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# MEAN VALUES CONNECTED WITH THE DEDEKIND ZETA FUNCTION

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*For a cubic extension $K_3/\mathbb{Q}$, which is not normal, new results on the behavior of mean values of the Dedekind zeta function of the field $K_3$ in the critical strip are obtained.*

*Let $M(m)$ denote the number of integral ideals of the field $K_3$ of norm $m$. For the sums*

$$\sum_{m\le x} M(m)^2 \quad \text{and} \quad \sum_{m\le x} M(m)^3$$

*asymptotic formulas are derived. Previously, only upper bounds for these sums were known. Bibliography: 23 titles.*

## 1

This is a preparatory section. In connection with the results presented here, see [1, 2]. Consider the group $S_3$. The elements of $S_3$ fall into the following three conjugacy classes: $C_1$ : (1); $C_2$ : $(1,2,3),(3,2,1)$; $C_3$ : $(1,2),(2,3),(3.1)$. Hence there are three simple characters: the one-dimensional characters $\psi_1$ (the principal character) and $\psi_2$ (the other character determined by the subgroup $C_1\cup C_2$), and the two-dimensional character $\psi_3$. $S_3$ is the Galois group of the non-Abelian extension $K_6$ of degree 6 of $\mathbb{Q}$, where $K_6$ is the normal closure of a cubic field $K_3$ over $\mathbb{Q}$ given by an irreducible polynomial $f(x) = x^3 + ax^2 + bx + c$ of discriminant $D$. The fields $K_2 = \mathbb{Q}(\sqrt{D})$ and $K_3$ are the intermediate extensions fixed under the subgroups $A_3$ and $\{(1),(1,2)\}$, respectively. The extensions $K_2/\mathbb{Q}$, $K_6/K_2$, and $K_6/K_3$ are Abelian. The Dedekind zeta functions (for the definition, see Sec. 2) satisfy the relations

$$\begin{aligned}
\zeta_{K_6}(s) &= L_{\psi_1}L_{\psi_2}L_{\psi_3}^2,\\
\zeta_{K_2}(s) &= L_{\psi_1}L_{\psi_2},\\
\zeta_{K_3}(s) &= L_{\psi_1}L_{\psi_3},\\
\zeta(s) &= L_{\psi_1},
\end{aligned}$$

where

$$\begin{aligned}
L_{\psi_2} =&L(s,\psi_2,K_6/\mathbb{Q}) = L(s,\chi,K_2/\mathbb{Q}),\\
L_{\psi_3} =&L(s,\psi_3,K_6/\mathbb{Q}) = L(s,\chi',K_6/K_2).
\end{aligned}$$

Here, the second column involves the Artin $L$-functions, and the third column involves the $L$-functions with Hecke characters (more exactly, $\chi(*) = (D/*)$).

Below, we assume that

$$K_2, K_3, K_6 \ \text{ are the fields indicated above, and } \ D<0; \tag{$*$}$$

$\varepsilon > 0$ is an arbitrary fixed number.

The function $L_{\psi_3}$ can also be interpreted in another way [3]. Let $\rho : S_3 \to GL_2(\mathbb{C})$ be the irreducible two-dimensional representation. Then $\rho$ gives rise to a cuspidal representation $\pi$ of $GL_2(\mathbb{A}_\mathbb{Q})$. Let

$$L(s,\pi) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

In particular, if $\rho$ is odd, i.e., $D<0$, then $L(s,\pi) = L(s,F)$, where $F$ is a holomorphic primitive cusp form of weight 1 with respect to $\Gamma_0(|D|)$,

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad q = e^{2\pi iz}.$$

As usual, $L(s,\pi)$ denotes the $L$-function of the representation $\pi$; $L(s,F)$ denotes the Hecke $L$-function of the form $F$. Thus, $L_{\psi_3} = L(s,F)$,

$$\zeta_{K_3}(s) = \zeta(s)L(s,F). \tag{1}$$

Formula (1) will be used below.

