Journal of Mathematical Sciences, Vol. 150, No. 5, 2008

EXISTENCE OF SOLUTIONS OF CERTAIN QUASILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$ WITHOUT CONDITIONS AT INFINITY

G. I. Laptev

UDC 513.8+517.9

Abstract. This paper deals with conditions for the existence of solutions of the equations

$$-\sum_{i=1}^{n} D_i A_i(x,u,Du) + A_0(x,u) = f(x), \quad x \in \mathbb{R}^n,$$

considered in the whole space $\mathbb{R}^n$, $n \geq 2$. The functions $A_i(x,u,\xi)$, $i = 1, \dots, n$, $A_0(x,u)$, and $f(x)$ can arbitrarily grow as $|x| \to \infty$. These functions satisfy generalized conditions of the monotone operator theory in the arguments $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. We prove the existence theorem for a solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ under the condition $p > n$.

1. Statement of the Problem and Formulation of the Result

This paper is devoted to the study of the solvability conditions of second-order, quasilinear, elliptic equations in the space $\mathbb{R}^n$, $n \geq 2$. We use the methods developed for monotone operators and also the compact operator method. The monotone operator theory was developed in the 1960s through the efforts of many mathematicians and enables a wide class of higher-order partial differential equations of elliptic type to be studied. The totalities of the method are given in [2,8,11]. We stress that the works mentioned are devoted to equations considered in a bounded domain. If the domain considered is not bounded, then it is assumed that the solution belongs to an appropriate Sobolev space $W^{m,p}(\Omega)$, which imposes certain conditions on the solution as $|x| \to \infty$. In recent years, there arose a considerable interest in solutions that can arbitrarily grow as $|x| \to \infty$. Especially, this remark refers to anti-coercive equations to which a vast literature is devoted, in particular, [9].

Coercive equations in $\mathbb{R}^N$ without conditions at infinity are studied in relatively few works. The subject originates from [1] in which the author proved the solvability of equations of the form

$$-\Delta u + |u|^{q-1}u = f(x), \quad q > 0, \quad x \in \mathbb{R}^n.$$

In [10], in an unbounded domain $\Omega \in \mathbb{R}^n$, the author studied the equations

$$\sum D_i(a_{ij} D_j u) - a(x)|u|^{p-1}u = f(x), \quad p > 1, \quad a \in L^1_{\mathrm{loc}}(\Omega), \quad a(x) \geq a_0 > 0.$$

The variational method for the equation

$$\Delta u = f(x,u), \quad x \in \mathbb{R}^n,$$

was applied in [5]. In [3], the author considered the Dirichlet problem for the equation

$$-\sum (|u_{x_i}|^{\alpha} u_{x_i})_{x_i} + c(x)u = f(x)$$

in an unbounded domain Ω with compact boundary $\partial\Omega$. Here,

$$\alpha > 0, \quad c(x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n), \quad c(x) \geq 0, \quad f(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^n).$$

Some generalizations of the latter equation were studied in [6, 7].

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 12, No. 4, pp. 133–147, 2006.

The present paper is devoted to the study of the solvability of the equation

$$-\sum_{i=1}^{n} D_i A_i(x, u, Du) + A_0(x, u) = f(x), \quad x \in \mathbb{R}^n. \tag{1}$$

Here,

$$D_i = \frac{\partial}{\partial x_i} \quad (i = 1, \dots, n), \quad Du = (D_1 u, \dots, D_n u), \quad n \geq 2.$$

No conditions on the behavior of the solution as $|x| \to \infty$ are imposed.

Let us list the conditions for the functions entering Eq. (1). It is assumed that the functions $A_i(x, u, \xi)$, $i = 1, \dots, n$, and $A_0(x, u)$ are defined for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $u \in \mathbb{R}$ and satisfy the Caratheodory conditions, i.e., they are measurable in x and continuous in u, ξ for almost all $x \in \mathbb{R}^n$. Moreover, these functions satisfy the following restrictions.

1A. Monotonicity condition in the principal part. For almost all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} [A_i(x, u, \xi) - A_i(x, u, \eta)](\xi_i - \eta_i) > 0 \quad (\xi \neq \eta),$$

where $\xi = (\xi_1, \dots, \xi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$.

2A. Coercivity condition. For almost all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} A_i(x, u, \xi)\xi_i + A_0(x, u)u \geq a(x)|\xi|^p + b(x)|u|^q + h(x),$$

where $n < p < q$ and the functions $a(x)$ and $b(x)$ are positive; moreover, $a(x)$, $a^{-1}(x)$, $b(x)$, $b^{-1}(x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, and $h(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

3A. Growth conditions. For $i = 1, \dots, n$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$,

$$|A_i(x, u, \xi)| \leq a_1(x)|\xi|^{p-1} + b_1(x)|u|^{q/p'} + h_1(x),$$

where $p + p' = pp'$, $a_1(x), b_1(x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, and $h_1(x) \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$;

$$|A_0(x, u)| \leq a_0(|u|)h_0(x),$$

where the function $a_0(t)$ increases and is continuous for $t \geq 0$, $h_0(x) \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^n)$.

1f. $f(x) \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^n)$.

