming problems with interval coefficients that deals with the uncertainty in all the coefficients of the model. The presented procedures provide a global view of the solutions in the best and worst case coefficient scenarios and allow performing the search for new solutions according to the achievement rates of the objective functions regarding both the upper and lower bounds. The main goal is to find solutions associated with the interval objective function values that are closer to their corresponding interval ideal solutions. It is also possible to find solutions with non-dominance relations regarding the achievement rates of the upper and lower bounds of the objective functions considering interval coefficients in the whole model.

1. Introduction

Interval programming methods have been used to tackle specific issues in multiple objective linear programming (MOLP): some deal with uncertainty in the objective functions, others handle uncertainty both in the objective functions and in the right-hand side (RHS) of the constraints, and others deal with uncertainty in all the coefficients of the model (see an illustrated overview of these methods in [8]). In the framework of an interactive approach Urli and Nadeau [11] have considered MOLP models with interval coefficients in the whole model. This approach uses a simple mathematical formulation and allows a strong integration of the decision-maker (DM) into the decision phases. The following assumptions are considered for the problem transformation: the DM is less satisfied when the lower bound of the objective function is closer to the lower bound of the target interval, and more satisfied when the upper bound of the objective function is closer to the upper bound of the target interval; the DM hopes that the lower bound of the (left and side) LHS of the constraints will not be larger than the upper bound of the RHS of the constraints and his/her satisfaction level will be even higher as much as the upper bound of the LHS will be closer to the upper bound of the RHS. This latter problem is then solved by an interactive approach derived from STEM [1]. Nevertheless, although the DM prefers the most favorable situation, the results obtained through the algorithm do not allow the DM to take into account the worst case and the best case "scenarios" in order to perceive the risk at stake (see [4]). In the next sections of this paper, we propose a new approach to addressing MOLP problems with interval coefficients that deals with the uncertainty in all the coefficients of the model. The methodology developed is aimed at making the most of distinct methodological approaches in order to provide effective decision support to DMs, paying also attention to the minimization of the computational efforts. It starts with obtaining two surrogate deterministic problems by considering the minimization of the worst possible deviation of the interval objective functions from their corresponding interval ideal solutions. The first compromise solution is obtained by solving the surrogate deterministic problem, which is chosen according to a more or less conservative stance of the DM with the narrowest version of the feasible region. Other solutions are computed in the framework of the method's interactive phases. During these interactive phases, additional information

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is shown to the DM: the interval objective function values obtained in each solution and the corresponding midpoint and width, which can be compared with the values obtained for the corresponding interval ideal solutions; the distance of the interval objective function values obtained in each solution to the corresponding interval ideal solutions and the comparison between the interval objective function values and the corresponding interval ideal solutions through the use of an acceptability index that allows comparing any interval numbers (since the main aim is to find the solutions that allow obtaining the interval objective function values that are closer to their corresponding interval ideal solutions when both the distance and the acceptability index are close to zero, the interval objective functions are close to the corresponding interval ideal solutions); the achievement rates of the bounds of the objective function values in each solution regarding the bounds of the corresponding interval ideal solutions (the closer these values are to 1, the closer the DM is to meet his/her aspiration levels). After providing this information to the DM, he/she is asked to reveal his/her satisfaction regarding the solution being analyzed. If the DM is not yet satisfied with the obtained solution, then the algorithm proceeds. Then the DM is asked to choose the objective function he/she wishes to improve. If the problem obtained with the additional constraints has an empty feasible region, then information is provided on the amount he/she should relax the different objective reference values in order to restore feasibility. In this phase the DM can also choose the objective function(s) he/she is willing to relax in order to improve the other objective function(s) and solve the problem with the additional constraints. If the DM wants to have a sensitivity measure of the efficient basis obtained for simultaneous and independent changes of the reference values considered for the objective functions, then the ranges of variation of these reference values must be computed according to the individual tolerance range approach [13]. On the other hand, if the DM wishes to have a sensitivity measure of the efficient basis obtained when changes occur in only one reference value for one objective function, then the range of variation of this reference value must be computed according to sensitivity analysis techniques. In each case, the DM might choose new reference values within the ranges of computed variation or outside these ranges, knowing that in the last option the efficient basis will be changed. The main advantage of these procedures is that it is no longer necessary to solve the entire problem all over again in order to obtain a new solution. The impacts of different thresholds on the constraints on the compromise solution can also be shown, allowing the DM to analyze distinct solutions with different coefficient sets. The exhaustiveness of the solution search process depends on the DM, who can decide to end the procedure when he/she considers to have gathered enough information about the problem.

The methodology herein presented is applied to a small illustrative example, some conclusions are made, and some flexibility proposals are suggested.

2. Obtaining the Surrogate Deterministic Problems

Let the MOLP problem with interval coefficients be given, without loss of generality, by

$$\begin{gathered}\max Z_k(\mathbf{x}) = \sum_{j=1}^{n} \left[c_{kj}^L, \quad c_{kj}^U\right] x_j, \quad k = 1, \ldots, p, \\ s.t. \sum_{j=1}^{n} \left[a_{ij}^L, \quad a_{ij}^U\right] x_j \leq \left[b_i^L, \quad b_i^U\right], \quad i = 1, \ldots, m, \\ x_j \geq 0, \quad j = 1, \ldots, n, \end{gathered} \tag{1}$$

where $\left[c_{kj}^L, \quad c_{kj}^U \right]$, $\left[a_{ij}^L, \quad a_{ij}^U \right]$, and $\left[b_i^L, \quad b_i^U \right]$, $k = 1, \ldots, p$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, are closed intervals.

Two surrogate problems are initially obtained by considering that the DM wants to minimize the worst possible deviation [6] of each interval objective function from an established interval goal and considering satisfaction thresholds of the constraints [11].

