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ON THE PHASE TRANSITION TEMPERATURE IN A VARIATIONAL PROBLEM OF ELASTICITY THEORY FOR TWO-PHASE MEDIA

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Dedicated to Nina Nikolaevna Uraltseva

We obtain sufficient conditions for the noncoincidence of the phase transition temperatures and illustrate this result by examples of problems in two-phase elastic media. We also indicate some cases where equilibrium states exist or not depending on the values of the temperature lying between the lower and upper phase transition temperatures. Bibliography: 10 *titles.*

1. Introduction

If the surface energy of the boundary of the interface of phases is not taken into account, the energy functional of a two-phase elastic medium is given by the formula [**1**]

$$I[u,\chi,t] = \int\limits_{\Omega} \{\chi(F^+(\nabla u) + g^+ \cdot u + t) + (1-\chi)(F^-(\nabla u) + g^- \cdot u)\}\, dx + \int\limits_{\partial\Omega} f \cdot u\, dS, \qquad (1.1)$$

where the measurable characteristic function χ is a distribution of the phases in a bounded domain $\Omega \subset R^m$, $m \geqslant 2$, with Lipschitz boundary, $g^\pm$ and f are fixed exterior force fields, $t \in R^1$ is the temperature (we assume that t is constant in the domain), and u is the replacement field. $\Omega \in R^m$, $m \geqslant 2$.

We denote by $R^{m\times m}$ the space of $m \times m$-matrices and by $R^{m\times m}_s$ the space of symmetric $m \times m$-matrices. To describe the quadratic strain energy density of each of the phases $F^\pm(M)$, $M \in R^{m\times m}$, we introduce a linear mapping $A^\pm : R^{m\times m}_s \to R^{m\times m}_s$ such that

$$(A^\pm \xi)_{ij} = a^\pm_{ijkl}\xi_{kl}, \quad \xi \in R^{m\times m}_s,$$

where the sum is taken with respect to related indices from 1 to m and

$$a^\pm_{ijkl} = a^\pm_{jikl} = a^\pm_{ijlk} = a^\pm_{klij}, \quad a^\pm_{ijkl}\xi_{ij}\xi_{kl} \geqslant \nu|\xi|^2.$$

Translated from *Problemy Matematicheskogo Analiza*, **41**, May 2009, pp. 37–48.

In the last inequality, $\xi \in R_s^{m\times m}$, ν is a positive constant, and $|\xi|^2 = \xi_{ij}\xi_{ij}$. Under these assumptions, $A^\pm$ are symmetric in $R_s^{m\times m}$ relative to the Hilbert–Schmidt inner product

$$\langle \xi^1, \xi^2\rangle = \operatorname{tr} \xi^1\xi^2, \quad \xi^1, \xi^2 \in R_s^{m\times m};$$

moreover, $A^\pm$ are invertible. We set

$$\begin{gathered} F^\pm(M) = \langle A^\pm(e(M)-\zeta^\pm), e(M)-\zeta^\pm\rangle, \\ M \in R^{m\times M}, \quad e(M) = \frac{M+M^*}{2}, \quad \zeta^\pm \in R_s^{m\times m}, \end{gathered} \tag{1.2}$$

where the matrices $\zeta^\pm$ are interpreted as the residual strain tensors. By (1.2), for $M = \nabla u$ the matrix $e(\nabla u)$ has entries

$$e_{ij}(\nabla u) = \frac{u^i_{x_j} + u^j_{x_i}}{2}.$$

It is the strain tensor for the displacement field u.

To describe the domain of the functional (1.1), we fix a measurable subset $\Gamma \subset \partial\Omega$ and a function $u_0 \in W_2^1(\Omega, R^m)$. We set

$$\begin{gathered} \mathbb{X} = \{u \in W_2^1(\Omega, R^m) : u(x) - u_0(x) = 0 \quad \text{for } x \in \Gamma\}, \\ \mathbb{Z} = \{\chi \in L_\infty(\Omega) : \chi(x) = \chi^2(x) \quad \text{almost everywhere in } \Omega\}. \end{gathered} \tag{1.3}$$

For the domain of the functional (1.1) we take the sets $\mathbb{X}$ and $\mathbb{Z}$.

By an *equilibrium state* of a two-phase elastic medium for fixed t we mean a solution to the variational problem

$$I[\widehat{u}, \widehat{\chi}, t] = \inf_{u\in\mathbb{X}, \chi\in\mathbb{Z}} I[u, \chi, t], \qquad \widehat{u} \in \mathbb{X}, \quad \widehat{\chi} \in \mathbb{Z}. \tag{1.4}$$

An equilibrium state is said to be *one-phase* if $\widehat{\chi} \equiv 0$ or $\widehat{\chi} \equiv 1$ and *two-phase* in the opposite case.

In the case $\Gamma = \varnothing$, it is easy to see that the following condition is necessary for the solvability of the problem (1.4):

$$g^+ = g^- \equiv g, \qquad \int\limits_\Omega g\cdot v\,dx + \int\limits_{\partial\Omega} f\cdot v\,dS = 0 \quad \text{for all functions } v(x) = Bx + x_0, \tag{1.5}$$

where B is a skew-symmetric $m\times m$-matrix and $x_0 \in R^m$.

