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A PRIORI PROPERTIES OF SOLUTIONS OF NONLINEAR EQUATIONS WITH DEGENERATE COERCIVITY AND L^1-DATA

A. A. Kovalevsky UDC 517.9

Abstract. A Dirichlet problem for a second-order nonlinear elliptic equation in the general divergent form with a right-hand side from L^1 is considered. The high-order coefficients in the equation are assumed to satisfy the degenerate coercivity condition. The main results concern a priori properties of summability and some estimates of entropy solutions of this problem.

1. Introduction

We consider the Dirichlet problem for the equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and $f \in L^1(\Omega)$. We suppose that the high-order coefficients in the equation have an arbitrary growth with respect to u and growth of order $p-1$ with respect to ∇u, $p \in (1, n)$, and satisfy the conditions of degenerate coercivity and strict monotonicity. The results of this paper mainly concern a priori properties of entropy solutions of problem (1.1). These results do not require any restrictions on the growth and sign of the low-order coefficient. In the only theorem of existence formulated at the end of the paper, it is supposed that the function $a_0(x, u, \nabla u)$ has growth of order $\sigma \in (0, p-1)$ with respect to u and of order p with respect to ∇u, but still no conditions on its sign are required.

Solvability and properties of solutions for equations of the form (1.1) with right-hand sides from $L^1(\Omega)$ or from some class of Radon measures were investigated in a number of works (e.g., see [1–4, 6–9, 12–14, 16]). However, the conditions imposed on the coefficients of equations in these works are less general than the assumptions made in the present paper. Moreover, as far as we know, the main results of the paper are new not only in this general form, but also in the particular cases considered earlier.

The paper is organized as follows. In Sec. 2, we give some necessary definitions and formulate the results concerning elements of the special functional set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ introduced in [3] for the investigation of second-order equations with right-hand sides belonging to L^1. Section 3 contains the problem statement for the Dirichlet problem under consideration and definitions of different types of its solutions. Also, in Sec. 3 the relations between the types of solutions are established and one result concerning the summability of entropy solutions is proved. Other a priori properties of summability and some estimates of entropy solutions are proved in Sec. 4. In Sec. 5, the theorem of existence of entropy solutions is formulated and, finally, in Sec. 6, we give some bibliographic comments.

It should be mentioned that the main results of this work were announced in [15].

2. Set of Functions $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$

Let $n \in \mathbb{N}$, $n \geq 2$, Ω be a bounded open set in $\mathbb{R}^n$, and $p \in (1, n)$.

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For any $k > 0$, let T_k be a function over $\mathbb{R}$ such that

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k \operatorname{sign} s & \text{if } |s| > k. \end{cases}$$

It is well known that if $\lambda \ge 1$, $u \in \overset{\circ}{W}{}^{1,\lambda}(\Omega)$, and $k > 0$, then $T_k(u) \in \overset{\circ}{W}{}^{1,\lambda}(\Omega)$ and for any $i \in \{1, \dots, n\}$, we have

$$D_i T_k(u) = D_i u \cdot 1_{\{|u|<k\}} \quad \text{almost everywhere in } \Omega. \tag{2.1}$$

By $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ denote the set of all functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in \overset{\circ}{W}{}^{1,p}(\Omega)$ for any $k > 0$.

Note that any function belonging to $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ is measurable. Indeed, if $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, then the function u is measurable since the functions $T_k(u)$, $k \in \mathbb{N}$, are measurable and the sequence $\{T_k(u)\}$ converges to u in the pointwise sense.

It is obvious that

$$\overset{\circ}{W}{}^{1,p}(\Omega) \subset \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega). \tag{2.2}$$

For any $u : \Omega \to \mathbb{R}$ and $x \in \Omega$, set $k(u, x) = \min\{l \in \mathbb{N} : |u(x)| \le l\}$.

Definition 2.1. Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and $i \in \{1, \dots, n\}$. Then by $\delta_i u$ denote the function over Ω such that for any $x \in \Omega$, we have

$$\delta_i u\,(x) = D_i T_{k(u,x)}(u)\,(x). \tag{2.3}$$

Proposition 2.1. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and $i \in \{1, \dots, n\}$. Then for any $k > 0$, we have*

$$D_i T_k(u) = \delta_i u \cdot 1_{\{|u|<k\}} \quad \textit{almost everywhere in } \Omega. \tag{2.4}$$

The proof for this proposition is simple. It is based on the definition of functions T_k and Definition 2.1.

It follows from Proposition 2.1 that if $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, then for any $i \in \{1, \dots, n\}$, we have $D_i T_k(u) \to \delta_i u$ almost everywhere in Ω. Hence, if $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, then functions $\delta_i u$, $i = 1, \dots, n$, are measurable.

Note also that it follows from (2.1), (2.2), and Proposition 2.1 that if $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$, then we have $\delta_i u = D_i u$ almost everywhere in Ω for any $i \in \{1, \dots, n\}$. Moreover, Eq. (2.1) and Proposition 2.1 imply that if $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega) \cap \overset{\circ}{W}{}^{1,1}(\Omega)$, then we have $\delta_i u = D_i u$ almost everywhere in Ω for any $i \in \{1, \dots, n\}$.

Definition 2.2. If $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, then by δu we denote the map from Ω to $\mathbb{R}^n$ such that we have $(\delta u\,(x))_i = \delta_i u\,(x)$ for any $x \in \Omega$ and $i \in \{1, \dots, n\}$.

Proposition 2.2. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, $\lambda \in [1, p]$, and $|\delta u| \in L^\lambda(\Omega)$. Then $u \in \overset{\circ}{W}{}^{1,\lambda}(\Omega)$ and we have $D_i u = \delta_i u$ almost everywhere in Ω for any $i \in \{1, \dots, n\}$.*

The proof of this proposition is based on Proposition 2.1 and the Sobolev inequality for functions belonging to $\overset{\circ}{W}{}^{1,\lambda}(\Omega)$, $\lambda \in [1, n)$.