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This section improves known results connected with the behavior of the Dedekind zeta function of the cubic field $K_3$ in the critical strip. Let $K_n$ be an algebraic number field over $\mathbb{Q}$ of degree $n$. The Dedekind zeta function of $K_n$ is defined by the relation

$$\zeta_{K_n}(s) = \sum_{\mathfrak{a}}{}' (N\mathfrak{a})^{-s},$$

where the summation runs over all nonzero integral ideals in $K_n$. If $r_1$ is the number of real conjugates of $K_n$, $2r_2$ is the number of imaginary conjugates, and $\Delta$ is the discriminant of $K_n$, then the functional equation for $\zeta_{K_n}(s)$ may be written as

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \Gamma^{r_1}\left(\frac{1}{2}s\right)\Gamma^{r_2}(s)B^{-s}\zeta_{K_n}(s),$$

with

$$B = 2^{r_2}\pi^{n/2}(|\Delta|)^{-1/2}.$$

The Dedekind zeta function admits analytic continuation to the entire complex plane, and it only has a simple pole at $s = 1$ with residue

$$\operatorname*{res}_{s=1}\zeta_{K_n}(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|\Delta|}},$$

where $h$ is the class number of $K_n$; $R$ is the regulator of the field, and $w$ the number of roots of unity. Let $s = \sigma + it$. The critical strip for $\zeta_{K_n}(s)$ is the strip $0 \le \sigma \le 1$, and the critical line is the line $s = 1/2 + it$.

As was proved by Kaufman [4, 5] and Heath–Brown [6],

$$\zeta_{K_n}\left(\frac{1}{2}+it\right) \ll t^{(n/6)+\varepsilon} \quad (t \ge 1). \tag{2}$$

Also the mean square

$$\int_1^T \left|\zeta_{K_n}(\sigma+it)\right|^2 dt, \quad 0 \le \sigma \le 1,$$

was estimated (see [7, 8]). For $K_3$ and the critical line, the upper bound obtained in [7] takes the form

$$\int_1^T \left|\zeta_{K_3}\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^{3/2}\log^3 T.$$

Let $\sigma(K_n)$ be the lower bound of the numbers $\sigma$ such that

$$\int_1^T \left|\zeta_{K_n}(\sigma+it)\right|^2 dt \ll T^{1+\varepsilon}.$$

In [7, 8], it was shown that

$$\sigma(K_n) \le 1 - \frac{1}{n}.$$

The theorem below improves the above results in the case of $K_3$.

**Theorem 1.** *In the case of* $K_3$*,*

(i) $\zeta_{K_3}\left(\frac{1}{2}+it\right) \ll t^{\frac{301}{615}+\varepsilon} \quad (r \ge 1)$;

(ii) $\int_1^T \left|\zeta_{K_3}\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^{\frac{5}{4}+\varepsilon}$;

(iii) $\sigma(K_3) \le \frac{5}{8}$.

The proof of assertion (i) immediately follows from the known estimates ($t \geq 1$)

$$\zeta\Big(\frac{1}{2}+it\Big) \ll t^{\frac{32}{205}+\varepsilon} \quad ([9]),$$

$$L\Big(\frac{1}{2}+it, F\Big) \ll t^{\frac{1}{3}+\varepsilon} \quad ([10]).$$

Pass to the proof of assertion (ii). First we obtain the estimate

$$\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^4 dt \ll T^{\frac{3}{2}+\varepsilon}. \tag{3}$$

We use the following known results:

$$\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^2 dt \sim CT\log T, \tag{4}$$

$$\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^6 dt \ll T^{2+\varepsilon}; \tag{5}$$

see [11] and [12], respectively.

In connection with (5), we note that this estimate was originally obtained by Jutila for a holomorphic cusp form $F(z)$ of even weight with respect to the full modular group. His proof can readily be extended to our case.

By virtue of (4) and (5), we have

$$\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^4 dt \leq \bigg\{\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^2 dt\bigg\}^{1/2} \bigg\{\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^6 dt\bigg\}^{1/2} \ll (T\log T)^{1/2}(T^{2+\varepsilon})^{1/2} \ll T^{\frac{3}{2}+\varepsilon},$$

and estimate (3) is proved.