The functional (1.1) with the domain (1.3) is not, in general, lower semicontinuous [**2**]. Therefore, traditional direct methods of the Calculus of Variations [**3**] cannot provide us with a positive answer to the question about the solvability of the problem (1.4). Furthermore, it is not reasonable to use the approach of [**4**], based on the study of the quasiconvex hull of the energy functional (1.1), since the integrand explicitly depends on u and x.

There are approaches to the study of the problem (1.4) based on the use of the explicit form of the functional (1.1). One approach allows us to reduce the variational problem (1.4) to a boundary value problem for a system of first order differential equations. In this direction, the solvability of the problem (1.4) is proved in the case of homogeneous isotropic two-phase media under some additional assumptions on the energy densities and boundary conditions and a number of model one-dimensional problems were also considered (cf. [**5**]–[**7**]).

The other approach is based on the notion of the phase transition temperatures. Keeping in mind formula (1.1) for the energy functional, it is reasonable to suggest that there exist numbers $t_\pm$ such that

$$\begin{aligned} & -\infty < t_- \leqslant t_+ < \infty \\ & \text{for } t < t_- \text{ there exist only one-phase equilibrium states with } \widehat{\chi} \equiv \chi^+ \equiv 1, \\ & \text{for } t > t_+ \text{ there exist only one-phase equilibrium states with } \widehat{\chi} \equiv \chi^- \equiv 0, \\ & \text{for } t = t_\pm \text{ there exist equilibrium states with } \widehat{\chi} = \chi^\pm \text{ respectively,} \\ & \text{for } t_- < t_+,\ t \in (t_-, t_+) \text{ there are no one-phase equilibrium states.} \end{aligned} \tag{1.6}$$

The numbers $t_\pm$ (if they exist) are called the *upper temperature* and *lower temperature* of phase transitions. The conjecture (1.6) asserts the existence of equilibrium states for $t \leqslant t_-$ and $t \geqslant t_+$, but not for $t \in (t_-, t_+)$. Therefore, for $t_- = t_+$ the problem (1.4) is always solvable, whereas for $t_- < t_+$ and $t \in (t_-, t_+)$ the solvability question remains open.

We set

$$I^\pm[u] = \int\limits_\Omega (F^\pm(\nabla u) + g^\pm \cdot u)\, dx + \int\limits_{\partial\Omega} f \cdot u\, dS, \quad u \in \mathbb{X}. \tag{1.7}$$

A sufficient condition for the existence of the phase transition temperatures was formulated in [**8**] in terms of the solutions $\widehat{u}^\pm$ to the variational problem

$$I^\pm[\widehat{u}^\pm] = \inf_{u \in \mathbb{X}} I^\pm[u], \quad \widehat{u}^\pm \in \mathbb{X}. \tag{1.8}$$

It is obvious that the variational problem (1.8) is uniquely solvable. The functions $\widehat{u}^\pm$ determine the equilibrium displacement fields for one-phase problems with energy densities (1.2).

The following assertion holds (cf. [**8**]).

$$\begin{aligned} & \text{if } g^+ - g^- \in L_\infty(\Omega, R^m),\ \widehat{u}^\pm \in W^1_\infty(\Omega, R^m), \\ & \text{then the phase transition temperatures } t_\pm \text{ exist.} \end{aligned} \tag{1.9}$$

There are examples with $t_- = t_+$, as well with $t_- < t_+$. To the first kind of examples we can relate the problem (1.4) with $g^+ = g^- = 0$, $\Gamma = \varnothing$, and $f = qn$, where q is a constant (the external hydrostatic pressure) and n is the field of unit normals to $\partial\Omega$ (cf. [**9**]). Examples of second kind are presented in [**4**]–[**8**] (except for several degenerate cases) with $\Gamma = \partial\Omega$.

From the unique solvability of the problem (1.8) it follows that

$$\begin{aligned} & \text{if one of the following equalities holds: } I^+[\widehat{u}^+] = I^+[\widehat{u}^-] \text{ or } I^-[\widehat{u}^-] = I^-[\widehat{u}^+], \\ & \text{then } \widehat{u}^+ = \widehat{u}^- \equiv \widehat{u}^0 \text{ and both equalities hold simultaneously.} \end{aligned} \tag{1.10}$$

The goal of this paper is to find sufficient conditions for the realization of the inequality $t_- < t_+$ and to illustrate the result obtained by a number of examples.

Theorem. *Suppose that the phase transition temperatures exist and one of the equalities in* (1.10) *is satisfied. Then the following assertions hold.*

(1) *If the following condition is satisfied*:

$$F^+(\nabla\widehat{u}^0) - F^-(\nabla\widehat{u}^0) + (g^+ - g^-) \cdot \widehat{u}^0 \neq \text{const} \quad \textit{almost everywhere in } \Omega, \tag{1.11}$$

then $t_- < t_+$.

(2) *If the following condition is satisfied*:

$$F^{+}(\nabla\widehat{u}^{0}) - F^{-}(\nabla\widehat{u}^{0}) + (g^{+} - g^{-})\cdot\widehat{u}^{0} = \mathrm{const} \quad \textit{almost everywhere in} \quad \Omega, \tag{1.12}$$

but

$$F_M^{+}(\nabla\widehat{u}^{0}) \neq F_M^{-}(\nabla\widehat{u}^{0}) \quad \textit{on a set of positive measure} \quad E\subset\Omega, \tag{1.13}$$

then $t_- < t_+$.