The possible deviation $D_k(\mathbf{x}) = [d_k^L(\mathbf{x}),\, d_k^U(\mathbf{x})]$ of $Z_k(\mathbf{x}) = \left[\sum_{j=1}^{n} c_{kj}^L x_j = Z_k^L(\mathbf{x}),\, \sum_{j=1}^{n} c_{kj}^U x_j = Z_k^U(\mathbf{x})\right]$ from an interval target $T_k = [t_k^L,\, t_k^U]$ is

$$D_k(\mathbf{x}) = |T_k(-)Z_k(\mathbf{x})| = \left|\left[t_k^L - \sum_{j=1}^{n} c_{kj}^U x_j, t_k^U - \sum_{j=1}^{n} c_{kj}^L x_j\right]\right|$$

$$= \begin{cases} (a) \quad \left[t_k^L - \sum_{j=1}^{n} c_{kj}^U x_j, t_k^U - \sum_{j=1}^{n} c_{kj}^L x_j\right] & \text{if } \ t_k^L - \sum_{j=1}^{n} c_{kj}^U x_j \geq 0, \\ (b) \quad \left[0, \sum_{j=1}^{n} c_{kj}^U x_j - t_k^L \vee t_k^U - \sum_{j=1}^{n} c_{kj}^L x_j\right] & \text{if } \ t_k^L - \sum_{j=1}^{n} c_{kj}^U x_j < 0 < t_k^U - \sum_{j=1}^{n} c_{kj}^L x_j, \\ (c) \quad \left[\sum_{j=1}^{n} c_{kj}^L x_j - t_k^U, \sum_{j=1}^{n} c_{kj}^U x_j - t_k^L\right] & \text{if } \ t_k^U - \sum_{j=1}^{n} c_{kj}^L x_j \leq 0. \end{cases} \tag{2}$$

In order to transform the interval constraints of (1) into deterministic constraints, we have used the approach shown in [11], which is based on the DM's degree of satisfaction relative to a nondeterministic constraint. The degree of satisfaction, μ, of an interval constraint of (1) is given by

$$\mu\left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right) = \begin{cases} 0 & \text{if } \ \sum_{j=1}^{n} a_{ij}^L x_j \geq b_i^U, \\ 1 & \text{if } \ \sum_{j=1}^{n} a_{ij}^U x_j \leq b_i{}^L, \\ \dfrac{b_i^U - \sum_{j=1}^{n} a_{ij}^L x_j}{(b_i^U - b_i^L) + \sum_{j=1}^{n} (a_{ij}^U - a_{ij}^L) x_j} & \text{otherwise,} \end{cases} \tag{3}$$

where $a_{ij} \in [\ a_{ij}^L,\ \ a_{ij}^U\ ]$ and $b_i \in [\ b_i^L,\ \ b_i^U\ ]$.

The solutions to problem (1) should individually satisfy each nondeterministic constraint with some satisfaction threshold, called individual satisfaction threshold on constraints and denoted by α_i, $i = 1, \ldots, m$. Therefore, each interval constraint has the following surrogate deterministic form for each $\alpha_i \in [0, 1]$, $i = 1, \ldots, m$:

$$\mu\left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right) \geq \alpha_i. \tag{4}$$

From relation (3), this constraint takes the form

$$\sum_{j=1}^{n} (a_{ij}^L + \alpha_i (a_{ij}^U - a_{ij}^L)) x_j \leq b_i^U - \alpha_i (b_i^U - b_i^L). \tag{5}$$

Reference points in general represent the aspiration levels of the DM for the objective functions. The ideal solution is frequently used as a reference point in MOLP problems since it represents the best value of each objective function in the feasible region. In this context, the individual optima obtained with the best and worst case coefficients scenario are respectively considered to be the bounds of the interval ideal solutions.

For each objective function $Z_k(\mathbf{x})$, $k = 1, \dots, p$, we solve the following LP problems which allow obtaining the best and worst optima, respectively [4]:

$$\begin{aligned} &\max Z_k^U(\mathbf{x}) \\ &s.t. \sum_{j=1}^{n} a_{ij}^L x_j \le b_i^U, \quad i = 1, \dots, m, \\ &x_j \ge 0, \quad j = 1, \dots, n, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\max Z_k^L(\mathbf{x}), \\ &s.t. \sum_{j=1}^{n} a_{ij}^U x_j \le b_i^L, \quad i = 1, \dots, m, \\ &x_j \ge 0, \quad j = 1, \dots, n. \end{aligned} \tag{7}$$

Problems (6) and (7) are denoted by $\beta = 0$ and $\beta = 1$, respectively, and the optimal solution of each problem is given by $\mathbf{x}_k^\beta$, $\beta = 0, 1$, $k = 1, \dots, p$. The interval goals are chosen by considering

$$t_k^U = Z_k^U(\mathbf{x}_k^0) = Z_k^{U^*}, \quad k = 1, \dots, p, \tag{8}$$

$$t_k^L = Z_k^L(\mathbf{x}_k^1) = Z_k^{L^*}, \quad k = 1, \dots, p. \tag{9}$$

The possible deviation of each interval objective function from the corresponding interval ideal solution is

$$\begin{aligned} D_k(\mathbf{x}) &= |[Z_k^{L^*} - Z_k^U(\mathbf{x}), Z_k^{U^*} - Z_k^L(\mathbf{x})]| \\ &= \begin{cases} (a) \quad [Z_k^{L^*} - Z_k^U(\mathbf{x}), Z_k^{U^*} - Z_k^L(\mathbf{x})] & \text{if} \quad Z_k^{L^*} - Z_k^U(\mathbf{x}) \ge 0, \\ (b) \quad [0, (Z_k^U(\mathbf{x}) - Z_k^{L^*}) \vee (Z_k^{U^*} - Z_k^L(\mathbf{x}))] & \text{if} \quad Z_k^{L^*} - Z_k^U(\mathbf{x}) < 0 < Z_k^{U^*} - Z_k^L(\mathbf{x}). \end{cases} \end{aligned} \tag{10}$$

In this case, line (c) of expression (2) can never occur because $Z_k^{U^*} - Z_k^L(\mathbf{x}) \ge 0$ since $Z_k^{U^*}$ is the best possible optimum that objective function k can attain.

