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THE THIRD BOUNDARY-VALUE PROBLEM FOR PARABOLIC DIFFERENTIAL-DIFFERENCE EQUATIONS

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1. Introduction

In [11, 16, 17], the first boundary-value problem for parabolic differential-difference equations with translations with respect to spatial variables was considered. In [12], it is found that problems of the above kind are related to nonlocal problems. The second boundary-value problem for parabolic differential-difference equations was originally considered in [10]. Regions of dimension $n > 1$ are considered in those papers. In [9], the first boundary-value problem was studied for $n = 1$. It is proved in the above papers that, unlike parabolic equations, the smoothness of solutions may be broken inside the region even in the case where the initial function is infinitely differentiable.

In [6, 17], the strong solvability is investigated. It turns out that the arising spaces of the initial data are related to the well-known Kato problem on the square root of the operator constructed with respect to a sectorial form (see [3, Ch. VI, Sec. 2, Remark 2.29]). In [9, 11], comprehensive classes of functional-differential operators satisfying the Kato hypothesis are provided.

In this paper, we study the solvability and the smoothness of strong solutions of the third boundary-value problem for parabolic differential-difference equations with translations with respect to spatial variables for $n \geq 2$. The methods applied are based on the theory of elliptic functional-differential equations (see [13, 18]).

Note that boundary-value problems for parabolic functional-differential equations arise in the theory of nonlinear optical two-dimensional feedback systems (see, e.g., [8, 14, 15, 22]).

Parabolic functional-differential equations with time delay have been studied by many authors; in [20, 21], the most general case, including variable delays in the higher derivatives is considered.

2. Problem Setting

1. Let Q be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. Let $\partial Q = \bigcup_{i=1}^{N_0} \overline{M}_i$ be its piecewise-smooth boundary, where M_i are $(n-1)$-dimensional C^∞-manifolds open and connected with respect to the topology

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of ∂Q. Suppose that Q satisfies the cone condition in a neighborhood of any point $x \in \mathrm{K} = \partial Q \setminus \bigcup\limits_i M_i$; in particular, it is diffeomorphic to a plane angle for $n = 2$.

Introduce bounded difference operators R_{ij}, $R_i \colon L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ as follows:

$$(R_{ij}u)(x) = \sum_{h \in M} a_{ijh} u(x+h) \quad (i,\, j = 1,\, \ldots,\, n),$$

$$(R_i u)(x) = \sum_{h \in M} a_{ih} u(x+h) \quad (i = 0,\, 1,\, \ldots,\, n).$$

Here a_{ijh} and a_{ih} are complex numbers, while the set M consists of a finite number of vectors with integer-valued coordinates.

Introduce linear operators I_Q, P_Q, R_{ijQ}, and R_{iQ}. The operator $I_Q \colon L_2(Q) \to L_2(\mathbb{R}^n)$ extends functions outside Q by the identical zero, the operator $P_Q \colon L_2(\mathbb{R}^n) \to L_2(Q)$ is the restriction of functions to Q, while the operators R_{ijQ}, $R_{iQ} \colon L_2(Q) \to L_2(Q)$ are defined as follows: $R_{ijQ} = P_Q R_{ij} I_Q$, $R_{iQ} = P_Q R_i I_Q$.

Consider the differential-difference equation

$$u_t(x,t) - \sum_{i,j=1}^{n} \big(R_{ijQ} u_{x_j}(x,\, t)\big)_{x_i} + \sum_{i=1}^{n} R_{iQ} u_{x_i}(x,\, t) + R_{0Q} u(x,\, t) = f(x,t) \quad ((x,t) \in Q_T) \tag{2.1}$$

with the boundary condition

$$\sum_{i,j=1}^{n} R_{ijQ} u_{x_j} \cos(\nu,\, x_i) + \sigma(x) u = 0 \qquad ((x,t) \in \Gamma_T) \tag{2.2}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q)\,, \tag{2.3}$$

where $Q_T = Q \times (0,\, T)$, $0 < T < \infty$, $\Gamma_T = (\partial Q \setminus \mathrm{K}) \times (0,\, T)$, ν is the unit vector of the outer normal to Γ_T, $f \in L_2(Q_T)$, $\varphi \in L_2(Q)$, $\sigma \in C(\partial Q)$, and $\sigma \geq 0$ in ∂Q.

Let $W_2^k(Q)$ be the Sobolev space of complex-valued functions from $L_2(Q)$ such that all their generalized derivatives up to the order k belong to $L_2(Q)$; the norm in $W_2^k(Q)$ is introduced as follows:

$$\|v\|_{W_2^k(Q)} = \left\{ \sum_{|\alpha| \leq k} \int_Q |D^\alpha v(x)|^2 \, dx \right\}^{1/2}.$$

Introduce in $L_2(Q)$ a sesquilinear form $a_R[v,\, w]$ with the domain $W_2^1(Q)$ as follows:

$$a_R[v,\, w] = \sum_{i,j=1}^{n} \big(R_{ijQ} v_{x_j}, w_{x_i}\big)_{L_2(Q)} + \sum_{i=1}^{n} \left(R_{iQ} v_{x_i}, w\right)_{L_2(Q)} + \left(R_{0Q} v, w\right)_{L_2(Q)} + (v, w)_{L_2(\partial Q)}. \tag{2.4}$$

The difference operators R_{ijQ}, R_{iQ}, $R_{0Q} \colon L_2(Q) \to L_2(Q)$ are bounded. Therefore, there exists a positive c_0 such that

$$|a_R[v,\, w]| \leq c_0 \|v\|_{W_2^1(Q)} \|w\|_{W_2^1(Q)} \qquad (v, w \in W_2^1(Q)). \tag{2.5}$$

The sesquilinear form $a_R[v,\, w]$ is continuous in $W_2^1(Q)$ with respect to w. Hence, there exists a linear bounded operator $A_R \colon W_2^1(Q) \to [W_2^1(Q)]'$ such that

$$\langle A_R v,\, \overline{w} \rangle = a_R[v,\, w] \qquad (v,\, w \in W_2^1(Q)), \tag{2.6}$$

where $[W_2^1(Q)]'$ is the space dual to $W_2^1(Q)$.

Definition 2.1. The operator A_R is called *strongly elliptic* if there exist $c_1 > 0$ and $c_2 \geq 0$ such that

$$\operatorname{Re} \langle A_R v,\, v \rangle \geq c_1 \|v\|^2_{W_2^1(Q)} - c_2 \|v\|^2_{L_2(Q)} \qquad (v \in W_2^1(Q)). \tag{2.7}$$

2. To formulate necessary and sufficient conditions of the strong ellipticity of the operator A_R in algebraic terms, we have to introduce the following notation. The additive group generated by the set M is denoted by G. Open connected components of the set $Q\setminus \bigcup_{h\in G}(\partial Q+h)$ are denoted by Q_r.

Definition 2.2. Any set Q_r is called a *subdomain*. The set of all subdomains Q_r is called a *partition of the domain* Q and is denoted by $\mathcal{R}$.

The partition $\mathcal{R}$ is decomposed in nonintersecting classes in the following natural way. Subdomains $Q_{r_1}, Q_{r_2} \in \mathcal{R}$ belong to the same class if there exists a vector $h \in G$ such that $Q_{r_2} = Q_{r_1} + h$. Denote the subdomains by Q_{sl}, where s is the class number ($s = 1, 2, \ldots$), while l is the number of the subdomain within the sth class. By virtue of the boundedness of Q, each class consists of a finite number $N = N(s)$ of subdomains Q_{sl} and we have $N(s) \le (\operatorname{diam} Q+1)^n$. In general, the set of classes is countable. Let h_{sl} denote a vector $h \in G$ such that $Q_{s1} + h = Q_{sl}$. Obviously, $h_{s1} = 0$.

If $v \in W_2^k(Q)$, then the function $R_{ijQ}v$ can be outside $W_2^k(Q)$. However, the following assertion is valid (cf. [13, Sec. 8, Lemma 8.15]).

Lemma 2.1. *Let $v \in W_2^k(Q'_{sl})$, where $Q'_{sl} \subset Q_{sl}$ and $Q'_{sl} = Q'_{s1} + h_{sl}$. Then $R_{ijQ}v \in W_2^k(Q'_{sl})$ and $(R_{ijQ}v)_{x_p}(x) = (R_{ijQ}v_{x_p})(x)$ ($x \in Q'_{sl}$, $p = 1, \ldots, n$).*

Introduce the matrices R_{ijs} and R_{is} of order $N(s) \times N(s)$ with elements

$$r_{km}^{ijs} = \begin{cases} a_{ijh}, & h = h_{sm} - h_{sk} \in M, \\ 0, & h_{sm} - h_{sk} \notin M, \end{cases} \tag{2.8}$$

$$r_{km}^{is} = \begin{cases} a_{ih}, & h = h_{sm} - h_{sk} \in M, \\ 0, & h_{sm} - h_{sk} \notin M. \end{cases} \tag{2.9}$$

Let R_s denote the block matrices $\|R_{ijs}\|_{i,j=1}^n$ of order $nN(s) \times nN(s)$ ($s = 1, 2, \ldots$). Since Q is a bounded domain, it follows from (2.8) that the number of different matrices R_{s_k} is finite ($k = 1, 2, \ldots, N_1$).

Lemma 2.2. *Let the matrices $R_{s_k} + R_{s_k}^*$ be positive definite for all $k = 1, 2, \ldots, N_1$, where $R_{s_k}^*$ are Hermitian-adjoint matrices. Then* (2.7) *holds for any v from $W_2^1(Q)$.*

Proof. By virtue of [18, Lemma 1], there exists a positive k_1 such that

$$\operatorname{Re} \sum_{i,j=1}^n \left(R_{ijQ}v_{x_j}, v_{x_i}\right)_{L_2(Q)} \ge k_1\|v\|_{W_2^1(Q)}^2 - k_1\|v\|_{L_2(Q)}^2 \quad (v \in W_2^1(Q)). \tag{2.10}$$

On the other hand, since the operators R_{iQ} are bounded ($i = 0, 1, \ldots, n$), we have

$$\begin{aligned} \left|\sum_{i=1}^n (R_{iQ}v_{x_i}, v)_{L_2(Q)} + (R_{0Q}v, v)_{L_2(Q)}\right| &\le k_2\|v\|_{W_2^1(Q)}\|v\|_{L_2(Q)} \\ &\le k_2\left(\varepsilon^2\|v\|_{W_2^1(Q)}^2 + \varepsilon^{-2}\|v\|_{L_2(Q)}^2\right) \end{aligned} \tag{2.11}$$

for all $\varepsilon > 0$ and $v \in W_2^1(Q)$, where k_2 is positive and does not depend on v and ε.

Taking into account the nonnegativity of the latter term of $(\sigma v, v)_{L_2(Q)}$ and choosing a positive ε such that $k_2\varepsilon^2 < k_1/2$, we obtain (2.7) with $c_1 = k_1/2$ and $c_2 = k_1 + k_2\varepsilon^{-2}$. □

Lemma 2.3. *Let* (2.7) *hold for any v from $W_2^1(Q)$. Then the matrices*

$$\sum_{i,j=1}^n \left(R_{ijs} + R_{ijs}^*\right)\xi_i\xi_j$$

are positive definite for all $s = 1, 2, \ldots$ and $0 \ne \xi \in \mathbb{R}^n$.

