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ON THE CHARACTERISTIC LIE ALGEBRAS FOR EQUATIONS $u_{xy} = f(u, u_x)$

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Abstract. A new approach to classification of integrable nonlinear equations is proposed. The method is based on description of the structure of the characteristic algebra. A basis of the characteristic algebra is constructed for the sinh-Gordon equation.

CONTENTS

1. Introduction

The symmetry method is one of the ways to classify the integrable equations. This approach is very effective for the evolutionary equations. However, the symmetry classification of hyperbolic equations encounters serious technical difficulties even in the simplest situations (e.g., see [7,8]). Therefore effective study of integrability for hyperbolic type systems requires another approach.

In the present paper, we solve the classification problem using the method based on the analysis of structure of the characteristic Lie algebras. A notion of the characteristic Lie algebra has been introduced in the paper [4] for hyperbolic type systems of the form

$$u^i_{xy} = f^i(u^1, u^2, \ldots, u^n), \quad i = 1, 2, \ldots, n.$$

In [4] it is shown that the above system is exactly integrable in quadratures if the algebra is finite dimensional, and the system is integrable by inverse scattering if the algebra admits a finite dimensional representation.

An important classification result was obtained in the paper [5] for the exponential systems

$$u^i_{xy} = \exp(a_{i_1} u^1 + \ldots + a_{i_n} u^n), \quad i = 1, 2, \ldots, n. \tag{1}$$

It was proved that the characteristic Lie algebra for system (1) is finite dimensional if and only if $A = (a_{ij})$ is the Cartan matrix of a simple Lie algebra. Also, we note the papers [1,2,6], where the integrability was analyzed using the characteristic Lie algebras and the method was applied to hyperbolic systems of form

$$u^i_{xy} = c^i_{jk} u^j u^k_x, \quad i = 1, 2, \ldots, n. \tag{2}$$

In particular, it was shown that system (2) possesses two characteristic algebras rather than one, and these algebras are "glued" to a unique Lie algebra using the zero curvature relations.

Recently the notion of characteristic Lie algebras has also been defined for discrete hyperbolic equations (see [3]).

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In this paper we consider the following nonlinear equations,

$$u_{xy} = f(u, u_x). \tag{3}$$

We show that a list of integrable equations can be obtained using the characteristic Lie algebras.

Consider the following set of independent variables u, u_1, $\overline{u}_1$, u_2, $\overline{u}_2$, …, u_n, $\overline{u}_n$, …, where $u_1 = u_x$, $\overline{u}_1 = u_y$, $u_2 = u_{xx}$, $\overline{u}_2 = u_{yy}$, …. Let us define the x-characteristic Lie algebra A for equation (3). To this end, we formulate the definition of a symmetry.

Definition 1. A function $F = F(u, u_1, \overline{u}_1, u_2, \overline{u}_2, \ldots, u_n, \overline{u}_n)$ is a *symmetry* of equation (3) if F satisfies the determining relation

$$D\overline{D}F = \frac{\partial f}{\partial u_1} DF + \frac{\partial f}{\partial u} F,$$

where D (respectively, $\overline{D}$) is the operator of total derivative with respect to x (respectively, y) by virtue of Eq. (3). For example, we have

$$\overline{D} = \sum_{k=0}^{\infty} \overline{u}_{k+1} \frac{\partial}{\partial \overline{u}_k} + \sum_{k=1}^{\infty} D^{k-1}(f) \frac{\partial}{\partial u_k}. \tag{4}$$

It is known (see [8]) that any symmetry F of equation (3) can be represented in the form

$$F = \varphi(u_1, u_2, \ldots, u_n) + \overline{\varphi}(u, \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n),$$

where φ and $\overline{\varphi}$ are two symmetries of equation (3).

We denote by $\Im$ the set of local analytic functions that depend on a finite number of variables from the set u, u_1, u_2, …, u_n, …, that is,

$$\Im = \langle \varphi = \varphi(u, u_1, u_2, \ldots, u_n),\ n = 1, 2, \ldots \rangle.$$

The operator $\overline{D}$ (see Eq. (4)) restricted onto this class of functions acts by the rule

$$\overline{D}\varphi = \overline{u}_1 \frac{\partial \varphi}{\partial u} + \sum_{k=1}^{\infty} D^{k-1}(f(u)) \frac{\partial \varphi}{\partial u_k}.$$

Further, let X_1 and X_2 be the vector fields defined through

$$X_1 = \sum_{k=1}^{\infty} D^{k-1}(f) \frac{\partial}{\partial u_k},\ X_2 = \frac{\partial}{\partial u}. \tag{5}$$

We note that

$$\overline{D} = \overline{u}_1 X_2 + X_1. \tag{6}$$

The x-characteristic Lie algebra for equation (3) is the algebra A generated by the elements X_1 and X_2.