Let $\varepsilon_k = Z_k^{L^*} - Z_k^U(\mathbf{x})$ be such that $\varepsilon_k = \varepsilon_k^+ - \varepsilon_k^-$, $\varepsilon_k^+ \ge 0, \varepsilon_k^- \ge 0$ and $\varepsilon_k^+ \varepsilon_k^- = 0$; then

$$D_k(\mathbf{x}) = \begin{cases} (a) \quad [\varepsilon_k^+ - \varepsilon_k^-, Z_k^{U^*} - Z_k^L(\mathbf{x})] & \text{if} \quad \varepsilon_k^+ - \varepsilon_k^- \ge 0, \\ (b) \quad [0, (\varepsilon_k^- - \varepsilon_k^+) \vee (Z_k^{U^*} - Z_k^L(\mathbf{x}))] & \text{if} \quad \varepsilon_k^+ - \varepsilon_k^- < 0 < Z_k^{U^*} - Z_k^L(\mathbf{x}). \end{cases} \tag{11}$$

Hence the following situations can occur:

(i) if $\varepsilon_k^+ = 0$, then $\varepsilon_k^- \ge 0$ and $D_k(\mathbf{x}) = [0, \varepsilon_k^- \vee (Z_k^{U^*} - Z_k^L(\mathbf{x}))]$;

(ii) if $\varepsilon_k^- = 0$, then $\varepsilon_k^+ \ge 0$ and $D_k(\mathbf{x}) = [\varepsilon_k^+, (Z_k^{U^*} - Z_k^L(\mathbf{x}))]$.

Therefore, $D_k(\mathbf{x}) = [\varepsilon_k^+, \varepsilon_k^- \vee (Z_k^{U^*} - Z_k^L(\mathbf{x}))]$, $k = 1, \dots, p$.

If the DM wants to minimize the worst possible deviation of each objective function from the corresponding interval target, then problem (1) has the following surrogate problem for a given α_i, $i = 1, \dots, m$:

$$\begin{aligned} &\min \max_{k=1,\dots,p} \lambda_k D_k(\mathbf{x}), \\ &s.t. \sum_{j=1}^{n} (a_{ij}^L + \alpha_i(a_{ij}^U - a_{ij}^L)) x_j \le b_i^U - \alpha_i(b_i^U - b_i^L), \quad i = 1, \dots, m, \\ &\varepsilon_k^+ - \varepsilon_k^- = Z_k^{L^*} - Z_k^U(\mathbf{x}), \quad k = 1, \dots, p, \\ &\varepsilon_k^+ \varepsilon_k^- = 0, \quad \varepsilon_k^+, \varepsilon_k^- \ge 0, \quad k = 1, \dots, p, \\ &x_j \ge 0, \quad j = 1, \dots, n, \end{aligned} \tag{12}$$

where λ_k are scaling factors taking account of the different orders of magnitude of the objective functions.

Let $\omega_k \in \{0,1\}$, $k=1,\ldots,p$, $s_k^+ = \omega_k \varepsilon_k^+$, $s_k^- = (1-\omega_k)\varepsilon_k^-$; then (12) can be rewritten as

$$
\begin{aligned}
&\min \max_{k=1,\ldots,p} \lambda_k D_k(\mathbf{x}),\\
&s.t. \sum_{j=1}^{n}(a_{ij}^L + \alpha_i(a_{ij}^U - a_{ij}^L))x_j \le {b_i}^U - \alpha_i(b_i^U - b_i^L), \quad i=1,\ldots,m,\\
&s_k^+ - s_k^- = Z_k^{L^*} - Z_k^U(\mathbf{x}), \quad k=1,\ldots,p,\\
&s_k^+ - M\omega_k \le 0, \quad k=1,\ldots,p,\\
&s_k^- + M\omega_k, \le M, \quad k=1,\ldots,p,\\
&s_k^+, s_k^- \ge 0, \quad k=1,\ldots,p,\\
&\omega_k \in \{0,1\}, \quad k=1,\ldots,p,\\
&x_j \ge 0, \quad j=1,\ldots,n,
\end{aligned}
\tag{13}
$$

where M is an arbitrary large number.

If the DM wants to minimize the lower bound of the worst possible deviation, then problem (1) has the following surrogate mixed integer linear programming (MILP) problem:

$$
\begin{aligned}
&\min d^L(\mathbf{x}) = d^L + \gamma \sum_{k=1}^{p}(s_k^+),\\
&s.t.\, \lambda_k s_k^+ \le d^L, \quad k=1,\ldots,p,\\
&\sum_{j=1}^{n}(a_{ij}^L + \alpha_i(a_{ij}^U - a_{ij}^L))x_j \le {b_i}^U - \alpha_i(b_i^U - b_i^L), \quad i=1,\ldots,m,\\
&s_k^+ - s_k^- = Z_k^{L^*} - Z_k^U(\mathbf{x}), \quad k=1,\ldots,p,\\
&s_k^+ - M\omega_k \le 0, \quad k=1,\ldots,p,\\
&s_k^- + M\omega_k \le M, \quad k=1,\ldots,p,\\
&s_k^+, s_k^- \ge 0, \quad k=1,\ldots,p,\\
&d^L \ge 0,\\
&\omega_k \in \{0,1\}, \quad k=1,\ldots,p,\\
&x_j \ge 0, \quad j=1,\ldots,n,
\end{aligned}
\tag{14}
$$

where γ is a very small number and M is an arbitrary large number.

However, if $Z_k^{L^*} - Z_k^U(\mathbf{x}) \le 0$, variable d^L of problem (14) becomes sign unrestricted.

