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CAPITAL ACCUMULATION FOR PRODUCTION IN A DYNAMIC SPATIAL MONOPOLY

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ABSTRACT. This paper characterises the dynamics of capacity accumulation in a spatial monopoly, contrasting the socially optimal behavior of a benevolent planner against that of a profit-seeking monopolist. In steady state, the monopolist always distorts its investment as compared with the social optimum, except for those situations where, under both monopoly and social planning, either the equilibrium is driven by the Ramsey golden rule or consumers' reservation price is sufficiently high to induce the profit-seeking firm to serve all of them.

CONTENTS

1. Introduction

The analysis of dynamic monopoly goes back to Evans [7] and Tintner [14], who analyzed the pricing behavior of a firm subject to an U-shaped variable cost curve (see [4] for a recent exposition of the original model of Evans, as well as later developments). The analysis of intertemporal capital accumulation appeared later on (see [6]).

A commonly accepted feature of monopoly power is that a firm endowed with it consistently distorts price (upwards) and output (downward) as compared with the perfect competition or, equivalently, social planning. To reassess this issue, I propose a monopoly model where the firm locates the product in a spatial market representing the space of consumer preferences, as in [9]. Supplying the market involves building up productive capacity, at the implicit cost of reinserting unsold output into the production process as additional capital, as in [13]. Accordingly, the natural question arises: given the profit incentive of the monopolist, to what extent shall we expect the firm to distort output as compared with the socially optimal policies?

To answer this question, I characterize the dynamics of capacity accumulation, contrasting the socially optimal behavior of a benevolent planner against the behavior of a profit-maximizing monopolist. I show that, in steady state, the monopolist generally distorts the equilibrium investment and the associated output choice as compared with the social optimum, except for a situation where a Ramsey-like equilibrium prevails under both regimes, i.e.:

1. There exists a long-run equilibrium, where the firm operates with a productive capacity which is driven only by demand conditions. In this situation, the profit-seeking monopolist invests too little resources, as compared with a benevolent planner.

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2. There exists a long-run equilibrium, where the firm operates with a productive capacity which is driven only by time discounting and depreciation. In this situation, the behavior of the firm in steady state is the same irrespective of the regime being considered. Accordingly, the distortion typically associated with monopoly power disappears.

The remainder of the paper is structured as follows. The set-up is laid out in Sec. 2. The capital accumulation problem is investigated, first, in Sec. 3 from the standpoint of the social planner, and then in Sec. 4 from the standpoint of a profit-seeking monopolist. Concluding comments are in Sec. 5.

2. The Model

The set-up shares the basic features of the demand side with [11, 12]. I consider a market for horizontally differentiated products, where consumers are uniformly distributed with unit density along the unit interval $[0,1]$. Consequently, the total mass of consumers is equal to 1. Let the market exist over continuous time $t \in [0,\infty)$. The market is served by a single firm, selling a single good, located at $\ell(t) \in [0,1]$. Product location is costless.[1] The generic consumer located at $a(t) \in [0,1]$ buys one unit of the good, if net surplus from purchase is non-negative:

$$U(t) = s - p(t) - [\ell(t) - a(t)]^2 \geq 0, \quad i = 1,2, \tag{1}$$

where $p(t)$ is the firm's mill price, and $s > 0$ is gross consumer surplus, that is, the reservation price that a generic consumer is willing to pay for the good. Therefore, s can be considered as a preference parameter which, together with the disutility of transportation, yields a measure of consumers' taste for the good. The term $[\ell(t) - a(t)]^2$ measures the disutility of transportation. The mill price is such that marginal consumers at distance $|\ell(t) - a(t)|$ from the store enjoy zero surplus, i.e.,

$$p(t) = s - [\ell(t) - a(t)]^2 . \tag{2}$$

Observe that, in line of principle, it could be possible to have $q(t) = a(t)$ (if $a(t) > \ell(t)$) or $q(t) = 1 - a(t)$ (if $a(t) < \ell(t)$). However, this situation would be clearly suboptimal for the monopolist, in that he could gain by relocating the product costlessly until demand becomes symmetric around $\ell(t)$. Therefore, the choice of location can be solved once and for all at $t = 0$ by setting $\ell(t) = 1/2$. The same location is also optimal for a benevolent social planner aiming at the maximization of the total surplus.[2] The demand $q(t)$ is then easily defined as the interval $[1 - a(t), a(t)]$, i.e., $q(t) = 2a(t) - 1 \in [0,1]$, provided that $a(t) \in (\ell(t), 1]$. If so, then $1 - a(t) \in [0, \ell(t))$.