(3) *If* (1.12) *holds and*

$$F_M^{+}(\nabla\widehat{u}^{0}) = F_M^{-}(\nabla\widehat{u}^{0}) \quad \textit{almost everywhere in} \quad \Omega, \tag{1.14}$$

then $g^+ = g^-$ *and* $t_- = t_+$.

The theorem is proved in Section 2. The corresponding examples are given in Section 3, whereas situations where the problem (1.4) does not has a solution for $t\in(t_-,t_+)$ are described in Section 4.

2. Proof of Theorem

We introduce the functions

$$i^{\pm}(t) = \inf_{u\in\mathbb{X}} I[u,\chi^{\pm},t] = \begin{cases} I^{+}[\widehat{u}^{+}] + t|\Omega|, \\ I^{-}[\widehat{u}^{-}] \end{cases} \tag{2.1}$$

and set

$$i_{\min}(t) = \min\{i^{+}(t), i^{-}(t)\}. \tag{2.2}$$

It is obvious that $i_{\min}(t)$ is a piecewise linear function and

$$i_{\min}(t) = \begin{cases} i^{+}(t) & \text{for} \quad t\leqslant t^{*}, \\ i^{-}(t) & \text{for} \quad t\geqslant t^{*}, \end{cases} \qquad t^{*} = -|\Omega|^{-1}(I^{+}[\widehat{u}^{+}] - I^{-}[\widehat{u}^{-}]). \tag{2.3}$$

Let

$$i(t) = \inf_{u\in\mathbb{X},\chi\in\mathbb{Z}} I[u,\chi,t]. \tag{2.4}$$

It is obvious that $i(t)\leqslant i_{\min}(t)$. Let

$$\begin{aligned} E_{=} &= \{t\in R^{1} : i(t) = i_{\min}(t)\}, \\ E_{<} &= \{t\in R^{1} : i(t) < i_{\min}(t)\}. \end{aligned} \tag{2.5}$$

As is known [**8**], the following assertions hold:

(a) if $t^{*}\in E_{=}$, then $E_{=} = R^{1}$,

(b) if $t^{*}\in E_{<}$, then one of the following cases is realized:

$$E_{<} = \begin{cases} R^{1} \\ (t_-,\infty) \\ (-\infty,t_+) \\ (t_-,t_+) \end{cases}, \quad -\infty < t_- < t_+ < \infty, \tag{2.6}$$

Furthermore, in case (a), there exists only the equilibrium state $\widehat{u} = \widehat{u}^+$, $\widehat{\chi} = \chi^+$ for $t < t^*$ and $\widehat{u} = \widehat{u}^-$, $\widehat{\chi} = \chi^-$. for $t > t^*$. The same assertions are valid in case (b) for $t < t^-$ and $t > t^+$ respectively.

Consequently, in case (a), the numbers $t_\pm = t^*$ are the phase transition temperatures in view of (1.6) and (2.6), whereas, in case (b), $E_< = (t_-, t_+)$ provided that the phase transition temperatures exist.

Therefore, if the phase transition temperatures exist, then the inequality $t_- < t_+$ is equivalent to the inequality $i(t^*) < i_{\min}(t^*)$ which, in turn, is equivalent to the following assertion:

$$\begin{gathered} \text{there exist } u \in \mathbb{X},\ \chi \in \mathbb{Z} \text{ such that } J[u,\chi] < 0, \\ J[u,\chi] = I[u,\chi,t^*] - I[\widehat{u}^+,\chi^+,t^*] = I[u,\chi,t^*] - I[\widehat{u}^-,\chi^-,t^*]. \end{gathered} \tag{2.7}$$

Lemma 1. *The following equalities hold*:

$$\begin{aligned} J[u,\chi] = &\int\limits_\Omega (1-\chi)\{(F^-(\nabla\widehat{u}^+) - F^+(\nabla\widehat{u}^+)) + (g^- - g^+)\cdot\widehat{u}^+ - t^*\}\,dx \\ &+ \int\limits_\Omega (1-\chi)\{(F^-_M(\nabla\widehat{u}^+) - F^+_M(\nabla\widehat{u}^+))\nabla h + (g^- - g^+)\cdot h\}\,dx \\ &+ \frac{1}{2}\int\limits_\Omega (\chi F^+_{MM} + (1-\chi)F^-_{MM})(\nabla h,\nabla h)\,dx, \quad u \in \mathbb{X}, \quad u = \widehat{u}^+ + h, \end{aligned} \tag{2.8}$$

$$\begin{aligned} J[u,\chi] = &\int\limits_\Omega \chi\{(F^+(\nabla\widehat{u}^-) - F^-(\nabla\widehat{u}^-)) + (g^+ - g^-)\cdot\widehat{u}^- + t^*\}\,dx \\ &+ \int\limits_\Omega \chi\{(F^+_M(\nabla\widehat{u}^-) - F^-_M(\nabla\widehat{u}^-))\nabla h + (g^+ - g^-)\cdot h\}\,dx \\ &+ \frac{1}{2}\int\limits_\Omega (\chi F^+_{MM} + (1-\chi)F^-_{MM})(\nabla h,\nabla h)\,dx, \quad u \in \mathbb{X},\ u = \widehat{u}^- + h. \end{aligned} \tag{2.9}$$

Proof. Expanding in the Taylor series, we find

$$F^\pm(\nabla u) = F^\pm(\nabla\widehat{u}^+) + F^\pm_M(\nabla\widehat{u}^+)\nabla h + \frac{1}{2}F^\pm_{MM}(\nabla h,\nabla h), \quad u = \widehat{u}^+ + h. \tag{2.10}$$