If the DM wants to minimize the upper bound of the worst possible deviation, then problem (1) has the following surrogate MILP problem:

$$
\begin{aligned}
\min d^U(\mathbf{x}) = d^U + \gamma \sum_{k=1}^{p} [s_k^- + Z_k^{U^*} - Z_k^L(\mathbf{x})], \\
s.t. \lambda_k s_k^- \le d^U, \quad k = 1, \ldots, p, \\
\lambda_k (Z_k^{U^*} - Z_k^L(\mathbf{x})) \le d^U, \quad k = 1, \ldots, p, \\
\sum_{j=1}^{n} (a_{ij}^L + \alpha_i (a_{ij}^U - a_{ij}^L)) x_j \le b_i{}^U - \alpha_i (b_i^U - b_i^L), \quad i = 1, \ldots, m, \\
s_k^+ - s_k^- = Z_k^{L^*} - Z_k^U(\mathbf{x}), \quad k = 1, \ldots, p, \\
s_k^+ - M\omega_k \le 0, \quad k = 1, \ldots, p, \\
s_k^- + M\omega_k \le M, \quad k = 1, \ldots, p, \\
s_k^+, s_k^- \ge 0, \quad k = 1, \ldots, p, \\
d^U \ge 0, \\
\omega_k \in \{0, 1\}, \quad k = 1, \ldots, p, \\
x_j \ge 0, \quad j = 1, \ldots, n,
\end{aligned}
\tag{15}
$$

where γ is a very small number and M is an arbitrary large number.

3. Interactive Steps

The first compromise solution is obtained by considering the minimum value range inequalities [9] ($\alpha_i = 1$ for all $i = 1, \ldots, m$) of problems (14) or (15) depending on the optimistic or pessimistic perspective of the DM, respectively. Let the first compromise solutions to problems (14) and (15) be given by $x^{1U'}$ and $x^{1U''}$, respectively. If the first compromise solution satisfies the DM, then the algorithm stops and one of the solutions, $x^{1U'}$ or $x^{1U''}$, is chosen; otherwise, the algorithm proceeds. The other compromise solutions are given by $x^m = x^{mU'}$ and/or $x^{mU''}$, $m = 2, 3, 4, \ldots$. The interactive phases are described below.

For each obtained compromise solution, the following information is presented to the DM:

(1) $Z_k(\mathbf{x}^m)$,

$$m[Z_k(\mathbf{x}^m)] = \frac{Z_k^L(\mathbf{x}^m) + Z_k^U(\mathbf{x}^m)}{2}$$

(the midpoint of the interval), and

$$w[Z_k(\mathbf{x}^m)] = Z_k^U(\mathbf{x}^m) - Z_k^L(\mathbf{x}^m)$$

(the width of the interval), which can be compared with the corresponding values obtained for their interval ideal solutions.

(2) The distance from Z_k^* to $Z_k(\mathbf{x}^m)$,

$$d(Z_k^*, Z_k(\mathbf{x}^m)) = \mathrm{Max}(|Z_k^{L^*} - Z_k^L(\mathbf{x}^m)|, |Z_k^{U^*} - Z_k^U(\mathbf{x}^m)|),$$

$k = 1, \ldots, p$, and the acceptability of $Z_k(\mathbf{x}^m)$ being inferior to $Z_k^* = [Z_k^{L*}, Z_k^{U*}]$,

$$A(Z_k(\mathbf{x}^m) \prec Z^{k*}) = \frac{\left(m[Z_k^*] - m[Z_k(\mathbf{x}^m)]\right)}{\left(\dfrac{w[Z_k(\mathbf{x}^m)]}{2} + \dfrac{w[Z_k^*]}{2}\right)}$$

(see [10]). When both the distance and the acceptability index are close to zero, the interval objective functions are close to their corresponding interval ideal solutions. That is, when $A(Z_k(\mathbf{x}^m) \prec Z^{k*})$ and $d(Z_k^*, Z_k(\mathbf{x}^m))$ are close to zero, this means that $Z_k(\mathbf{x}^m)$ is close to Z^{k*}.

(3) The achievement rate of $Z_k(\mathbf{x}^m)$ with respect to $Z_k(\mathbf{x}^m)$. The achievement rate is

$$tc_k^L = 1 - \frac{(Z_k^{L*} - Z_k^L(\mathbf{x}^m))}{(Z_k^{L*} - m_k^L)}$$

with respect to $Z_k^L(\mathbf{x}^m)$, and it is

$$tc_k^U = 1 - \frac{(Z_k^{U*} - Z_k^U(\mathbf{x}^m))}{(Z_k^{U*} - m_k^U)}$$

with respect to $Z_k^U(\mathbf{x}^m)$, where m_k^L and m_k^U are the worst optimum values obtained from the expanded pay-off table. The extended pay-off table is the pay-off table with the optimum values of problems (6) and (7). The closer the values of tc_k^L and tc_k^U are to 1, the closer the DM is to meet his/her aspiration level Z^{k*}. In general, $0 < tc_k^U < 1$; however, tc_k^L can be greater than 1, precisely, when α_i is relaxed. A value greater than 1 corresponds to a deviation from the goal considered and not to an improvement of the achievement solution rate.

(4) The impact of different values for α_i on the compromise solution if the DM wishes to analyze other solutions with distinct coefficients sets.

After providing this information to the DM, he/she is asked to reveal his/her satisfaction regarding the compromise solution being analyzed. If the DM is satisfied with the solution, then the algorithm stops. Otherwise, the algorithm proceeds with the search for new solutions. For this purpose, the DM may choose the objective function which he/she wishes to improve, and, if possible, sets the maximum improvement level, Δ_k^{L*} or Δ_k^{U*}. If the DM is not able to specify Δ_k^{L*} or Δ_k^{U*}, the improvement level can be automatically set in the following way:

$$\Delta_k^{L*} = Z_k^{L*} - Z_k^L(\mathbf{x}^m) \quad \text{and} \quad \Delta_k^{U*} = Z_k^{U*} - Z_k^U(\mathbf{x}^m).$$

If the improvement selected by the DM leads to an empty feasible region, then information is provided on the amount he/she must relax the different reference values of the objective functions considering the values of the binary variables equal to the ones obtained in the solution without the introduction of that improvement in order to restore the feasibility of the problem. In this context, the concept of "elastic programming" introduced by Brown and Graves [2] is used to widen the feasible region. This method consists in the addition of extra variables (the "elastic variables," e_k) which allow constraints to be relaxed thus enlarging the feasible region. A linear programming problem is solved basing on "the smallest variable cost model" (see [2, 3] and [7]). According to this model, the resistance to "stretch" implied by the elastic term is supplied by creating a new objective: minimize the sum of the total variable cost of all changes (i.e., of the elastic variables). If this problem leads to a solution where the elastic variables are positive, then an optimal change of the RHS is obtained according to the model used herein.