Recall the following classical result of Ingham [13] :

$$\int_1^T \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4 dt \sim \frac{T\log^4 T}{2\pi^2}.$$

Using (3) and Ingham's asymptotics, we derive

$$\int_1^T \Big|\zeta_{K_3}\Big(\frac{1}{2}+it, F\Big)\Big|^2 dt = \int_1^T \Big|\zeta\Big(\frac{1}{2}+it\Big)L\Big(\frac{1}{2}+it, F\Big)\Big|^2 dt$$

$$\leq \bigg\{\int_1^T \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4 dt\bigg\}^{1/2} \bigg\{\int_1^T \Big|L\Big(\frac{1}{2}+it, F\Big)\Big|^4 dt\bigg\}^{1/2} \ll (T\log^4 T)^{1/2}(T^{3/2+\varepsilon})^{1/2} \ll T^{\frac{5}{4}+\varepsilon},$$

and assertion (ii) is proved.

In order to prove assertion (iii), we use the inequalities

$$\int_1^T |\zeta(\sigma+it)|^4 dt \ll T \quad \Big(\sigma > \frac{1}{2}\Big),$$

$$\int_1^T |L(\sigma+it, F)|^4 dt \ll T^{1+\varepsilon} \quad \Big(\sigma \geq \frac{5}{8}\Big).$$

The latter inequality is due to Ivic [14], who proved it in the case of a holomorphic cusp form $F$ of even weight with respect to the full modular group; his proof can be extended to our case.

As above, the assertion desired is obtained by using the Cauchy inequality.

For completeness, we state one more fact (cf. [8]), which is an immediate corollary of the general Matsumoto result [15, Theorem 2].

**Proposition 1.** *For* $\frac{1}{2} < \sigma \leq 1$,

$$\int_1^T \left|\zeta_{K_3}(\sigma+it)\right|^2 dt = c(\sigma)T + O\big(T^{3(1-\sigma)+\varepsilon}\big).$$

*Proof.* Represent $\zeta_{K_3}(s)$ in the form (see Sec. 3)

$$\zeta_{K_3}(s) = \sum_{m=1}^{\infty} M(m)m^{-s}. \tag{6}$$

Consider the expansion

$$\sum_{m\leq x}{}' M(m) = Ax + R(x),$$

where $A > 0$ is a constant, and, for $\frac{1}{3} < \sigma_1 < 1$,

$$R(x) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \zeta_{K_3}(s)\frac{x^s}{s}\,ds.$$

Let $\beta$ be the lower bound for the numbers $b$ such that

$$\int_X^{2X} |R(x)|^2 dx \ll X^{1+2b+\varepsilon}.$$

By Theorem 2 in [15], for $\max\left\{\beta, \frac{1}{2}\right\} < \sigma \leq 1$,

$$\int_1^T \left|\zeta_{K_3}(\sigma+it)\right|^2 dt = c(\sigma)T + O\big(T^{2(1-\sigma)/(1-\beta_1)+\varepsilon}\big),$$

where $\beta_1 = \max\{\beta, 0\}$. As is known [16], $\beta \leq \frac{1}{3}$, and the result follows.

## 3

Consider representation (6), where $M(m)$ is the number of nonzero integral ideals of the field $K_3$ of norm $m$ (the same notation is preserved for an arbitrary field $K_n$). In the present section, new facts on the distribution of values of this function are presented. Note that the function $M(m)$ is multiplicative.

Factorization (1) readily implies the formula

$$M(m) = \sum_{d|m} a(d),$$

where $a(d)$ is the $d$th Fourier coefficient of $F$. In particular,

$$M(p) = 1 + a(p).$$

The values of $M(p)$ can be computed using the results in [17]. More exactly, the following assertion is valid.