Proposition 2.3. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and $v \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then*

(1) $u - v \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$;
(2) *if $k > 0$ and $i \in \{1, \dots, n\}$, then $D_i T_k(u - v) = \delta_i u - \delta_i v$ almost everywhere in $\{|u - v| < k\}$.*

Proof. It is obvious that there exists a set $E \subset \Omega$ of zero measure such that for any $x \in \Omega \backslash E$, we have

$$|v(x)| \le \|v\|_{L^\infty(\Omega)}. \tag{2.5}$$

Choose some $k > 0$ and let $k_1 = k + \|v\|_{L^\infty(\Omega)}$. For any $j \in \mathbb{N}$, set

$$u_j = T_j(u) - v. \tag{2.6}$$

Now let $j \in \mathbb{N}$, $j \geq k_1$, and $x \in \{|u_j| < k\} \backslash E$. Then

$$|T_j(u(x)) - v(x)| < k. \tag{2.7}$$

If $|u(x)| > j$, then by (2.5) and (2.7) we get

$$j = |T_j(u(x))| \leq |T_j(u(x)) - v(x)| + |v(x)| < k_1.$$

Hence, $j < k_1$, which contradicts the initial assumption about j. Therefore, $|u(x)| \leq j$. Then inequality (2.7) implies that $|u(x) - v(x)| < k$. Combining this with inequality (2.5), we obtain that $|u(x)| < k_1$. Then, by virtue of (2.6), we get

$$u_j(x) = T_j(u(x)) - v(x) = u(x) - v(x) = T_{k_1}(u(x)) - v(x).$$

Thus,

$$u_j = T_{k_1}(u) - v \quad \text{almost everywhere in } \{|u_j| < k\}.$$

Then, for any $i \in \{1, \ldots, n\}$, we get

$$D_i u_j = D_i T_{k_1}(u) - D_i v \quad \text{almost everywhere in } \{|u_j| < k\}.$$

From the above reasoning, we conclude that the sequence $\{T_k(u_j)\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(\Omega)$. Hence, since we have $T_k(u_j) \to T_k(u - v)$ strongly in $L^p(\Omega)$, we see that $T_k(u - v) \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Since $k > 0$ is arbitrary, we obtain $u - v \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$. This proves statement (1) of the proposition.

Now we prove statement (2). Let $k > 0$ and $i \in \{1, \ldots, n\}$. Set $k_1 = k + \|v\|_{L^\infty(\Omega)}$. From (2.5), we get

$$\{|u - v| < k\} \backslash E \subset \{|u| < k_1\}. \tag{2.8}$$

Then

$$T_k(u - v) = T_{k_1}(u) - v \quad \text{almost everywhere in } \{|u - v| < k\}.$$

Therefore,

$$D_i T_k(u - v) = D_i T_{k_1}(u) - D_i v \quad \text{almost everywhere in } \{|u - v| < k\}. \tag{2.9}$$

Moreover, by virtue of Proposition 2.1 and embedding (2.8), we have

$$D_i T_{k_1}(u) = \delta_i u \quad \text{almost everywhere in } \{|u - v| < k\}.$$

Combining this with (2.9), we derive the relation

$$D_i T_k(u - v) = \delta_i u - \delta_i v \quad \text{almost everywhere in } \{|u - v| < k\},$$

which proves statement (2). □

Example 2.1. Let $y \in \Omega$, $\rho > 0$, and let B_1 and B_2 be closed balls with centers at y and radii ρ and $\rho/2$, respectively. Suppose that $B_1 \subset \Omega$ and let function $\varphi \in C^1(\Omega)$ be such that $0 \leq \varphi \leq 1$ in Ω, $\varphi = 1$ in B_2, and $\varphi = 0$ in $\Omega \backslash B_1$. Let $\lambda \geq n$ and u be a function over Ω such that

$$u(x) = \begin{cases} |x - y|^{-\lambda}\, \varphi(x) & \text{if} \quad x \in \Omega \backslash \{y\}, \\ \quad 0 & \text{if} \quad x = y. \end{cases}$$

Then $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega) \setminus L^1(\Omega)$.

This example and embedding (2.2) show that the set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ is wider than the space $\overset{\circ}{W}{}^{1,p}(\Omega)$.

Now we formulate one general result for the summability of functions $u : \Omega \to \mathbb{R}$. This result depends on a qualified estimate of measures of the sets $\{|u| \geq k\}$, $k \in \mathbb{N}$, and will be used to establish the summability of elements of the set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ satisfying some family of integral estimates.

Lemma 2.1. *Let u be a measurable function over Ω, $M > 0$, $\gamma > 0$, and for any $k \in \mathbb{N}$, let*

$$\operatorname{meas}\{|u| \geq k\} \leq Mk^{-\gamma}. \tag{2.10}$$

Then, for any $\lambda \in (0, \gamma)$, we have $u \in L^{\lambda}(\Omega)$ and

$$\int_{\Omega} |u|^{\lambda}\, dx \leq 2^{\gamma+(\gamma+\lambda)/(\gamma-\lambda)} M + \operatorname{meas}\Omega. \tag{2.11}$$

Proof. Fix some $\lambda \in (0, \gamma)$ and set $\lambda_1 = 2/(\gamma - \lambda)$. By virtue of estimate (2.10), for any $k \in \mathbb{N}$, we have

$$\int_{\{k^{\lambda_1} \leq |u| < (k+1)^{\lambda_1}\}} |u|^{\lambda}\, dx \leq 2^{\gamma+\lambda_1\lambda} M k^{-2}.$$

Hence, summing up with respect to k on both sides of this inequality, we conclude that $u \in L^{\lambda}(\Omega)$ and estimate (2.11) is true. □

Next, set $p^* = np/(n-p)$. Recall (e.g., see [10]) that $\overset{\circ}{W}{}^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and there exists a positive constant $c_{n,p}$ depending only on n and p such that for any function $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$, we have

$$\left(\int_{\Omega} |u|^{p^*}\, dx\right)^{1/p^*} \leq c_{n,p}\left(\int_{\Omega} |\nabla u|^{p}\, dx\right)^{1/p}. \tag{2.12}$$

Lemma 2.2. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, $M \geq 1$, and $0 < \theta < p$. Suppose that the following inequality holds for any $k \geq 1$:*

$$\int_{\{|u|<k\}} |\delta u|^{p}\, dx \leq Mk^{\theta}. \tag{2.13}$$

Then, for any $k \geq 1$, we have

$$\operatorname{meas}\{|u| \geq k\} \leq c_{n,p}^{p^*} M^{n/(n-p)}\, k^{-n(p-\theta)/(n-p)}, \tag{2.14}$$

$$\operatorname{meas}\{|\delta u| \geq k\} \leq (c_{n,p}^{p^*} + 1) M^{n/(n-\theta)}\, k^{-n(p-\theta)/(n-\theta)}. \tag{2.15}$$

Proof. Let $k \geq 1$. We have $T_k(u) \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Then, from inequality (2.12), it follows from Proposition 2.1 and inequality (2.13) that

$$\int_{\Omega} |T_k(u)|^{p^*}\, dx \leq c_{n,p}^{p^*}\left(\int_{\{|u|<k\}} |\delta u|^{p}\, dx\right)^{p^*/p} \leq c_{n,p}^{p^*} M^{n/(n-p)}\, k^{\theta n/(n-p)}. \tag{2.16}$$

Since $|T_k(s)| = k$ for $s \in \mathbb{R}$, $|s| \geq k$, we have

$$k^{p^*} \operatorname{meas}\{|u| \geq k\} \leq \int_{\Omega} |T_k(u)|^{p^*}\, dx.$$

Combining this with (2.16), we derive inequality (2.14).