From the first equality in (2.7) we find

$$\begin{aligned} J[u,\chi] = &\int\limits_\Omega \chi\{F^+(\nabla\widehat{u}^+) + F^+_M(\nabla\widehat{u}^+)\nabla h + \frac{1}{2}F^+_{MM}(\nabla h,\nabla h) + g^+\cdot\widehat{u}^+ + g^+\cdot h + t^*\}\,dx \\ &+ \int\limits_\Omega (1-\chi)\{F^-(\nabla\widehat{u}^+) + F^-_M(\nabla\widehat{u}^+)\nabla h + \frac{1}{2}F^-_{MM}(\nabla h,\nabla h) + g^-\cdot\widehat{u}^+ + g^-\cdot h\}\,dx \\ &+ \int\limits_{\partial\Omega} f\cdot\widehat{u}^+\,dS + \int\limits_{\partial\Omega} f\cdot h\,dS - \int\limits_\Omega (F^+(\nabla\widehat{u}^+) + g^+\cdot\widehat{u}^+ + t^*)\,dx - \int\limits_{\partial\Omega} f\cdot\widehat{u}^+\,dS. \end{aligned}$$

Taking into account the identity

$$\int\limits_{\Omega} \big(F_M^+(\nabla\widehat{u}^+)\nabla h + g^+\cdot h\big)\,dx + \int\limits_{\partial\Omega} f\cdot h\,dS = 0,$$

we obtain (2.8). Replacing (2.10) with

$$F^{\pm}(\nabla u) = F^{\pm}(\nabla\widehat{u}^-) + F_M^{\pm}(\nabla\widehat{u}^-)\nabla h + \frac{1}{2}F_{MM}^{\pm}(\nabla h,\nabla h), \quad u = \widehat{u}^- + h,$$

after simple calculations we arrive at (2.9). □

If we wish to determine the sign of $J[u,\chi]$, then for the zero approximation it is natural to determine the sign of the first term on the right-hand side of (2.8) with $\chi\equiv 0$ or the first term on the right-hand side of (2.9) with $\chi\equiv 1$. Taking into account the definition of t^*, we find

$$\begin{gathered}\int\limits_{\Omega}\{(F^-(\nabla\widehat{u}^+) - F^+(\nabla\widehat{u}^+)) + (g^- - g^+)\cdot\widehat{u}^+ - t^*)\}\,dx = I^-[\widehat{u}^+] - I^-[\widehat{u}^-] \geqslant 0,\\ \int\limits_{\Omega}\{(F^+(\nabla\widehat{u}^-) - F^-(\nabla\widehat{u}^-)) + (g^+ - g^-)\cdot\widehat{u}^- + t^*)\}\,dx = I^+[\widehat{u}^-] - I^+[\widehat{u}^+] \geqslant 0.\end{gathered} \tag{2.11}$$

The last inequalities are valid because for any sign $\pm$ the functions $\widehat{u}^{\pm}$ are minimizers of the functionals $I^{\pm}[u]$ over the set $\mathbb{X}$. Thus, it is reasonable to assume that the inclusion $t^*\in E_<$ holds under the condition (1.10).

Now, we conclude the proof of the theorem.

(1) By the second equality in (2.11), the integral on the left-hand side of the inequality (1.11) is equal to $-|\Omega|t^*$. Consequently, the constant on the right-hand side of (1.11) can be equal only to $-t^*$. By the assumption (1.11), there exists a set of positive measure $E\subset\Omega$ such that

$$F^+(\nabla\widehat{u}^0(x)) - F^-(\nabla\widehat{u}^0(x)) + (g^+(x) - g^-(x))\cdot\widehat{u}^0(x) + t^* < 0, \quad x\in E.$$

Taking for χ the characteristic function of this set in (2.9) and setting $h=0$, we obtain $t_- < t_+$ in view of (2.7).

(2) As was already mentioned, the constant in (1.12) should coincide with $-t^*$. Then, by (2.7) and (2.9), a sufficient condition for the validity of the inequality $t_- < t_+$ is the existence of functions $h_0\in C_0^\infty(\Omega,R^m)$ and $\chi_0\in\mathbb{Z}$ such that

$$\int\limits_{\Omega}\chi_0\{(F_M^+(\nabla\widehat{u}^0) - F_M^-(\nabla\widehat{u}^0))\nabla h_0 + (g^+ - g^-)\cdot h_0\}\,dx \neq 0. \tag{2.12}$$

Indeed, if the relation (2.12) holds, then, replacing h_0 with $-h_0$ if necessary, we see that the left-hand side of (2.12) is negative. If we multiply h_0 by a sufficiently small positive number λ, then the right-hand side of (2.9) becomes negative for $\chi=\chi_0$ and $h=\lambda h_0$.

To prove the inequality (2.12), we assume the contrary. Let

$$\begin{gathered}\int\limits_{\Omega}\chi\{\Phi\nabla h + \varphi\cdot h\}\,dx = 0,\\ \Phi = F_M^+(\nabla\widehat{u}^0) - F_M^-(\nabla\widehat{u}^0), \quad \varphi = g^+ - g^- \quad \text{for all } \chi\in\mathbb{Z},\ h\in C_0^\infty(\Omega,R^m).\end{gathered} \tag{2.13}$$

Now, we arrive to a contradiction between the assumption (2.13) and the condition (1.13).