Here and in what follows, we use the traditional notation for the spaces of Lebesgue integrable functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. All measurable functions $u(x)$, $x \in \Omega$, with finite norm

$$\|u\|^p_{L^p(\Omega)} = \int_{\Omega} |u(x)|^p \, dx, \quad 1 \leq p < \infty,$$

compose the Banach space $L^p(\Omega)$. All measurable functions $u(x)$, $x \in \mathbb{R}^n$, with finite norm $\|u\|_{L^p(\Omega)}$ for any bounded domain $\Omega \in \mathbb{R}^n$ compose the space $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, which is no longer a Banach space. The Sobolev space $W^{1,p}(\Omega)$ consists of all measurable functions $u(x)$, $x \in \Omega$, having measurable partial derivatives $Du(x)$ with finite norm

$$\|u\|_{W^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}.$$

We say that $u(x) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ if the function $u(x)$ is measurable on $\mathbb{R}^n$ and, moreover, $u(x) \in W^{1,p}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^n$. If $p > 1$, then the duality between the spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$ is denoted by $(f, u) = \int_{\Omega} f(x)u(x)\, dx$, $u \in L^p(\Omega)$, $f \in L^{p'}(\Omega)$. Also, by (f, u) we denote the duality between the spaces $W^{1,p}(\Omega)$ and its dual $\left(W^{1,p}(\Omega)\right)^*$. The space of continuously differential compactly supported functions on $\mathbb{R}^n$ is denoted by $C^1_0(\mathbb{R}^n)$.

Definition 1. A function $u(x) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ is called a solution of Eq. (1) if the following identity holds for any function $\psi(x) \in C_0^1(\mathbb{R}^n)$:

$$\int\limits_{\mathbb{R}^n} \left(\sum_{i=1}^n A_i(x,u,Du) D_i\psi + A_0(x,u)\psi \right) dx = \int\limits_{\mathbb{R}^n} f\psi \, dx.$$

Now let us formulate the main result of the paper.

Theorem 1. *Let Conditions* 1A–3A *and* 1f *hold. Then Eq.* (1) *has a solution in the sense of Definition* 1.

Remark 1. We can take the right-hand side of Eq. (1) in the form

$$f(x) = \sum_{i=1}^n D_i f_i(x) + f_0(x),$$

where $f_i \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$ and $f_0 \in L^{q'}_{\text{loc}}(\mathbb{R}^n)$. Such a form of the function $f(x)$ formally looks more general than that in Condition 1f. In this case, the functions $D_i f_i(x)$ must be included into the summands $D_i A_i(x,u,Du)$ of Eq. (1). Conditions 1A–3A allow us to do this.

2. Approximations by Bounded Domains

In the space $\mathbb{R}^n$, we consider a bounded domain Ω with Lipschitzian boundary $\partial\Omega$, which ensures the possibility of application of the Sobolev embedding theorems. Let two numbers $p, q \in (1,\infty)$ be fixed. The main role for what follows is played by the space

$$X = \{u(x) \in L^q(\Omega),\ Du(x) \in L^p(\Omega)\}$$

with the norm

$$\|u\|_X = \|Du\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}.$$

We can give a more clear characteristic of the space introduced.

Lemma 1. *If $q > p > n$, then $X = W^{1,p}(\Omega)$.*

Proof. In the case where Ω is a bounded domain, the following inequality holds for any $q \geq 1$:

$$\|u\|_{L^q(\Omega)} \leq c_1 \|u\|_{C(\overline{\Omega})}.$$

Since $p > n$, the space $W^{1,p}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$, i.e.,

$$\|u\|_{C(\overline{\Omega})} \leq c_2 \|u\|_{W^{1,p}(\Omega)}.$$

Combining the presented inequalities, we obtain

$$\|u\|_X = \|Du\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} + c\|u\|_{W^{1,p}(\Omega)} = (1+c)\|u\|_{W^{1,p}(\Omega)}.$$

This means that $W^{1,p}(\Omega) \subset X$. Conversely, since $q > p$, it follows that

$$\|u\|_{L^p(\Omega)} \leq c\|u\|_{L^q(\Omega)},$$

whence

$$\|u\|_{W^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \leq \|Du\|_{L^p(\Omega)} + c\|u\|_{L^q(\Omega)} \leq (1+c)\|u\|_X.$$

Hence $X \subset W^{1,p}(\Omega)$, which completes the proof of the lemma. □

Now we pass to the description of the operators of the problem. Our goal is to show that under Conditions 1A–3A, the differential expression

$$Au = -\sum_{i=1}^n D_i A_i(x,u,Du) + A_0(x,u) \tag{2}$$

defines a bounded, continuous operator from $W^{1,p}(\Omega)$ into the dual space $\left(W^{1,p}(\Omega)\right)^*$ for any bounded domain Ω. Let us project Conditions 1A–3A and 1f on a bounded domain Ω. In other words, we assume that the functions $A_i(x,u,\xi)$, $i=1,\dots,n$, $A_0(x,u)$, and also $f(x)$ introduced early are defined only for $x\in\Omega$. In the case considered, we can represent Conditions 1A–3A and 1f as follows.

$1A_\Omega$. For almost all $x\in\Omega$ and all $u\in\mathbb{R}$, and $\xi,\eta\in\mathbb{R}^n$,

$$\sum_{i=1}^{n}[A_i(x,u,\xi)-A_i(x,u,\eta)](\xi_i-\eta_i)>0 \quad (\xi\neq\eta).$$

$2A_\Omega$. For almost all $x\in\Omega$ and all $u\in\mathbb{R}$, and $\xi\in\mathbb{R}^n$,

$$\sum_{i=1}^{n}A_i(x,u,\xi)\xi_i+A_0(x,u)u\geq c_\Omega(|\xi|^p+|u|^q)+h(x),$$

where $n<p<q$, the constant c_Ω is positive, and $h(x)\in L^1(\Omega)$.