Next, set

$$k_1 = M^{1/(n-\theta)}\, k^{(n-p)/(n-\theta)}.$$

Since $k_1 \geq 1$, analogously to (2.14), we have

$$\operatorname{meas}\{|u| \geq k_1\} \leq c_{n,p}^{p^*} M^{n/(n-p)}\, k_1^{-n(p-\theta)/(n-p)}. \tag{2.17}$$

Moreover, from inequality (2.13), we obtain

$$\int_{\{|u|<k_1\}} |\delta u|^{p}\, dx \leq Mk_1^{\theta}. \tag{2.18}$$

It is obvious that

$$\operatorname{meas}\{|\delta u| \geq k\} \leq \operatorname{meas}\{|u| \geq k_1\} + \operatorname{meas}\{|u| < k_1,\, |\delta u| \geq k\}. \tag{2.19}$$

If $x \in \{|u| < k_1,\, |\delta u| \geq k\}$, then $k \leq |\delta u|\,(x)$, Therefore, taking inequality (2.18) into account, we get

$$k^p \operatorname{meas}\{|u| < k_1, |\delta u| \geq k\} \leq \int\limits_{\{|u|<k_1\}} |\delta u|^p \, dx \leq M k_1^{\theta}.$$

Combining this with inequalities (2.17) and (2.19) yields

$$\operatorname{meas}\{|\delta u| \geq k\} \leq c_{n,p}^{p^*} M^{n/(n-p)}\, k_1^{-n(p-\theta)/(n-p)} + M k^{-p}\, k_1^{\theta}. \tag{2.20}$$

Observe that, by the definition of k_1, we have

$$M^{n/(n-p)}\, k_1^{-n(p-\theta)/(n-p)} = M^{n/(n-\theta)} k^{-n(p-\theta)/(n-\theta)},$$
$$M k^{-p} k_1^{\theta} = M^{n/(n-\theta)} k^{-n(p-\theta)/(n-\theta)}.$$

Combining these equations with (2.20), we derive inequality (2.15). □

Lemmas 2.1 and 2.2 imply the following result.

Lemma 2.3. *Let* $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, $M \geq 1$, *and* $0 < \theta < p$. *Suppose that the following inequality holds for any* $k \geq 1$:

$$\int\limits_{\{|u|<k\}} |\delta u|^p \, dx \leq M k^{\theta}.$$

Let $0 < \lambda < n(p-\theta)/(n-\theta)$. *Then*

$$\int\limits_{\Omega} |u|^{\lambda(n-\theta)/(n-p)} \, dx \leq C_1 M^{n/(n-p)},$$

$$\int\limits_{\Omega} |\delta u|^{\lambda} \, dx \leq C_2 M^{n/(n-\theta)},$$

where C_1 *and* C_2 *are positive constants depending only on* n, p, $\operatorname{meas}\Omega$, θ, *and* λ.

3. Dirichlet Problem for Equations with L^1-data. Types of Solutions and Relations between Them

For any $i \in \{1, \ldots, n\}$, let a_i be a Carathéodory function over $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Assume that for any $k > 0$, there exist $\overline{c}_k > 0$ and $\overline{g}_k \in L^1(\Omega)$ $(\overline{g}_k \geq 0)$ on Ω such that the following inequality is true for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, $|s| \leq k$, and $\xi \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} |a_i(x, s, \xi)|^{p/(p-1)} \leq \overline{c}_k |\xi|^p + \overline{g}_k(x). \tag{3.1}$$

Also, assume that there exist $p_1 \in [\,0, p-1)$, $p_2 \in [\,0, p-p_1)$, $c_1, c_2 > 0$, and $g_1 \in L^1(\Omega)$ such that $g_1 \geq 0$ on Ω and the following inequality is true for almost all $x \in \Omega$ and any $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} a_i(x, s, \xi)\xi_i \geq \frac{c_1 |\xi|^p}{(1+|s|)^{p_1}} - c_2 p_2 (1+|s|)^{p_2} - g_1(x). \tag{3.2}$$

Finally, assume that the following inequality is true for almost all $x \in \Omega$ and all $s \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^n$ $(\xi \neq \xi')$:

$$\sum_{i=1}^{n} [\, a_i(x, s, \xi) - a_i(x, s, \xi')](\xi_i - \xi_i') > 0. \tag{3.3}$$

Let a_0 be a Carathéodory function over $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and $f \in L^1(\Omega)$.

Consider the following Dirichlet problem:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f \quad \text{in } \Omega, \tag{3.4}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

Definition 3.1. We call a function $u \in \overset{\circ}{W}{}^{1,1}(\Omega)$ a *weak solution* of problem (3.4), (3.5) if

(1) $a_i(x, u, \nabla u) \in L^1(\Omega)$ for any $i \in \{1, \ldots, n\}$;
(2) $a_0(x, u, \nabla u) \in L^1(\Omega)$;
(3) for any function $v \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \left\{ \sum_{i=1}^{n} a_i(x, u, \nabla u) D_i v + a_0(x, u, \nabla u) v \right\} dx = \int_\Omega f v \, dx.$$

Definition 3.2. We call a function $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ a *$\mathcal{T}$-solution* of problem (3.4), (3.5) if

(1) $a_i(x, u, \delta u) \in L^1(\Omega)$ for any $i \in \{1, \ldots, n\}$;
(2) $a_0(x, u, \delta u) \in L^1(\Omega)$;
(3) for any function $v \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \left\{ \sum_{i=1}^{n} a_i(x, u, \delta u) D_i v + a_0(x, u, \delta u) v \right\} dx = \int_\Omega f v \, dx.$$

Proposition 3.1. *Let u be a $\mathcal{T}$-solution of problem* (3.4), (3.5) *and let $|\delta u| \in L^1(\Omega)$. Then u is a weak solution of problem* (3.4), (3.5).