This assertion follows from [13, Sec. 9, Theorem 9.1].

In the sequel, we assume that the operator A_R corresponding to (2.1) is strongly elliptic. In this case, it is natural to say that problem (2.1)–(2.3) is the third boundary-value problem for a parabolic differential-difference equation.

3. Weak Solutions

1. Consider the space $\mathcal{V} = L_2\left(0,\, T;\; W_2^1(Q)\right)$. Its dual space is $\mathcal{V}' = L_2\left(0,\, T;\; \left[W_2^1(Q)\right]'\right)$.

Define a bounded operator $L_R\colon \mathcal{V} \to \mathcal{V}'$ as follows:

$$L_R v(\cdot,\, t) = A_R v(\cdot,\, t) \quad \text{for almost all } t \in (0,\, T).$$

Introduce the Hilbert space

$$\mathcal{W} = \left\{ v \in \mathcal{V} :\; v_t \in \mathcal{V}' \right\}$$

with the norm

$$\|v\|_{\mathcal{W}} = \left\{ \int_0^T \|v\|^2_{W_2^1(Q)}\, dt + \int_0^T \|v_t\|^2_{\left[W_2^1(Q)\right]'}\, dt \right\}^{1/2}.$$

Here the derivatives are understood in the sense of distributions over Q_T.

Let $f \in \mathcal{V}'$ and $\varphi \in L_2(Q)$.

Definition 3.1. A function $u \in \mathcal{W}$ is called a *weak solution of problem* (2.1)–(2.3) if it satisfies the equation

$$\frac{du}{dt} + L_R u = f \quad \text{for almost all } \; t \in (0,\, T) \tag{3.1}$$

and the initial condition

$$u|_{t=0} = \varphi. \tag{3.2}$$

Note that, by virtue of [5, Ch. 1, Theorem 3.1 and Proposition 2.1], we have $\mathcal{W} \subset C\left([0,\, T],\, L_2(Q)\right)$, i.e., the trace $u|_{t=0} \in L_2(Q)$ is well defined.

In the sequel, we assume that $c_2 = 0$ in (2.7). Otherwise, we set $u = z\, e^{c_2 t}$. Then problem (3.1), (3.2) is equivalent to the problem $\dfrac{dz}{dt} + (L_R + c_2 I)z = e^{-c_2 t} f$, $z|_{t=0} = \varphi$.

Theorem 3.1. *Let the operator A_R be strongly elliptic. Then problem* (2.1)–(2.3) *has a unique weak solution $u \in \mathcal{W}$ for all $f \in \mathcal{V}'$ and $\varphi \in L_2(Q)$.*

The proof follows from inequalities (2.5) and (2.7) and [4, Ch. 3, Theorem 1.2].

2. Suppose that $f \in L_2(Q_T)$ and $\varphi \in L_2(Q)$. Then we can formulate the definition of the weak solution of problem (2.1)–(2.3) by means of an integral identity.

Let $W_2^{k,0}(Q_T)$ be the space of functions $u \in L_2(Q_T)$ such that all their generalized derivatives with respect to x up to the kth order belong to $L_2(Q_T)$; the norm in $W_2^{k,0}(Q_T)$ is defined as follows:

$$\|v\|_{W_2^{k,0}(Q_T)} = \left\{ \sum_{|\alpha| \le k} \int_{Q_T} \left|D_x^\alpha v(x,\, t)\right|^2\, dx\, dt \right\}^{1/2}.$$

It is easy to see that $\mathcal{V} = W_2^{1,0}(Q_T)$ due to the Fubini theorem.

Definition 3.2. A function $u \in W_2^{1,0}(Q_T)$ is called a *weak solution of problem* (2.1)–(2.3) if

$$\int\limits_{Q_T} \left\{ -u\bar{v}_t + \sum_{i,j=1}^{n} (R_{ijQ} u_{x_j}) \bar{v}_{x_i} + \sum_{i=1}^{n} (R_{iQ} u_{x_i}) \bar{v} + (R_{0Q} u)\bar{v} \right\} dx\, dt + \int\limits_{\Gamma_T} \sigma u \bar{v}\, dS dt = \int\limits_{Q_T} f\bar{v}\, dx\, dt + \int\limits_{Q} \varphi \bar{v}|_{t=0}\, dx \quad (3.3)$$

for all $v \in W_2^1(Q_T)$ such that $v|_{t=T} = 0$.

Identity (3.3) can be obtained from Eq. (2.1) by means of formal integration by parts.

Theorem 3.2. *If the operator* A_R *is strongly elliptic,* $f \in L_2(Q_T)$*, and* $\varphi \in L_2(Q)$*, then Definitions* 3.1 *and* 3.2 *are equivalent to each other.*

Proof. Taking into account [5, Ch. 3, Theorems 4.1 and 4.2], we see that it suffices to prove that the set $\mathcal{V}_1 = \big\{v \in W_2^1(Q_T) : v|_{t=T} = 0\big\}$ is dense in the space $\mathcal{V}_2 = \left\{ v \in \mathcal{V} : \dfrac{dv}{dt} \in \mathcal{V}',\, v|_{t=T} = 0 \right\}$.

Let $v \in \mathcal{V}_2$. Consider the functions $\xi_k \in C^\infty[0, T]$ such that $0 \le \xi_k(t) \le 1$, $|\xi_k'(t)| \le C\, k$, and

$$\xi_k(t) = \begin{cases} 1, & 0 \le t \le T - \dfrac{2}{k}, \\ 0, & T - \frac{1}{k} \le t \le T. \end{cases}$$

It is easy to check that $\xi_k\, v \to v$ in $\mathcal{V}_2$ as $k \to \infty$. Smoothing out the functions $\xi_k\, v$ with respect to t, we obtain a sequence of $W_2^1(Q)$-valued functions $v_k(t)$ infinitely differentiable with respect to t and such that their supports are subsets of $[0, T)$. By construction, $v_k \to v$ in $\mathcal{V}_2$ and $v_k \in \mathcal{V}_1$. □

4. Strong Solutions

1. To prove the existence of a strong solution, i.e., a solution differentiable with respect to t, we use the semigroup theory.

Definition 4.1. A strongly continuous operator semigroup $\{T_t\}$ $(t \ge 0)$ in a Hilbert space H is called *contractive* if $\|T_t\| \le 1$ $(t \ge 0)$.

Introduce the notation $\Delta_\omega = \{z \in \mathbb{C} : |\arg z| < \omega\}$, where $0 < \omega < \pi$.

Definition 4.2. A family of linear bounded operators $\{T_z\}$ $(z \in \Delta_\omega)$ in H is called an *analytic semigroup* in Δ_ω if

(1) the function $z \to T_z$ is analytic in Δ_ω;
(2) we have $T_0 = I$ and $\lim\limits_{z \to 0,\, z \in \Delta_\omega} T_z x = x$ $(x \in H)$;
(3) the relation $T_{z_1 + z_2} = T_{z_1} T_{z_2}$ $(z_1, z_2 \in \Delta_\omega)$ is valid.

A semigroup $\{T_t\}$ $(t \ge 0)$ is called *analytic* if there exists an angle Δ_ω such that there exists an analytic continuation T_z of the operator-function T_t to Δ_ω.

Definition 4.3. A sesquilinear form $a[v, w]$ is called a *sectorial form with vertex* γ if its range is a subset of an angle of the kind

$$\{\zeta \in \mathbb{C} : |\arg(\zeta - \gamma)| \le \theta\}, \qquad 0 \le \theta < \frac{\pi}{2}, \qquad \gamma \in \mathbb{R}.$$

Lemma 4.1. *Let the operator* A_R *be strongly elliptic. Then the form* $a_R[v, w]$ *is sectorial with vertex* 0.

Proof. By assumption, we have $c_2 = 0$ in (2.7). It follows from (2.5) and (2.7) that

$$|\operatorname{Im} a_R[v, v]| \le |a_R[v, v]| \le c_0 \|v\|^2_{W_2^1(Q)} \le k \operatorname{Re} a_R[v, v],$$

where $k = c_0/c_1$.

This means that there exists $0 \le \theta < \dfrac{\pi}{2}$ such that the inequality

$$|\operatorname{Im} a_R[v, v]| \le (\tan\theta) \operatorname{Re} a[v, v]$$

holds for any $v \in W_2^1(Q)$. This yields the assertion of the lemma. □

Definition 4.4. We say that a linear operator A is *m-sectorial with top γ* if there exists $\alpha \in \mathbb{R}$ such that the operator $B = A + \alpha I$ satisfies the following conditions:

$$\text{there exists a bounded operator } (B + \lambda I)^{-1} \text{ for } \operatorname{Re}\lambda > 0 \text{ and}$$
$$\|(B + \lambda I)^{-1}\| \le (\operatorname{Re}\lambda)^{-1}, \ \operatorname{Re}\lambda > 0.$$

There exists $\theta \in [0, \frac{\pi}{2})$ such that the numerical range of the operator A is a subset of the angle

$$|\arg(\zeta - \gamma)| \le \theta.$$

The space $W_2^1(Q)$ is complete. Hence, the form $a_R[v, w]$ is closed. It follows from Lemma 4.1 and the first representation theorem (see [3, Ch. VI, Sec. 2, Theorem 2.1]) that there exists an m-sectorial operator $\mathcal{A}_\mathcal{R} \colon \mathcal{D}(\mathcal{A}_\mathcal{R}) \to L_2(Q)$, $\mathcal{D}(\mathcal{A}_\mathcal{R}) \subset L_2(Q)$, with vertex 0 and such that

$$a_R[v, w] = (\mathcal{A}_\mathcal{R} v, w)_{L_2(Q)} \qquad (v \in \mathcal{D}(\mathcal{A}_\mathcal{R}),\ w \in W_2^1(Q)); \tag{4.1}$$

moreover, $\mathcal{D}(\mathcal{A}_\mathcal{R})$ is dense in $W_2^1(Q)$. Note that (4.1) can be written as

$$A_R v = \mathcal{A}_\mathcal{R} v \qquad (v \in \mathcal{D}(\mathcal{A}_\mathcal{R})) \tag{4.2}$$

(by virtue of (2.6)).

Introduce a scalar product in $\mathcal{D}(\mathcal{A}_\mathcal{R})$ as follows:

$$(v, w)_{\mathcal{D}(\mathcal{A}_\mathcal{R})} = (\mathcal{A}_R v, \mathcal{A}_R w)_{L_2(Q)} + (v, w)_{L_2(Q)}.$$

Since the operator $\mathcal{A}_R$ is closed, it follows that $\mathcal{D}(\mathcal{A}_\mathcal{R})$ is a Hilbert space.