Let a point $x_0 \in \Omega$ belong to the intersection of the Lebesgue sets for the functions Φ and φ, and let both functions are finite at this point. Let χ_δ, $\delta \in (0, \delta_0]$, be the characteristic function of a ball $B_\delta \subset \overline{B}_\delta \subset \Omega$. Then for such functions χ_δ and the function $h \in C_0^\infty(\Omega, R^m)$ defined by the formula

$$h(x) = \Phi(x_0)(x - x_0), \quad x \in B_{\delta_0}(x_0),$$

the equality (2.13) takes the form

$$\int\limits_{|x-x_0|<\delta} \{\Phi(x)\Phi(x_0) + \varphi(x) \cdot \Phi(x_0)(x - x_0)\}\, dx = 0. \tag{2.14}$$

Since x_0 is a Lebesgue point of $\Phi(x)$, we have

$$\frac{1}{|B_\delta|} \int\limits_{B_\delta(x_0)} \Phi(x)\Phi(x_0)\, dx \to \Phi^2(x_0) \quad \text{as } \delta \to 0.$$

Since x_0 is a Lebesgue point of $\varphi(x)$ and the functions Φ and φ are finite at this point, we get

$$\frac{1}{|B_\delta|} \int\limits_{B_\delta(x_0)} |\varphi(x) \cdot \Phi(x_0)(x - x_0)|\, dx \leqslant \frac{|\Phi(x_0)|\delta}{|B_\delta|} \int\limits_{B_\delta(x_0)} |\varphi(x) - \varphi(x_0)|\, dx + |\varphi(x_0)||\Phi(x_0)|\delta \to 0$$

as $\delta \to 0$. Then (2.14) implies $|\Phi(x_0)| = 0$ for almost all $x_0 \in \Omega$. Therefore, $F_M^+(\nabla\widehat{u}^0) = F_M^-(\nabla\widehat{u}^0)$ for almost all $x \in \Omega$, which contradicts the assumption (1.13).

(3) By the definition of $\widehat{u}^0$, the following equalities hold:

$$\int\limits_\Omega \{F_M^+(\nabla\widehat{u}^0)\nabla h + g^+ \cdot h\}\, dx = \int\limits_\Omega \{F_M^-(\nabla\widehat{u}^0)\nabla h + g^- \cdot h\}\, dx = 0$$

for all $h \in C_0^\infty(\Omega, R^m)$. By (1.14), these equalities imply

$$\int\limits_\Omega (g^+ - g^-) \cdot h\, dx = 0 \quad \text{for all} \quad h \in C_0^\infty(\Omega, R^m).$$

Consequently, $g^+ = g^-$.

From (1.12), (1.14), and the last equality we find that the right-hand side of (2.9) is nonnegative for all χ and h. □

3. Examples

We give some examples to illustrate the possibilities of the above theorem.

Example 1. Let

$$A^+ = A^- \equiv A, \quad \zeta^+ \neq \zeta^-, \quad g^+ = g^- \equiv g, \quad \Gamma_0 = \partial\Omega. \tag{3.1}$$

If

$$g \in L_p(\Omega, R^m), \quad u_0 \in W_p^2(\Omega, R^m), \quad p > m, \qquad \partial\Omega \in C^2, \tag{3.2}$$

then the phase transition temperatures $t_\pm$ exist and $t_- < t_+$.

Proof. The equilibrium one-phase displacement fields $\widehat{u}^\pm$ are found from the identities

$$\int\limits_\Omega \big(F_M^\pm(\nabla\widehat{u}^\pm)\nabla h + g \cdot h\big)\, dx = 0 \quad \text{for all} \quad h \in \mathbb{X}_0 = \overset{\circ}{W}{}_2^1(\Omega, R^m),$$

from which we find

$$\int\limits_{\Omega} \left(2a_{ijkl}\widehat{u}^{\pm k}_{x_l} h^i_{x_j} + g^i h^i\right) dx - 2a_{ijkl}\zeta^{\pm}_{kl} \int\limits_{\Omega} h^i_{x_j}\, dx = 0.$$

Since the second term on the left-hand side of the previous equality vanishes, we conclude that

$$\widehat{u}^+ = \widehat{u}^- \equiv \widehat{u}^0. \tag{3.3}$$

If (3.2) holds, then $\widehat{u}^0 \in C^1(\overline{\Omega}, R^m)$. Therefore, the existence of the phase transition temperatures follows from (1.9).

We prove that $t_- < t_+$. By (3.3), the equalities (1.10) hold. Since

$$F^{\pm}_{M_{ij}}(\nabla \widehat{u}^{\pm}) = F^{\pm}_{M_{ij}}(\nabla \widehat{u}^0) = 2a_{ijkl}(\widehat{u}^{0k}_{x_l} - \zeta^{\pm}_{kl}),$$

we have

$$F^{+}_{M_{ij}}(\nabla \widehat{u}^0) - F^{-}_{M_{ij}}(\nabla \widehat{u}^0) = -2a_{ijkl}(\zeta^+ - \zeta^-)_{kl}. \tag{3.4}$$

Since

$$a_{ijkl}(\zeta^+ - \zeta^-)_{kl}(\zeta^+ - \zeta^-)_{ij} > 0,$$

the right-hand side of (3.4) does not vanish.