Proof. Since $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and $|\delta u| \in L^1(\Omega)$ by Proposition 2.2, we see that $u \in \overset{\circ}{W}{}^{1,1}(\Omega)$ and we have $D_i u = \delta_i u$ almost everywhere in Ω for any $i \in \{1, \ldots, n\}$. Hence, taking Definition 3.2 into account, we see that conditions (1)–(3) of Definition 3.1 are fulfilled. Therefore, u is a weak solution of problem (3.4), (3.5). ∎

Next, observe that if $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, $v \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$, $k > 0$, and $i \in \{1, \ldots, n\}$, then the function $a_i(x, u, \delta u)(\delta_i u - \delta_i v)$ is summable over the set $\{|u - v| < k\}$. This follows from Proposition 2.1 and inequality (3.1).

Definition 3.3. We call a function $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ an *entropy solution* of problem (3.4), (3.5) if

(1) $a_0(x, u, \delta u) \in L^1(\Omega)$;
(2) for any $v \in C_0^\infty(\Omega)$ and any $k > 0$, we have

$$\int_{\{|u-v|<k\}} \left\{ \sum_{i=1}^{n} a_i(x, u, \delta u)(\delta_i u - \delta_i v) \right\} dx + \int_\Omega a_0(x, u, \delta u) \, T_k(u - v) \, dx \le \int_\Omega f \, T_k(u - v) \, dx.$$

Lemma 3.1. *Let u be an entropy solution of problem* (3.4), (3.5). *Then the inequality in condition* (2) *of Definition* 3.3 *holds for any $v \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$ and any $k > 0$.*

One can prove this lemma, approximating a function $v \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$ by a sequence of smooth functions uniformly bounded over Ω, applying the inequality from condition (2) of Definition 3.3 to the functions of this sequence, and then proceeding to the appropriate limit, taking into account inequality (3.1) and Propositions 2.1 and 2.3.

Proposition 3.2. *Let u be an entropy solution of problem* (3.4), (3.5). *Then*

(1) *for any* λ *such that* $0 < \lambda < n(p-1-p_1)/(n-p)$, *the function* $|u|^\lambda$ *is summable over* Ω;
(2) *for any* λ *such that* $0 < \lambda < n(p-1-p_1)/(n-1-p_1)$, *the function* $|\delta u|^\lambda$ *is summable over* Ω.

Proof. Set $\sigma_1 = p_1 p_2/(p-p_2)$. By virtue of the inequalities $p_1 \geq 0$ and $0 \leq p_2 < p - p_1$, we have $\sigma_1 \geq 0$. Moreover, we have

$$\frac{pp_1}{p-p_2} = p_1 + \sigma_1. \tag{3.6}$$

Set

$$M_1 = \frac{2^{p_1} c_2 p_2}{c_1}, \qquad M_2 = \frac{2^{p_1}}{c_1} \int\limits_{\Omega} [\, g_1 + |f| + |a_0(x,u,\delta u)|] dx,$$

$$M_3 = 2^{p_2} M_1 \operatorname{meas}\Omega + \left[(2c_{n,p})^{p_2} M_1 (1 + \operatorname{meas}\Omega)\right]^{p/(p-p_2)}.$$

Let $k \geq 1$. By Definition 3.3, we have

$$\int\limits_{\{|u|<k\}} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u)\delta_i u \right\} dx \leq \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)]\, T_k(u)\, dx.$$

It follows from the latter inequality and from inequality (3.2) that

$$c_1 \int\limits_{\{|u|<k\}} \frac{|\delta u|^p}{(1+|u|)^{p_1}}\, dx \leq c_2 p_2 \int\limits_{\{|u|<k\}} (1+|u|)^{p_2} dx + \int\limits_{\Omega} g_1\, dx + \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)]\, T_k(u)\, dx.$$

Hence, taking into account that $(1+|u|)^{p_1} \leq (2k)^{p_1}$ in the set $\{|u| < k\}$ and $|T_k(u)| \leq k$ in Ω, we obtain

$$\int\limits_{\{|u|<k\}} |\delta u|^p\, dx \leq M_1 k^{p_1} \int\limits_{\{|u|<k\}} (1+|u|)^{p_2}\, dx + M_2 k^{p_1+1}. \tag{3.7}$$

To estimate the first term on the right-hand side of inequality (3.7), suppose that $p_2 > 0$. Using the Hölder inequality, inequality (2.12), Proposition 2.1, and the Young inequality, we derive

$$2^{p_2} k^{p_1} M_1 \int\limits_{\Omega} |T_k(u)|^{p_2}\, dx \leq 2^{p_2} k^{p_1} M_1 (\operatorname{meas}\Omega)^{(p^*-p_2)/p^*} \left(\int\limits_{\Omega} |T_k(u)|^{p^*}\, dx \right)^{p_2/p^*}$$

$$\leq 2^{p_2} k^{p_1} M_1 (1+\operatorname{meas}\Omega) c_{n,p}^{p_2} \left(\int\limits_{\Omega} |\nabla T_k(u)|^p\, dx \right)^{p_2/p}$$

$$= (2c_{n,p})^{p_2} M_1 (1+\operatorname{meas}\Omega) k^{p_1} \left(\int\limits_{\{|u|<k\}} |\delta u|^p\, dx \right)^{p_2/p}$$

$$\leq \left[\, (2c_{n,p})^{p_2} M_1 (1+\operatorname{meas}\Omega)\right]^{p/(p-p_2)} k^{p_1 p/(p-p_2)} + \frac{p_2}{p} \int\limits_{\{|u|<k\}} |\delta u|^p\, dx.$$

Then, taking relation (3.6) and the equation $T_k(u) = u$ in the set $\{|u| < k\}$ into account, we get

$$M_1 k^{p_1} \int\limits_{\{|u|<k\}} (1+|u|)^{p_2}\, dx \leq M_3 k^{p_1+\sigma_1} + \frac{p_2}{p} \int\limits_{\{|u|<k\}} |\delta u|^p\, dx. \tag{3.8}$$

It is obvious that the latter inequality is also true in the case where $p_2 = 0$. Inequalities (3.7) and (3.8) imply that

$$\int\limits_{\{|u|<k\}} |\delta u|^p\, dx \leq \frac{p}{p-p_2} \left[M_3 k^{p_1+\sigma_1} + M_2 k^{p_1+1} \right]. \tag{3.9}$$

If $\sigma_1 \le 1$, then from inequality (3.9), inclusion $p_1 \in [0, p-1)$, and Lemma 2.3 we derive that statements (1) and (2) are true.