**Proposition 2.** *Consider the field $K_3$. Then $M(p)=1$ if $(-D/p)=-1$. If $(-D/p)=1$, then $p=\mathfrak{p}\mathfrak{p}'$ in $K_2$; in this case, $M(p)=3$ if $\mathfrak{p}$ splits completely in $K_6$, and $M(p)=0$ if $\mathfrak{p}$ remains prime in $K_6$.*

For any field $K_n$, the asymptotics

$$\sum_{m\le x} M(m) = cx + O\big(x^{1-2/(n+1)}\big)$$

is classical (Landau).

Pass to the problem on the mean square of the function $M(m)$, which was first considered by Chandrasekharan and Narasimhan in [7]. Further results were obtained by Chandrasekharan and Good in [8]. In [7], the following estimate was proved: for any field $K_n$,

$$\sum_{m\le x} M(m)^2 \ll x\big(\log x\big)^{n-1}.$$

If $K_n$ is normal, then we have the asymptotics [8] ($l\ge 2$ integer)

$$\sum_{m\le x} M(m)^l = xP(\log x) + O\big(x^{1-2n^{-l}+\varepsilon}\big),$$

where $P$ is a suitable polynomial of degree $n^{l-1}-1$.

We obtain the asymptotics for the field $K_3$ (which is nonnormal).

**Theorem 2.** *For the field $K_3$, we have the relation*

$$\sum_{m\le x} M(m)^2 = C_1 x\log x + C_2 x + O\big(x^{\frac{9}{11}+\varepsilon}\big),$$

*where $C_1>0$ and $C_2$ are constants.*

*Proof.* Note that the function

$$T(s)=\sum_{m=1}^{\infty} M(m)^2 m^{-s}$$

can be expressed by the Euler product

$$T(s)=\prod_p \big(1+M(p)^2p^{-s}+M(p^2)^2p^{-2s}+M(p^3)^2p^{-3s}+\cdots\big).$$

By comparing it with the Euler product of the product of $L$-functions

$$\zeta(s)L(s,F)^2\,L(s,F\times F)$$

and using the formula

$$M(p)^2 = 1+2a(p)+a(p)^2,$$

we obtain ($\sigma>\frac{1}{2}$)

$$T(s)=\zeta(s)L(s,F)^2L(s,F\times F)\cdot B(s), \tag{7}$$

where $L(s,F\times F)$ is the Rankin-Selberg convolution $L$-function of the form $F$ with itself [18], and

$$B(s)=\prod_p(1+A_2p^{-2s}+A_3p^{-3s}+\cdots).$$

Using (7), one can readily show that the function $T(s)$ admits an analytic continuation into the half-plane $\sigma>\frac{1}{2}$ having as its only singularity a pole of second order at $s=1$, because each of the functions $\zeta(s)$ and $L(s,F\times F)$ has a simple pole at $s=1$, and $L(1,F)\ne 0$, $B(1)\ne 0$. It is known [19] that on the half-line $t\ge 1$,

$$\zeta\Big(\frac{1}{2}+it\Big)\ll t^{\frac{1}{6}+\varepsilon},$$
$$L\Big(\frac{1}{2}+it,F\Big)\ll t^{\frac{1}{3}+\varepsilon},$$
$$L\Big(\frac{1}{2}+it,F\times F\Big)\ll t^{\frac{11}{12}+\varepsilon}.$$

Therefore, by the convexity property in the strip $\frac{1}{2}+\varepsilon \le \sigma \le 1+\varepsilon$, the growth estimate

$$T(s) \ll \big(|t|+1\big)^{\frac{7}{2}(1-\sigma)+4\varepsilon} \tag{8}$$

holds for $|t| \to \infty$. By using the well-known inversion formula for Dirichlet series, we obtain

$$\sum_{m\le x} M(m)^2 = \frac{1}{2\pi i}\int\limits_{1+\varepsilon-iT}^{1+\varepsilon+iT} T(s)\frac{x^s}{s}\,ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).$$

Shifting the path of integration to the line $\sigma = \frac{1}{2}+\varepsilon$ and performing necessary computations with the use of estimate (8), we derive

$$\sum_{m\le x} M(m)^2 = C_1 x\log x + C_2 x + O\big(x^{\frac{1}{2}+\varepsilon}T^{\frac{7}{4}+\varepsilon} + x^{1+\varepsilon}/T\big).$$

The proof of Theorem 2 is completed by setting $T = x^{\frac{2}{11}}$.