I assume a constant marginal production cost. For the sake of simplicity, and without further loss of generality, I normalize it to zero. Instantaneous revenues (profits) are

$$R(t) = p(t)q(t) = \left[s - \left(\frac{1}{2} - a(t)\right)^2\right][2a(t) - 1]. \tag{3}$$

Instantaneous consumer surplus is

$$CS(t) = \int_{1-a}^{a} \left[s - p(t) - \left(\frac{1}{2} - m\right)^2\right] dm = \frac{(2a-1)^3}{6}. \tag{4}$$

Therefore, instantaneous social welfare amounts to

$$SW(t) = R(t) + CS(t) = \frac{(2a(t) - 1)\left[12s - 1 + 4a(t)\left(1 - a(t)\right)\right]}{12}. \tag{5}$$

In the remainder, I will consider the following scenario. Production requires physical capital k, which can be accumulated over time to create capacity. At any instant t, the output level is $y(t) = f(k(t))$,

[1] The monopolist's R&D investment for product innovation is investigated in [12]. R&D for product innovation in a duopoly is analyzed by Harter [8] and Lambertini [10].

[2] For the sake of brevity, the proof of these claims is omitted, since it is well known from the existing literature (see, e.g., [2]).

where $f' \equiv \partial f(k(t))/\partial k(t) > 0$ and $f'' \equiv \partial^2 f(k(t))/\partial k(t)^2 < 0$. A reasonable assumption is that $q(t) \leq y(t)$, i.e., the level of sales is at most equal to the quantity produced. Excess output is reintroduced into the production process, yielding accumulation of capacity according to the following process:

$$\frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t), \tag{6}$$

where δ denotes the rate of depreciation of capital. The cost of capital is represented by the opportunity cost of intertemporal relocation of unsold output.[3] Let the initial state be $k(0) = k_0 > 0$.

I shall first investigate the behavior of a social planner running the firm so as to maximize the net discounted welfare flow, and then contrast the behavior of a profit-seeking monopolist against the social planning benchmark.

3. Social Planning

The objective of a benevolent social planner is

$$\max_{a(t)} \int_0^\infty e^{-\rho t} SW(t)dt = \int_0^\infty e^{-\rho t} \frac{(2a(t)-1)[12s-1+4a(t)(1-a(t))]}{12} dt \tag{7}$$

such that

$$\frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t), \tag{8}$$

where $\rho \in [0, \infty)$ denotes time discounting. In choosing the optimal location of the marginal consumer(s) at any t, the planner indeed maximizes discounted social welfare with respect to output. The corresponding Hamiltonian is

$$\mathcal{H}^{\text{sp}}(t) = e^{-\rho t} \cdot \left\{ \frac{(2a(t)-1)[12s-1+4a(t)(1-a(t))]}{12} + \lambda(t)[f(k(t)) - 2a(t) + 1 - \delta k(t)] \right\}, \tag{9}$$

where $\lambda(t) = \beta(t)e^{\rho t}$ and $\beta(t)$ is the co-state variable associated with $k(t)$. The value of $\beta(t)$ is the marginal value at $t = 0$ of an additional unit of the capital at the time t. Superscript "sp" denotes "*social planning*." In the remainder, I will focus upon $\lambda(t)$, which measures the marginal value, as of time t, of an additional unit of the capital at the same date. For brevity, I will label it as the shadow price of the capital.