Theorem 4.1. *The operator $(-\mathcal{A}_\mathcal{R})$ generates an analytic contractive semigroup.*

Proof. The operator $\mathcal{A}_\mathcal{R}$ is m-sectorial and its vertex is 0. It follows from [3, Ch. IX, Sec. 1, Theorem 1.24] that the operator $(-\mathcal{A}_\mathcal{R})$ generates an analytic contractive semigroup. □

The next auxiliary result, which follows from [5, Ch. 2, Theorem 9.1], is used to investigate the smoothness of strong solutions of problem (2.1)–(2.3) (Definition 4.5).

Lemma 4.2. *The equation*

$$\mathcal{A}_R w = f_0 \tag{4.3}$$

has a unique solution $w \in \mathcal{D}(\mathcal{A}_\mathcal{R})$ for any function $f_0 \in L_2(Q)$ and

$$\|w\|_{W_2^1(Q)} \le C \|f_0\|_{L_2(Q)}, \tag{4.4}$$

where a positive C does not depend on f_0.

2. Now we investigate the existence and uniqueness of the strong solution of problem (2.1)–(2.3).

Introduce Hilbert space

$$\mathcal{W}(\mathcal{A}_\mathcal{R}) = \{w \in L_2(0, T; \mathcal{D}(\mathcal{A}_\mathcal{R})) : w_t \in L_2(Q_T)\}$$

with the scalar product

$$(v, w)_{\mathcal{W}(\mathcal{A}_\mathcal{R})} = \int_0^T (\mathcal{A}_\mathcal{R} v, \mathcal{A}_\mathcal{R} w)_{L_2(Q)}\, dt + \int_0^T (v, w)_{L_2(Q)}\, dt + \int_0^T (v_t, w_t)_{L_2(Q)}\, dt,$$

where the derivatives are treated in the sense of distributions over Q_T.

Definition 4.5. A weak solution $u(x, t)$ of problem (2.1)–(2.3) is called a *strong solution* if $u \in \mathcal{W}(\mathcal{A}_{\mathcal{R}})$.

Let X and Y be Hilbert spaces such that X is continuous and densely embedded into Y. For any $\psi \in Y$ and any positive t, we define the functional

$$K(t,\, \psi;\, X,\, Y) = \inf \left(\|\psi_1\|_X^2 + t^2\|\psi_2\|_Y^2\right)^{1/2}.$$

Here $\psi_1 \in X$ and $\psi_2 \in Y$ are such that $\psi = \psi_1 + \psi_2$.

Introduce the interpolation space

$$[X,\, Y]_{1/2} = \left\{\psi \in Y : \int_0^\infty t^{-2} K^2(t,\, \psi;\, X,\, Y)\, dt < \infty\right\}$$

with the norm

$$\|\psi\|_{[X,\, Y]_{1/2}} = \left\{\|\psi\|_Y^2 + \int_0^\infty t^{-2} K^2(t,\, \psi;\, X,\, Y)\, dt\right\}^{1/2}$$

(see, e.g., [19] for detailed treatment of interpolation theory).

Theorem 4.2. *For any $f \in L_2(Q_T)$ and $\varphi \in [\mathcal{D}(\mathcal{A}_{\mathcal{R}}),\, L_2(Q)]_{1/2}$, problem* (2.1)–(2.3) *has a unique solution. It is defined by the relation*

$$u(x,\, t) = T_t\varphi(x) + \int_0^t T_{t-s} f(x,\, s)\, ds, \tag{4.5}$$

where $\{T_t\}$ $(t \geq 0)$ is an analytic semigroup generated by $(-\mathcal{A}_{\mathcal{R}})$.

Proof. Let us treat problem (2.1)–(2.3) as an abstract Cauchy problem for a parabolic equation in the space $L_2(Q)$. By virtue of [1, Ch. 1, Theorem 3.7], problem (2.1)–(2.3) has a unique strong solution if and only if the following condition is satisfied:

$$\int_0^T \|\mathcal{A}_{\mathcal{R}} T_t \varphi\|_{L_2(Q)}^2 dt < \infty. \tag{4.6}$$

That solution is presented by (4.5). Due to Theorem 4.1, the semigroup $\{T_t\}$ generated by the operator $(-\mathcal{A}_{\mathcal{R}})$ is analytic and contractive. Then [19, Ch. 1, Theorem 1.14.5] implies that inequality (4.6) holds if and only if $\varphi \in [D(\mathcal{A}_{\mathcal{R}}),\, L_2(Q)]_{1/2}$. ☐

5. Smoothness of Strong Solutions in Cylindrical Subdomains

1. Consider the sets

$$\mathcal{K} = \bigcup_{h_1,\, h_2 \in G} \left\{\overline{Q} \cap (\partial Q + h_1) \cap \overline{[(\partial Q + h_2) \backslash (\partial Q + h_1)]}\right\}$$

and

$$\mathcal{K}^\varepsilon = \{x \in \mathbb{R}^n : \rho(x,\, \mathcal{K}) < \varepsilon\},$$

where $\varepsilon > 0$. For simplicity, we assume that $\mathrm{K} \subset \mathcal{K}$.

We need the following auxiliary results (see [13, Sec. 7, Lemmas 7.3–7.5]).

Lemma 5.1.

(1) *Let $x^0 \in \partial Q_{sl} \cap \partial Q$. Suppose that there exists a sequence $\{x^k\}$ such that $x^k \to x^0$ as $k \to \infty$ and $x^k \in \overline{Q}_{s_k l_k}$, $(s_k,\, l_k) \neq (s,\, l)$. Then $x^0 \in \mathcal{K}$.*

(2) *Let* $x^0 \in Q \cap \partial Q_{pl} \cap \partial Q_{qm}$, $(p,\ l) \neq (q,\ m)$. *Suppose that there exists a sequence* $\{x^k\}$ *such that* $x^k \to x^0$ *as* $k \to \infty$ *and* $x^k \in \overline{Q}_{s_k l_k}$, $(s_k,\ l_k) \neq (p,\ l), (q,\ m)$. *Then* $x^0 \in \mathcal{K}$.

Let Γ_p denote connected (with respect to the topology of ∂Q) components of the set $\partial Q \setminus \mathcal{K}$. Obviously, $\Gamma_p \in C^\infty$.

Lemma 5.2. *Let there exist* $h \in G$ *such that* $(\Gamma_p + h) \cap \overline{Q} \neq \varnothing$. *Then either* $\Gamma_p + h \subset Q$ *or there exists* $\Gamma_r \subset \partial Q \setminus \mathcal{K}$ *such that* $\Gamma_p + h = \Gamma_r$.

By virtue of Lemma 5.1, we can decompose the set $\{\Gamma_p + h : \Gamma_p + h \subset \overline{Q},\ p = 1, 2, \dots;\ h \in G\}$ into classes as follows: sets $\Gamma_{p_1} + h_1$ and $\Gamma_{p_2} + h_2$ belong to the same class if

(1) there exists $h \in G$ such that $\Gamma_{p_1} + h_1 = \Gamma_{p_2} + h_2 + h$;
(2) if $\Gamma_{p_1} + h_1,\ \Gamma_{p_2} + h_2 \subset \partial Q$, then the directions of the inner normals to ∂Q at the points $x \in \Gamma_{p_1} + h_1$ and $x - h \in \Gamma_{p_2} + h_2$ coincide.

The set $\Gamma_p \subset \partial Q$ belongs to at most one class, while the set $\Gamma_p + h \subset Q$ can belong to no more than two classes. Denote the set $\Gamma_p + h$ by Γ_{rj}, where r is the class number, while j is the number of the element within the given class ($1 \le j \le J = J(r)$). Without loss of generality, we assume that $\Gamma_{r1}, \dots, \Gamma_{rJ_0} \subset Q$, $\Gamma_{r,J_0+1}, \dots, \Gamma_{r,J} \subset \partial Q$ $(0 \le J_0 = J_0(r) < J(r))$.

To prove the smoothness theorem for strong solutions of problem (2.1)–(2.3) in cylindrical subdomains, we have to investigate the smoothness of generalized solutions of the second boundary-value problem for strongly elliptic differential-difference equations.

Lemma 5.3. *Suppose that* $\mu_{n-1}(\mathcal{K} \cap \partial Q) = 0$, $f_0 \in L_2(Q)$, *and* w *satisfies* (4.3). *Let* $\varepsilon > 0$. *Then* $w \in W_2^2\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right)$ $(s = 1,\ 2,\ \dots,\ l = 1, \dots, N(s))$ *and*

$$\|w\|_{W_2^2\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right)} \le c\|f_0\|_{L_2(Q)}, \tag{5.1}$$

where $c = c(\varepsilon) > 0$ *does not depend on* f_0.

Proof. The first part of the lemma follows from [2, Sec. 14, Theorem 2]. However, we will need inequality (5.1). This is why we present the complete proof.

I. It follows from the proof of [13, Sec. 11, Theorem 11.1] that if a subdomain Ω_{sl} is such that $\overline{\Omega}_{sl} \subset Q_{sl}$, then $w \in W_2^2(\Omega_{sl})$ and

$$\|w\|_{W_2^2(\Omega_{sl})} \le k_1(\|w\|_{W_2^1(Q)} + \|f_0\|_{L_2(Q)}),$$

where $k_1 = k_1(\Omega_{sl}) > 0$ does not depend on f_0 and w. It follows from the latter inequality and from (4.4) that

$$\|w\|_{W_2^2(\Omega_{sl})} \le k_2\|f_0\|_{L_2(Q)}, \tag{5.2}$$

where $k_2 = k_2(\Omega_{sl}) > 0$ does not depend on f_0 and w.

Therefore, it suffices to prove that for any $y \in \partial Q_{pi} \setminus \mathcal{K}$, there exists a ball $B_\delta(y)$ such that $w \in W_2^2(Q_{pi} \cap B_\delta(y))$ and

$$\|w\|_{W_2^2(Q_{pi} \cap B_\delta(y))} \le k_3\|f_0\|_{L_2(Q)}, \tag{5.3}$$

where $k_3 = k_3(Q_{pi}, \delta) > 0$ does not depend on f_0 and w.

II. Fix a class of subdomains $s = p$. Let h_{pl} be a vector satisfying the condition $Q_{pl} = Q_{p1} + h_{pl}$ $(l = 1, \dots, N(p))$, $h_{p1} = 0$. Introduce points $y^1, \dots, y^{N(p)}$ such that $y^l = y^i - h_{pi} + h_{pl}$, where $y^i = y$. It follows from the definition of the sets Γ_{sl} that there exists a unique r such that $J(r) = N(p)$ and we have (after the corresponding renumbering of the sets Q_{pl} and Γ_{rl}) the following inclusions: $y^l \in \Gamma_{rl} \subset \partial Q_{pl} \setminus \mathcal{K}$, $\Gamma_{rl} \subset Q$ for $1 \le l \le J_0 = J_0(r)$, and $\Gamma_{rl} \subset \partial Q$ for $J_0 + 1 \le l \le J(r)$.

There exists a unique subdomain $Q_{qm} \neq Q_{p1}$ such that $\Gamma_{r1} \subset \partial Q_{qm}$. Change the numeration of subdomains of the qth class to obtain the inclusion $\Gamma_{rl} \subset \partial Q_{ql}$ $(l = 1, \dots, J_0)$.