Let L_n be the linear space of commutators of length $n-1$, $n = 2, 3, \ldots$. For example, the space L_2 is the linear span of the vector fields X_1 and X_2. The space L_3 is generated by the element $[X_1, X_2]$, etc. Then we can represent the characteristic Lie algebra A in the form $A = \bigcup_{i=2}^{\infty} L_i$. The y-characteristic Lie algebra $\overline{A}$ for equation (3) is defined similarly, $\overline{A} = \bigcup_{i=2}^{\infty} \overline{L}_i$.

In this paper we analyze the dimensions of the spaces L_n and $\overline{L}_m$ with $n, m = 3, 4, 5, 6$. We show that the right-hand sides of equations (3) are completely determined if the growth of the dimensions is not greater than one. The list of equations we obtain coincides with the known list of integrable equations. We obtain a complete description of the characteristic Lie algebra for the sinh-Gordon equation, see Theorem 1 in Sec. 3.

2. The Klein–Gordon Equations

In this section, we consider the following equations:

$$u_{xy} = f(u). \tag{7}$$

First we recall the result of the paper [7]. It was shown that nonlinear equation (7) which possesses higher symmetries can be reduced to one of the following equations:

$$u_{xy} = \mathrm{e}^{u}, \tag{8}$$

$$u_{xy} = \sin u, \tag{9}$$

$$u_{xy} = \mathrm{e}^{u} + \mathrm{e}^{-2u}. \tag{10}$$

The total derivative operator D acts on the set of functions $\Im$ by the rule

$$D = \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.$$

The following statement is valid.

Lemma 1. *Let Z be the vector field*

$$Z = \alpha_1 \frac{\partial}{\partial u_1} + \alpha_2 \frac{\partial}{\partial u_2} + \alpha_3 \frac{\partial}{\partial u_3} + \ldots, \quad \alpha_i = \alpha_i(u,\, u_1,\, u_2,\, \ldots),\ i = 1, 2, 3, \ldots.$$

Then $[D, Z] = 0$ if and only if $Z = 0$.

Proof. We have

$$[D, Z] = \left(D(\alpha_1) \frac{\partial}{\partial u_1} + D(\alpha_2) \frac{\partial}{\partial u_2} + D(\alpha_3) \frac{\partial}{\partial u_3} + \ldots \right) - \left(\alpha_1 \frac{\partial}{\partial u} + \alpha_2 \frac{\partial}{\partial u_1} + \alpha_3 \frac{\partial}{\partial u_2} + \ldots \right) = 0.$$

Hence we conclude that

$$\alpha_1 = 0, \quad D(\alpha_1) - \alpha_2 = 0, \quad D(\alpha_2) - \alpha_3 = 0, \ldots$$

and thus $\alpha_i = 0$ for $i = 1, 2, 3, \ldots$. □

Further, since D and $\overline{D}$ commute and in view of $[D, \overline{D}] = f X_2 + \overline{u}_1 [D, X_2] + [D, X_1]$ we arrive at

$$[D, X_1] = -f X_2, \quad [D, X_2] = 0. \tag{11}$$

We note that the operators X_1 and X_2 are linearly independent if $f(u) \neq 0$. Let $X_3 = [X_2, X_1]$. Using the Jacobi identity and Eq. (11), we obtain

$$[D, X_3] = -f_u X_2. \tag{12}$$

By definition, we set

$$\mathcal{L}_n = \bigcup_{i=2}^{n} L_i, \quad n = 3, 4, \ldots.$$

Lemma 2. *The dimension of the linear space $\mathcal{L}_3$ is equal to 2 if and only if*

$$X_3 - c X_1 = 0,$$

where $c = \mathrm{const}$. Thus the right-hand side of Eq. (7) acquires the form

$$f(u) = \alpha \mathrm{e}^{cu},$$

where $\alpha = \mathrm{const} \neq 0$.

Proof. Assume that $\dim \mathcal{L}_3 = 2$. Recall that

$$X_3 = f_u \frac{\partial}{\partial u_1} + f_{uu} u_1 \frac{\partial}{\partial u_2} + \dots,$$

and hence $X_3 = c(u)X_1$. Lemma 1 and formulas (11)–(12) yield

$$[D, X_3 - cX_1] = -f_u X_2 - D(c)X_1 + cfX_2 = 0.$$

The above relation is equivalent to the following system of the equations:

$$f_u - cf = 0, \quad D(c) = 0.$$

Hence $c = \text{const}$ and $f = \alpha e^{cu}$. □

This argument shows that the nonlinear equation (7) with the two-dimentional characteristic Lie algebra A is reduced to the Liouville equation (8).