The first-order conditions are[4]

$$\mathcal{H}^{\text{sp}}(t) = 0 \Leftrightarrow 2[1-a(t)]\,a(t) + 2s - \frac{1}{2} - 2\lambda(t) = 0; \tag{10}$$

$$-\frac{\partial \mathcal{H}^{\text{sp}}(t)}{\partial k(t)} = \frac{d\lambda(t)}{dt} - \rho\lambda(t) \Leftrightarrow \frac{d\lambda(t)}{dt} = \left[\rho + \delta - f'(k(t))\right]\lambda(t); \tag{11}$$

$$\lim_{t\to\infty} \beta(t) \cdot k(t) = 0. \tag{12}$$

From (10), I obtain[5] $a(t) = 1/2 + \sqrt{s - \lambda(t)}$. In combination with (2) and $\ell(t) = 1/2$, this implies the following lemma.

Lemma 3.1. *Under social planning, the market price of the final good and the shadow price of the capital coincide, i.e.,* $p(t) = \lambda(t)$.

[3] The adoption of technology (6) in the economic literature goes back to Ramsey [13]. For an application to a dynamic oligopoly model with either price or quantity competition, see [3].

[4] Second-order conditions are omitted. They hold in both cases analyzed in the paper.

[5] I consider $a(t) \in (1/2, 1]$, so that the smaller solution of (10) can be excluded.

Now we can differentiate $a(t)$ with respect to time to obtain

$$\frac{da(t)}{dt} = \frac{-\frac{d\lambda(t)}{dt}}{2\sqrt{s-\lambda(t)}}. \tag{13}$$

Using (11), expression (13) can be rewritten as follows:

$$\frac{da(t)}{dt} = -\frac{[\rho + \delta - f'(k(t))]\lambda(t)}{2\sqrt{s-\lambda(t)}}. \tag{14}$$

Moreover, using $\lambda(t) = a(t)[1 - a(t)] + s_0 - 1/4$, we simplify (14) as follows:

$$\frac{da(t)}{dt} = -\frac{\left[a(t) - (a(t))^2 + s - \frac{1}{4}\right][\rho + \delta - f'(k(t))]}{2[a(t) - 1]}. \tag{15}$$

Therefore,

$$\frac{da(t)}{dt} \propto \left[a(t) - (a(t))^2 + s - \frac{1}{4}\right][f'(k(t)) - \rho - \delta]. \tag{16}$$

The right-hand side of (16) is equal to zero at

$$f'(k(t)) = \rho + \delta; \tag{17}$$

$$a(t) = \frac{1}{2} \pm \sqrt{s}. \tag{18}$$

The critical point (17) denotes the situation where the marginal product of capital is just sufficient to cover discounting and depreciation. The smaller solution in (18) can be omitted on the basis of the assumption that $a(t) \in (1/2, 1]$. Therefore, the long run equilibrium output is either $q^{\mathrm{sp}}(t) = 2\sqrt{s}$ or the quantity corresponding to the capacity $f'^{-1}(\rho+\delta)$. It is also worth noting that, in correspondence with $a(t) = 1/2 + \sqrt{s}$, we have

(i) $q^{\mathrm{sp}}(t) \leq 1$ for all $s \leq 1/4$, and
(ii) $p^{\mathrm{sp}}(t) = 0$, i.e., the planner sells the product at marginal cost.

I am now able to draw a phase diagram in the space $\{k, q\}$, in order to characterize the steady state equilibrium (to ease the exposition, the indication of time is omitted in the rest of the discussion). The locus $\dot{q} \equiv dq/dt = 0$ is given by $q^{\mathrm{sp}} = 2\sqrt{s}$ and $f'(k) = \rho + \delta$ in Fig. 1. It is easy to show that the horizontal locus $q^{\mathrm{sp}} = 2\sqrt{s}$ denotes the usual equilibrium solution we are well accustomed to from the static model. The two loci partition the space $\{k, q\}$ into four regions, where the dynamics of q is defined by (15) with $a = (q + 1)/2$, as indicated by the vertical arrows. The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by horizontal arrows, derive from (8). Steady states, denoted by A and C along the horizontal arm, and B along the vertical one, are identified by intersections between loci.