In the case of a normal field $K_n$, after the appearence of [4–6] the asymptotics due to Chandrasekharan and Good can be improved for $n \ge 4$. For simplicity, we restrict our considerations to the case $l = 2$.

**Theorem 3.** *For a normal field $K_n$ we have the relation*

$$\sum_{m\le x} M(m)^2 = xP_n(\log x) + O\big(x^{1-\frac{3}{n^2+6}+\varepsilon}\big),$$

*where $P_n$ is a suitable polynomial of degree $n-1$.*

Let, as above,

$$T(s) = \sum_{m=1}^{\infty} M(m)^2 n^{-s}.$$

Consider the factorization (see [7])

$$T(s) = \zeta_{K_n}^n(s)U(s), \tag{9}$$

where $U(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$. By virtue of (2), on the half-line $t \ge 1$,

$$\zeta_{K_n}^n\Big(\frac{1}{2}+it\Big) \ll t^{\frac{n^2}{6}+\varepsilon}.$$

Using (9), we show that $T(s)$ admits an analytic continuation into the half-plane $\sigma > \frac{1}{2}$ having as its only singularity a pole of order $n$ at $s=1$. In the strip $\frac{1}{2}+\varepsilon \le \sigma \le 1+\varepsilon$, for $|t| \to \infty$, the following growth estimate is valid:

$$T(s) \ll \big(|t|+1\big)^{\frac{n^2(1-\sigma)}{3}+\frac{n^2}{2}\varepsilon}. \tag{10}$$

Using the inversion formula for Dirichlet series and estimate (10), we prove Theorem 3.

In the case of the field $K_3$, we can also treat the sum $\sum\limits_{m\le x} M(m)^3$.

**Theorem 4.** *For the field $K_3$, we have the relation*

$$\sum_{m\le x} M(m)^3 = xP_5(\log x) + O\big(x^{73/79+\varepsilon}\big),$$

*where $P_5$ is a suitable polynomial of degree 4.*

We begin the proof with the notation

$$T^{(3)}(s) = \sum_{m=1}^{\infty} M(m)^3 m^{-s}.$$

We have

$$M(p)^3 = 1 + 3a(p) + 3a(p)^2 + a(p)^3. \tag{11}$$

By virtue of (11), in the half-plane $\sigma > \frac{1}{2}$ we have the relation

$$T^{(3)}(s) = \zeta(s)L(s,F)^3 L(s,F\times F)^3 R^{(3)}(s)B'(s),$$

where (cf. [20, 21])

$$R^{(3)}(s) = L(s,F)^2 L(s,\mathrm{sym}\,^3F)\cdot\prod_p K(s);$$

$$B'(s) = \prod_p (1 + A_2' p^{-2s} + A_3' p^{-3s} + \cdots);$$

$L(s,\mathrm{sym}\,^3F)$ is the symmetric cube $L$-function of the form $F$ (see [22, 23]), and

$$K(s) = 1 + N_2 p^{-2s} + \cdots + N_6 p^{-6s}.$$

Consequently,

$$T^{(3)}(s) = \zeta(s)L(s,F)^5 L(s,F\times F)^3 L(s,\mathrm{sym}\,^3F)B''(s), \tag{12}$$

where

$$B''(s) = \prod_p (1 + A_2'' p^{-2s} + A_3'' p^{-3s} + \cdots).$$

Using (12), we show that $T^{(3)}(s)$ admits an analytic continuation into the half-plane $\sigma > \frac{1}{2}$ having as its only singularity a pole of order 5 at $s = 1$. Note that although in most cases the function $L(s,\mathrm{sym}\,^3F)$ is entire [23], in our case it has a simple pole at $s = 1$.