If the DM wishes to obtain other compromise solutions that generate the same efficient basis, then the extreme bounds for simultaneous and independent changes of the reference values considered for the objective functions are computed according to the individual tolerance range (ITR) approach [13] and sensitivity analysis [5]. If the DM wants to obtain other solutions with the same efficient basis changing more than one reference point simultaneously, he/she should consider the values obtained from the ITR approach. On the other hand, if the DM wants to obtain a solution where only a reference point is changed, then he/she should consider the values obtained from sensitivity analysis.

The ITR approach uses two vectors with upper ($\hat{\mathbf{b}}^+$) and lower ($\hat{\mathbf{b}}^-$) bounds for simultaneous and independent changes of the RHS of the constraints. Consider, without loss of generality, that the constraints that lead to an empty feasible region are of the type "$\geq$."

Let $\hat{b}_k^L = Z_k^L(\mathbf{x}^m) + \Delta_k^{L*} - e_k$ and $\hat{b}_k^U = Z_k^U(\mathbf{x}^m) + \Delta_k^{U*} - e_k$, $k = 1, \ldots, p$, where e_k is an elastic variable. Consider $\hat{b}_k = \hat{b}_k^L$ or $\hat{b}_k^U$. Vector $\hat{\mathbf{b}}^-$ is defined for each k, $k = 1, \ldots, p$, as

$$\hat{b}_k^- = \begin{cases} \hat{b}_k - \underset{i}{\text{Min}}\{(|\hat{b}_k|\rho_i) : B_{ik}^{-1} > 0\}, & \\ -\infty & \text{if } \quad B_{ik}^{-1} \leq 0 \quad \text{for all} \quad i, \\ 0 & \text{if } \quad \hat{b}_k = 0 \quad \text{and} \quad B_{ik}^{-1} > 0 \quad \text{for some} \quad i, \end{cases} \tag{16}$$

where $\rho_i = \dfrac{B_{i.}^{-1}\hat{\mathbf{b}}}{\sum\limits_{j=1}^{m+2p} |B_{ij}^{-1}\hat{b}_j|}$ (see [12]), $\mathbf{B}_{i.}^{-1}$ is the i^{th} row of the inverse matrix of the basis, B_{ij}^{-1} (B_{ik}^{-1}) is the element located on the i^{th} row and on the j^{th} (k^{th}) column of the inverse matrix of the basis, and $\hat{\mathbf{b}}$ is the RHS with the original reference points.

On the other hand, vector $\hat{\mathbf{b}}^+$ is defined for each k, $k = 1, \ldots, p$, as

$$\hat{b}_k^+ = \begin{cases} \hat{b}_k + \underset{i}{\text{Min}}\{(|\hat{b}_k|\rho_i) : B_{ik}^{-1} < 0\}, & \\ +\infty & \text{if } \quad B_{ik}^{-1} \geq 0 \quad \text{for all} \quad i, \\ 0 & \text{if } \quad \hat{b}_k = 0 \quad \text{and} \quad B_{ik}^{-1} < 0 \quad \text{for some} \quad i. \end{cases} \tag{17}$$

The obtained results are shown to the DM. If the DM wants to obtain other solutions with the same efficient basis by changing more than one reference point, then he/she should consider the values belonging to the range of variation computed according to the ITR approach. On the other hand, if the DM wants to obtain a solution where only a reference point is changed, then he/she should consider the values obtained from sensitivity analysis. Hence the new efficient solution can be computed in the following way: $\mathbf{x}_B = B^{-1}[\mathbf{b}]$, where B^{-1} is the inverse matrix of the basis, $\mathbf{x}_B$ is the vector of basic variables, and $[\mathbf{b}]$ is the changed RHS vector (i.e., the RHS vector with the new reference values). This new compromise solution is operated like the previous ones and the procedure stops when the DM considers that the obtained solution is a satisfactory compromise.

The flowchart of the algorithm herein suggested is illustrated in Fig. 1. In the next section, a small example will be given in order to illustrate the proposed approach.

4. Illustrative Example

Let us consider the following MOLP with interval coefficients [11]:

$$\begin{aligned}
&\max Z_1(\mathbf{x}) = [0.8, 1.2]x_1 + [-0.5, 0.2]x_2, \\
&\max Z_2(\mathbf{x}) = [-0.3, 0.2]x_1 + [0.7, 1.2]x_2, \\
&\max Z_3(\mathbf{x}) = [0.8, 1.1]x_1 + [0.9, 1.2]x_2, \\
&s.t. [1.5, 2.8]x_1 + [0.5, 1.2]x_2 \leq [7, 9], \\
&[0.5, 1.5]x_1 + [2, 4]x_2 \leq [13.5, 16], \\
&x_1 \geq 0, \quad x_2 \geq 0.
\end{aligned}$$

In order to obtain the surrogate deterministic problems, it is necessary to determine the interval goals T_k, $k = 1, \ldots, 3$, by solving LP problems (6) and (7).