$3A_\Omega$. For almost all $x\in\Omega$, all $u\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, and certain constants $C_{1\Omega}$, $C_{2\Omega}$,

$$|A_i(x,u,\xi)|\leq C_{1\Omega}|\xi|^{p-1}+C_{2\Omega}|u|^{q/p'}+h_1(x),\quad i=1,\dots,n;$$
$$|A_0(x,u)|\leq a_0(u)h_0(x),\quad a_0\in C(\mathbb{R}),\quad h_0(x)\in L^{p'}(\Omega).$$

1f. $f(x)\in L^{q'}(\Omega)$.

All the written conditions are consequences of Conditions 1A–3A and 1f. Therefore, Condition $1A_\Omega$ is obvious. In Condition $2A_\Omega$, the constant $c_\Omega>0$ arises as a consequence of the condition $a^{-1}(x),b^{-1}(x)\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. The constants $C_{1\Omega}$ and $C_{2\Omega}$ appear as a consequence of the local boundedness of the functions $a_1(x)$ and $b_1(x)$ in Condition 3A.

For comparison, let us write conditions from [4, 16.16 (3), p. 119] in the case $p>n$ considered:

$$\begin{gathered}2|A_i(x,u,\xi)|\leq c_1(|u|)(g_1(x)+|\xi|^{p-1}),\quad g_1\in L^{p'}(\Omega),\quad i=1,\dots,n;\\ |A_0(x,u)|\leq c_0(|u|)g_0(x),\quad g_0\in L^1(\Omega),\end{gathered}\tag{3}$$

where c_0 and c_1 are nonnegative continuous functions. Under conditions (3), the formal differential operator (2) defines a bounded continuous operator A defined on the space $W^{1,p}(\Omega)$ and assuming its values in the dual space $\big(W^{1,p}(\Omega)\big)^*$ [4, Theorem 16.14, p. 115]. This operator is defined by the form

$$(Au,v)=\int_\Omega\bigg(\sum_{i=1}^{n}A_i(x,u,Du)D_iv+A_0(x,u)v\bigg)dx,\quad u,v\in W^{1,p}(\Omega).\tag{4}$$

Obviously, Conditions $3A_\Omega$ imply conditions (3), and, therefore, the following assertion holds.

Lemma 2. *Under Conditions* $3A_\Omega$*, the formal differential operator* (2) *defines a bounded continuous operator* A *defined on the space* $W^{1,p}(\Omega)$ *and assuming the values in the dual space* X^**. This operator is defined by form* (4)*.*

We now turn to the corresponding differential equation

$$-\sum_{i=1}^{n}D_iA_i(x,u,Du)+A_0(x,u)=f(x),\quad x\in\Omega,\tag{5}$$

which is considered in a bounded domain $\Omega\in\mathbb{R}^n$. According to Lemma 2, we can represent this equation in the operator form $Au=f$, where the operator A is defined by (4). Recall that the specification of a form defines certain boundary conditions on $\partial\Omega$, which we do not explicitly write now, since they do not play a special role in the case considered.

We can also describe Conditions $1A_\Omega$ and $2A_\Omega$ as the properties of the operators A introduced above. Precisely, Condition $1A_\Omega$ defines an operator that is monotone in the principal part, and Condition $2A_\Omega$

implies its coercivity, which means the fulfillment of the following relation as $|u\|_{W^{1,p}(\Omega)} \to \infty$:

$$\frac{(Au,u)}{\|u\|_{W^{1,p}(\Omega)}} = \|u\|^{-1}_{W^{1,p}(\Omega)} \int\limits_{\Omega} \left(\sum_{i=1}^{n} A_i(x,u,Du) D_i u + A_0(x,u)u \right) dx \to \infty.$$

Therefore, the operator $A\colon X \to X^*$ introduced in Lemma 2 is a bounded continuous operator monotone in the principal part. For each function $f \in \left(W^{1,p}(\Omega)\right)^*$, the equation $Au = f$ has a solution $u \in W^{1,p}(\Omega)$ [4, Theorem 29.11, p. 207]. By Lemma 1, the identity $W^{1,p}(\Omega) = X$ holds, and the structure of the function $f(x)$ from Condition 1f shows that $f \in X^*$. All that was presented above leads to the following assertion.

Theorem 2. *In $\mathbb{R}^n$, let Ω be a bounded domain with Lipschitzian domain $\partial\Omega$ and let Conditions* $1\mathrm{A}_\Omega$–$3\mathrm{A}_\Omega$ *and* $1\mathrm{f}_\Omega$ *hold. Then the operator equation $Au = f$, where the operator A is defined in Lemma 2, has a solution $u \in X$ satisfying the following identity for any function $v \in X$:*

$$\int\limits_{\Omega} \left(\sum_{i=1}^{n} A_i(x,u,Du) D_i v + A_0(x,u)v \right) dx = \int\limits_{\Omega} f v \, dx.$$

3. Estimates for Solutions in Expanding Domains

As bounded domains Ω from the previous section, we choose the balls $B_N = \{x \in \mathbb{R}^n \colon |x| < N\}$ of integer radii $N = 1, 2, 3, \ldots$. Then we can reformulate Theorem 2 as follows.