In the strip $\frac{1}{2} + \varepsilon \le \sigma \le 1 + \varepsilon$, for $|t| \to \infty$, the growth estimate

$$T^{(3)}(s) \ll \big(|t| + 1\big)^{\frac{67}{6}(1-\sigma)+12\varepsilon}$$

is valid. Using the inversion formula for Dirichlet series and this estimate, we prove Theorem 4.

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## REFERENCES


1. *Algebraic Number Theory* [Russian translation], J. W. S. Cassels and A. Fröhlich (eds.), Mir, Moscow (1969).
2. H. Cohn, *A Classical Invitation to Algebraic Numbers and Class Fields*, New York etc. (1978).
3. P. Deligne and J.-P. Serre, "Formes modulaires de poids 1," *Ann. Sci. École Norm. Sup.*, Sér. 4, **7**, 507–530 (1974).
4. R. M. Kaufman, "A. F. Lavrik's truncated equations," *Zap. Nauchn. Semin. LOMI*, **76**, 124–158 (1978).
5. R. M. Kaufman, "Estimation of the Hecke $L$-functions on the critical line," *Zap. Nauchn. Semin. LOMI*, **91**, 40–51 (1979).
6. D. R. Heath-Brown, "The growth rate of the Dedekind zeta-function on the critical line," *Acta Arithm.*, **49**, 323–339 (1988).
7. K. Chandrasekharan and R. Narasimhan, "The approximate functional equation for a class of zeta-functions," *Math. Ann.*, **152**, 30–64 (1963).
8. K. Chandrasekharan and A. Good, "On the number of integral ideals in Galois extensions," *Monatsh. Math.*, **95**, 99–109 (1983).
9. M. N. Huxley, "Exponential sums and the Riemann zeta function V," *Proc. London Math. Soc.*, **90**, 1–41 (2005).
10. C. S. Yoganandra, "Transformation formula for exponential sums involving Fourier coefficients modular forms," *Proc. Indian Acad. Sci. (Math. Sci.)*, **103**, No. 1, 1–25 (1993).
11. A. Good, "Approximative Funktionalgleichungen und Mittelwertsätze für Dirichletreihen, die Spitzenformen assoziiert sind," *Comment. Math. Helvet.*, **50**, 327–361 (1975).
12. M. Jutila, *Lectures on a Method in the Theory of Exponential Sums*, Bombay (1987).

13. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, New York (1986).
14. A. Ivić, "On zeta-functions associated with Fourier coefficients of cusp forms," in: *Proc. Amalfi Conference on Analytic Number Theory* (Maiori, 1989), Salerno (1992), pp. 231–246.
15. K. Matsumoto, "Lifting and mean value theorems for automorphic $L$-functions," *Proc. London Math. Soc.*, **90**, 297–320 (2005).
16. K. Chandrasekharan and R. Narasimhan, "On the mean value of the error term of a class of arithmetical functions," *Acta Math.*, **112**, 41–67 (1964).
17. M. Koike, "Higher reciprocity law, modular forms of weight 1 and elliptic curves," *Nagoya Math. J.*, **98**, 109–115 (1985).
18. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, *AMS*, Providence, Rhode Island (2004).
19. A. Sankaranarayanan, "Fundamental properties of symmetric square $L$-functions. I," *Illinois J. Math.*, **46**, 23–43 (2002).
20. P. B. Garrett, "Decomposition of Eisenstein series: Rankin triple product," *Ann. Math.*, **125**, 209–235 (1987).
21. O. M. Fomenko, "Fourier coefficients of cusp forms and automorphic $L$-functions," *Zap. Nauchn. Semin. POMI*, **237**, 194–226 (1997).
22. C. J. Moreno and F. Shahidi, "The $L$-function $L_3(s, \pi_\Delta)$ is entire," *Invent. Math.*, **79**, 247–251 (1985).
23. H. M. Kim and F. Shahidi, "Symmetric cube $L$-functions for $GL_2$ are entire," *Ann. Math.*, **150**, 645–662 (1999).