The first problem allows obtaining the best optimum values with the maximum value range inequalities ($\beta = 0$); the second problem allows obtaining the worst optimum values with the minimum value range inequalities ($\beta = 1$). The obtained optimal solutions are $\mathbf{x}_k^\beta$, $k = 1, 2, 3$ and $\beta = 0, 1$: $\mathbf{x}_1^0 = (6.0000, 0.0000)^T$, $\mathbf{x}_2^0 = (0.0000, 8.0000)^T$, $\mathbf{x}_3^0 = (3.6364, 7.0909)^T$, $\mathbf{x}_1^1 = (2.5000, 0.0000)^T$, $\mathbf{x}_2^1 = (0.000, 3.3750)^T$, $\mathbf{x}_3^1 = (1.2553, 2.9043)^T$. This information is displayed in Table 1, which contains

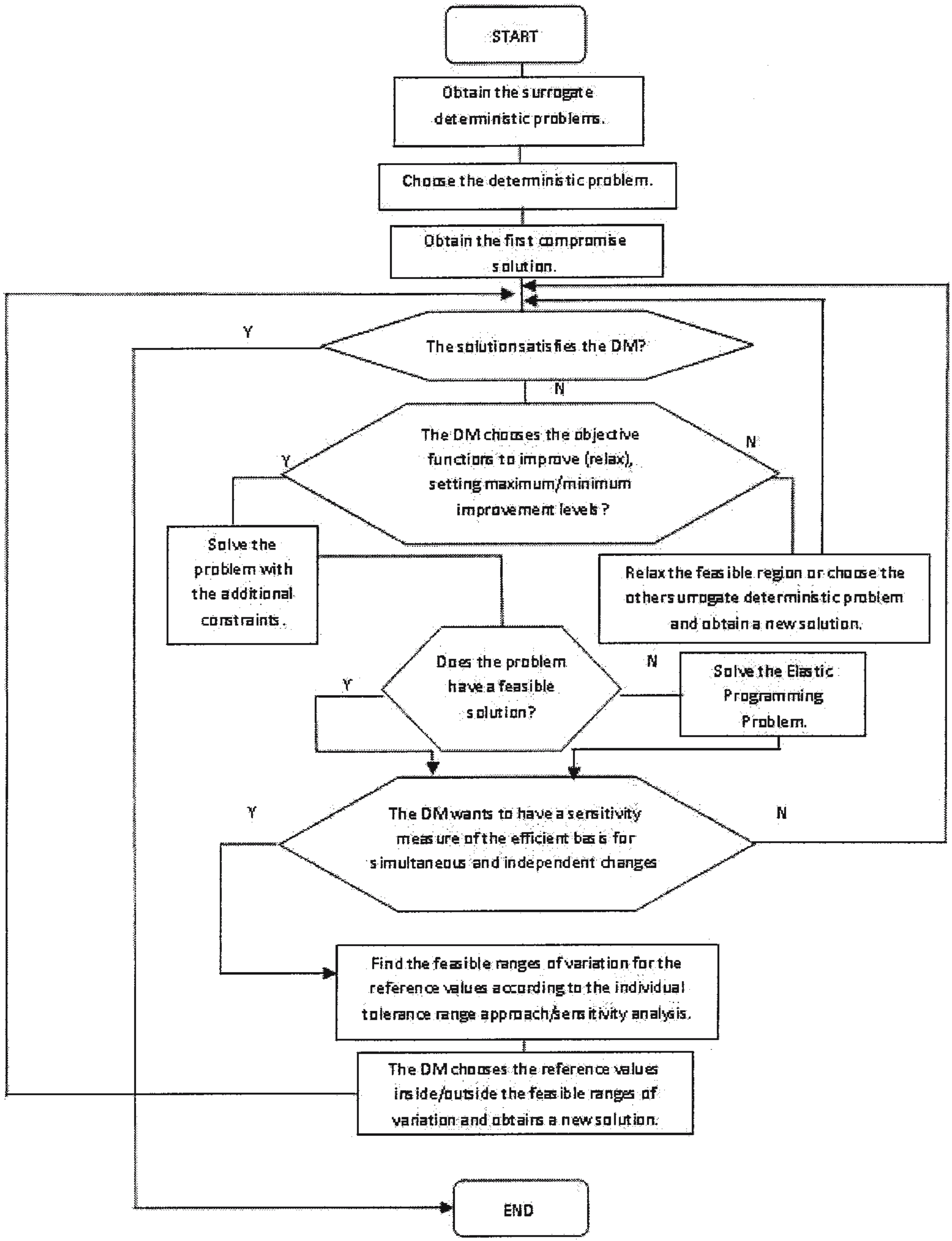

Fig. 1. Flowchart of the algorithm proposed.

the values of $Z_k^U(\mathbf{x}_k^\beta)$ and $Z_k^L(\mathbf{x}_k^\beta)$ for $\beta = 0, 1$ and $k = 1, 2, 3$. From Table 2 we obtain the goals t_k^U and t_k^L.

Table 1. Values of $Z_k^U(\mathbf{x}_k^\beta)$ and $Z_k^L(\mathbf{x}_k^\beta)$.

	$\mathbf{x}_1^0$	$\mathbf{x}_1^1$	$\mathbf{x}_2^0$	$\mathbf{x}_2^1$	$\mathbf{x}_3^0$	$\mathbf{x}_3^1$
$Z_1^U(\mathbf{x}_k^\beta)$	**7.2000**	3.0000	1.6000	0.6750	5.7818	2.0872
$Z_1^L(\mathbf{x}_k^\beta)$	4.8000	**2.0000**	-4.0000	-1.6875	-0.6364	-0.4479
$Z_2^U(\mathbf{x}_k^\beta)$	1.2000	0.5000	**9.6000**	4.0500	9.2364	3.7362
$Z_2^L(\mathbf{x}_k^\beta)$	-1.8000	-0.7500	5.6000	**2.3625**	3.8727	1.6564
$Z_3^U(\mathbf{x}_k^\beta)$	6.6000	2.7500	9.6000	4.0500	**12.5091**	4.8660
$Z_3^L(\mathbf{x}_k^\beta)$	4.8000	2.0000	7.2000	3.0375	9.2909	**3.6181**

Table 2. Values of the interval targets.