Now let

$$\sigma_1 > 1. \tag{3.10}$$

Then, by virtue of the inequalities $p_1 \ge 0$ and $0 \le p_2 < p - p_1$, we have

$$\sigma_1 < p_2, \tag{3.11}$$

and, therefore,

$$p_1 + \sigma_1 < p. \tag{3.12}$$

From (3.9), (3.10), (3.12), and Lemma 2.3 we see that

$$\text{the function } |u|^\lambda \text{ is summable over } \Omega \text{ for any } \lambda \text{ such that } 0 < \lambda < \frac{n(p - p_1 - \sigma_1)}{n-p}. \tag{3.13}$$

Set

$$\sigma_j = p_2 - \frac{n(p - p_1 - \sigma_{j-1})}{n-p}, \quad j = 2, 3, \dots. \tag{3.14}$$

For any $j \in \mathbb{N}$, we claim that

$$\sigma_j < p_2. \tag{3.15}$$

Let us prove this by induction. Since (3.11) is true, inequality (3.15) holds for $j = 1$. Suppose that inequality (3.15) holds for some $j \in \mathbb{N}$. Using relations (3.14) and the latter assumption, we obtain

$$\begin{aligned}\sigma_{j+1} = p_2 - \frac{n(p - p_1 - \sigma_j)}{n-p} &= p_2 - \frac{n(p-p_1)}{n-p} + \frac{n\sigma_j}{n-p} \\ &< p_2 - \frac{n(p-p_1)}{n-p} + \frac{np_2}{n-p} = p_2 - \frac{n(p - p_1 - p_2)}{n-p}.\end{aligned}$$

Hence, taking into account that $p_2 < p - p_1$, we see that inequality (3.15) holds for $j+1$. Therefore, inequality (3.15) holds for any $j \in \mathbb{N}$.

Next, let $j \in \mathbb{N}$, $j \ge 2$. By virtue of (3.14) and (3.15) we have

$$\begin{aligned}\sigma_j = p_2 - \frac{n(p-p_1)}{n-p} + \frac{n\sigma_{j-1}}{n-p} &= \sigma_{j-1} + p_2 - \frac{n(p-p_1)}{n-p} + \frac{p\sigma_{j-1}}{n-p} \\ &< \sigma_{j-1} + p_2 - \frac{n(p-p_1)}{n-p} + \frac{pp_2}{n-p} = \sigma_{j-1} - \frac{n(p-p_1-p_2)}{n-p}.\end{aligned}$$

Thus, for any $j \in \mathbb{N}$, $j \ge 2$, the following inequality is true:

$$\sigma_j < \sigma_{j-1} - \frac{n(p - p_1 - p_2)}{n-p}. \tag{3.16}$$

This implies that for any $j \in \mathbb{N}$, $j \ge 2$, we have

$$\sigma_j < \sigma_1 - (j-1)\,\frac{n(p-p_1-p_2)}{n-p}.$$

Therefore, there exist numbers $j \in \mathbb{N}$ such that $\sigma_j < 1$. Set

$$m = \min\{j \in \mathbb{N} : \ \sigma_j < 1\}.$$

Then $\sigma_m < 1$. Hence, by (3.10), we see that $m \ge 2$. Moreover, it is obvious that $\sigma_{m-1} \ge 1$; therefore,

$$1 - \sigma_m \le \sigma_{m-1} - \sigma_m\,. \tag{3.17}$$

Set

$$\alpha = \frac{p_2 - 1}{p - p_1 - p_2}\,. \tag{3.18}$$

By virtue of (3.10) and (3.11), we have $\alpha > 0$.

Choose a number ε such that

$$0 < \varepsilon < \min\left\{ 1,\ \frac{1-\sigma_m}{\alpha^m(\sigma_{m-1}-\sigma_m)} \right\} \tag{3.19}$$

and for any $j \in \mathbb{N}$, $j \geq 2$, set

$$\beta_j = \sigma_j + \varepsilon\alpha^j(\sigma_{j-1} - \sigma_j). \tag{3.20}$$

Let $j \in \mathbb{N}$, $2 \leq j \leq m$. It is obvious that $\varepsilon\alpha^j > 0$. If $\alpha \leq 1$, then, since we have $\varepsilon < 1$ by (3.19), we get $\varepsilon\alpha^j < 1$. If $\alpha > 1$, then, using (3.17) and (3.19), we obtain $\varepsilon\alpha^j < \alpha^{j-m} \leq 1$. Thus, $\varepsilon\alpha^j < 1$ in any case and, therefore, $\varepsilon\alpha^j \in (0,1)$.

Now from (3.20) and (3.16) we derive that for any $j \in \mathbb{N}$, $2 \leq j \leq m$, we have

$$\sigma_j < \beta_j < \sigma_{j-1}\,. \tag{3.21}$$

Combining this with relations (3.14) and (3.15), we see that for any $j \in \mathbb{N}$, $2 \leq j \leq m$, we have

$$0 < p_2 - \beta_j < \frac{n(p-p_1-\sigma_{j-1})}{n-p}\,. \tag{3.22}$$

Then it follows from (3.22) and (3.13) that

$$\text{the function } |u|^{p_2-\beta_2} \text{ is summable over } \Omega. \tag{3.23}$$

We claim that for any $j \in \mathbb{N}$, $2 \leq j \leq m$,

$$\text{the function } |u|^{p_2-\beta_j} \text{ is summable over } \Omega. \tag{3.24}$$

Clearly, it is true if $m = 2$. Let $m > 2$. Let us apply the method of induction. By virtue of (3.23), statement (3.24) holds for $j = 2$. Suppose that statement (3.24) holds for some $j \in \mathbb{N}$, $2 \leq j \leq m-1$. By the definition of the number m, we have

$$\sigma_j \geq 1. \tag{3.25}$$

It follows from the latter inequality and from (3.21) that

$$\beta_j > 1. \tag{3.26}$$

Choose some $k \geq 1$. Combining inequalities (3.7) and (3.26), we obtain

$$\int\limits_{\{|u|<k\}} |\delta u|^p\,dx \leq 2^{p_2} M_1 k^{p_1+\beta_j} \int\limits_{\{|u|<k\}} (1+|u|)^{p_2-\beta_j}\,dx + M_2 k^{p_1+1}$$

$$\leq \left\{ 2^{p_2} M_1 \int\limits_\Omega (1+|u|)^{p_2-\beta_j}\,dx + M_2 \right\} k^{p_1+\beta_j},$$

where the integral of the function $(1+|u|)^{p_2-\beta_j}$ over Ω is finite due to the assumption. Hence, taking inequality $0 < p_1 + \beta_j < p$ into account and applying Lemma 2.3, we conclude that

$$\text{the function } |u|^\lambda \text{ is summable over } \Omega \text{ for any } \lambda \text{ such that } 0 < \lambda < \frac{n(p-p_1-\beta_j)}{n-p}. \tag{3.27}$$