Assume that $X_4 = [X_2, X_3]$ and $X_5 = [X_1, X_3]$. Using the Jacobi identity and (11), (12) we obtain

$$[D, X_4] = -f_{uu} X_2, \quad [D, X_5] = f_u X_3 - fX_4. \tag{13}$$

Further on, we assume that the dimension of the linear space $\mathcal{L}_3$ is equal to 3. We claim that the cases $\dim \mathcal{L}_4 = 3$ cannot be realized. Indeed, if $\dim \mathcal{L}_4 = 3$, then

$$X_4 = c_1 X_1 + c_2 X_3 \quad \text{and} \quad X_5 = \bar{c}_1 X_1 + \bar{c}_2 X_3, \tag{14}$$

where $c_i = c_i(u, u_1, u_2, \dots)$ and $\bar{c}_i = \bar{c}_i(u, u_1, u_2, \dots)$, $i = 1, 2$. By lemma 1 and formulas (11)–(13), the first relation (14) is equivalent to

$$D(c_1) = 0, \quad c_1 f - f_{uu} + c_2 f_u = 0, \quad D(c_2) = 0.$$

Therefore $c_1, c_2 = \text{const}$ and

$$f_{uu} - c_2 f_u - c_1 f = 0. \tag{15}$$

The second relation (14) is equivalent to the following system of equations:

$$D(\bar{c}_1) + c_1 f = 0, \quad \bar{c}_1 f + \bar{c}_2 f_u = 0, \quad D(\bar{c}_2) + c_2 f - f_u = 0.$$

The third equation implies that $\bar{c}_2 = \text{const}$, and hence $f_u = c_2 f$. Thus the reasonings are reduced to the case $\dim \mathcal{L}_3 = 2$, which has already been considered.

Now assume that $\dim \mathcal{L}_4 = 4$. Using Lemma 1 and formulas (11)–(13), we obtain the following two cases. First we have

$$X_4 = c_1 X_1 + c_2 X_3 + c_3 X_5$$

and therefore,

$$D(c_1) - c_1 c_3 f = 0, \quad f_{uu} - c_1 f - c_2 f_u = 0, \quad D(c_2) + c_3 f_u - c_2 c_3 f = 0. \tag{16}$$

Secondly, we have

$$X_5 = \bar{c}_1 X_1 + \bar{c}_2 X_3 + \bar{c}_3 X_4;$$

then we obtain

$$D(\bar{c}_1) = 0, \quad \bar{c}_1 f + \bar{c}_2 f_u + \bar{c}_3 f_{uu} = 0, \quad D(\bar{c}_2) - f_u = 0, \quad D(\bar{c}_3) + f = 0. \tag{17}$$

The first and the third equations in system (16) imply that $c_1, c_2 = \text{const}$; indeed, otherwise we have $f_u = c_2 f$ and thence $\dim \mathcal{L}_3 = 2$. We note also that the function f satisfies the equation $f_{uu} - c_2 f_u - c_1 f = 0$. Finally, if system (17) is valid, then $f = 0$.

Now we formulate the assertion.

Lemma 3. *The dimension of the space $\mathcal{L}_4$ generated by the operators X_1, X_2, X_3, X_4, and X_5 is equal to 4 if and only if the function f satisfies the equation*

$$f_{uu} - pf_u - qf = 0, \tag{18}$$

where $p, q = \text{const}$ and $f_u \neq \beta f$. Then we also have $X_4 = pX_3 + qX_1$.

In what follows we assume that the assumptions of Lemma 3 hold. Let us introduce the operators of length 4: we set

$$X_6 = [X_1, X_5] \quad \text{and} \quad X_7 = [X_2, X_5].$$

Using the Jacobi identity, it is easily shown that $X_7 = pX_5$. Therefore $\dim \mathcal{L}_5 \leq 5$.

Remark 1. If $X_7 = 0$, then $p = 0$ and equality (18) acquires the form

$$f_{uu} - qf = 0.$$

Then equation (7) is reduced to the sinh-Gordon equation (9).

Using formulas (11)–(13), we conclude that

$$[D, X_6] = (f_u - 2pf)X_5. \tag{19}$$

It can easily be verified that $\dim \mathcal{L}_5 = 5$.

Now we introduce the operators of length 5: we set

$$X_8 = [X_3, X_5], \quad X_9 = [X_1, X_6], \quad X_{10} = [X_2, X_6].$$

As $X_8 = -pX_6 + X_{10}$, $\dim \mathcal{L}_6 \leq 7$.