Z_k^*	t_k^L	t_k^U	m$[Z_k^*]$	w$[Z_k^*]$
Z_1^*	2.0000	7.2000	4.6000	5.2000
Z_2^*	2.3625	9.6000	5.9813	7.2375
Z_3^*	3.6181	12.5091	8.0636	8.8910

Consider hypothetically that the DM has a pessimistic stance leading to the choice of the surrogate problem (15). In order to obtain the first compromise solution, the narrowest feasible region is considered (i.e., $\alpha_1 = \alpha_2 = 1$). The following solution is obtained: $\mathbf{x}^{1U''} = (1.2553, 2.9043)^T$. The information regarding the first compromise solution is shown to the DM (Table 3),

Table 3. Information regarding solution $\mathbf{x}^{1U''}$.

	$Z_k^L(\mathbf{x}^{1U''})$	$Z_k^U(\mathbf{x}^{1U''})$	$m[Z_k(\mathbf{x}^{1U''})]$	$w[Z_k(\mathbf{x}^{1U''})]$	$A_\prec$	dist	tc_k^L	tc_k^U
Z_1	-0.4479	2.0872	0.8197	2.5351	0.9774	5.1128	0.5920	0.2164
Z_2	1.6564	3.7362	2.6963	2.0798	0.7051	5.8638	0.8304	0.3556
Z_3	3.6181	4.8660	4.2420	1.2479	0.7538	7.6431	1.0000	0.0890

where $A_\prec = A(Z_k(\mathbf{x}^{1U''}) \prec Z_k^*)$ and dist $= d(Z_k^*, Z_k(\mathbf{x}^{1U''}))$.

Let us assume that the DM wishes to change the satisfaction thresholds of the constraints by choosing $\alpha_1 = 0.5$ and $\alpha_2 = 1$. The new obtained solution corresponds to $\mathbf{x}^{2U''} = (2.1796, 2.5576)^T$. The information regarding solution $\mathbf{x}^{2U''}$ is displayed in Table 4,

Table 4. Information regarding solution $\mathbf{x}^{2U''}$.

	$Z_k^L(\mathbf{x}^{2U''})$	$Z_k^U(\mathbf{x}^{2U''})$	$m[Z_k(\mathbf{x}^{2U''})]$	$w[Z_k(\mathbf{x}^{2U''})]$	$A_\prec$	dist	tc_k^L	tc_k^U
Z_1	0.4649	3.1271	1.7960	2.6622	0.7133	4.0729	0.7441	0.3758
Z_2	1.1365	3.5051	2.3208	2.3686	0.7621	6.0949	0.7055	0.3302
Z_3	4.0456	5.4667	4.7562	1.4212	0.6415	7.0424	1.2642	0.1328

where $A_\prec = A(Z_k(\mathbf{x}^{2U''}) \prec Z_k^*)$ and dist $= d(Z_k^*, Z_k(\mathbf{x}^{2U''}))$.

In what concerns the previous solution, there is a significant improvement of the achievement rates of the first objective function regarding both the upper and lower bounds and an improvement of the achievement rate of the upper bound of the third objective function; there is also a reduction of the distance between the interval values of the objective functions and their corresponding interval ideal solutions (see either the acceptability index or the distance between the intervals for both objective functions). On the other hand, there is a deterioration of the achievement rates of the second objective function regarding both the upper and lower bounds and also a deviation (hence, a deterioration) of the achievement rate of the lower bound of the third objective function from the value 1 (the aimed value).

Let us now consider that the DM wants to choose the objective functions that he/she wishes to improve/relax according to the values given in Table 5, but considering the previous satisfaction thresholds of the constraints.

Table 5. Maximum/Minimum improvement of the objective functions.

$Z_k^L(\mathbf{x}^{2U''}) + \Delta_k^{L*}$	$Z_k^U(\mathbf{x}^{2U''}) + \Delta_k^{U*}$
1	0
2	0
4	0

Since the problem becomes infeasible with the additional imposed constraints, an elastic programming problem is solved with $\omega_k = 0,\ k = 1, 2, 3$ deduced from the solution obtained for the previous problem. The solution obtained through the elastic programming problem is $\mathbf{x}^{3U''} = (2.7215, 2.3544)^T$. The information regarding this new solution is shown in Table 6,

Table 6. Information regarding solution $\mathbf{x}^{3U''}$.

	$Z_k^L(\mathbf{x}^{3U''})$	$Z_k^U(\mathbf{x}^{3U''})$	$m[Z_k(\mathbf{x}^{3U''})]$	$w[Z_k(\mathbf{x}^{3U''})]$	$A_{\prec}$	dist	tc_k^L	tc_k^U
Z_1	1.0000	3.7362	2.3684	2.7367	0.5624	3.4633	0.8333	0.4692
Z_2	0.8316	3.3696	2.1006	2.5380	0.7940	6.2304	0.6322	0.3153
Z_3	4.2962	5.8190	5.0576	1.5228	0.5573	6.6901	1.4191	0.1584

where $A_{\prec} = A(Z_k(\mathbf{x}^{3U''}) \prec Z_k^*)$ and dist $= d(Z_k^*, Z_k(\mathbf{x}^{3U''}))$.

A comparison with the previous solution allows us to conclude that there is an improvement of the achievement rate of the first objective function regarding both the upper and lower bounds and of the achievement rate of the upper bound of the third objective function; on the other hand, the achievement rate of the second objective function regarding both the upper and lower bounds and the achievement rate of the lower bound of the third objective function are worsened. There is also an increase of the distance between the value of the second interval objective function and its corresponding interval ideal solution. On the contrary, the first and third objective functions are closer to their corresponding interval targets (the acceptability index and the distance between these intervals are reduced).

Let us suppose that the DM wants to have a sensitivity measure of the efficient basis regarding the reference values considered for the objective functions. The ranges of variation of these reference values are computed according to distinct approaches (see Table 7).

Table 7. Ranges of variation of the objective function reference values.