Lemma 3. *Let Conditions* 1A–3A *and* 1f *hold. For each $N = 1, 2, 3, \ldots$, in the ball B_N, there exists a solution $u_N \in X_N = W^{1,p}(B_N)$ of the problem*

$$\int\limits_{B_N} \left(\sum_{i=1}^{n} A_i(x,u_N,Du_N) D_i v + A_0(x,u_N)v \right) dx = \int\limits_{B_N} f v \, dx, \quad v \in X_N. \tag{6}$$

Thus, we have defined a sequence $\{u_N,\ N \in \mathbb{N}\}$ of solutions of problem (6). Fix an integer $m \in \mathbb{N}$ and introduce a truncating function $\varphi \in C_0^1(\overline{B_{m+2}})$ with the condition $\varphi(x) > 0$ for $|x| < m+2$. In identity (6), we set $v = u_N\varphi$ for $N \ge m+2$ and represent the obtained result in the form

$$\int\limits_{B_N} \left(\sum_{i=1}^{n} A_i(x,u_N,Du_N)(D_i u_N)\varphi + A_0(x,u_N)u_N\varphi \right) dx = \int\limits_{B_N} f u_N \varphi \, dx - \int\limits_{B_N} \sum_{i=1}^{n} A_i(x,u_N,Du_N) u_N D_i\varphi \, dx. \tag{7}$$

We stress the following important fact. With account for the factor $\varphi \in C_0^1(\overline{B_{m+2}})$, all the integrals in identity (7) are calculated over a fixed bounded set $\operatorname{supp}\varphi \subset \overline{B_{m+2}}$. Let us estimate the left-hand side of identity (7) from below using the coercivity Condition $2\mathrm{A}_\Omega$ for $\Omega = B_{m+2}$:

$$\int\limits_{B_N} \left(\sum_{i=1}^{n} A_i(x,u_N,Du_N)(D_i u_N)\varphi + A_0(x,u_N)u_N\varphi \right) dx \ge c_m \int\limits_{B_{m+2}} (|Du_N|^p + |u_N|^q)\varphi(x) \, dx - C_{0m}. \tag{8}$$

Here, $c_m = c_\Omega > 0$ for $\Omega = B_{m+2}$ and $C_{0m} = \int\limits_{B_{m+2}} h(x)\varphi(x)\,dx$. Let us estimate the right-hand of identity (7). We begin with the last integral and represent it in the form

$$\int\limits_{B_N} \sum_{i=1}^{n} A_i(x,u_N,Du_N) u_N D_i\varphi \, dx = \int\limits_{B_{m+2}} \sum_{i=1}^{n} A_i(x,u_N,Du_N)\varphi^{1/p'} u_N \varphi^{1/q} \varphi^{-(1/p'+1/q)} D_i\varphi \, dx.$$

Apply the Hölder inequality with three factors with exponents p', q, and $s = \frac{qp}{q-p}$ to the last integral. Note that the necessary condition

$$\frac{1}{p'} + \frac{1}{q} + \frac{1}{s} = 1$$

holds, and, moreover,

$$\left(\frac{1}{p'}+\frac{1}{q}\right)s = s-1;$$

therefore,

$$\begin{aligned}
&\Bigl|\int\limits_{B_N}\sum_{i=1}^{n} A_i(x,u_N,Du_N)u_N D_i\varphi\,dx\Bigr|\\
&\quad\le \Bigl(\int\limits_{B_{m+2}}\sum_{i=1}^{n}|A_i(x,u_N,Du_N)|^{p'}\varphi\,dx\Bigr)^{1/p'}\Bigl(\int\limits_{B_{m+2}}|u_N|^q\varphi\,dx\Bigr)^{1/q}\Bigl(\int\limits_{B_{m+2}}\varphi^{-s+1}|D\varphi|^s dx\Bigr)^{1/s}\\
&\quad\le \varepsilon\int\limits_{B_{m+2}}\sum_{i=1}^{n}|A_i(x,u_N,Du_N)|^{p'}\varphi\,dx+\varepsilon\int\limits_{B_{m+2}}|u_N|^q\varphi\,dx+C(\varepsilon)\int\limits_{B_{m+2}}\varphi^{-s+1}|D\varphi|^s\,dx. \qquad (9)
\end{aligned}$$

Here, we have used the Young inequality with an arbitrary exponent $\varepsilon > 0$. According to the growth Condition $3\mathrm{A}_\Omega$ for $\Omega = B_{m+2}$, for $i = 1,\dots,n$, we can estimate the summands

$$\begin{aligned}
&\int\limits_{B_{m+2}}|A_i(x,u_N,Du_N)|^{p'}\varphi\,dx\\
&\quad\le \int\limits_{B_{m+2}}\bigl(C_{1m}|Du_N|^{p/p'}+C_{2m}|u_N|^{q/p'}+h_1(x)\bigr)^{p'}\varphi\,dx \le C_{3m}\int\limits_{B_{m+2}}(|Du_N|^p+|u_N|^q)\varphi\,dx + C_{4m}, \quad (10)
\end{aligned}$$

where $C_{4m} = \int\limits_{B_{m+2}}|h_1(x)|^{p'}\varphi(x)\,dx$. Here and in what follows, the symbols C with subscripts stands for the constants independent of the number $N \ge m+2$, although it is possible that they depend on a given m. Substituting estimate (10) in (9), we obtain

$$\begin{aligned}
&\Bigl|\int\limits_{B_{m+2}}\sum_{i=1}^{n} A_i(x,u_N,Du_N)u_N D_i\varphi\,dx\Bigr|\\
&\quad\le \varepsilon n C_{3m}\int\limits_{B_{m+2}}(|Du_N|^p+|u_N|^q)\varphi\,dx+\varepsilon\int\limits_{B_{m+2}}|u_N|^q\varphi\,dx+\varepsilon n C_{4m}+C(\varepsilon)\int\limits_{B_{m+2}}|D\varphi|^s\varphi^{-s+1}\,dx.
\end{aligned}$$