It is worth noting that the situation illustrated in Fig. 1 is only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho+\delta$ is independent of demand parameters, while the horizontal locus $q^{\mathrm{sp}} = 2\sqrt{s}$ shifts upwards (downwards) as s increases (decreases). Therefore, we obtain one out of five possible regimes:

1. there exist three steady state points with $k_A^{\mathrm{sp}} < k_B^{\mathrm{sp}} < k_C^{\mathrm{sp}}$ (this is the situation depicted in Fig. 1);
2. there exist two steady state points with $k_A^{\mathrm{sp}} = k_B^{\mathrm{sp}} < k_C^{\mathrm{sp}}$;
3. there exist three steady state points with $k_B^{\mathrm{sp}} < k_A^{\mathrm{sp}} < k_C^{\mathrm{sp}}$;
4. there exist two steady state points with $k_B^{\mathrm{sp}} < k_A^{\mathrm{sp}} = k_C^{\mathrm{sp}}$;
5. there exists a unique steady state point, corresponding to B.

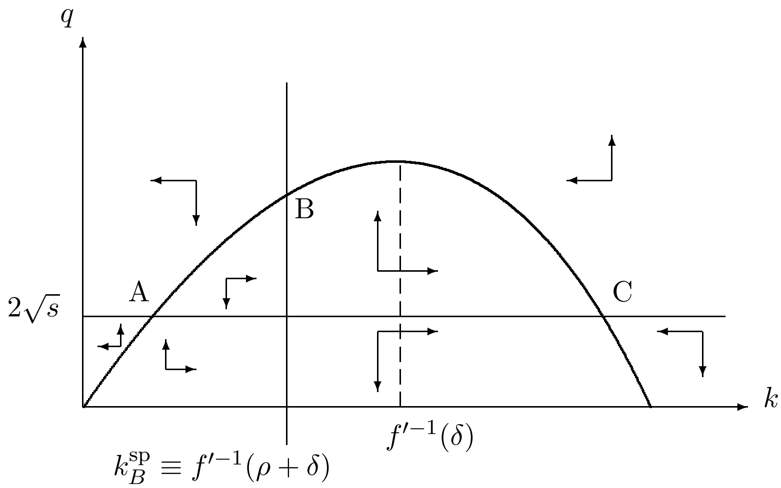

Fig. 1. Capital accumulation for production under social planning.

An intuitive explanation for the above classification can be provided in the following terms. The vertical locus $f'(k) = \rho + \delta$ identifies a constraint on optimal capital embodying the monopolist's intertemporal preferences. Accordingly, the maximum steady state output would be that corresponding to (i) $\rho = 0$, and (ii) a capacity such that $f'(k) = \delta$. Yet, a positive discounting (i.e., impatience) induces the planner to install a smaller steady-state capacity, much the same as happens in the well-known Ramsey model [13].[6] On this basis, define this level of k as the *optimal capital constraint*, and label it by $\widehat{k}$. This is the level of capacity associated with the so-called *Ramsey golden rule*. When the reservation price s is very large, points A and C either do not exist (regime 5) or fall to the right of B (regimes 2, 3, and 4). Under these circumstances, the capital constraint is operative and the planner chooses the capital accumulation corresponding to B. As we will see below, this is completely consistent with the dynamic properties of the steady-state points.

Note that both steady-state points located along the horizontal locus entail the same levels of sales. Consequently, the point C is surely inefficient in the sense that it requires a higher amount of capital. At the point A, $dSW(t)/dq_i(t) = 0$, i.e., the marginal instantaneous social welfare is zero.[7]

Now we come to the stability analysis of the above system. The joint dynamics of a (or q) and k, can be described by the linearization of (15) and (8) around $(k^{\rm sp}, a^{\rm sp})$, to obtain the following:

$$\begin{bmatrix} \dot{k} \\ \dot{a} \end{bmatrix} = \Xi \begin{bmatrix} (k - k^{\rm sp}) \\ (a - a^{\rm sp}) \end{bmatrix}, \tag{19}$$

$$\Xi \equiv \begin{bmatrix} f'(k) - \delta & -2 \\ \dfrac{\left(s - \dfrac{1}{4} + a(1-a)\right)}{2a-1} f''(k) & \dfrac{(4a(1-a) - 4s - 1)}{2(2a-1)^2}(f'(k) - \rho - \delta) \end{bmatrix}.$$

[6]For a detailed exposition of the Ramsey model, see [1].