We claim that

$$p_2 - \beta_{j+1} < \frac{n(p-p_1-\beta_j)}{n-p}\,. \tag{3.28}$$

Indeed, using relations (3.14), we obtain

$$p_2 - \beta_{j+1} = p_2 - \sigma_{j+1} + \sigma_{j+1} - \beta_{j+1} = \frac{n(p-p_1-\sigma_j)}{n-p} + \sigma_{j+1} - \beta_{j+1}$$

$$= \frac{n(p-p_1-\beta_j)}{n-p} + \frac{n(\beta_j-\sigma_j)}{n-p} + \sigma_{j+1} - \beta_{j+1}, \tag{3.29}$$

and, by virtue of (3.20), we have

$$\frac{n(\beta_j-\sigma_j)}{n-p}+\sigma_{j+1}-\beta_{j+1}=\varepsilon\alpha^j\left[\frac{n}{n-p}(\sigma_{j-1}-\sigma_j)-\alpha(\sigma_j-\sigma_{j+1})\right]. \tag{3.30}$$

Inequalities (3.15) and (3.25) imply that

$$\sigma_{j-1}-\sigma_j<p_2-1. \tag{3.31}$$

Moreover, combining relations (3.16) and (3.18), we obtain

$$\frac{n(p_2-1)}{n-p}<\alpha(\sigma_j-\sigma_{j+1}). \tag{3.32}$$

It follows from relations (3.29)–(3.32) that inequality (3.28) is true.

In addition, observe that $p_2-\beta_{j+1}>0$ because of inequalities (3.22). Hence, taking inequality (3.28) into account, we derive by statement (3.27) that the function $|u|^{p_2-\beta_{j+1}}$ is summable over Ω. Therefore, statement (3.24) is true for $j+1$.

Now we conclude that statement (3.24) holds for any $j\in\mathbb{N}$, $2\le j\le m$.

In particular, this implies that the function $|u|^{p_2-\beta_m}$ is summable over Ω. Moreover, by virtue of (3.20) and (3.19), we have $\beta_m<1$. Then, using inequality (3.7), for any $k\ge 1$ we establish an estimate

$$\int\limits_{\{|u|<k\}}|\delta u|^p\,dx\le\left\{2M_1\int\limits_\Omega(1+|u|)^{p_2-\beta_m}\,dx+M_2\right\}k^{p_1+1}.$$

Hence, taking inequality $0<p_1+1<p$ into account and applying Lemma 2.3, we conclude that statements (1) and (2) of the proposition are true. □

Proposition 3.3. *Let* u *be an entropy solution of problem* (3.4), (3.5). *Let the function* $(1+|u|)^{p_2}$ *be summable over* Ω *and* $a_i(x,u,\delta u)\in L^1(\Omega)$ *for any* $i\in\{1,\dots,n\}$. *Then* u *is a* $\mathcal{T}$*-solution of problem* (3.4), (3.5).

Proof. By the assumptions of the proposition, we have $u\in\mathring{\mathcal{T}}^{1,p}(\Omega)$ and conditions (1) and (2) of Definition 3.2 are fulfilled.

We claim that condition (3) of Definition 3.2 holds. Let $v\in C_0^\infty(\Omega)$. Choose $k>\max\limits_\Omega|v|$ and set $E_m=\{\,|u-T_m(u)+v|<k\}$ for any $m\in\mathbb{N}$.

Let $m\in\mathbb{N}$. Since $T_m(u)-v\in\mathring{W}^{1,p}(\Omega)\cap L^\infty(\Omega)$, by Lemma 3.1 we have

$$\int\limits_{E_m}\left\{\sum_{i=1}^n a_i(x,u,\delta u)(\delta_i u-D_iT_m(u)+D_iv)\right\}dx\le\int\limits_\Omega[\,f-a_0(x,u,\delta u)]\,T_k(u-T_m(u)+v)\,dx. \tag{3.33}$$

Using Proposition 2.1 and inequality (3.2), we obtain

$$\begin{aligned}\int\limits_{E_m}&\left\{\sum_{i=1}^n a_i(x,u,\delta u)(\delta_i u-D_iT_m(u)+D_iv)\right\}dx\\&=\int\limits_{E_m}\left\{\sum_{i=1}^n a_i(x,u,\delta u)D_iv\right\}dx+\int\limits_{E_m\cap\{|u|\ge m\}}\left\{\sum_{i=1}^n a_i(x,u,\delta u)\delta_i u\right\}dx\\&\ge\int\limits_{E_m}\left\{\sum_{i=1}^n a_i(x,u,\delta u)D_iv\right\}dx-c_2p_2\int\limits_{E_m\cap\{|u|\ge m\}}(1+|u|)^{p_2}\,dx-\int\limits_{\{|u|\ge m\}}g_1\,dx.\end{aligned}$$

It follows from the latter relations and from inequality (3.33) that for any $m \in \mathbb{N}$, we have

$$\int\limits_{E_m} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i v \right\} dx \le \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)]\, T_k(u - T_m(u) + v)\, dx + c_2 p_2 \int\limits_{\{|u| \ge m\}} (1+|u|)^{p_2}\, dx + \int\limits_{\{|u| \ge m\}} g_1\, dx. \quad (3.34)$$

It is obvious that $\bigcup\limits_{m=1}^{\infty} E_m = \Omega$. Moreover, we have $E_m \subset E_{m+1}$ for any $m \in \mathbb{N}$. Indeed, let $m \in \mathbb{N}$ and $x \in E_m$. The inclusion $x \in E_{m+1}$ is obvious in the case where $|u(x)| \le m+1$. Suppose that $|u(x)| > m+1$. In this case, if $u(x) > 0$, then $u(x) > T_{m+1}(u(x))$ and

$$-k < v(x) < u(x) - T_{m+1}(u(x)) + v(x) = u(x) - T_m(u(x)) - 1 + v(x) < k. \quad (3.35)$$

If $u(x) < 0$, then $u(x) < T_{m+1}(u(x))$ and

$$-k < u(x) - T_m(u(x)) + v(x) + 1 = u(x) - T_{m+1}(u(x)) + v(x) < v(x) < k.$$

Combining this with (3.35), we see that if $|u(x)| > m+1$, then $x \in E_{m+1}$ as well. Therefore, $E_m \subset E_{m+1}$.