If (1.11) holds, then $t_- < t_+$ by assertion (1) of the theorem. If (1.12) holds, then $t_- < t_+$ by assertion (2) of the theorem. □

Example 2. Let

$$g^+ = g^- \equiv 0, \quad \Gamma_0 = \partial\Omega, \quad u_0 = 0. \tag{3.5}$$

Then the phase transition temperatures exist and

$$\begin{aligned} t_- < t_+ \quad &\text{if } A^+\zeta^+ \neq A^-\zeta^-, \\ t_- = t_+ \quad &\text{if } A^+\zeta^+ = A^-\zeta^-. \end{aligned} \tag{3.6}$$

Proof. As above, we prove that

$$\widehat{u}^+ = \widehat{u}^- \equiv 0. \tag{3.7}$$

Therefore, the existence of the phase transition temperatures is guaranteed by (1.9).

From (3.7) we obtain the validity of (1.10) and (1.12). It is obvious that $F^{\pm}_M(0) = -2A^{\pm}\zeta^{\pm}$. Thus, the assertions in (3.6) are consequences of assertions (2) and (3) of the theorem. □

Example 3. Let

$$A^+ = A^- \equiv A, \quad \zeta = \zeta^+ - \zeta^- \neq 0, \quad g^+ = g^- \equiv g. \tag{3.8}$$

We assume that for $B = A\zeta$ there exists a vector $n_0 \in R^m$ such that $Bn_0 = 0$, $|n_0| = 1$. Suppose that a part Γ_1 of $\partial\Omega$ is plane and n_0 is the unit outward normal to Γ_1. Let

$$\Gamma = \partial\Omega \setminus \Gamma_1, \quad f = qn. \tag{3.9}$$

Then the inequality $t_- < t_+$ holds provided that the phase transition temperatures exist.

Proof. In our case,

$$I^{\pm}[u] = \int\limits_{\Omega} \left(F^{\pm}(\nabla u) + g \cdot u\right) dx + q \int\limits_{\Gamma_1} u \cdot n_0\, dS.$$

Therefore, the one-phase equilibrium states $\widehat{u}^{\pm} \in \mathbb{X}$ are found from the relations

$$\int\limits_{\Omega} \left(2a_{ijkl}\widehat{u}^{\pm k}_{x_l} v^i_{x_j} + g^i v^i\right) dx + (q\delta_{ij} - 2a_{ijkl}\zeta^{\pm}_{kl}) n^j_0 \int\limits_{\Gamma_1} v^i \, dS = 0, \quad v \in W^1_2(\Omega, R^m),$$
$$v\big|_{\Gamma} = 0. \tag{3.10}$$

Under the above assumptions, the factor in the second term on the left-hand side of (3.10) is independent of the choice of sign $\pm$. Therefore, (3.3) holds. Then, under the condition

$$\widehat{u}^0 \in W^2_p(\Omega, R^m), \quad p > m, \tag{3.11}$$

(1.9) implies the existence of the phase transition temperatures $t_{\pm}$. The inequality $t_- < t_+$ is established on the basis of the above theorem in the same way as in Example 1.

The inclusion (3.11) can be realized if we choose u_0 providing the boundary condition on Γ in such a way that $u_0 \in W^2_p(\Omega, R^m)$, $p > m$, $2a_{ijkl}u^k_{0x_l} n^j_0 + (q\delta_{ij} - 2a_{ijkl}\zeta^{\pm}_{kl}) n^j_0 = 0$, $x \in \Gamma_1$, and set $g^i = 2(a_{ijkl}u^k_{0x_l})_{x_j}$. In this case, $\widehat{u}^0 \equiv u_0$ and the inclusion (3.11) holds. □

4. Example of Nonexistence of Solutions for $t \in (t_-, t_+)$

We consider the functional (1.1) with

$$g^+ = g^- = 0, \quad \Gamma_0 = \partial\Omega, \quad u_0 = 0, \quad t = 0. \tag{4.1}$$

Let us construct the residual strain tensors. We set $N_{\pm} = \{x \in R^m : \zeta^{\pm} x = 0\}$ and assume that

$$N_+ = N_- \equiv N, \quad \dim N = m - 1. \tag{4.2}$$

Let

$$l \in R^m, \quad |l| = 1, \quad l \perp N. \tag{4.3}$$

Since the matrices $\zeta^{\pm}$ are symmetric, we have

$$\zeta^{\pm} l = \alpha_{\pm} l, \quad \alpha_{\pm} \in R^1.$$

We fix $\zeta^{\pm}$ by the requirement

$$\zeta^+ l = l, \quad \zeta^- l = -l. \tag{4.4}$$

It is obvious that the functions

$$u^{\pm}(x) = \zeta^{\pm} x - x^{\pm}_0, \qquad x,\, x^{\pm}_0 \in R^m \tag{4.5}$$

satisfy the equation

$$e(\nabla u^{\pm}) = \zeta^{\pm}. \tag{4.6}$$

Using the formula

$$x = \widetilde{x} + sl, \qquad \widetilde{x} \in N, \quad s \in R^1, \tag{4.7}$$

we write the functions $u^{\pm}$ in the form

$$u^{\pm}(x) = \pm sl - x^{\pm}_0. \tag{4.8}$$

We fix $\delta > 0$. We choose vectors $x^{\pm}_0$ such that

$$u^-(x) = 0, \quad x \in \{N + 2\delta l\},$$
$$u^-(x) = u^+(x), \quad x \in \{N + \delta l\}, \tag{4.9}$$
$$u^+(x) = 0, \quad x \in N.$$