[7]The point A corresponds to the optimal quantity characterizing the static version of the model [2].

The stability properties of the system in a neighborhood of the steady state depend upon the trace and determinant of the (2×2)-matrix Ξ. In studying the system, we restrict the consideration to steady-state points. The trace of Ξ is

$$\operatorname{tr}(\Xi) = f'(k) - \delta + \frac{(4a(1-a) - 4s - 1)}{2(2a-1)^2}(f'(k) - \rho - \delta) \tag{20}$$

yielding $\operatorname{tr}(\Xi) = \rho > 0$ in correspondence with both $a = 1/2 + \sqrt{s}$ and $f'(k) = \rho + \delta$. Together with the evaluation of the determinant $\Delta(\Xi)$ at the same points, we obtain the following classification.

Regime 1. In A, $\Delta(\Xi) < 0$, and hence this is a saddle point. In B, $\Delta(\Xi) > 0$, and, therefore, B is an unstable focus. In C, $\Delta(\Xi) < 0$, and this is again a saddle point, with the horizontal line as the stable arm.

Regime 2. In this regime, A coincides with B, and, therefore, we have only two steady states which are both saddle points. In $A = B$, the saddle path approaches the saddle point from the left only, while in C the stable arm is again the horizontal line.

Regime 3. Here, B is a saddle; A is an unstable focus; C is a saddle point, as in Regimes 1 and 2.

Regime 4. Here, points A and C coincide. B remains a saddle, while $A = C$ is a saddle whose converging arm proceeds from the right along the horizontal line.

Regime 5. There exists a unique steady-state point B, which is a saddle.

We can sum up as follows. The unique efficient and unstable steady-state point is B if $k_B^{\mathrm{sp}} \equiv \widehat{k} < k_A$, while it is A if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is $q^{\mathrm{sp}} = 2\sqrt{s}$ if the equilibrium is at the point A, or the level corresponding to the optimal capital constraint $\widehat{k}$ if the equilibrium is at the point B. The reason is that, if the capacity at which marginal instantaneous profit is zero is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and planner's decisions in each period are solely driven by the unconstrained maximization of instantaneous social welfare.

The above discussion produces the following proposition.

Proposition 3.1. *If $k_B^{\mathrm{sp}} \equiv \widehat{k} > k_A$, the steady-state output level is*

- $q^{\mathrm{sp}} = 2\sqrt{s}$ *for all* $s \in [0, 1/4)$, *and partial market coverage is obtained*;
- $q^{\mathrm{sp}} = 1$ *for all* $s \geq 1/4$, *and full market coverage is obtained.*

Otherwise, if $k_B^{\mathrm{sp}} \equiv \widehat{k} < k_A$, the steady-state output is $q^{\mathrm{sp}} = f\big(\widehat{k}\big)$, and

- *partial market coverage is obtained* (i) *for all* $s \in [0, 1/4)$ *or* (ii) *for all* $s \geq 1/4$, *iff* $f\big(\widehat{k}\big) < 1$;
- *full market coverage is obtained iff* $s \geq 1/4$ *and* $f\big(\widehat{k}\big) \geq 1$.

It is worth noting, in particular, that if the Ramsey golden rule applies, whereby $q^{\mathrm{sp}} = f\big(\widehat{k}\big)$, the fact that the reservation price s is higher than $1/4$ is not sufficient to ensure the full market coverage, since this indeed depends on the specific properties of productive technology.