It is easy to estimate the remaining summand of identity (7):

$$\Bigl|\int\limits_{B_{m+2}} f u_N\varphi\,dx\Bigr| \le \int\limits_{B_{m+2}}|f|\varphi^{1/q'}|u_N|\varphi^{1/q}\,dx \le \varepsilon\int\limits_{B_{m+2}}|u_N|^q\varphi\,dx + C(\varepsilon)\int\limits_{B_{m+2}}|f|^{q'}\varphi\,dx.$$

Substituting all the presented estimates in identity (7), we obtain the inequality

$$(c_m-\varepsilon n C_{3m})\int\limits_{B_{m+2}}|Du_N|^p\varphi\,dx+(c_m-2\varepsilon-\varepsilon n C_{3m})\int\limits_{B_{m+2}}|u_N|^q\varphi\,dx \le C_{5m}+C(\varepsilon)\int\limits_{B_{m+2}}\frac{|D\varphi|^s}{\varphi^{s-1}}\,dx.$$

Choosing $\varepsilon > 0$ sufficiently small but fixed here, we obtain the estimate

$$\int\limits_{B_{m+2}}(|Du_N|^p+|u_N|^q)\varphi\,dx \le C_{6m}\Bigl(1+\int\limits_{B_{m+2}}\frac{|D\varphi|^s}{\varphi^{s-1}}\,dx\Bigr). \qquad (11)$$

Recall that the function $\varphi(x)$ is compactly supported, and, therefore, we need to justify the existence of the last integral in estimate (11). Choose a function $\psi \in C_0^1(\overline{B_{m+2}})$ with the condition $\psi(x) > 0$ for

$x \in B_{m+2}$ such that $\psi(x) = 0$ only on the boundary of the ball B_{m+2}. Let us verify that it suffices to assume that $\varphi(x) = \psi^s(x)$. Indeed, on the set $|x| < m+2$, we obtain

$$\frac{|D\varphi|^s}{\varphi^{s-1}} = \frac{|s\psi^{s-1}D\psi|^s}{\psi^{s(s-1)}} = s^s|D\psi|^s, \quad \psi \in C_0^1(\overline{B}_{m+2}).$$

The required fraction $|D\varphi|^s\varphi^{-s+1}$ becomes a continuous, compactly supported function, and, therefore,

$$\int\limits_{B_{m+2}} \frac{|D\varphi|^s}{\varphi^{s-1}}\,dx \le s^s \int\limits_{B_{m+2}} |D\varphi|^s\,dx \equiv C_{7m}.$$

Finally, we obtain from estimate (11) that

$$\int\limits_{B_{m+2}} (|Du_N|^p + |u_N|^q)\psi^s dx \le C_m, \tag{12}$$

where the constant C_m is independent of the number $N \ge m+2$. Choose a function $\psi(x)$ such that $\psi(x) \equiv 1$ for $|x| \le m+1$. Then estimate (12) implies the inequality

$$\int\limits_{B_{m+1}} (|Du_N|^p + |u_N|^q)\,dx \le C_m, \quad N \ge m+2. \tag{13}$$

Let us formulate the result obtained.

Lemma 4. *Let Conditions* 1A–3A *and* 1f *hold. Fix a natural number m. Then for the sequence of solutions u_N, $N = 1, 2, 3, \ldots$, constructed in Lemma* 3*, estimate* (13) *holds, in which the constant C_m can depend on m but not on $N \ge m+2$.*

We use estimate (13) in several variants. So, it directly follows from it that $\|Du_N\|_{L^p(B_{m+1})} \le C_m$ and $\|u_N\|_{L^q(B_{m+1})} \le C_m$ for $N \ge m+2$. This implies that for $N \ge m+2$,

$$\|u_N\|_{X_{m+1}} = \|Du_N\|_{L^p(B_{m+1})} + \|u_N\|_{L^q(B_{m+1})} \le C_m, \tag{14}$$

and inequality (10) implies the estimate

$$\|A_i(x, u_N, Du_N)\|_{L^{p'}(B_{m+1})} \le C_m. \tag{15}$$

It follows from (14) and the reflexivity of the space $X_{m+1} = W^{1,p}(B_{m+1})$ that there exist a function $u^{(m)} \in X_{m+1}$ and a subsequence of integers $K_m \subset \mathbb{N}$ such that $u_k \to u^{(m)}$ weakly in X_{m+1} as $k \to \infty$, $k \in K_m$, and $Du_k \to Du^{(m)}$ weakly in $L^p(B_{m+1})$. Therefore, since $k \to \infty$, $k \in K_m$,

$$u_k \rightharpoonup u^{(m)} \text{ in } X_{m+1}, \quad Du_k \rightharpoonup Du^{(m)} \text{ in } L^p(B_{m+1}). \tag{16}$$

The space $X_{m+1} = W^{1,p}(B_{m+1})$ is compactly embedded in $L^p(B_{m+1})$ and also in $C(\overline{B_{m+1}})$, since we consider the case where $p > n$. Therefore, we can assume that the following strong convergence holds as $k \to \infty$, $k \in K_m$:

$$\|u_k - u^{(m)}\|_{L^p(B_{m+1})} \to 0, \quad \|u_k - u^{(m)}\|_{C(\overline{B_{m+1}})} \to 0. \tag{17}$$

Thus, we have proved the following assertion.