Furthermore, it follows easily that formulas (12)–(14) and (19) imply

$$[D, X_9] = -fX_{10} + (f_u - 2pf)X_6, \quad [D, X_{10}] = (q - 2p^2)fX_5. \tag{20}$$

If $\dim \mathcal{L}_6 = 5$, we have the following system of relations:

$$X_9 = c_1X_1 + c_2X_3 + c_3X_5 + c_4X_6, \quad X_{10} = \overline{c}_1X_1 + \overline{c}_2X_3 + \overline{c}_3X_5 + \overline{c}_4X_6.$$

By Lemma 1 and formulas (11)–(13), (19), (20), the first relation is equivalent to

$$\begin{gathered} D(c_1) - qc_3f = 0, \\ c_1f + c_2f_u = 0, \\ D(c_3) + c_3f_u - pc_3f = 0, \\ D(c_3) + c_4f_u - 2pc_4f = 0, \\ D(c_4) - f_u + 2pf = 0. \end{gathered}$$

The fifth equation implies that $c_4 = 0$ $f_u = 2pf$. Then $X_3 = 2pX_1$, and hence $\dim \mathcal{L}_3 = 2$. As $\dim \mathcal{L}_3 = 3$, the assumption $\dim \mathcal{L}_6 = 5$ does not hold.

Lemma 4. *The dimension of the space $\mathcal{L}_6$ is equal to 6 if and only if*

$$X_{10} = 0.$$

Proof. Assume that $\dim \mathcal{L}_6 = 6$. Then we obtain two possible cases: First we have

$$X_9 = c_1X_1 + c_2X_3 + c_3X_5 + c_4X_6 + c_5X_{10}$$

and therefore,

$$\begin{gathered} D(c_1) - qc_3f = 0, \quad c_1f + c_2f_u = 0, \quad D(c_2) + c_3f_u - pc_3f = 0, \\ D(c_3) + c_4f_u - 2pc_4f + c_5f_{uu} - c_5pf_u - 2c_5p^2f = 0, \quad D(c_4) - f_u + 2pf = 0, \quad D(c_5) + f = 0. \end{gathered} \tag{21}$$

Secondly, we have

$$X_{10} = \overline{c}_1X_1 + \overline{c}_2X_3 + \overline{c}_3X_5 + \overline{c}_4X_6 + \overline{c}_5X_9$$

and hence,

$$\begin{gathered} D(\overline{c}_1) - \overline{c}_3qf - \overline{c}_1\overline{c}_5f = 0, \quad \overline{c}_1f + \overline{c}_2f_u = 0, \quad D(\overline{c}_2) + \overline{c}_3f_u - \overline{c}_3pf - \overline{c}_2\overline{c}_5f = 0, \\ D(\overline{c}_3) - (q - 2p^2)f + \overline{c}_4(f_u - 2pf) - \overline{c}_3\overline{c}_5f = 0, \quad D(\overline{c}_4) - \overline{c}_4\overline{c}_5f = 0, \quad D(\overline{c}_5) - \overline{c}_5^2f = 0. \end{gathered} \tag{22}$$

Consider the last equation in system (21); we see that $f = 0$. Let us rewrite system (22) in the form

$$\overline{c}_3q = 0, \quad \overline{c}_1f + \overline{c}_2f_u = 0, \quad \overline{c}_3(f_u - pf) = 0, \quad -(q - 2p^2)f + \overline{c}_4(f_u - 2pf) = 0,$$

where $\overline{c}_1$, $\overline{c}_2$, $\overline{c}_3$, $\overline{c}_4 = \text{const}$ and $\overline{c}_5=0$.

If $\overline{c}_3 \neq 0$, then the function f satisfies the equation $f_u = pf$, and hence $\dim \mathcal{L}_2 = 2$. If $\overline{c}_3 = 0$, then $\overline{c}_4 = 0$ (indeed, we have $\dim \mathcal{L}_2 = 2$ otherwise) and from the fourth equation it follows that $q = 2p^2$. According to Eq. (21), we have $X_{10} = 0$. Thus, the necessity is proved.

Now we prove the sufficiency. Let $X_{10} = 0$. Since $X_8 = -pX_6$, we obtain $\dim \mathcal{L}_6 \leq 6$. If $\dim \mathcal{L}_6 = 5$, then X_9 is a linear combination of the operators X_1, X_3, X_5, and X_6, but we have shown above that $\dim \mathcal{L}_3 = 2$ in this case. □

Assume that the condition $q = 2p^2$ in Eq. (18) holds; then Eq. (7) can be transformed to the Tsitseika equation (10).