4. Profit-Seeking Monopoly

The objective of the monopolist is

$$\max_{a(t)} \int_0^{\infty} e^{-\rho t} R(t)\, dt = \int_0^{\infty} e^{-\rho t} \left[s - \left(\frac{1}{2} - a(t) \right)^2 \right] [2a(t) - 1]\, dt, \tag{21}$$

such that

$$\frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t), \tag{22}$$

where $\rho \in [0, \infty)$ denotes the same discount rate as for the planner. The corresponding Hamiltonian function is

$$\mathcal{H}^m(t) = e^{-\rho t} \cdot \left\{ \left[s - \left(\frac{1}{2} - a(t) \right)^2 \right] [2a(t) - 1] + \lambda(t)[f(k(t)) - 2a(t) + 1 - \delta k(t)] \right\}, \tag{23}$$

where superscript m denotes "*monopoly*" and, again, $\lambda(t) = \beta(t)e^{\rho t}$; $\beta(t)$ is the co-state variable associated with $k(t)$.

The solution of the monopolist's problem is similar to that of the planner as is shown in Sec. 3, and the detailed calculations are confined in Appendix 1. However, one specific result is the following lemma.

Lemma 4.1. *Under monopoly, the shadow price of the capital is*

$$\lambda(t) = 3a(t)\,[1 - a(t)] + s - \frac{3}{4},$$

which is less than the market price $p(t) = s - [\ell(t) - a(t)]^2$ *evaluated at* $\ell(t) = 1/2$*, for all admissible* $a(t) \neq 1/2$*. At* $a(t) = 1/2$*, we have* $\lambda(t) = p(t)$*.*

According to Lemma 4.1, the value attached by the monopolist to a current unit of sales is larger than the shadow price of a further unit of capital which would increase productive capacity in the future. Since any increase in productive capacity requires some unsold output, Lemma 4.1 says that we should expect to observe cases where the monopolist undersupplies the market in steady state, as compared with the planner.

The steady state output is either $q^m(t) = 2\sqrt{s/3}$ or the quantity corresponding to the capacity $\widehat{k} = f'^{-1}(\rho + \delta)$ (the Ramsey equilibrium is the same as under planning).

The main results can be stated as follows.

Proposition 4.1. *If* $\widehat{k} > k_A^m$*, the steady-state output level is*

- $q^m = 2\sqrt{s/3}$ *for all* $s \in [0, 3/4)$*, and partial market coverage is obtained*;
- $q^m = 1$ *for all* $s \geq 3/4$*, and full market coverage is obtained.*

  Otherwise, if $\widehat{k} < k_A^m$*, the steady-state output is* $q^m = f(\widehat{k})$*, where* $\widehat{k}$ *is the capital level at which* $f'(k) = \rho + \delta$*, and*

- *partial market coverage is obtained* (i) *for all* $s \in [0, 3/4)$ *or* (ii) *for all* $s \geq 3/4$ *iff* $f(\widehat{k}) < 1$;
- *full market coverage is obtained iff* $s \geq 3/4$ *and* $f(\widehat{k}) \geq 1$.

Now it is possible to assess the performances of the two regimes comparatively. Propositions 3.1 and 4.1 imply the following corollary.

Corollary 4.1. *If* $\widehat{k} > k_A$ *under both regimes, then* $q^m < q^{\text{sp}} = 1$ *for all* $s \in [1/4, 3/4)$*, i.e., the planner covers the whole market while the monopolist does not.*

Clearly, if $\widehat{k} > k_A$ under both regimes and $s \geq 3/4$, the output distortion generated by the market power vanishes and full coverage takes place in both cases since consumers are rich enough to make it irrational for the profit-seeking monopolist to price anyone of them out of consumption. Another relevant corollary of Propositions 3.1 and 4.1 is as follows.

Corollary 4.2. *If* $\widehat{k} < k_A$ *under both regimes, then equilibrium output is* $q^m = q^{\text{sp}} = f(\widehat{k})$*.*

The reason for the output distortion in monopoly (as in Corollary 4.1) is provided by Lemmas 3.1 and 4.1, stressing the difference between unit prices of output and capital in the two regimes. In the situation considered in Corollary 4.2, welfare is the same under both regimes, except for the fact that the distribution of total surplus differs, due to the different pricing policies adopted, i.e., monopoly pricing versus marginal cost pricing.

5. Concluding Remarks

I have investigated the optimal capacity accumulation decisions of a single-product firm operating in a spatial market with a uniform consumer distribution, comparing the steady-state behavior of a profit-seeking monopolist versus that of a benevolent social planner.