The conditions (4.9) are realized for $x_0^+ = 0$, $x_0^- = -2\delta l$. For such $x_0^\pm$ the functions (4.8) take the form

$$u^+(x) = sl, \quad u^-(x) = (2\delta - s)l. \tag{4.10}$$

For x in the representation (4.7) we introduce the composed function

$$u_\delta(x) = \begin{cases} 0, & s \leqslant 0, \\ sl, & s \in [0, \delta], \\ (2\delta - s)l, & s \in [\delta, 2\delta], \\ 0, & s \geqslant 2\delta. \end{cases} \tag{4.11}$$

We divide the space R^m into strips

$$\begin{gathered} \Pi_k = \{x \in R^m : \text{in the representation (4.7) the number } s \in [2k\delta, 2(k+1)\delta]\} \\ k = \ldots, -2, -1, 0, 1, 2, \ldots \end{gathered} \tag{4.12}$$

For every strip Π_k we define the function

$$u_\delta^k(x) = u_\delta(x + 2k\delta l).$$

Then the function

$$\widehat{u}_\delta(x) = \Sigma_{k=-\infty}^{\infty} u_\delta^k(x) \tag{4.13}$$

possesses the properties

$$\widehat{u}_\delta \in W^1_{2,\mathrm{loc}}(R^m, R^m), \qquad |\widehat{u}_\delta(x)| \leqslant C\delta \quad \text{for all} \quad x \in R^m. \tag{4.14}$$

Furthermore, by (4.6),

$$\begin{gathered} e(\nabla\widehat{u}_\delta) = \widehat{\chi}_\delta \zeta^+ + (1 - \widehat{\chi}_\delta)\zeta^-, \\ \widehat{\chi}_\delta = \begin{cases} 1, & x \in \Pi_k^+,\ \Pi_k^+ = \{x \in \Pi_k : s \in [2k\delta, (2k+1)\delta]\}, \\ 0, & x \in \Pi_k^-,\ \Pi_k^- = \{x \in \Pi_k : s \in ((2k+1)\delta, 2(k+1)\delta)\}. \end{cases} \end{gathered} \tag{4.15}$$

By the assumption (4.1) and property (4.15), we have

$$I[\widehat{u}_\delta, \widehat{\chi}_\delta, 0] = 0. \tag{4.16}$$

Lemma 1. *If the assumption* (4.1) *holds for the tensors* $\zeta^\pm$ *satisfying the conditions* (4.2) *and* (4.4)*, then*

$$\inf_{u \in \mathbb{X}, \chi \in \mathbb{Z}} I[u, \chi, 0] = 0. \tag{4.17}$$

Proof. For any sufficiently small positive number ρ we introduce the function φ_ρ such that

$$\varphi_\rho \in C_0^\infty(\Omega), \qquad \varphi_\rho(x) = 1 \quad \text{for} \quad \operatorname{dist}(x, \partial\Omega) > \rho, \qquad |\nabla\varphi_\rho(x)| \leqslant \frac{C}{\rho}. \tag{4.18}$$

Since

$$e(\nabla(\varphi_\rho \widehat{u}_\delta)) = \varphi_\rho e(\nabla\widehat{u}_\delta) + e(\nabla\varphi_\rho \otimes \widehat{u}_\delta), \quad (\nabla\varphi_\rho \otimes \widehat{u}_\delta)_{ij} = \varphi_{\rho x_i} \widehat{u}_\delta^j, \tag{4.19}$$

from (4.15) and (4.18) for some positive constant $C \neq C(\rho, \delta)$ it follows that

$$\begin{gathered} e(\nabla(\varphi_\rho \widehat{u}_\delta)) = \widehat{\chi}_\delta \zeta^+ + (1 - \widehat{\chi}_\delta)\zeta^-, \quad \operatorname{dist}(x, \partial\Omega) > \rho, \\ |e(\nabla(\varphi_\rho \widehat{u}_\delta))| \leqslant C(\delta/\rho + 1), \quad \operatorname{dist}(x, \partial\Omega) \leqslant \rho. \end{gathered}$$

Therefore,

$$\widehat{\chi}_\delta F^+(\nabla(\varphi_\rho \widehat{u}_\delta)) + (1-\widehat{\chi}_\delta)F^-(\nabla(\varphi_\rho \widehat{u}_\delta)) = 0, \quad \mathrm{dist}(x, \partial\Omega) > \rho,$$

$$\widehat{\chi}_\delta F^+(\nabla(\varphi_\rho \widehat{u}_\delta)) + (1-\widehat{\chi}_\delta)F^-(\nabla(\varphi_\rho \widehat{u}_\delta)) \leqslant C(\delta/\rho + 1)^2, \quad \mathrm{dist}(x, \partial\Omega) \leqslant \rho.$$

Thus,

$$I[\widehat{u}_\delta \varphi_\rho, \widehat{\chi}_\delta, 0] \leqslant C(\delta/\rho + 1)^2 \rho.$$

Setting $\rho = \delta$ in the last inequality and taking into account the inclusion $\widehat{u}_\delta \varphi_\rho \in \mathbb{X}$ and the fact that the functional $I[u, \chi, 0]$ is nonnegative, we obtain (4.17). □

If, under the condition (4.1), the functional $I[u, \chi, 0]$ possesses an equilibrium state $\widehat{u}$, $\widehat{\chi}$, then, by the lemma,

$$e(\nabla \widehat{u}) = \widehat{\chi}\zeta^+ + (1-\widehat{\chi})\zeta^-. \tag{4.20}$$

Conversely, any solution $\widehat{u} \in \mathbb{X}$, $\widehat{\chi} \in \mathbb{Z}$ to the problem (4.20) is an equilibrium state of the functional $I[u, \chi, 0]$.