Introduce the points $z^1, \dots, z^{N(q)}$ as follows: $z^l = y^1 + h_{ql}$. By construction, $z^l \in \partial Q_{ql} \setminus \mathcal{K}$ and $z^l = y^l \in Q$ for $1 \le l \le J_0$, $z^l \in \partial Q$ for $J_0 + 1 \le l \le N(q)$, and $\left(\bigcup\limits_{l>J_0} \{y^l\}\right) \cap \left(\bigcup\limits_{l>J_0} \{z^l\}\right) = \varnothing$.

Consider the balls $B_{4\delta}(x^{sl})$, where $x^{pl} = y^l$, $x^{ql} = z^l$ $(l = 1, \dots, N(s);\ s = p,\ q.)$ Using Lemma 5.1 and the fact that $\mathrm{K} \subset \mathcal{K}$, we can choose $\delta > 0$ so small that $4\delta < \min\limits_{s,\ l} \min\{\rho(x^{sl},\ \mathcal{K}), 1/2\}$, the sets $\partial Q_{sl} \cap B_{4\delta}(x^{sl})$ are connected and belong to the class C^∞, $B_{4\delta}(x^{sl}) \subset \Gamma_{rl} \cup Q_{pl} \cup Q_{ql}$ for $1 \le l \le J_0$, and $B_{4\delta}(x^{sl}) \cap Q = B_{4\delta}(x^{sl}) \cap Q_{sl}$ for $J_0 + 1 \le l \le N(s)$ $(s = p,\ q)$.

By definition, the function w satisfies the integral identity

$$\sum_{i,j=1}^{n} \int\limits_{Q} R_{ijQ} w_{x_j}\ \overline{v}_{x_i}\ dx + \sum_{i=1}^{n} \int\limits_{Q} R_{iQ} w_{x_i}\ \overline{v}\ dx + \int\limits_{Q} R_{0Q} w\ \overline{v}\ dx + \int\limits_{\partial Q} \sigma w\ \overline{v}\ dS = \int\limits_{Q} f_0\ \overline{v}\ dx \tag{5.4}$$

for all $v \in W_2^1(Q)$.

Introduce the function

$$\xi(x) = \sum_{l=1}^{N(p)} \eta(x + h_{pl}) + \sum_{l=J_0+1}^{N(q)} \eta(x + h_{ql}),$$

where $\eta \in \dot{C}^\infty(\mathbb{R}^n)$, $0 \le \eta(x) \le 1$, $\eta(x) = 1$ for $x \in B_\delta(y^1)$, and $\eta(x) = 0$ for $x \notin B_{2\delta}(y^1)$.

Let $v = \xi v_0$, $v_0 \in W_2^1(Q)$, in (5.4). Then, using the Leibnitz formula, we get

$$\sum_{s,l} \int\limits_{\Omega_{sl}} \sum_{i,j=1}^{n} (R_{ijQ} w_{x_j})\ \xi \overline{v_0}_{x_i}\ dx + \sum_{s,l} \int\limits_{\Omega_{sl}} \sum_{i=1}^{n} (R_{iQ} w_{x_i})\ \xi \overline{v_0}\ dx + \sum_{s,l} \int\limits_{\Omega_{sl}} (R_{0Q} w)\ \xi \overline{v_0}\ dx$$
$$+ \int\limits_{\partial Q} \sigma w\ \xi \bar{v}_0\ dS = \sum_{s,l} \int\limits_{\Omega_{s,l}} f_0\ \xi \overline{v_0}\ dx - \sum_{s,l} \int\limits_{\Omega_{s,l}} \sum_{i,j=1}^{n} (R_{ijQ} w_{x_j})\ \xi_{x_i} \overline{v_0}\ dx,$$

where $\Omega_{sl} = Q_{sl} \cap B_{4\delta}(x^{sl})$. Hereinafter we can sum over $l = 1, \dots, N(s)$, $s = p,\ q$.

Applying the Leibnitz formula to the first and second terms again, we get

$$a_R[\xi w,\ v_0] = \sum_{s,l} \int\limits_{\Omega_{s,l}} f_0\ \xi \overline{v_0}\ dx - \sum_{s,l} \int\limits_{\Omega_{s,l}} \sum_{i,j=1}^{n} (R_{ijQ} w_{x_j})\ \xi_{x_i} \overline{v_0}\ dx$$
$$+ \sum_{s,l} \int\limits_{\Omega_{s,l}} \sum_{i,j=1}^{n} (R_{ijQ} w)\ \xi_{x_j} \overline{v_0}_{x_i}\ dx + \sum_{s,l} \int\limits_{\Omega_{s,l}} \sum_{i=1}^{n} (R_{iQ} w)\ \xi_{x_i} \overline{v_0}\ dx. \tag{5.5}$$

Let $L_2\left(\bigcup\limits_{l} Q_{sl}\right)$ be the subspace of functions from $L_2(Q)$ vanishing outside $\bigcup\limits_{l} Q_{sl}$ $(l = 1, \dots, N(s))$. The operator of the orthogonal projection on $L_2\left(\bigcup\limits_{l} Q_{sl}\right)$ is denoted by P_s. Introduce an isomorphism $U_s \colon L_2\left(\bigcup\limits_{l} Q_{sl}\right) \to L_2^{N(s)}(Q_{s1}) = \prod\limits_{l=1}^{N(s)} L_2(Q_{s1})$ as follows:

$$\big(U_s u\big)_l(x) = u(x + h_{sl}) \quad (x \in Q_{s1}). \tag{5.6}$$

It follows from [13, Sec. 8, Lemma 8.6] that the operators $R_{ijQs} = U_s R_{ijQ} U_s^{-1}$ and $R_{iQs} = U_s R_{iQ} U_s^{-1}$ are the operators of multiplication by the matrices $R_{ij\,s}$ and $R_{i\,s}$ respectively. Then (5.5) implies that

$$a_R[\xi w,\ v_0] = \sum_s \int \Bigg\{ (\eta\ F_s,\ V_s^0) - \sum_{i,j=1}^n (\eta_{x_i} R_{ijs} W_{s x_j},\ V_s^0) + \sum_{i,j=1}^n (\eta_{x_j} R_{ijs} W_s,\ V^0_{s\ x_i}) + \sum_{i=1}^n (\eta_{x_i} R_{is} W_s,\ V_s^0) \Bigg\}\, dx, \quad (5.7)$$

where $W_s = \big(U_s P_s w\big)(x)$, $V_s^0 = \big(U_s P_s v_0\big)(x)$, and $F_s = \big(U_s P_s f_0\big)(x)$ for $x \in \Omega_s = \Omega_{s1}$. Hereinafter we integrate over the set Ω_s.

Without loss of generality, we assume that $y^1 = 0$, while the equation of the surface $\Gamma_{p1} \cap B_{4\delta}(y^1)$ has the form $x_n = 0$. Let $W^1_{2,\,\delta}$ be the space of vector-functions $V = \big(V_p,\ V_q\big)$, $V_s \in \prod_{l=1}^{N(s)} W_2^1(\Omega_s)$, such that $\operatorname{esssupp} V_s \subset \overline{\Omega_s \cap B_{2\delta}(y^1)}$ and $SV \in W_2^1(Q)$, where $(SV)(x) = V_{sl}(x - h_{sl})$ for $x \in \Omega_{sl}$ and $(SV)(x) = 0$ for $x \notin \bigcup_{s,l} \Omega_{sl}$.

Define the operators $\delta^r_{\pm t}$ as follows:

$$\delta^r_{\pm t} W = \frac{W(x_1, \dots,\ x_r \pm t, \dots,\ x_n) - W(x_1, \dots,\ x_r, \dots,\ x_n)}{\pm t},$$
$$\mathcal{D}(\delta^r_{\pm t}) = \Big\{ W \in L_2^{N(s)}(\Omega_s) :\ \operatorname{esssupp} W \subset \overline{\Omega_s \cap B_{3\delta}(y^1)} \Big\}.$$

Assign $v_0 = SV^0$ and $V^0 = \delta^r_{-t} V^1$, where $V^1 \in W^1_{2,\,3\delta/2}$, $1 \le r \le n-1$, and $0 < t < \delta$. By construction, we have $v_0 \in W_2^1(Q)$. Since the operators $-\delta^r_{-t}$ and δ^r_t are formally adjoint to each other, it follows that (5.7) takes the form

$$a_R[\delta^r_t \xi w,\ v_0] = \sum_s \int \Bigg\{ -(\eta\ F_s,\ \delta^r_{-t} V_s^1) + \sum_{i,j=1}^n (\eta_{x_i} R_{ijs} W_{s x_j},\ \delta^r_{-t} V_s^1) + \sum_{i,j=1}^n (R_{ijs} \delta^r_t(\eta_{x_j} W_s),\ V^1_{s\ x_i}) + \sum_{i=1}^n (R_{is} \delta^r_t(\eta_{x_i} W_s),\ V_s^1) \Bigg\}\ dx. \quad (5.8)$$

Note that no translations orthogonal to the plane $x_n = 0$ are used.

Assign $V_s^1 = \delta^r_{t_1}(\eta W_s)$ $(0 < t_1 < \delta)$; obviously, $V^1 \in W^1_{2,\,3\delta/2}$. Denote the right-hand side of (5.8) by I. Due to the Cauchy–Bunyakovskii inequality and [7, Ch. 3, Sec. 3, Theorem 4], we get

$$|I| \le k_4 \Big(\sum_s \|V_s^1\|^2_{1,N} \Big)^{1/2} \sum_s \big(\|W_s\|_{1,N} + \|F_s\|_{0,N} \big), \quad (5.9)$$

where $\|V_s^1\|_{k,N} = \left\{ \sum_{l=1}^{N(s)} \|(V_s^1)_l\|^2_{W_2^k(\Omega_s)} \right\}^{1/2}$, $k = 0,\ 1$.

Now we assign $t_1 = t$, i.e., $V_s^1 = \delta^r_t(\eta W_s)$. Due to the strong ellipticity condition, we have

$$\operatorname{Re} a_R[v_1,\ v_1] \ge c_1 \|v_1\|^2_{W_2^1(Q)} = c_1 \sum_s \|V_s^1\|^2_{1,N}, \quad (5.10)$$

where $v_1 = SV^1$.

It follows from (5.8)–(5.10) and Lemma 4.2 that

$$\left(\sum_s \|V_s^1\|_{1,N}^2\right)^{1/2} \le k_5 \sum_s \big(\|W_s\|_{1,N} + \|F_s\|_{0,N}\big) \le k_6 \sum_s \|F_s\|_{0,N}. \tag{5.11}$$

Now, by virtue of [7, Ch. 3, Sec. 3, Theorem 4], we obtain that $(\eta W_s)_{x_i x_r} \in L_2^{N(s)}(\Omega_s)$ for all $i = 1, \dots, n$, $r = 1, \dots, n-1$. Hence, $W_{p_{x_i x_r}} \in L_2^{N(p)}(Q_{p1} \cap B_\delta(y^1))$ and the estimate $\|W_{p_{x_i x_r}}\|_{0,N} \le k_7 \|f_0\|_{L_2(Q)}$ is valid.