Now we can conclude that $\operatorname{meas}(\Omega \backslash E_m) \to 0$. Then, since functions $a_i(x,u,\delta u)$ are summable over Ω by the assumptions of the proposition, we obtain

$$\int\limits_{E_m} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i v \right\} dx \to \int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i v \right\} dx. \quad (3.36)$$

Next, by statement (1) of Proposition 3.2, we have $\operatorname{meas}\{|u| \ge m\} \to 0$. Hence, since the functions $(1+|u|)^{p_2}$ and g_1 are summable, we derive

$$\int\limits_{\{|u| \ge m\}} (1+|u|)^{p_2}\, dx \to 0, \qquad \int\limits_{\{|u| \ge m\}} g_1\, dx \to 0. \quad (3.37)$$

Finally, we have

$$\int\limits_{\Omega} [\, f - a_0(x,u,\delta u)]\, T_k(u - T_m(u) + v)\, dx \to \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)] v\, dx. \quad (3.38)$$

This is true because $T_k(u - T_m(u) + v) \to v$ in Ω and the functions f and $a_0(x,u,\delta u)$ are summable over Ω.

By virtue of inequality (3.34) and relations (3.36)–(3.38), we see that for any function $v \in C_0^\infty(\Omega)$, we have

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i v \right\} dx \le \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)] v\, dx.$$

Therefore, the following relation holds for any function $v \in C_0^\infty(\Omega)$:

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i v \right\} dx = \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)] v\, dx.$$

Thus, condition (3) of Definition 3.2 holds and we conclude that u is a $\mathcal{T}$-solution of problem (3.4), (3.5). □

Corollary 3.1. *Let u be an entropy solution of problem* (3.4), (3.5). *Let the functions* $(1+|u|)^{p_2}$ *and* $|\delta u|$ *be summable over Ω and $a_i(x,u,\delta u) \in L^1(\Omega)$ for any $i \in \{1,\dots,n\}$. Then u is a weak solution of problem* (3.4), (3.5).

This result follows from Propositions 3.3 and 3.1.

Before we proceed further, note one useful remark.

Remark 3.1. If $p_1 < (p-1)/(n-p+1)$, then we have

$$1 < \frac{n(p-1-p_1)}{(n-1-p_1)(p-1)}\,. \tag{3.39}$$

Proposition 3.4. *Let*

$$p_1 < \frac{p-1}{n-p+1}, \tag{3.40}$$

$$p_2 < \frac{n(p-1-p_1)}{n-p}, \tag{3.41}$$

$0 < \overline{p} < p^*(p-1-p_1)/(p-1)$, $\overline{c} > 0$, $\overline{g} \in L^1(\Omega)$, *and* $\overline{g} \geq 0$ *in* Ω. *For almost all* $x \in \Omega$, *any* $s \in \mathbb{R}$, *and* $\xi \in \mathbb{R}^n$, *let the following inequality hold*:

$$\sum_{i=1}^{n} |a_i(x,s,\xi)|^{p/(p-1)} \leq \overline{c}\,(|s|^{\overline{p}} + |\xi|^p) + \overline{g}(x). \tag{3.42}$$

Let u *be an entropy solution of problem* (3.4), (3.5). *Then*

(1) $a_i(x,u,\delta u) \in L^\lambda(\Omega)$ *for any number* λ *such that*

$$1 \leq \lambda < \min\left\{\frac{p^*(p-1-p_1)}{\overline{p}(p-1)},\ \frac{n(p-1-p_1)}{(n-1-p_1)(p-1)}\right\} \tag{3.43}$$

and any $i \in \{1,\dots,n\}$;

(2) u *is a* $\mathcal{T}$*-solution of problem* (3.4), (3.5).

Proof. First, observe that inequality (3.39) holds by virtue of inequality (3.40) and Proposition 3.1. This inequality and the assumption on $\overline{p}$ imply that the set of numbers λ satisfying inequality (3.43) is not empty.

Let λ satisfy inequality (3.43) and let $i \in \{1,\dots,n\}$. By virtue of (3.42) and the inequality $\lambda(p-1) < p$, we have

$$|a_i(x,u,\delta u)|^\lambda \leq (\overline{c}+1)\left[|u|^{\lambda(p-1)\overline{p}/p} + |\delta u|^{\lambda(p-1)}\right] + \overline{g} + 1 \quad \text{almost everywhere in } \Omega. \tag{3.44}$$

Inequalities (3.43) imply that

$$\lambda(p-1)\overline{p}/p < n(p-1-p_1)/(n-p), \qquad \lambda(p-1) < n(p-1-p_1)/(n-1-p_1).$$

Then the functions $|u|^{\lambda(p-1)\overline{p}/p}$ and $|\delta u|^{\lambda(p-1)}$ are summable over Ω by Proposition 3.2. Combining this with (3.44), we deduce that $a_i(x,u,\delta u) \in L^\lambda(\Omega)$, which proves statement (1). It follows from this statement that we have $a_i(x,u,\delta u) \in L^1(\Omega)$ for any $i \in \{1,\dots,n\}$. Moreover, by inequality (3.41) and Proposition 3.2, the function $(1+|u|)^{p_2}$ is summable over Ω.

Thus, all the conditions of Proposition 3.3 are fulfilled, and, by this, statement (2) of the proposition is true. ☐

Corollary 3.2. *Let* $p > 2 - 1/n$,

$$p_1 < \min\left\{\frac{n}{n-1}\left(p-2+\frac{1}{n}\right),\ \frac{p-1}{n-p+1}\right\}, \tag{3.45}$$

$$p_2 < \frac{n(p-1-p_1)}{n-p},$$

$0 < \overline{p} < p^*(p-1-p_1)/(p-1)$, $\overline{c} > 0$, $\overline{g} \in L^1(\Omega)$, *and* $\overline{g} \geq 0$ *in* Ω. *For almost all* $x \in \Omega$, *all* $s \in \mathbb{R}$, *and all* $\xi \in \mathbb{R}^n$, *the following inequality holds*:

$$\sum_{i=1}^{n} |a_i(x,s,\xi)|^{p/(p-1)} \leq \overline{c}(|s|^{\overline{p}} + |\xi|^p) + \overline{g}(x).$$

Let u *be an entropy solution of problem* (3.4), (3.5). *Then* u *is a weak solution of problem* (3.4), (3.5).