Lemma 5. *Let Conditions* 1A–3A *and* 1f *hold, and let a natural number m be fixed. Then there exist a function $u^{(m)} \in W^{1,p}(B_{m+1})$ and a subsequence $K_M \subset \mathbb{N}$ such that relations* (16), (17) *hold.*

4. Proof of Theorem 1

Our next goal is to perform the passage to the limit in identities (6), which are used only on the set K_m constructed in Lemma 5. These identities have the form

$$\int\limits_{B_k} \left(\sum_{i=1}^{n} A_i(x, u_k, Du_k) D_i v + A_0(x, u_k) v \right) dx = (f, v), \quad v \in X_k, \quad k \in K_m. \tag{18}$$

Fix a function $\varphi \in C_0^1(\overline{B_{m+1}})$ and substitute the product $v = (u_k - u^{(m)})\varphi$ in identities (6); this is possible if we assume that the function $\varphi(x)$ is extended by zero outside the ball B_{m+1}, so that $v \in X_k$ for all $k \geq m+1$. Let us write the resulting identities in detail taking into account that, in fact, all the integrals are calculated only over the set B_{m+1} for the subscripts $k \in K_m$:

$$\int\limits_{B_{m+1}} \sum_{i=1}^{n} A_i(x, u_k, Du_k)(D_i u_k - D_i u^{(m)})\varphi \, dx + \int\limits_{B_{m+1}} \sum_{i+1}^{n} A_i(x, u_k, Du_k)(u_k - u^{(m)}) D\varphi \, dx$$
$$+ \int\limits_{B_{m+1}} A_0(x, u_k)(u_k - u^{(m)})\varphi \, dx = (f, (u_k - u^{(m)})\varphi). \tag{19}$$

According to the construction of the subset $K_m \subset \mathbb{N}$, the weak convergence $u_k \to u^{(m)}$ in $W^{1,p}(B_{m+1})$ holds, and hence $(f, (u_k - u^{(m)})\varphi) \to 0$ as $k \to \infty$, $k \in K_m$.

Let us estimate other summands of the system of relations (19) taking into account the strong convergence (17):

$$\left| \int\limits_{B_{m+1}} \sum_{i=1}^{n} A_i(x, u_k, Du_k)(u_k - u^{(m)}) D\varphi \, dx \right|$$
$$\leq \max_{|x| \leq m+1} |D\varphi| \sum_{i=1}^{n} \|A_i(x, u_k, Du_k)\|_{L^{p'}(B_{m+1})} \|u_k - u^{(m)}\|_{L^p(B_{m+1})}$$
$$\leq C\|u_k - u^{(m)}\|_{L^p(B_{m+1})} \to 0 \quad (k \to \infty, \quad k \in K_m).$$

Here, we have used estimate (15) for $\|A_i\|_{L^{p'}(B_{m+1})}$. Furthermore, we have

$$\left| \int\limits_{B_{m+1}} A_0(x, u_k)(u_k - u^{(m)})\varphi \, dx \right|$$
$$\leq \max_{|x| \leq m+1} |\varphi(x)| \, \|u_k - u^{(m)}\|_{C(\overline{B_{m+1}})} \int\limits_{B_{m+1}} |A_0(x, u_k)| \, dx$$
$$\leq C\|u_k - u^{(m)}\|_{C(\overline{B_{m+1}})} a_0(\|u_k\|_{C(\overline{B_{m+1}})}) \to 0 \quad (k \to \infty, \quad k \in K_m).$$

Here, we have used the following sufficiently obvious chain of estimates on the set B_{m+1}:

$$a_0(|u_k|) \leq a_0(\max_{|x| \leq m+1} |u_k|) = a_0(\|u_k\|_{C(\overline{B_{m+1}})}) \leq C,$$

since

$$\|u_k\|_{C(\overline{B_{m+1}})} \leq C\|u_k\|_{W^{1,p}(B_{m+1})}$$

in the case $p > n$ considered.

We use the convergence presented above in identities (19); as a result, we obtain the following relation as $k \to \infty$, $k \in K_m$:

$$\int_{B_{m+1}} \sum_{i=1}^{n} A_i(x, u_k, Du_k)(D_i u_k - D_i u^{(m)})\varphi \, dx \to 0.$$

Represent the obtained relation in the form

$$\int_{B_{m+1}} \sum_{i=1}^{n} (A_i(x, u_k, Du_k) - A_i(x, u_k, Du^{(m)})(D_i u_k - D_i u^{(m)})\varphi \, dx$$
$$+ \int_{B_{m+1}} A_i(x, u_k, Du^{(m)})(D_i u_k - D_i u^{(m)})\varphi \, dx \to 0 \quad (k \to \infty, \quad k \in K_m). \quad (20)$$

The growth Condition $3\mathrm{A}_\Omega$ for $\Omega = B_{m+1}$ implies

$$|A_i(x, u_k, Du^{(m)})| \le C_{1m}|Du^{(m)}|^{p-1} + C_{2m}|u_k|^{q/p'} + h_1(x), \quad x \in B_{m+1}.$$

Since the function $u^{(m)}$ is fixed, we can represent the latter inequality in the form

$$|A_i(x, u_k, Du^{(m)})| \le h_2(x) + C|u_k|^{q/p'}, \quad x \in B_{m+1}, \quad h_2 \in L^{p'}(B_{m+1}).$$

The estimate obtained shows that the Nemytskii operator $A_i\big(x, u_k(x), Du^{(m)}(x)\big)$ with respect to the argument $u_k(x)$ is a bounded continuous operator from the space $L^q(B_{m+1})$ into $L^{p'}(B_{m+1})$.