Tolerance approach		**ITR approach**		**Sensitivity analysis**	
$\hat{b}^-$	$\hat{b}^+$	$\hat{b}^-$	$\hat{b}^+$	$\hat{b}^-$	$\hat{b}^+$
1.0000	1.0000	$-\infty$	1.0000	$-\infty$	1.0001
0.8316	0.8316	0.8258	0.8316	0.7864	0.8317
4.0000	4.0000	$-\infty$	4.1226	$-\infty$	4.2962

After observing Table 7, consider that the DM decides to change the reference values within the range of variation that leads to a different efficient basis. For instance, these values could be (-2, 2, 3) for Z_1, Z_2, and Z_3, respectively, according to the information in Table 7 regarding the ITR approach since these values are being changed simultaneously.

The new obtained solution is $\mathbf{x}^{4U''} = (0.6444, 3.1333)^T$. The information regarding this solution is given in Table 8,

Table 8. Information regarding solution $\mathbf{x}^{4U''}$.

	$Z_k^L(\mathbf{x}^{4U''})$	$Z_k^U(\mathbf{x}^{4U''})$	$m[Z_k(\mathbf{x}^{4U''})]$	$w[Z_k(\mathbf{x}^{4U''})]$	$A_{\prec}$	dist	tc_k^L	tc_k^U
Z_1	-1.0511	1.4000	0.1744	2.4511	1.1568	5.8000	0.4915	0.1111
Z_2	2.0000	3.8889	2.9444	1.8889	0.6655	5.7111	0.9129	0.3724
Z_3	3.3356	4.4689	3.9022	1.1333	0.8303	8.0402	0.8254	0.0600

where $A_{\prec} = A(Z_k(\mathbf{x}^{4U''}) \prec Zk^*)$ and dist $= d(Z_k^*, Z_k(\mathbf{x}^{4U''}))$.

In this solution, the achievement rates of the first and third objective functions regarding both the upper and lower bounds reach the lower levels as compared with the values attained until now. On the other hand, the achievement rates of the second objective function regarding both bounds reach their highest levels as compared with the achievement rates obtained for this indicator until now.

In this example, it is possible to observe that all the analyzed solutions have a non-dominance relation regarding the achievement rates of the upper and lower bounds of the objective functions meaning that it is not possible to improve the achievement rate of one objective function (regarding its upper or lower bounds) without worsening at least the achievement rate of another objective function (regarding its upper or lower bounds).

If the last obtained solution satisfies the DM, then the interactive procedure stops. Otherwise, it can continue searching for new solutions, precisely by considering a similar analysis as the one previously made.

5. Conclusion

Generally speaking, the coefficients of mathematical programming models are not exactly known. Hence, it is convenient to extend traditional mathematical programming models considering their intrinsic uncertainty. Moreover, in most of real-world problems multiple axes of evaluation are at stake. This paper proposes an interactive procedure for tackling uncertainty in MOLP models with interval coefficients. The approach herein suggested is not very demanding from the point of view of both the information required from the DM in each interaction and the computational efforts involved. This method allows obtaining a simple mathematical surrogate formulation leading to a strong integration of the DM in the decision procedure; operates uncertainty in all the coefficients of the model; provides a global view of the solutions in the best and worst case coefficients scenario; performs the search of new solutions according to the achievement rates of the objective functions regarding both the upper and lower bounds, taking always into account the best and worst case coefficients scenario; identifies the interval solution that is closest to the interval ideal solution; enables one to review the search options of the solutions considered by the DM; finds solutions with non-dominance relations regarding the achievement rates of the upper and lower bounds of the objective functions.

Also, it is possible to use the algorithm in a more flexible manner by: changing the interval goals; considering reference values for the objective functions explicitly out of ITR approach or the sensitivity analysis of ranges of variation; introducing penalties associated with the objective functions of the elastic programming problems, using the DM's preferences in the solution search process; considering satisfaction thresholds for each coefficient and not for the constraint as a whole.

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REFERENCES

1. R. Benayoun, J. de Montgolifer, J. Tergny, and O. Larichev, “Linear programming and multiple objective functions: STEP method (STEM),” *Math. Programming*, **1**, No. 3, 366–375 (1971).

2. G. G. Brown and G. W. Graves, *Elastic Programming: A New Approach to Large – Scale Mixed Integer Optimization*, ORSA TIMS, Las Vegas (1975).
3. J. W. Chinneck and E. W. Dravnieks, "Locating minimal infeasible constraint sets in linear programs," *ORSA J. Comput.*, **3**, No. 2, 57–168 (1991).
4. J. W. Chinneck and K. Ramadan, "Linear programming with interval coefficients," *J. Oper. Res. Soc.*, **51**, 209–220 (2000).
5. T. Gal, *Postoptimal Analysis, Parametric Programming and Related Topics*, McGraw-Hill (1979).
6. M. Inuiguchi and Y. Kume, "Goal programming problems with interval coefficients and target intervals," *Eur. J. Oper. Res.*, **52**, 345–360 (1991).
7. K. G. Murty, S. N. Kabadi, and R. Chandrasekarn, "Infeasibility analysis for linear systems. A survey," *Arabian J. Sci. Eng.*, **25**, (1C), 3–18 (2000).
8. C. Oliveira and C. H. Antunes, "Multiple objective linear programming models with interval coefficients – an illustrated overview," *Eur. J. Oper. Res.*, **181**, 1434–1463 (2007).
9. T. Shaocheng, "Interval number and fuzzy number linear programming," *Fuzzy Sets Syst.*, **66**, 301–306 (1994).
10. A. Sengupta and K. Pal, "On comparing interval numbers," *Eur. J. Oper. Res.*, **127**, 28–43 (2000).
11. B. Urli and R. Nadeau, "An interactive method to multiobjective linear programming problems with interval coefficients," *INFOR*, **30**, No. 2, 127–137 (1992).
12. R. E. Wendell, "Using bounds on the data in linear programming: the tolerance approach to sensitivity analysis. Mathematical Programming," *Math. Programming*, **29**, 304–322 (1984).
13. F. R. Wondolowski, "A generalization of Wendell's tolerance approach to sensitivity analysis in linear programming," *Decision Sci.*, **22**, 792–810 (1991).

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