3. Characteristic Algebra for the sinh-Gordon Equation

In this section, we obtain the description of the x-characteristic Lie algebra A for the equation

$$u_{xy} = e^u + e^{-u}. \tag{23}$$

Let us introduce the multiple commutators

$$X_{i_1 \dots i_n} = \operatorname{ad}_{i_1} \dots \operatorname{ad}_{i_{n-1}} X_{i_n}, \quad \operatorname{ad}_j Y = [X_j, Y].$$

Then the linear space L_n is the linear span of the elements $X_{i_1 \dots i_n}$, where $i_k = 1, 2, k = 1, \dots, n$.

Consider the elements

$$Y_n = X_{1\dots121}, \quad Z_n = X_{21\dots121}.$$

Theorem 1. *Consider the sinh-Gordon equation* (23); *we have*

$$\dim L_n = \begin{cases} 2 \text{ if } n = 2k, \\ 1 \text{ if } n = 2k-1, \end{cases} \qquad k = 3, 4, \dots. \tag{24}$$

The linear space L_{2k-1} is generated by the vector field $X_{1\dots121}$, and the space L_{2k} is generated by the vector fields $X_{1\dots121}$ and $X_{21\dots121}$.

Proof. Recall that the elements X_1, X_2, and X_3 were defined in Sec. 2. For convenience, set $X_4 = [X_1, X_3]$. Then the space L_5 is the linear span of the elements $[X_1, X_4]$ and $[X_2, X_4]$. Using the Jacobi identity and the relations

$$\begin{gathered} [D, X_1] = -(\mathrm{e}^u + \mathrm{e}^{-u})X_2, \quad [D, X_2] = 0, \quad [D, X_3] = -(\mathrm{e}^u - \mathrm{e}^{-u})X_2, \\ [D, X_4] = -(\mathrm{e}^u + \mathrm{e}^{-u})X_1 + (\mathrm{e}^u - \mathrm{e}^{-u})X_3, \end{gathered} \tag{25}$$

we obtain

$$[D, [X_2, X_4]] = 0.$$

Continuing this line of reasoning and using Lemma 1, we obtain $[X_2, X_4] = 0$. Hence L_5 is generated by the element $[X_1, X_4] = X_{1121}$.

We set $X_5 = [X_1, X_4]$. Then we have $[D, X_5] = (e^u + e^{-u})X_4$. The space L_6 is the linear span of the elements $[X_1, X_5]$, $[X_2, X_5]$, and $[X_3, X_4]$. Using the Jacobi identity, we obtain

$$[X_3, X_4] = [X_2, X_5].$$

This implies that L_6 is generated by the elements $[X_1, X_5] = X_{11121}$ and $[X_2, X_5] = X_{21121}$.

Further let $[X_1, X_5] = X_6$. Then we have

$$[D, X_6] = -(\mathrm{e}^u - \mathrm{e}^{-u})X_5 + (\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_5], \quad [D, X_{21121}] = (\mathrm{e}^u + \mathrm{e}^{-u})X_4.$$

The linear span L_7 is generated by the elements $[X_1, X_6]$, $[X_2, X_6]$, $[X_3, X_5]$, $[X_1, X_{21121}]$, and $[X_2, X_{21121}]$. Using the Jacobi identity, we deduce that

$$[X_3, X_5] = -[X_1, X_{21121}] + [X_2, X_6].$$

It is easily shown that

$$[D, [X_2, X_{21121}]] = (\mathrm{e}^u - \mathrm{e}^{-u})X_4 = [D, X_5].$$

Consequently, $[X_2, X_{21121}] = X_5$. We note also that

$$[D, [X_1, X_{21121}]] = -(\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_{21121}] + (\mathrm{e}^u + \mathrm{e}^{-u})X_5,$$
$$[D, [X_2, X_6]] = -(\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_{21121}] + (\mathrm{e}^u + \mathrm{e}^{-u})X_5,$$

whence we have $[X_1, X_{21121}] = [X_2, X_6]$ and $[X_2, X_6] = 0$. This proves that L_7 is generated by the element $[X_1, X_6] = X_{111121}$.