III. To prove that $W_{p_{x_n x_n}} \in L_2^{N(p)}(Q_{p1} \cap B_\delta(y^1))$, we assign $V_q^0 = 0$ and take an arbitrary vector-function V_p^0 from $\prod_{l=1}^{N(p)} C^\infty(Q_{p1} \cap B_\delta(y^1))$. Then W_p is a generalized solution of the following system of differential equations:

$$-\sum_{i,j=1}^n R_{ijp} W_{p_{x_i x_j}} + \sum_i R_{i,p} W_{p_{x_i}} + R_{0p} W_p = F_p; \tag{5.12}$$

here $F_p \in L_2^{N(p)}(Q_{p1} \cap B_\delta(y^1))$. By virtue of Lemma 2.3, the matrix $R_{nnp} + R_{nnp}^*$ is positive definite. Therefore, there exists an inverse matrix R_{nnp}^{-1}. Hence, we have

$$W_{p_{x_n x_n}} = R_{nnp}^{-1} \left(-\sum_{i+j<2n} R_{ijp} W_{p_{x_i x_j}} + \sum_{i \le n} R_{ip} W_{p_{x_i}} + R_{0p} W_p - F_p \right).$$

Thus, $W_{p_{x_n x_n}}$ belongs to $L_2^{N(p)}(Q_{p1} \cap B_\delta(y^1))$ and the inequality $\|W_{p_{x_n x_n}}\|_{0,N} \le k_8 \|f_0\|_{L_2(Q)}$ holds.

Therefore, w belongs to $W_2^2(Q_{pi} \cap B_\delta(y))$ and the inequality $\|w\|_{W_2^2(Q_{pi} \cap B_\delta(y))} \le k_9 \|f_0\|_{L_2(Q)}$ holds. □

2. Now we investigate the smoothness of strong solutions of parabolic differential-difference equations.

Let $W_2^{2k,k}(Q_T)$ denote the space of functions $v \in L_2(Q_T)$ such that all their generalized derivatives up to the kth order with respect to t and all their generalized derivatives up to the $2k$th order with respect to x belong to $L_2(Q_T)$. The norm in $W_2^{2k,k}(Q_T)$ is defined as follows:

$$\|v\|_{W_2^{2k,k}(Q_T)} = \left\{ \sum_{2\beta + |\alpha| \le 2k} \int_{Q_T} \left| \frac{\partial^{|\alpha|+\beta} v(x, t)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \partial t^\beta} \right|^2 dx\, dt \right\}^{1/2}.$$

Theorem 5.1. *Let $\mu_{n-1}(\mathcal{K} \cap \partial Q) = 0$, $f \in L_2(Q_T)$, $\varphi \in [D(\mathcal{A}_\mathcal{R}), L_2(Q)]_{1/2}$, and u be a strong solution of (2.1)–(2.3). Then $u \in W_2^{2,1}\left(\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right) \times (0, T)\right)$ for any positive ε and any $s = 1, 2, \dots$, $l = 1, \dots, N(s)$.*

Proof. It follows from the definition of the strong solution and from Eq. (2.1) that

$$\mathcal{A}_\mathcal{R} u(\cdot, t) = F(\cdot, t),$$

where $F(\cdot, t) = f(\cdot, t) - u_t(\cdot, t) \in L_2(Q)$ for almost all $t \in (0, T)$.

It follows from Lemma 5.3 that $u(\cdot, t) \in W_2^2\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right)$ for almost all $t \in (0, T)$. Moreover, the inequality

$$\|u(\cdot,\ t)\|_{W_2^2\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right)} \le c \|F(\cdot,\ t)\|_{L_2(Q)}$$

holds for almost all $t \in (0, T)$.

Squaring both sides of the latter inequality and integrating by parts from 0 to T, we get

$$\|u\|^2_{W_2^{2,0}\left(\left(Q_{sl} \backslash \overline{\mathcal{K}^\varepsilon}\right) \times (0, T)\right)} \le 2c^2 \left(\|f\|^2_{L_2(Q_T)} + \|u_t\|^2_{L_2(Q_T)} \right).$$

Thus, we obtain the assertion of the theorem. $\square$

Lemma 5.4. *Let u be a strong solution of* (2.1)–(2.3). *Then u satisfies* (2.1) *a.e. in* $(Q_{sl}\setminus\overline{\mathcal{K}^\varepsilon})\times(0,\,T)$.

Proof. It follows from Lemma 2.1 and Theorem 5.1 that $R_{ijQ}u_{x_j} \in W_2^1\big((Q_{sl}\setminus\overline{\mathcal{K}^\varepsilon})\times(0,\,T)\big)$. Then, integrating (3.3) by parts, we have

$$\int_0^T\int_{Q_{sl}}\left(u_t-\sum_{i,j=1}^n (R_{ijQ}u_{x_j})_{x_i}+\sum_{i=1}^n R_{iQ}u_{x_i}+R_{0Q}u\right)\bar v\,dx\,dt=\int_0^T\int_{Q_{sl}} f\bar v\,dx\,dt$$

for all $v\in C^\infty(\overline{Q}_{sl}\times[0,\,T])$ such that $\operatorname{supp} v\subset(Q_{sl}\setminus\overline{\mathcal{K}^\varepsilon})\times[0,\,T)$. This implies that u satisfies (2.1) a.e. in $(Q_{sl}\setminus\overline{\mathcal{K}^\varepsilon})\times(0,\,T)$. $\square$

Corollary 5.1. *Let the assumptions of Theorem* 5.1 *be satisfied. Let u be a strong solution of problem* (2.1)–(2.3). *Then u satisfies boundary condition*

$$\left.\left(\sum_{i,j=1}^n R_{ijQ}u_{x_j}\cos(\nu,\,x_i)+\sigma(x)u\right)\right|_{(\partial Q\setminus\overline{\mathcal{K}^\varepsilon})\times(0,\,T)}=0 \tag{5.13}$$

for any $\varepsilon>0$.

Proof. Let us prove that for any $(x_0,\,t_0)\in\big(\partial Q\setminus\overline{\mathcal{K}^\varepsilon}\big)\times(0,\,T)$, there exists a positive δ such that u satisfies boundary condition (5.13) in $\Gamma_{T,\,\delta}=\Gamma_\delta\times(0,\,T)$, where $\Gamma_\delta=B_\delta(x_0)\cap\partial Q$.

It follows from Lemma 5.3 that there exists a positive δ and indices s and l such that $\omega=B_\delta(x_0)\cap Q\subset Q_{sl}\setminus\mathcal{K}^\varepsilon$ and $\Gamma_\delta\in C^\infty$. It follows from Theorem 5.1 and Lemma 2.1 that $R_{ijQ}u_{x_j}\in W_2^1\big(\omega\times(0,\,T)\big)$. Assign

$$H^1_\omega(Q_T)=\big\{v\in W_2^1(Q_T):\ v(x,\,t)=0\ (x\in Q\setminus\overline{\omega}),\ v|_{t=T}=0\big\}.$$

Then, integrating (3.3) by parts and taking into account that, by virtue of Lemma 5.4, the function u satisfies (2.1) a.e. in $\omega\times(0,\,T)$, we obtain the relation

$$\int_{\Gamma_{T,\,\delta}}\left(\sum_{i,j=1}^n R_{ijQ}u_{x_j}\cos(\nu,\,x_i)+\sigma(x)u\right)\bar v\,dx\,dt=0$$

for all $v\in H^1_\omega(Q_T)$. This implies that u satisfies (5.13) in $\Gamma_{T,\,\delta}$. $\square$

3. The following example shows that the smoothness of solutions can be broken at boundaries of neighboring cylindrical subdomains. As we see from Example 5.2, Theorem 5.1 is, in general, not valid for $\varepsilon=0$.

Example 5.1. Consider the problem

$$u_t(x,\ t)-\sum_{i=1,2}(R_Qu_{x_i}(x,\ t))_{x_i}=f(x,\ t)\quad((x,\ t)\in Q_T), \tag{5.14}$$

$$\sum_{i=1,2}R_Qu_{x_i}\cos(\nu,\ x_i)+\sigma u=0\quad((x,\ t)\in\Gamma_T), \tag{5.15}$$

$$u|_{t=0}=\varphi(x)\quad(x\in Q), \tag{5.16}$$

where $Q=(0,\,2)\times(0,\,2)$, $\Gamma_T=(\partial Q\setminus\mathrm{K})\times(0,\,T)$, the set K consists of the points $(0,\,0)$, $(0,\,2)$, $(2,\,0)$, and $(2,\,2)$, and $R_Q=P_QRI_Q$, where $Rv(x)=v(x_1,\ x_2)+\dfrac12v(x_1+1,\ x_2)+\dfrac12v(x_1-1,\ x_2)$. The partition $\mathcal{R}$ of the set Q consists of the following (single) class of subdomains: $Q_{11}=(0,\,1)\times(0,\,2)$ and

$Q_{12} = (1,\ 2) \times (0,\ 2)$. The set $\mathcal{K}$ consists of the following six points: $(s-1,\ m)$ $(s = 1,\ 2,\ 3;\ m = 0,\ 2)$. The function σ is such that

$$\sigma|_{x_1=0} = \xi(x_2), \quad \sigma|_{x_1=2} = \sigma|_{x_2=0} = \sigma|_{x_1=2} = 0,$$

where $\xi \in \dot{C}^\infty(\mathbb{R})$, $0 \le \xi(x_2) \le 1$, $\xi(x_2) = 1$ for $x_2 \in (1/2,\ 3/2)$, and $\xi(x_2) = 0$ for $x_2 \notin (1/4,\ 7/4)$.

The matrix

$$R_1 = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

is positive definite. Thus, the operator A_R corresponding to problem (5.14)–(5.16) is strongly elliptic. Introduce a function $v(x)$ as follows:

$$v(x) = \begin{cases} \left(\dfrac{4}{3}x_1 + 1\right)\eta(x_1)\xi(x_2) & (x \in Q_{11}), \\ -\dfrac{2}{3}(x_1 - 1)\eta(x_1 - 1)\xi(x_2) & (x \in Q_{12}), \end{cases}$$

where $\eta \in \dot{C}^\infty(-1/3,\ 1/3)$ and $\eta(x_1) = 1$ for $x_1 \in (-1/4,\ 1/4))$.

It is easy to check that $v \in W_2^1(Q)$, $v \in W_2^2(Q_{1i})$ $(i = 1,\ 2)$, $-\sum\limits_{i=1,2} (R_Q v_{x_i})_{x_i} \in L_2(Q)$, and $\left.\left(\sum\limits_{i=1,2} R_Q v_{x_i} \cos(\varkappa,\ x_i) + \sigma u\right)\right|_{(\partial Q \setminus \mathrm{K})} = 0$, where $\varkappa$ is the unit vector of the outer normal to $\partial Q \setminus \mathrm{K}$. Therefore, the function $u(x,\ t) = tv(x)$ is a strong solution of problem (5.14)–(5.16) for $f(x,\ t) = v(x) - t \sum\limits_{i=1,2} (R_Q v_{x_i}(x))_{x_i} \in L_2(Q_T)$ and $\varphi(x) = 0$. By construction, we have

$$u_{x_1}\big|_{x_1=1-0} \ne u_{x_1}\big|_{x_1=1+0}.$$

Thus, the smoothness of strong solutions may be broken at the boundary between the neighboring subdomains $Q_{11} \times (0,\ T)$ and $Q_{12} \times (0,\ T)$.