It follows from (17) that $u_k \to u^{(m)}$ in $C(\overline{B_{m+1}})$ and hence in $L^q(B_{m+1})$, and, therefore, by the properties of the Nemytskii operator, the following convergence holds for all $i = 1, \dots, n$ as $k \to \infty$, $k \in K_m$:

$$A_i(x, u_k, Du^{(m)}) \to A_i(x, u^{(m)}, Du^{(m)}) \text{ in } L^{p'}(B_{m+1}). \quad (21)$$

Recall that $Du_k \to Du^{(m)}$ weakly in $L^p(B_{m+1})$ by construction. In combination with the strong convergence (21), this means that as $k \to \infty$, $k \in K_m$, for $i = 1, \dots, n$,

$$\int_{B_{m+1}} A_i(x, u_k, Du^{(m)})(D_i u_k - D_i u^{(m)})\varphi \, dx \to 0.$$

Then (20) implies that as $k \to \infty$, $k \in K_m$,

$$\int_{B_{m+1}} \sum_{i=1}^{n} \big(A_i(x, u_k, Du_k) - A_i(x, u_k, Du^{(m)})\big)(D_i u_i - D_i u^{(m)})\varphi \, dx \to 0. \quad (22)$$

We have chosen the functions $\varphi(x)$ satisfying the condition $\varphi(x) > 0$ for $x \in B_{m+1}$. Now, we also assume that $\varphi(x) \equiv 1$ for $|x| \le m$. According to the monotonicity Condition 1A, the function under the sign of the integral in (22) is nonnegative, and, therefore, the following relation holds for $k \in K_m$:

$$\overline{\lim_{k\to\infty}} \int_{B_m} \sum_{i=1}^{n} \big(A_i(x, u_k, Du_k) - A_i(x, u_k, Du^{(m)})\big)(D_i u_k - D_i u^{(m)}) \, dx \le 0. \quad (23)$$

As was shown in [8, Chap. 2, Sec. 2, Lemma 2.2, p. 196], (23) implies that $Du_k(x) \to Du^{(m)}(x)$ almost everywhere in the domain B_m and that the weak convergence $A_i(x, u_k, Du_k) \to A_i(x, u^{(m)}, Du^{(m)})$ holds in the space $L^{p'}(B_m)$ as $k \to \infty$, $k \in K_m$, so that the following relation holds for every function $v \in W^{1,p}(B_m)$:

$$\int_{B_m} \sum_{i=1}^{n} A_i(x, u_k, Du_k) D_i v \, dx \to \int_{B_m} \sum_{i=1}^{n} A_i(x, u^{(m)}, Du^{(m)}) D_i v \, dx. \quad (24)$$

In particular, the convergence $u_k \to u^{(m)}$ in $C(\overline{B_{m+1}})$ implies that the following relation holds for every function $v \in W^{1,p}(B_m)$ as $k \to \infty$, $k \in K_m$:

$$\int\limits_{B_m} A_i(x, u_k) v \, dx \to \int\limits_{B_m} A_i(x, u^{(m)}) v \, dx. \tag{25}$$

We use the presented relations for passing to the limit in identities (18); for this purpose, we fix a natural $m \in \mathbb{N}$ and choose a function $v \in C_0^1(B_m)$ such that for $k \in K_m$, identities (18) become

$$\int\limits_{B_m} \left(\sum_{i=1}^{n} A_i(x, u_k, Du_k) D_i v + A_0(x, u_k) v \right) dx = \int\limits_{B_m} f v \, dx, \quad v \in C_0^1(B_m).$$

Letting $k \in K_m$ tend to ∞ in these identities and applying relations (24) and (25), we obtain the following relation for $v \in C_0^1(B_m)$ in the limit:

$$\int\limits_{B_m} \left(\sum_{i=1}^{n} A_i(x, u^{(m)}, Du^{(m)}) D_i v + A_0(x, u^{(m)}) v \right) dx = \int\limits_{B_m} f v \, dx. \tag{26}$$

Now, let us formulate the obtained result.

Theorem 3. *Let Conditions* 1A–3A *and* 1f *hold. If the natural number $m \in \mathbb{N}$ is fixed, then the function $u^{(m)} \in W^{1,p}(B_{m+1})$ introduced in Lemma* 5 *satisfies identity* (26)*.*