Set $X_{i+1} = [X_1, X_i]$; then we see that

$$[D, X_{i+1}] = (\mathrm{e}^u - \mathrm{e}^{-u})X_i - (\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_i], \tag{26}$$
$$[D, [X_2, [X_2, X_i]]] = (\mathrm{e}^u - \mathrm{e}^{-u})X_{i-1} + (\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_{i-1}], \tag{27}$$
$$[X_j, X_{i-j+3}] = 0, \quad j = 3, 4, \ldots, \tag{28}$$
$$[X_j, X_{21\ldots121}] = 0, \quad j = 3, 4, \ldots. \tag{29}$$

Note that

$$\begin{aligned}[D, [X_2, X_{i+1}]] &= -(\mathrm{e}^u + \mathrm{e}^{-u})[X_2, [X_2, X_i]] + (\mathrm{e}^u + \mathrm{e}^{-u})X_i,\\ [D, [X_1, [X_2, X_i]]] &= -(\mathrm{e}^u + \mathrm{e}^{-u})[X_2, [X_2, X_i]] + (\mathrm{e}^u + \mathrm{e}^{-u})X_i.\end{aligned} \tag{30}$$

This shows that

$$[X_2, X_{i+1}] = [X_1, [X_2, X_i]]. \tag{31}$$

Assume that L_{2k-1} is generated by the element $[X_1, X_{2k-2}]$. Note that

$$[X_2, X_{2k-2}] = 0.$$

Then the linear span L_{2k} is generated by the elements $[X_1, X_{2k-1}]$, $[X_2, X_{2k-1}]$, $[X_3, X_{2k-2}]$, …, $[X_j, X_{21\ldots121}]$. Using formulas (28) and (29), we see that that all these elements except the first two are equal to zero. Thence L_{2k} is generated by the elements $[X_1, X_{2k-1}]$, $[X_2, X_{2k-1}]$, and

$$[D, [X_2, X_{2k-1}]] = -(\mathrm{e}^u + \mathrm{e}^{-u})[X_2, [X_2, X_{2k-2}]] + (\mathrm{e}^u + \mathrm{e}^{-u})X_{2k-2} = (\mathrm{e}^u + \mathrm{e}^{-u})X_{2k-2}.$$

Now assume that L_{2k} is generated by the elements $[X_1, X_{2k-1}]$, $[X_2, X_{2k-1}]$, then L_{2k+1} is the linear span of the elements $[X_1, X_{2k}]$, $[X_2, X_{2k}]$, $[X_1, [X_2, X_{2k-1}]]$, $[X_2, [X_2, X_{2k-1}]]$, $[X_3, X_{2k-1}]$, …, $[X_j, X_{21\ldots121}]$. Relations (28) and (29) imply that

$$[X_3, X_{2k-1}] = [X_4, X_{2k-2}] = \ldots = [X_j, X_{21\ldots121}] = 0.$$

From Eqs. (26), (27) we obtain the equalities

$$[D, [X_2, [X_2, X_{2k-1}]]] = (\mathrm{e}^u - \mathrm{e}^{-u})X_{2k-2} + (\mathrm{e}^u + \mathrm{e}^{-u})[X_2, X_{2k-2}] = [D, X_{2k-1}].$$

We have proved that $[X_2, [X_2, X_{2k-1}]] = X_{2k-1}$.

Now, relations (30) and (31) acquire the form

$$[X_2, X_{2k}] = [X_1, [X_2, X_{2k-1}]] = -(\mathrm{e}^u + \mathrm{e}^{-u})X_{2k-1} + (\mathrm{e}^u + \mathrm{e}^{-u})X_{2k-1} = 0.$$

Hence L_{2k+1} is generated by the element $[X_1, X_{2k}]$. We also see that equalities (24) hold. This is proved by induction. □

We have proved that the basis of the x-characteristic Lie algebra A consists of the elements X_1, X_2, X_{21}, Y_3, Y_4, Y_5, Z_5, Y_6, Y_7, Z_7, …, Y_{2n}, Y_{2n+1}, Z_{2n+1}, ….

Finally, we note that in the paper [6] the sinh-Gordon equation is represented as a quadratic system. Another basis of the characteristic algebra is found in that paper.

4. The Equations $u_{xy} = f(u, u_x)$

The x-characteristic Lie algebra A for equation (3) is generated by the vector fields

$$X_1 = \sum_{i=1}^{\infty} D^{i-1}(f)\frac{\partial}{\partial u_i} \quad \text{and} \quad X_2 = \frac{\partial}{\partial u}.$$

The y-characteristic Lie algebra $\overline{A}$ is generated by the fields

$$Y_1 = u_1\frac{\partial}{\partial u} + \sum_{i=1}^{\infty} \overline{D}^{i-1}(f)\frac{\partial}{\partial \overline{u}_i} \quad \text{and} \quad Y_2 = \frac{\partial}{\partial u_1}.$$

Let us recall that

$$A = \bigcup_{i=2}^{\infty} L_i, \quad \overline{A} = \bigcup_{i=2}^{\infty} \overline{L}_i, \quad \mathcal{L}_n = \bigcup_{i=2}^{n} L_i, \quad \overline{\mathcal{L}}_n = \bigcup_{i=2}^{n} \overline{L}_i, \quad n = 3, 4, \dots,$$

where L_n (respectively, $\overline{L}_n$) is the linear span of the vector fields $X_{i_1 i_2 \dots i_n}$ (respectively, $Y_{i_1 i_2 \dots i_n}$); see Sec. 3.