Example 5.2. Consider the problem

$$u_t(x,\ t) - \sum_{i=1,2} (R_Q u_{x_i}(x,\ t))_{x_i} = f(x,\ t) \quad ((x,\ t) \in Q_T), \tag{5.17}$$

$$\sum_{i=1,2} R_Q u_{x_i} \cos(\nu,\ x_i) = 0 \quad ((x,\ t) \in \Gamma_T), \tag{5.18}$$

$$u|_{t=0} = \varphi(x) \quad (x \in Q), \tag{5.19}$$

where the domain $Q \subset \mathbb{R}^2$ has the boundary $\partial Q \in C^\infty$ coinciding with the boundary of the square $(0,\ 4/3) \times (0,\ 4/3)$ outside the disks $B_{1/8}\big((i4/3,\ j4/3)\big)$ $(i, j = 0,\ 1)$, $R_Q = P_Q R I_Q$, and the difference operator acts as follows: $Rv(x) = 2v(x_1,\ x_2) + v(x_1 + 1,\ x_2 + 1) + v(x_1 - 1,\ x_2 - 1)$.

Introduce the following notation:

$$\Gamma_{12} = \{x \in \partial Q : x_1 < 1/3,\ x_2 < 1/3\}, \quad \Gamma_{11} = \Gamma_{12} + (1, 1),$$

$$\Gamma_{22} = \{x \in \partial Q : 1 < x_1,\ 1 < x_2\}, \quad \Gamma_{21} = \Gamma_{22} - (1, 1).$$

The partition $\mathcal{R}$ consists of the following two classes of subdomains:

(1) the domains Q_{11} bounded by the curves $\overline{\Gamma}_{12}$ and $\overline{\Gamma}_{21}$, and the domain Q_{12} bounded by the curves $\overline{\Gamma}_{11}$ and $\overline{\Gamma}_{22}$,
(2) the domain $Q_{21} = Q \setminus \big(\overline{Q}_{11} \cup \overline{Q}_{12}\big)$.

The set $\mathcal{K}$ is a subset of ∂Q; it consists of the following four points: $g^1 = (1/3,\, 0)$, $g^2 = (4/3,\, 1)$, $g^3 = (0,\, 1/3)$, and $g^4 = (1,\, 4/3)$. It is easy to prove that the operator A_R is strongly elliptic.

Introduce the function

$$v(x) = \begin{cases} 10u_1(x_1 - 1/3,\, x_2) + u_2(x_1 - 1/3,\, x_2) & (x \in Q_{11}), \\ u_1(x_1 - 4/3,\, x_2 - 1) + 10u_2(x_1 - 4/3,\, x_2 - 1) & (x \in Q_{12}), \\ b(u_1(x_1 - 1/3,\, x_2) + u_2(x_1 - 4/3,\, x_2 - 1)) & (x \in Q_{21}), \end{cases}$$

where u_1 and u_2 are defined as follows: $u_1(r,\, \varphi) = \xi(r) r^\lambda \cos \lambda\varphi$, $u_2(r,\, \varphi) = \xi(r) r^\lambda \cos \lambda(\varphi - 3\pi/2)$; $\xi \in \dot{C}^\infty(\mathbb{R})$, $0 \le \xi(r) \le 1$, $\xi(r) = 1$, if $r \le 1/8$, $\xi(r) = 0$, if $r \ge 1/6$; $\lambda\pi = 2 \arccos 2/7$, $b = 99/14$; $(r,\, \varphi)$ are the polar coordinates.

Therefore, we have

$$R_Q v(x) = \begin{cases} 21u_1(x_1 - 1/3,\, x_2) + 12u_2(x_1 - 1/3,\, x_2) & (x \in Q_{11}), \\ 12u_1(x_1 - 4/3,\, x_2 - 1) + 21u_2(x_1 - 4/3,\, x_2 - 1) & (x \in Q_{12}), \\ 2b(u_1(x_1 - 1/3,\, x_2) + u_2(x_1 - 4/3,\, x_2 - 1)) & (x \in Q_{21}). \end{cases}$$

It is easy to check that $v \in W_2^1(Q)$ and $-\sum\limits_{i=1,2} \big(R_Q v_{x_i}\big)_{x_i} \in L_2(Q)$ (because $0 < \lambda < 1$). Therefore, the function $u(x,\, t) = tv(x)$ is a strong solution of problem (5.17)–(5.19) with the following data: $f(x,\, t) = v(x) - t \sum\limits_{i=1,2} \big(R_Q v_{x_i}\big)_{x_i} \in L_2(Q)$ and $\varphi(x) = 0$. Now we prove that $u \notin W_2^{2,1}\big((Q_{11} \cap B_\delta(g^1)) \times (0,\, T)\big)$ for any $\delta > 0$. Let us prove that

$$u_{x_1}\big|_{x_1 = 1/3+0,\, x_2 \le 1/8} \ne u_{x_1}\big|_{x_1 = 1/3-0,\, x_2 \le 1/8}.$$

It suffices to prove that

$$b u_{1\varphi}\big|_{\varphi = \pi/2 - 0,\, r \le 1/8} \ne \big(10 u_{1\varphi} + u_{2\varphi}\big)\big|_{\varphi = \pi/2 + 0,\, r \le 1/8}. \tag{5.20}$$

Relation (5.20) is equivalent to the inequality

$$-b \sin \frac{\lambda\pi}{2} \ne -10 \sin \frac{\lambda\pi}{2} + \sin \lambda\pi. \tag{5.21}$$

Since $\lambda\pi = 2 \arccos 2/7$ and $b = 99/14$, it follows that Condition (5.21) is satisfied. Hence, we proved that $u \notin W_2^{2,1}\big((Q \cap B_{1/8}(g^1)) \times (0,\, T)\big)$.

Thus, the smoothness of strong solutions of problem (2.1)–(2.3) may be broken near points of the set $\mathcal{K}$.

6. Smoothness of Strong Solutions at Boundaries between Neighboring Cylindrical Subdomains

In this section, we use the notation introduced in the proof of Lemma 5.3.

For simplicity, we assume that $y^1 = 0$ and

$$Q_{p1} \cap B_{4\delta}(0) = \big\{x \in \mathbb{R}^n :\, |x| < 4\delta,\ x_n < 0\big\},$$
$$\partial Q_{p1} \cap B_{4\delta}(0) = \big\{x \in \mathbb{R}^n :\, |x| < 4\delta,\ x_n = 0\big\}.$$

Introduce the sets $\gamma_{sl}^T = \gamma_{sl} \times (0,\, T)$, where $\gamma_{sl} = \partial Q_{sl} \cap B_{2\delta}(x^{sl})$.

By virtue of Lemma 5.4, the strong solution u of problem (2.1)–(2.3) satisfies Eq. (2.1) a.e. in $(Q_{sl} \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\, T)$. Multiply both sides of (2.1) by a function $v \in C^\infty(Q_T)$ such that supp $v \subset \bigcup\limits_l B_{2\delta}(x^{pl}) \times [0,\, T)$ $(l = 1,\, \dots,\, J_0)$ and integrate over the regions $Q_{sl} \cap B_{2\delta}(x^{pl})$. Then, summing with respect to $s = p$, q and integrating by parts, we have

$$\int_0^T \int_{B_{2\delta}(x^{pl})} \left\{ -u\bar{v}_t + \sum_{i,j=1}^n (R_{ijQ}u_{x_j})\bar{v}_{x_i} + \sum_{i=1}^n (R_{iQ}u_{x_i})\bar{v} + (R_{0Q}u)\bar{v} \right\} dx\,dt$$

$$- \int_0^T \sum_{j=1}^n \left(R_{njQ}u_{x_j}\big|_{\gamma_{pl}^T} - R_{njQ}u_{x_j}\big|_{\gamma_{ql}^T} \right) dt = \int_0^T \int_{B_{2\delta}(x^{pl})} f\bar{v}\,dx\,dt + \int_{B_{2\delta}(x^{pl})} \varphi\bar{v}|_{t=0}\,dx.$$

Since u is a generalized solution of problem (2.1)–(2.3) and v is selected arbitrarily, it follows that

$$\sum_{j=1}^n \left(R_{njQ}u_{x_j}\big|_{\gamma_{pl}^T} - R_{njQ}u_{x_j}\big|_{\gamma_{ql}^T} \right) = 0 \quad (l = 1, \ldots, J_0). \tag{6.1}$$

Note that either $N(p) \neq J_0$ or $N(q) \neq J_0$. For definiteness, we assume that $N(q) \neq J_0$. Due to Corollary 5.1, the function u satisfies Condition (2.2) in γ_{sl}^T for $l > J_0$, i.e., we have

$$\sum_{j=1}^n (R_{njQ}u_{x_j} + \sigma u)\big|_{\gamma_{pl}^T} = \sum_{j=1}^n (-R_{njQ}u_{x_j} + \sigma u)\big|_{\gamma_{ql}^T} = 0 \quad (l = J_0 + 1, \ldots, N(s);\ s = p,\ q). \tag{6.2}$$

Let A_{js} denote the matrices obtained from the matrices R_{njs} by deleting the latter $N(s) - J_0$ rows. Let B_{js} denote the matrices obtained from the matrices R_{njs} by deleting the initial J_0 rows. Introduce the following vector-functions:

$$V_s = (U_s P_s u)\big|_{\gamma_{s1}^T}; \quad W_{js} = \big(U_s P_s u_{x_j}\big)\big|_{\gamma_{s1}^T} \quad (j = 1, \ldots, n).$$

Let L'_{js} (L'_s) denote the matrix (the vector) obtained from the matrix L_{js} (the vector L_s) by deleting the latter $N(s) - J_0$ columns (elements). Let L''_{js} (L''_s) denote the matrix (the vector) obtained from the matrix L_{js} (the vector L_s) by deleting the initial J_0 columns (elements).

Also, introduce the square matrices σ^s of order $(N(s) - J_0)$ such that their elements are defined as follows:

$$\sigma_{ii}^p = \sigma(x + h_{p,J_0+i})\big|_{x\in\gamma_{p1}}, \quad \sigma_{ii}^q = -\sigma(x + h_{q,J_0+i})\big|_{x\in\gamma_{q1}} \quad (i = 1, \ldots, N(s) - J_0),$$

and $\sigma_{ij}^s = 0$ for $i \neq j$.