Lemma 2. *If $\zeta^\pm$ satisfy the conditions* (4.2) *and* (4.4)*, then the problem* (4.20) *does not have a solution $\widehat{u} \in \mathbb{X}$, $\widehat{\chi} \in \mathbb{Z}$.*

Proof. From (4.20) we find

$$\widehat{u}^i_{x_j} + \widehat{u}^j_{x_i} = 2(\widehat{\chi}\zeta^+_{ij} + (1-\widehat{\chi})\zeta^-_{ij}). \tag{4.21}$$

Let τ be a unit vector in N. Multiplying both sides of (4.21) by $\tau_i\tau_j$, $\tau_i l_j$, $l_i l_j$ and taking the sum over repeated indices, we obtain the relations

$$\frac{\partial(\widehat{u}\cdot\tau)}{\partial\tau} = 0, \quad \frac{\partial(\widehat{u}\cdot\tau)}{\partial l} + \frac{\partial(\widehat{u}\cdot l)}{\partial\tau} = 0, \quad \frac{\partial(\widehat{u}\cdot l)}{\partial l} = 2\widehat{\chi} - 1. \tag{4.22}$$

We fix $\tau \in N$ and denote by S the subspace R^m orthogonal to this vector. It is obvious that $l \in S$. We set

$$\Omega_x = \{r \in R^1 : \Omega \cap \{x + r\tau\}\}, \quad x \in S.$$

Since Ω_x is an open set on the line, it is either empty or the union of at most countable family of open disjoint intervals l_j.

The function $\widehat{u} \in \mathbb{X} = \overset{\circ}{W}{}^1_2(\Omega, R^m)$. Consequently, for almost all $x \in S$ such that $\Omega_x \neq \varnothing$ the restriction of $\widehat{u}\cdot\tau$ onto the interval l_j belongs to the space $\overset{\circ}{W}{}^1_2(l_j)$ for every j and the Sobolev derivative with respect to $r \in l_j$ of this restriction coincides with the restriction of the Sobolev derivative $\partial\widehat{u}\cdot\tau/\partial\tau$ to this interval.

The above arguments show that from the first equality in (4.22) it follows that $\widehat{u}\cdot\tau = 0$ on the set of full measure $E_\tau \subset \Omega$. Then, by the same reasons, the second equality in (4.22) leads to the relation $\widehat{u}\cdot l = 0$ on the set of full measure $E'_\tau \subset \Omega$. Repeating the arguments for the elements of the basis $\tau^1, \ldots, \tau^{m-1}$ for the space N, we conclude that $\widehat{u} = 0$ almost everywhere in Ω. The last equality contradicts the third equation in (4.22). □

We note that functions of type $\widehat{u}_\delta$ are traditionally used in the study of variational problems of elastic media with microstructure [**10**].

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References

1. M. A. Grinfel'd, *Methods of Continuum Mechanics in the Theory of Phase Transitions* [in Russian], Nauka, Moscow (1990).
2. V. G. Osmolovskii, "Criterion for the lower semicontinuity of the energy functional of a two-phase elastic medium" [in Russian], *Probl. Mat. Anal.* **26**, 215–254 (2003); English transl.: *J. Math. Sci. (New York)* **117**, No. 3, 4211–4236 (2003).
3. B. Dacorogna, *Direct Methods in the Calculus of Variations*, Berlin (1989).
4. B. Dacorogna, G. Pisante, and A. Ribero, "On non quasiconvex problem of the calculus of variations," Sect. Math. EPFL, 1015 Lausanne, Switzerland. February 4, 2005.
5. V. G. Osmolovskii, "Exact solutions to the variational problem of the phase transition theory in continuum mechanics" [in Russian], *Probl. Mat. Anal.* **27**, 171–206 (2004); English transl.: *J. Math. Sci. (New York)* **120**, No. 2, 1167–1190 (2004).
6. V. G. Osmolovskii, "Existence of equilibrium states in the one-dimensional phase transition problem" [in Russian], *Vest. S. Peterburg. State Univ. Ser. 1*, No. 3, 54-65 (2006).
7. V. G. Osmolovskii, "On the solvability of a variational problem about phase transitions in continuum mechanics" [in Russian], *Probl. Mat. Anal.* **38** 61–72 (2008); English transl.: *J. Math. Sci. (New York)* **156**, No. 4, no. 2, 632–643 (2009).
8. V. G. Osmolovskii, "Existence of phase transition temperatures of a nonhomogeneous anisotropic two-phase elastic medium" [in Russian], *Probl. Mat. Anal.* **31** 59-66 (2005); English transl.: *J. Math. Sci. (New York)* **132**, No. 4, 441–450 (2006).
9. V. G. Osmolovskii, "Dependence of the temperature of phase transitions on the size of the domain" [in Russian], *Zap. Nauchn. Semin. POMI* **310**, 98–114 (2004); English transl.: *J. Math. Sci. (New York)* **132**, No. 3, 304–312 (2006).
10. S. Muller, "Variational models for microstructure and phase transitions," Preprint. Max-Planck-Institut, Leipzig, Lecture notes N2, (1998).

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