The classification of equations (3) is based on the following lemma.

Lemma 5. *Let*

$$X = \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial u_i}, \;\; \alpha_i = \alpha_i(u, \overline{u}_1, u_1, u_2, \dots, u_{n_i}), \;\; i = 1, 2, \dots,$$

$$Y = \sum_{i=1}^{\infty} \overline{\alpha}_i \frac{\partial}{\partial \overline{u}_i}, \;\; \overline{\alpha}_i = \overline{\alpha}_i(u, u_1, \overline{u}_1, \overline{u}_2, \dots, \overline{u}_{n_i}), \;\; i = 1, 2, \dots.$$

Then $[D, X] = 0$ *and* $[\overline{D}, Y] = 0$ *if and only if* $X = 0$ *and* $Y = 0$*, respectively.*

The analysis of the linear spaces $\overline{\mathcal{L}}_n$, $n = 3, 4$ provides the following result.

$\dim \overline{\mathcal{L}}_3 = 2$: Then equation (3) has the form

$$u_{xy} = u_x R(u). \tag{32}$$

$\dim \overline{\mathcal{L}}_4 = 3$: Then three cases are possible; we have either

$$u_{xy} = R(u_x), \quad R' - \frac{u_x}{R} = \lambda, \quad \lambda = \text{const}, \tag{33}$$

or

$$u_{xy} = e^u R(u_x), \quad RR' - u_x = 0, \tag{34}$$

or

$$u_{xy} = s(u)u_x + B, \quad B = \text{const} \neq 0. \tag{35}$$

$\dim \overline{\mathcal{L}}_4 = 4$: Then we have either

$$u_{xy} = s(u)u_x + B(u), \tag{36}$$

or

$$u_{xy} = s(u)R(u_x), \quad R' - \alpha\frac{u_x}{R} = \lambda, \quad \lambda = \text{const}, \tag{37}$$

or

$$u_{xy} = e^u R(u_x), \tag{38}$$

where the function R satisfies the system

$$c_1' R + 2c_1 R' = \lambda u_x c_1^2 R^2, \quad c_1(1 - \lambda R)(u_x R' - R) + R'' = 0.$$

Under the same assumption we may have also either

$$u_{xy} = e^u R(u_x), \tag{39}$$

where the function R satisfies the equations

$$R'' + c_3\lambda(R - u_x R') = 0, \quad c_3(R' - c_3 u_x \lambda R) + c_3' R = 0,$$

or, finally,

$$u_{xy} = R(u_x). \tag{40}$$

The equations $u_{xy} = \mathrm{e}^u R(u_x)$ can be rewritten as

$$\upsilon_y = \mathrm{e}^u, \quad u_x = \varphi(\upsilon).$$

This equation is reduced to

$$\upsilon_{xy} = \upsilon_x \varphi(\upsilon). \tag{41}$$

The problem of integration of equations (32)–(34) and (38)–(40) is reduced to integration of the ordinary differential equations; therefore these cases will not be considered. Now we analyze equations (36) and (37).

Recall that the linear space $\overline{\mathcal{L}}_4$ is generated by the operators

$$Y_1, \quad Y_2, \quad Y_3 = [Y_2, Y_1], \quad Y_4 = [Y_2, Y_3], \quad \text{and} \quad Y_5 = [Y_1, Y_3].$$

We have $Y_4 = 0$ for equation (36), and for equation (37) we have

$$Y_4 = \frac{\alpha}{R^2(u_1)}(Y_1 - u_1 Y_3). \tag{42}$$

Further, let $Y_6 = [Y_2, Y_5]$, $Y_7 = [Y_1, Y_5]$, $Y_8 = [Y_2, Y_7]$, $Y_9 = [Y_1, Y_7]$, and $Y_{10} = [Y_3, Y_5]$. We see that $Y_6 = 0$ and $Y_{10} = Y_8$ for equation (36). Similarly, for equation (37) we have

$$Y_6 = -\frac{\alpha u_1}{R^2} Y_5, \quad Y_{10} = \frac{\alpha u_1}{R^2} Y_7 + Y_8.$$

Therefore $\dim \overline{\mathcal{L}}_5 = 5$, and $\dim \overline{\mathcal{L}}_6 \leq 7$. Assume that $\dim \overline{\mathcal{L}}_6 = 6$; then equation (36) is reduced to equation (32). Under the same assumption, for equation (37) we obtain

$$s'' = 0 \quad \text{and} \quad \alpha = 2\lambda^2. \tag{43}$$

If $s = u$, then equations (37) and (43) imply that

$$u_{xy} = 3uR(u_x), \quad (u_x - R)(R + 2u_x)^2 = 1. \tag{44}$$

Equation (44) is related [9] to the equation $\upsilon_{xy} = e^{\upsilon} + e^{-2\upsilon}$ by the differential substitution

$$\upsilon = -\frac{1}{2}\ln(u_x - R).$$

Now we consider the x-characteristic Lie algebra and pay special attention to the linear spaces $\mathcal{L}_3$ and $\mathcal{L}_4$. Then we obtain the following assertion.