Then Conditions (6.1) and (6.2) have the following form:

$$\sum_{j=1}^n (A_{jp}W_{jp} - A_{jq}W_{jq}) = 0, \tag{6.3}$$

$$\sum_{j=1}^n B_{js}W_{js} + \sigma^s V''_s = 0 \quad (s = p,\ q). \tag{6.4}$$

Introduce the vector-function

$$Z = \begin{pmatrix} W'_{np} - W'_{nq} \\ W''_{np} \end{pmatrix}. \tag{6.5}$$

By construction, we have

$$A'_{jp} = A'_{jq} \qquad (j = 1, \ldots, n). \tag{6.6}$$

Since $u \in W_2^{1,0}(Q_T)$, it follows that $V'_p = V'_q$. By virtue of Theorem 5.1, we have

$$W'_{jp} = W'_{jq} \qquad (j = 1, \ldots, n - 1). \tag{6.7}$$

Using (6.6) and (6.7), we can write (6.3) and (6.4) as follows:

$$A_{np}Z = A''_{nq}W''_{nq} - \sum_{j<n}\left(A''_{jp}W''_{jp} - A''_{jq}W''_{jq}\right), \tag{6.8}$$

$$B_{np}Z = -B'_{np}W'_{nq} - \sum_{j<n}\left(B'_{jp}W'_{jp} + B''_{jp}W''_{jp}\right) - \sigma^p V''_p, \tag{6.9}$$

$$B''_{nq}W''_{nq} = -B'_{nq}W'_{nq} - \sum_{j<n}\left(B'_{jq}W'_{jq} + B''_{jq}W''_{jq}\right) - \sigma^q V''_q. \tag{6.10}$$

The strong ellipticity of the operator A_R implies that R_{nnq} is positive definite. Therefore, there exists an inverse matrix $\left(B''_{nq}\right)^{-1}$. Then (6.10) implies that

$$W''_{nq} = -\left(B''_{nq}\right)^{-1} B'_{nq}W'_{nq} - \sum_{j<n}\left(\left(B''_{nq}\right)^{-1} B'_{jq}W'_{jq} + \left(B''_{nq}\right)^{-1} B''_{jq}W''_{jq}\right) - \left(B''_{nq}\right)^{-1}\sigma^q V''_q. \tag{6.11}$$

Using (6.11) in (6.8), we get

$$\begin{aligned} A_{np}Z = {} & -A''_{nq}\left(B''_{nq}\right)^{-1} B'_{nq}W'_{nq} - A''_{nq}\left(B''_{nq}\right)^{-1}\sigma^q V''_q \\ & - \sum_{j<n}\left(A''_{jp}W''_{jp} + A''_{nq}\left(B''_{nq}\right)^{-1} B'_{jq}W'_{jq} + (A''_{nq}\left(B''_{nq}\right)^{-1} B''_{jq} - A''_{jq})W''_{jq}\right). \end{aligned} \tag{6.12}$$

Introduce the following vector-functions H^j of dimension m_j ($j = 1, \dots, n$):

$$H^j = \begin{pmatrix} W'_{jp} \\ W''_{jp} \\ W''_{jq} \end{pmatrix}, \quad m_j = N(p) + N(q) - J_0 \quad (j = 1, \dots, n-1),$$

$$H^n = W'_{nq}, \quad (m_n = J_0).$$

Introduce the following vector-function of dimension $N(p) - N(q) - 2J_0$:

$$V = \begin{pmatrix} V''_p \\ V''_q \end{pmatrix}$$

We will also need the block matrices

$$T^j = \begin{pmatrix} A''_{nq}\left(B''_{nq}\right)^{-1} B'_{jq} & A''_{jp} & A''_{nq}\left(B''_{nq}\right)^{-1} B''_{jq} - A''_{jq} \\ B'_{jp} & B''_{jp} & O \end{pmatrix},$$

$$G^j = \begin{pmatrix} \left(B''_{nq}\right)^{-1} B'_{jq} & O & \left(B''_{nq}\right)^{-1} B''_{jq} \end{pmatrix} \quad (j = 1, \dots, n-1),$$

$$T^n = \begin{pmatrix} A''_{nq}\left(B''_{nq}\right)^{-1} B'_{nq} \\ B'_{np} \end{pmatrix}, \quad G^n = \left(B''_{nq}\right)^{-1} B'_{nq},$$

$$S_1 = \begin{pmatrix} O & A''_{nq}\left(B''_{nq}\right)^{-1}\sigma^q \\ \sigma^p & O \end{pmatrix}, \quad S = \begin{pmatrix} O & \left(B''_{nq}\right)^{-1}\sigma^q \end{pmatrix}.$$

Then Eqs. (6.8)–(6.10) can be written as follows:

$$R_{nnp}Z = -\sum_{j=1}^{n} T^j H^j - S_1 V, \tag{6.13}$$

$$W''_{nq} = -\sum_{j=1}^{n} G^j H^j - SV. \tag{6.14}$$

Let Λ^j_{lk} and $\tilde{\Lambda}_{lk}(x')$ denote the matrices obtained from R_{nnp} by replacing the lth column by the kth columns of the matrices T^j and S_1, respectively. Here $x' = (x_1, \ldots, x_{n-1})$.

Theorem 6.1. *Let* $1 \le l \le J_0$*. Then the strong solution* u *of problem* (2.1)–(2.3) *belongs to the space* $W_2^{2,1}(B_\delta(y^l) \times (0,\ T))$ *for all* $f \in L_2(Q_T)$ *and* $\varphi \in [D(\mathcal{A}_\mathcal{R}), L_2(Q)]_{1/2}$ *if and only if*

$$\det \Lambda^j_{lk} = 0 \qquad (j = 1, \ldots, n;\ k = 1, \ldots, m_j), \tag{6.15}$$

$$\det \tilde{\Lambda}_{lk}(x') = 0 \qquad (k = 1, \ldots, N(p) + N(q) - 2J_0;\ |x'| < \delta). \tag{6.16}$$

Proof. Introduce the matrices

$$\Lambda^j = \left\|\det \Lambda^j_{ik} / \det R_{nnp}\right\| \quad (i = 1, \ldots, N(p);\ k = 1, \ldots, m_j);$$

$$\tilde{\Lambda} = \left\|\det \tilde{\Lambda}_{ik} / \det R_{nnp}\right\| \quad (i = 1, \ldots, N(p);\ k = 1, \ldots, N(p) + N(q) - 2J_0).$$

Then it follows from (6.13) that

$$Z = -\sum_{j=1}^{n} \Lambda^j H^j - \tilde{\Lambda} V. \tag{6.17}$$

I (Sufficiency). Suppose that Conditions (6.15) and (6.16) are satisfied. Due to Theorem 5.1, $u \in W_2^{2,1}\big((Q_{sl} \cap B_\delta(x^{sl})) \times (0,\ T)\big)$ $(s = p, q)$, while it follows from (6.15)–(6.17) that

$$u_{x_n}\big|_{\gamma^T_{pl}} = u_{x_n}\big|_{\gamma^T_{ql}} \quad (l = 1, \ldots, J_0).$$

This implies that $u \in W_2^{2,1}\big(B_\delta(x^{sl}) \times (0,\ T)\big)$.

II (Necessity). Let

$$u(x,\ t) = \begin{cases} t\big(U_s^{-1} v_s\big)(x) & \left(x \in \bigcup\limits_l Q_{sl},\ t \in (0,\ T),\ s = p, q\right), \\ 0 & \left(x \in Q \setminus \bigcup\limits_{s,l} Q_{sl},\ t \in (0,\ T),\ s = p, q\right), \end{cases} \tag{6.18}$$

where $v_s(x) = \big(A_s(x')x_n + B_s(x')\big)\eta(x_n)$ for $x = (x', x_n) \in Q_{s1}$, $A_s(x')$ and $B_s(x')$ are smooth vector-functions of order $N(s)$, vanishing for $|x'| > 2\delta/3$, $\eta(x_n) \in \dot{C}^\infty(-\delta, \delta)$, and $\eta(x_n) = 1$ for $x_n \in (-\delta/3,\ \delta/3)$. Then, we obviously have

$$V_s = B_s(x'),\ \ W_{js} = \big(B_s(x')\big)_{x_j}\ \ (j < n),\ \ W_{ns} = A_s(x').$$

Note that $u \in W_2^1(Q_T)$ if and only if $B'_p = B'_q$. In the sequel, we assume that $\varphi(x) = 0$.

(a) Let $\det \Lambda^b_{lr} \ne 0$ $(b < n,\ 1 \le r \le N(p)+N(q)-J_0)$. Let e be a vector of dimension $N(p)+N(q)-J_0$ defined as follows: $e_k = \delta_{kr}$, where δ_{kr} is the Kronecker symbol. Let

$$A_p = -\sum_{j=1}^{n-1} \big(x_b \xi(x')\big)_{x_j} \Lambda^j_r - \big(x_b \xi(x')\big)\tilde{\Lambda} h, \qquad B_p = \big(x_b \xi(x')\big) e^1;$$

$$A'_q = 0,\ A''_q = -\sum_{j=1}^{n-1} \big(x_b \xi(x')\big)_{x_j} G^j_r - \big(x_b \xi(x')\big) S h, \quad B_q = \big(x_b \xi(x')\big) e^2.$$

Here h is the vector of dimension $N(p)+N(q)-2J_0$ with coordinates $h_k = e_{k+J_0}$, e^1 is the vector of dimension $N(p)$ with coordinates $e^1_k = e_k$, and e^2 is the vector of dimension $N(q)$ with coordinates $(e_1, \dots, e_{J_0}, e_{N(p)+1}, \dots, e_{N(p)+N(q)-J_0})$; Λ^j_r and G^j_r are the rth columns of the matrices Λ^j and G^j respectively; $\xi \in \dot{C}^\infty(\mathbb{R}^{n-1})$, $\xi(x') = 1$ for $x' \in \gamma_{p1} \cap B_{\delta/3}(0)$ and $\xi(x') = 0$ for $x' \notin \gamma_{p1} \cap B_{2\delta/3}(0)$.

By construction, $u \in W^1_2(Q_T) \cap W^{2,1}_2(Q_{sl} \times (0,\ T))$ $(l = 1, \dots, N(s),\ s = p,\ q)$ and $\operatorname{supp} u \subset \overline{B_{4\delta}(x^{sl})} \times [0,\ T]$. It is easy to check that conditions (6.14) and (6.17) are satisfied. This means that there exists $f \in L_2(Q_T)$ such that $u \in \mathcal{W}(\mathcal{A}_R)$ is a strong solution of problem (2.1)–(2.3). However, we have

$$u_{x_n}\big|_{\gamma^T_{pl}} = (W_{np})_l \neq (W_{nq})_l = u_{x_n}\big|_{\gamma^T_{ql}}.$$

Therefore, $u \notin W^{2,1}_2(B_\delta(x^{pl}) \times (0,\ T))$.

(b) Let $\det \Lambda^n_{lr} \neq 0$ $(1 \le r \le J_0)$ and

$$A'_p = \big(e - (\Lambda^n_r)'\big)\xi(x'),\ \ A''_p = -(\Lambda^n_r)''\xi(x'), \qquad B_p = 0;$$
$$A'_q = e\xi(x'),\ \ A''_q = -G^n_r\xi(x'), \qquad B_q = 0.$$

Here e is the vector of dimension J_0 with coordinates $e_k = \delta_{kr}$.