Now let us construct a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ that is a solution of the initial Eq. (1). Fix a natural number $m = 1$ and, in accordance with Lemma 5, construct the function $u^{(1)}(x) \in W^{1,p}(B_2)$ and the infinite set of natural numbers $K_1 \subset \mathbb{N}$ such that $u_k(x) \to u^{(1)}(x)$ weakly in the space X_2 as $k \to \infty$, $k \in K_1$. Moreover, according to Theorem 3, identity (26) with $m = 1$ holds for the limit function $u^{(1)}(x)$. Then we fix a number $m = 2$ and, in accordance with Lemma 5, construct the function $u^{(2)}(x) \in X_3$ and the infinite set of natural numbers $K_2 \in \mathbb{N}$ satisfying the additional condition $K_2 \subset K_1$ such that $u_k(x) \to u^{(2)}(x)$ weakly in X_3 as $k \to \infty$, $k \in K_2$. According to Theorem 3, the limit function $u^{(2)}(x)$ satisfies identity (26) with $m = 2$. With account for the condition $K_2 \subset K_1$, on the set $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, both limit functions coincide, i.e., $u^{(2)}(x) = u^{(1)}(x)$ for $|x| < 1$. Now extract an infinite subset $K_3 \subset K_2$ from the set K_2 such that according to Lemma 5, $u_k(x) \to u^{(3)}(x)$ weakly in X_4 as $k \to \infty$, $k \in K_3$, and, moreover, the limit function $u^{(3)}(x)$ satisfies the identity (26) for $m = 3$. With account for the condition $K_3 \subset K_2$, the relation $u^{(3)}(x) = u^{(2)}(x)$, $|x| < 2$, holds on the set B_2. Clearly, we can perform the described construction for all $m = 1, 2, 3, \ldots$; as a result, we construct a sequence of functions $u^{(1)}(x), u^{(2)}(x), u^{(3)}(x), \ldots$, and, moreover, each of the function is an extension of the previous to a wider set, precisely, $u^{(m+1)}(x) = u^{(m)}(x)$ for $|x| < m$. As a result, as $m \to \infty$, we define a unit function $u(x)$, $x \in \mathbb{R}^n$, such that

$$u(x) = u^{(m)}(x) \in X_{m+1}, \quad |x| < m. \tag{27}$$

This means that $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, and, by relations (27) and identities (26), the function $u(x)$ satisfies each of the following relations for $m = 1, 2, 3, \ldots$:

$$\int\limits_{B_m} \left(\sum_{i=1}^{n} A_i(x, u, Du) D_i v + A_0(x, u) v \right) dx = \int\limits_{B_m} f v \, dx, \quad v \in C_0^1(B_m). \tag{28}$$

Fix a compactly supported function $\psi(x) \in C_0^1(\mathbb{R}^n)$. Its support is contained in a certain ball $\overline{B}_m = \{x \in \mathbb{R}^n : |x| \le m\}$, $m \in \mathbb{N}$, and, therefore, $\psi(x) \in C_0^1(B_m)$, i.e., we can substitute this function for $v(x)$ in identity (28), which leads to the relation

$$\int\limits_{\mathbb{R}^n} \left(\sum_{i=1}^{n} A_i(x, u, Du) D_i \psi + A_0(x, u) \psi \right) dx = \int\limits_{\mathbb{R}^n} f \psi \, dx.$$

According to Definition 1, such a relation means that the constructed function $u(x)$, $x \in \mathbb{R}^n$, is a solution of the initial Eq. (1). This completes the proof of Theorem 1.

Let us present an example of an equation for which the conditions of Theorem 1 hold:

$$-\sum_{i=1}^{n} D_i(a_i(x)|D_i u|^{p-2} D_i u + a_{n+1}(x)|u|^r u) + a_0(x)|u|^{q-2} u = f(x), \quad x \in R^n,$$

where the functions $a_i(x)$ are positive and $a_i, a_i^{-1} \in L^\infty_{\text{loc}}(R^n)$ for all $i = 0, 1, \dots, n+1$. We assume that $n < p < q$ and $r + 1 \le \frac{q}{p'}$.

Acknowledgment. This work was supported by the Russian Foundation for Basic Research (project No. 04-01-00536) and by the Program of Ministry of Education (project No. E02-1.0-144).

REFERENCES

1. H. Brezis, "Semilinear equation in $\mathbb{R}^N$ without condition at infinity," *Appl. Math. Optim.*, **12**, 271–282 (1984).
2. Yu. A. Dubinskii, "Nonlinear higher-order parabolic equations," in: *Progress in Science and Technology, Series on Contemporary Problems in Mathematics* [in Russian], Vol. 37, All-Union Institute for Scientific and Technical Information, USSR Academy of Sciences, Moscow (1990), pp. 89–166.
3. A. L. Gladkov, "Dirichlet problem for certain degenerate elliptic equations in unbounded domains," *Differ. Uravn.*, **29**, 267–273 (1993).
4. A. Kufner and S. Fučik, *Nonlinear Differential Equations* [Russian translation], Nauka, Moscow (1988).
5. I. Kuzin and S. Pokhozhaev, *Entire Solutions of Semilinear Elliptic Equations*, Birkhäuser (1997).
6. G. I. Laptev, "Solvability of quasilinear elliptic second-order differential equations in $\mathbb{R}^n$ without condition at infinity," *Adv. Math. Research*, **4**, 1–18 (2003).
7. F. Leoni, "Nonlinear elliptic equations in $\mathbb{R}^n$ with "absorbing" zero-order terms," *Adv. Differential Equations*, **5**, No. 4–5, 681–722 (2000).
8. J. L. Lions, *Some Methods for Solving Nonlinear Boundary-Value Problems* [Russian translation], Mir, Moscow (1972).
9. E. Mitidieri and S. I. Pokhozhaev, "A priori estimates and the absence of solutions of nonlinear partial equations and inequalities," *Tr. Mat. Inst. Steklova*, **234** (2001).
10. O. Oleinik, *Some Asymptotic Problems in the Theory of Partial Differential Equations*, Cambridge Univ. Press (1996).
11. I. V. Skrypnik, *Methods for Studying Nonlinear Elliptic Boundary-Value Problems* [in Russian], Nauka, Moscow (1990).

G. I. Laptev
Moscow State Social University
E-mail: glaptev@yandex.ru