$\dim \mathcal{L}_3 = 2$: Then equation (3) acquires the form

$$u_{xy} = \mathrm{e}^{\alpha u} R(u_x), \quad \alpha = \mathrm{const}\,. \tag{45}$$

$\dim \mathcal{L}_4 = 3$: Then we have

$$u_{xy} = s(u) u_x, \tag{46}$$

where the function s satisfies the system

$$s'' - c_1 s - c_3 s' = 0, \quad \overline{c}_1 s + \overline{c}_3 s' = 0,$$
$$\overline{c}_1' + c_1 s = 0, \quad \overline{c}_3' + c_3 s - s' = 0, \quad c_i = \mathrm{const}_i, \overline{c}_i = \overline{c}_i(u), i = 1, 3.$$

$\dim \mathcal{L}_4 = 4$: In this case, we have

$$u_{xy} = s(u) u_x$$

where the function s satisfies either the system

$$\begin{gathered} s'' - c_1 s - c_3 s' = 0, \quad c_1 = \alpha c_5, \\ c_5' = c_5^2 s, \quad c_3' + c_5 s' - c_3 c_5 s = 0, \quad c_i = c_i(u),\ i = 1, 3, 5, \end{gathered} \tag{47}$$

or s satisfies the equations

$$s'' + \overline{c}_1 s + \overline{c}_3 s' = 0, \quad \overline{c}_3' = s', \\ \overline{c}_5' = -s, \quad \overline{c}_1 = \text{const}, \overline{c}_3 = \overline{c}_3(u), \overline{c}_5 = \overline{c}_5(u). \tag{48}$$

Under the same assumption $\dim \mathcal{L}_4 = 4$ we may also have

$$u_{xy} = s(u)R(u_x), \tag{49}$$

where the function s is such that

$$s'' - c_3 s' - c_1 s = 0, \quad c_1, c_3 = \text{const}\,. \tag{50}$$

Equations (46)–(48) belong to the class defined by Eq. (41).

Let us consider equations(49) and (50) such that $\dim \overline{\mathcal{L}}_4 = 4$ and such that the function $R(u_x)$ is a solution of the equation

$$R' - \frac{u_x}{R} = \beta, \quad \beta = \text{const}\,. \tag{51}$$

Note that $X_4 = c_1 X_1 + c_3 X_3$ and set

$$X_6 = [X_2, X_5], \quad X_7 = [X_1, X_5], \quad X_8 = [X_2, X_7], \quad X_9 = [X_1, X_7], \quad X_{10} = [X_3, X_5].$$

It is easily shown that $X_6 = c_3 X_5$ and $X_{10} = -c_3 X_7 + X_8$.

Assume that $X_4 = 0$; then we have $s(u) = u$, and if $X_6 = 0$, then we obtain $s(u) = \sin u$.

We see that $\dim \mathcal{L}_5 = 5$. The condition $\dim \mathcal{L}_6 = 6$ for equations (49)–(51) is equivalent to $s(u) = u$. Then we have also $X_8 = -X_1 + uX_3 + \beta X_5$.

If $\lambda = 0$ for the function $s = \sin u$, then equations (49)–(51) are related to the equation $\upsilon_{xy} = \sin \upsilon$ using the differential substitution $\upsilon = \arcsin u_x + u$. If $s = u$, then equations (49)–(51) are related to the equation $\upsilon_{xy} = \sin \upsilon$ by using the differential substitution $\upsilon = \arcsin u_x$.

The structure of the x-characteristic Lie algebra for the equations

$$u_{xy} = u\sqrt{1 - u_x^2} \quad \text{and} \quad u_{xy} = \sin u \sqrt{1 - u_x^2} \quad (\lambda = 0)$$

is similar to the x-characteristic algebra of the sinh-Gordon equation (9), and the structure of the y-characteristic algebra for equation (44) coincides with the structure of the characteristic algebra for the Tsitseika equation (10).

We conclude that the list of integrable equations we have obtained coincides with the known list.

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