As in case (a), there exists $f \in L_2(Q_T)$ such that $u \in \mathcal{W}(\mathcal{A}_R)$ is a strong solution of problem (2.1)–(2.3). However, we have

$$u_{x_n}\big|_{\gamma^T_{pl}} = (W_{np})_l \neq (W_{nq})_l = u_{x_n}\big|_{\gamma^T_{ql}}.$$

Hence, $u \notin W^{2,1}_2(B_\delta(x^{pl}) \times (0,\ T))$.

(c) Let $\det \tilde{\Lambda}_{lr}(x') \neq 0$ $(1 \le r \le N(p)+N(q)-2J_0)$ for $x' = x'_0$. Without loss of generality, we assume that $x'_0 = 0$. Let

$$A_p = -\sum_{j=1}^{n-1} \Lambda^j_{J_0+r}\xi_{x_j}(x') - \tilde{\Lambda}_r\xi(x'), \qquad B'_p = 0,\ B''_p = e^1\xi(x');$$
$$A'_q = 0,\ A''_q = -\sum_{j=1}^{n-1} G^j_{J_0+r}\xi_{x_j}(x') - S_r\xi(x'), \quad B'_q = 0,\ B''_q = e^2\xi(x').$$

Here e^1 is the vector of dimension $N(p)-J_0$ with coordinates $e^1_k = \delta_{kr}$, e^2 is the vector of dimension $N(q)-J_0$ with coordinates $e^2_k = \delta_{k+N(p)-J_0,r}$, while $\tilde{\Lambda}_r$ and S_r are the rth columns of the matrices $\tilde{\Lambda}$ and S respectively.

As in cases (a) and (b), there exists $f \in L_2(Q_T)$ such that $u \in \mathcal{W}(\mathcal{A}_R)$ is a strong solution of problem (2.1)–(2.3). However, we have

$$u_{x_n}\big|_{\gamma^T_{pl}} = (W_{np})_l \neq (W_{nq})_l = u_{x_n}\big|_{\gamma^T_{ql}}.$$

Hence, $u \notin W^{2,1}_2(B_\delta(x^{pl}) \times (0,\ T))$. □

Example 6.1. Consider the problem

$$u_t(x,\ t) - \sum_{i=1,2} (R_Q u_{x_i}(x,\ t))_{x_i} = f(x,\ t) \quad ((x,\ t) \in Q_T), \tag{6.19}$$

$$\sum_{i=1,2} R_Q u_{x_i} \cos(\nu,\ x_i) + \sigma u = 0 \quad ((x,\ t) \in \Gamma_T), \tag{6.20}$$

$$u|_{t=0} = \varphi(x) \quad (x \in Q), \tag{6.21}$$

where $Q = (0,\ 2)\times(0,\ 1)$, $Rv(x) = v(x_1,\ x_2) + \gamma v(x_1+1,\ x_2) + \gamma v(x_1-1,\ x_2)$, $0 < |\gamma| < 1$, $R_Q = P_Q R I_Q$, $f \in L_2(Q_T)$, and $\varphi \in [\mathcal{D}(\mathcal{A}_{\mathcal{R}}),\ L_2(Q)]_{1/2}$.

Represent the boundary-value conditions (6.20) as follows:

$$- u_{x_1}(0+0,\ x_2,\ t) - \gamma u_{x_1}(1+0,\ x_2,\ t) + \sigma|_{x_1=0}$$
$$= u_{x_1}(2-0,\ x_2,\ t) + \gamma u_{x_1}(1-0,\ x_2,\ t) + \sigma|_{x_1=2} = 0 \ \ (0 < x_2 < 1), \quad (6.22)$$

$$- u_{x_2}(x_1,\ 0,\ t) - \gamma u_{x_2}(x_1+1,\ 0,\ t) + \sigma|_{x_2=0}$$
$$= u_{x_2}(x_1,\ 1,\ t) + \gamma u_{x_2}(x_1+1,\ 1,\ t) + \sigma|_{x_2=1} = 0 \ \ (0 < x_1 < 1), \quad (6.23)$$

$$- u_{x_2}(x_1,\ 0,\ t) - \gamma u_{x_2}(x_1-1,\ 0,\ t) + \sigma|_{x_2=0}$$
$$= u_{x_2}(x_1,\ 1,\ t) + \gamma u_{x_2}(x_1-1,\ 1,\ t) + \sigma|_{x_2=1} = 0 \ \ (1 < x_1 < 2). \quad (6.24)$$

The partition $\mathcal{R}$ consists of the following (single) class of subdomains: $Q_{11} = (0,\ 1) \times (0,\ 1)$, $Q_{12} = (1,\ 2) \times (0,\ 1)$. The set $\mathcal{K}$ consists of the following six points: $(s-1,\ m)$ $(s = 1,\ 2,\ 3;\ m = 0,\ 1)$.

Since the matrix

$$R_1 = \begin{pmatrix} 1 & \gamma & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & \gamma & 1 \end{pmatrix}$$

is positive definite, it follows that the operator A_R corresponding to problem (6.19)–(6.21) is strongly elliptic.

We have $Q_{21} = Q_{12}$ and $Q_{22} = Q_{11}$. The matrices $R_{111},\ R_{112},\ T^1,\ T^2,\ \Lambda^1_{11},\ \Lambda^2_{1k}$ $(k = 1,\ 2,\ 3)$, $\tilde{\Lambda}_{11}$, and $\tilde{\Lambda}_{12}$ introduced above have the forms

$$R_{111} = R_{112} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix},\ T^1 = \begin{pmatrix} \gamma^2 \\ \gamma \end{pmatrix},\ T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda^1_{11} = \begin{pmatrix} \gamma^2 & \gamma \\ \gamma & 1 \end{pmatrix},\ \Lambda^2_{1k} = \begin{pmatrix} 0 & \gamma \\ 0 & 1 \end{pmatrix},\ \tilde{\Lambda}_{11} = \begin{pmatrix} 0 & \gamma \\ \sigma|_{x_1=2} & 1 \end{pmatrix},\ \tilde{\Lambda}_{12} = \begin{pmatrix} -\gamma\sigma|_{x_1=0} & \gamma \\ 0 & 1 \end{pmatrix}.$$

Obviously, we have $\det \Lambda^1_{11} = \det \Lambda^2_{1k} = 0$. The smoothness-preserving conditions of Theorem 6.1 are formulated as follows: $\det \tilde{\Lambda}_{11} = \det \tilde{\Lambda}_{12} = 0$, i.e., $\sigma|_{x_1=0} = \sigma|_{x_1=2} = 0$ for $0 < x_2 < 1$. We assume that those conditions are satisfied, while the functions $\sigma|_{x_2=0}$ and $\sigma|_{x_2=1}$ are sufficiently smooth and their supports are subsets of $(0,\ 2)$. Then Theorem 6.1 implies that

$$u_{x_1}\big|_{x_1=1-0} = u_{x_1}\big|_{x_1=1+0}. \quad (6.25)$$

Thus, $u \in W^1_2(Q_T) \cap W^{2,1}_2((Q \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\ T))$. Let us prove that $u \in W^{2,1}_2(Q_T)$. Since u belongs to $W^{2,1}_2((Q \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\ T))$ and satisfies condition (6.1), it follows that $R_Q u_{x_i} \in W^1_2((Q \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\ T))$. condition (6.1) and Lemma 2.1 imply that $(R_Q u_{x_i})_{x_i} = R_Q u_{x_i x_i}$ $(i = 1,\ 2)$, i.e., $\mathcal{A}_R u = -R_Q \Delta u$ in $(Q \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\ T)$. It follows from (6.22) and (6.25) that

$$u_{x_1}\big|_{x_1=0} = -\gamma u_{x_1}\big|_{x_1=1} = u_{x_1}\big|_{x_1=2}. \quad (6.26)$$

Make the change of variables: $x_1' = x_1 - 1$ in (6.24) and $x_1'' = x_1 + 1$ in (6.23) (note that $|\gamma| < 1$). It is easy to see that conditions (6.23) and (6.24) are equivalent to the following conditions:

$$u_{x_2}\big|_{x_2=0} = \begin{cases} \dfrac{\sigma|_{x_2=0}(x_1) - \gamma\sigma|_{x_2=0}(x_1+1)}{1-\gamma^2} & (0 < x_1 < 1), \\[2ex] \dfrac{\sigma|_{x_2=0}(x_1) - \gamma\sigma|_{x_2=0}(x_1-1)}{1-\gamma^2} & (1 < x_1 < 2); \end{cases} \quad (6.27)$$

$$u_{x_2}\big|_{x_2=1} = \begin{cases} -\dfrac{\sigma|_{x_2=1}(x_1) - \gamma\sigma|_{x_2=1}(x_1+1)}{1-\gamma^2} & (0 < x_1 < 1), \\[2ex] -\dfrac{\sigma|_{x_2=1}(x_1) - \gamma\sigma|_{x_2=1}(x_1-1)}{1-\gamma^2} & (1 < x_1 < 2). \end{cases} \quad (6.28)$$

Since the matrix R_{111} is nondegenerate, it follows that the operator $R_Q\colon L_2(Q) \to L_2(Q)$ has a bounded inverse operator R_Q^{-1}.

Thus, problem (6.19)–(6.21) is equivalent to the problem of finding a generalized solution $u \in W_2^1(Q_T) \cap W_2^{2,1}((Q \setminus \overline{\mathcal{K}^\varepsilon}) \times (0,\ T))$ of the equation

$$u_t - \Delta u = \big(R_Q^{-1}(f - u_t)\big)(x,\ t) + u_t \quad ((x,\ t) \in Q_T) \tag{6.29}$$

with nonlocal boundary-value conditions (6.26)–(6.28).

The smoothness theorem for generalized solutions of the third mixed problem for parabolic equations (see [7, Ch. VI, Sec. 2, Theorem 4]) implies that $u \in W_2^{2,1}(Q^\rho \times (0, T))$ $(0 < \rho < 1/4)$, where $Q^\rho = (\rho,\ 2-\rho) \times (0,\ 1)$. Therefore, the function $w = \xi(x_1)u(x_1+1,\ x_2) + \xi(x_1-2)u(x_1-1,\ x_2)$ belongs to $W_2^{2,1}(Q_T)$, where $\xi \in \dot{C}^\infty(-1/4,\ 1/4)$ and $\xi(x_1) = 1$ for $x_1 \in (-1/8,\ 1/8)$. This yields that $v_t - \Delta v \in L_2(Q)$, $v_{x_1}\big|_{x_1=0} = v_{x_1}\big|_{x_1=2} = 0$, $v_{x_2}\big|_{x_2=0} = (u_{x_2} + \gamma w_{x_2})\big|_{x_2=0}$, $v_{x_2}\big|_{x_2=1} = (u_{x_2} + \gamma w_{x_2})\big|_{x_2=1}$, and $v|_{t=0} = \varphi + w|_{t=0}$, where $v = u + \gamma w$.

It follows from the smoothness of generalized solutions of the third mixed problem for parabolic equations in cylindrical domains that $v \in W_2^{2,1}(Q_T)$, i.e., $u \in W_2^{2,1}(Q_T)$.

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