It turns out that the monopolist distorts capital accumulation (and thereby also sales), whenever partial market coverage is obtained at the equilibrium and the output falls short of that associated with the Ramsey golden rule. These distortions disappear when the steady-state equilibrium is dictated only by the conditions driving intertemporal capital accumulation. Accordingly, it turns out that there are admissible market configurations where monopoly power, although admittedly misaligned with social incentives, is not as evil as we are usually led to believe on the basis of the static economic models dealing with this issue.

Appendix

Capital Accumulation Under Monopoly

The monopolist's problem outlined in Sec. 3 is solved as follows. By (23), the necessary conditions for a path to be optimal are

$$\frac{\partial \mathcal{H}^m(t)}{\partial a(t)} = 0 \Leftrightarrow 6\,[1-a(t)]\,a(t) + 2s - \frac{3}{2} - 2\lambda(t) = 0; \tag{a1}$$

$$-\frac{\partial \mathcal{H}^m(t)}{\partial k(t)} = \frac{d\lambda(t)}{dt} - \rho\lambda(t) \Leftrightarrow \frac{d\lambda(t)}{dt} = \left[\rho + \delta - f'(k(t))\right]\lambda(t); \tag{a2}$$

$$\lim_{t\to\infty} \beta(t)\cdot k(t) = 0\,. \tag{a3}$$

From (a1), we obtain

$$a(t) = \frac{1}{2} + \sqrt{\frac{s-\lambda(t)}{3}} \tag{a4}$$

which yields $\lambda(t) = 3a(t)[1-a(t)] + s - 3/4$. This, in combination with (2) and $\ell(t) = 1/2$, proves Lemma 4.1. Moreover, we can differentiate (a4) with rspect to the time to obtain

$$\frac{da(t)}{dt} = \frac{-\dfrac{d\lambda(t)}{dt}}{2\sqrt{3[s-\lambda(t)]}}. \tag{a5}$$

Thanks to (a2), expression (a5) can be simplified as follows:

$$\frac{da(t)}{dt} = -\frac{[\rho+\delta-f'(k(t))]\,\lambda(t)}{2\sqrt{3\,[s-\lambda(t)]}}. \tag{a6}$$

From (a1), $\lambda(t) = 3a(t)\,[1-a(t)] + s - 3/4$, and (a6) takes the form

$$\frac{da(t)}{dt} = -\frac{\left[3a(t) - 3(a(t))^2 + s - \dfrac{3}{4}\right][\rho+\delta-f'(k(t))]}{2\sqrt{3\left[\dfrac{3}{4} - 3a(t) - 3(a(t))^2\right]}}. \tag{a7}$$

Hence we have that

$$\frac{da(t)}{dt} =\propto \left[3a(t) - 3(a(t))^2 + s - \frac{3}{4}\right][f'(k(t)) - \rho - \delta]. \tag{a8}$$

The expression on the right-hand side of (a8) is zero if

$$f'(k(t)) = \rho + \delta; \tag{a9}$$

$$a(t) = \frac{1}{2} \pm \sqrt{\frac{s}{3}}. \tag{a10}$$

Critical point (a9) defines the situation where the marginal product of the capital is just sufficient to cover discounting and depreciation. The smaller solution in (a10) can be omitted on the basis of the assumption that $a(t) \in (1/2\,,\,1]$. Therefore, the long run equilibrium output is either $q^m(t) = 2\sqrt{s/3}$, where superscript m denotes "*monopoly*," or the quantity corresponding to the capacity $f'^{-1}(\rho + \delta)$. Observe that the Ramsey equilibrium is the same as under social planning. It is also worth noting that $q^m(t) \leq 1$ for all $s \leq 3/4$. At $q^m(t) = 2\sqrt{s/3}$, the optimal price is $p^m(t) = 2s/3$. The discussion and graphical illustration of steady states are omitted since they are similar to those proposed above for the case of social planning, except for the case where the demand-driven long-run equilibrium at $q^m(t) = 2\sqrt{s/3}$ obviously involves a smaller quantity (and a higher price) than observed at the corresponding equilibrium under planning.

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