Proof. Since all the conditions of Proposition 3.4 are fulfilled, it follows by this proposition that u is a $\mathcal{T}$-solution of problem (3.4), (3.5). By virtue of (3.45), we have

$$p_1 < \frac{n}{n-1}\left(p - 2 + \frac{1}{n}\right).$$

Then $1 < n(p-1-p_1)/(n-1-p_1)$. Combining this with statement (2) of Proposition 3.2, we see that $|\delta u| \in L^1(\Omega)$. Then, by Proposition 3.1, we conclude that u is a weak solution of problem (3.4), (3.5). ☐

4. A Priori Properties of Summability and Estimates for Entropy Solutions

Let us begin with the following auxiliary result.

Lemma 4.1. *Let* $h \in C^1(\mathbb{R})$ *and* $h(0) = 0$. *Let* $u \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$. *Then* $h(u) \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $D_i h(u) = h'(u) D_i u$ *almost everywhere in* Ω *for any* $i \in \{1, \dots, n\}$.

Proof. By the continuity of the function h in $\mathbb{R}$ and the inclusion $u \in L^\infty(\Omega)$, we obtain $h(u) \in L^\infty(\Omega)$.

Next, set $m = \|u\|_{L^\infty(\Omega)} + 1$, and let $\{u_j\}$ be a sequence of functions belonging to $C_0^\infty(\Omega)$ such that

$$u_j \to u \quad \text{strongly in } W^{1,p}(\Omega), \tag{4.1}$$

$$u_j \to u \quad \text{almost everywhere in } \Omega, \tag{4.2}$$

$$\forall\, j \in \mathbb{N} \quad |u_j| \leq m \quad \text{in } \Omega. \tag{4.3}$$

Since $h \in C^1(\mathbb{R})$ and $h(0) = 0$, we have

$$\{h(u_j)\} \subset C_0^1(\Omega). \tag{4.4}$$

From relations (4.2) and (4.3) and the inclusion $h(u) \in L^\infty(\Omega)$, we deduce that

$$h(u_j) \to h(u) \quad \text{strongly in } L^p(\Omega). \tag{4.5}$$

Choose some $i \in \{1, \dots, n\}$. Since $u \in L^\infty(\Omega)$, $D_i u \in L^p(\Omega)$, and the function h' is continuous in $\mathbb{R}$, we have $h'(u) D_i u \in L^p(\Omega)$. Taking inequalities (4.3) into account, we obtain (for any $j \in \mathbb{N}$)

$$\int_\Omega |D_i h(u_j) - h'(u) D_i u|^p \, dx \leq 2^p \int_\Omega |h'(u_j) - h'(u)|^p |D_i u|^p \, dx + \big(2 \max_{[-m,m]} |h'|\big)^p \int_\Omega |D_i u_j - D_i u|^p \, dx.$$

Combining this with relations (4.1)–(4.3), we see that

$$D_i h(u_j) \to h'(u) D_i u \quad \text{strongly in } L^p(\Omega). \tag{4.6}$$

It follows from relations (4.4)–(4.6) that the generalized derivative $D_i h(u)$ exists and $D_i h(u) = h'(u) D_i u$ almost everywhere in Ω.

Thus, we conclude that $h(u) \in W^{1,p}(\Omega)$. Moreover, relations (4.5) and (4.6) imply that $h(u_j) \to h(u)$ strongly in $W^{1,p}(\Omega)$. Hence, using (4.4), we see that $h(u) \in \overset{\circ}{W}{}^{1,p}(\Omega)$. ☐

Proposition 4.1. *Let* $p_2 = 0$ *and* $g_1 = 0$ *in* Ω. *Let* u *be an entropy solution of problem* (3.4), (3.5). *Let* $h \in C^1(\mathbb{R})$, $h(0) = 0$, *the function* h *be bounded in* $\mathbb{R}$, *and* $h' \geq 0$ *in* $\mathbb{R}$. *Then the function* $|\delta u|^p h'(u)/(1+|u|)^{p_1}$ *is summable in* Ω *and the following inequality holds*:

$$\int_\Omega \frac{|\delta u|^p}{(1+|u|)^{p_1}} h'(u) \, dx \leq \frac{1}{c_1} \int_\Omega [\, f - a_0(x,u,\delta u)]\, h(u) \, dx. \tag{4.7}$$

Proof. Choose some $k > \sup\limits_{\mathbb{R}} |h|$. For any $m \in \mathbb{N}$, set

$$v_m = T_m(u) - h(T_m(u)), \tag{4.8}$$

$$E_m = \{\, |u - T_m(u) + h(T_m(u))| < k \,\}. \tag{4.9}$$

Let $m \in \mathbb{N}$. Since $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$, we have $T_m(u) \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $h(T_m(u)) \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$ by Lemma 4.1 and, for any $i \in \{1, \dots, n\}$, we have

$$D_i h(T_m(u)) = h'(T_m(u)) D_i T_m(u) \quad \text{almost everywhere in } \Omega. \tag{4.10}$$

Clearly, $v_m \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, by Lemma 3.1, we have

$$\int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u)(\delta_i u - \delta_i v_m) \right\} dx \le \int\limits_\Omega [\, f - a_0(x, u, \delta u)]\, T_k(u - v_m)\, dx. \tag{4.11}$$

Using Proposition 2.1, inequality (3.2), relation (4.10), the inclusion $\{|u| < m\} \subset E_m$, and the nonnegativity of the function h' in $\mathbb{R}$, we obtain

$$\begin{aligned}
&\int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u)(\delta_i u - \delta_i v_m) \right\} dx \\
&\quad = \int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u)(\delta_i u - D_i T_m(u)) \right\} dx + \int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u) D_i h(T_m(u)) \right\} dx \\
&\quad \ge \int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u) D_i h(T_m(u)) \right\} dx = \int\limits_{E_m} \left\{ \sum_{i=1}^n a_i(x, u, \delta u) D_i T_m(u) \right\} h'(T_m(u))\, dx \\
&\quad = \int\limits_{\{|u|<m\}} \left\{ \sum_{i=1}^n a_i(x, u, \delta u) \delta_i u \right\} h'(u)\, dx \ge c_1 \int\limits_{\{|u|<m\}} \frac{|\delta u|^p}{(1+|u|)^{p_1}} h'(u)\, dx.
\end{aligned}$$

Combining this with inequality (4.11), we get (for any $m \in \mathbb{N}$)

$$\int\limits_{\{|u|<m\}} \frac{|\delta u|^p}{(1+|u|)^{p_1}}\, h'(u)\, dx \le \frac{1}{c_1} \int\limits_\Omega [\, f - a_0(x, u, \delta u)]\, T_k(u - v_m)\, dx. \tag{4.12}$$

Taking into account that $T_k(u - v_m) \to h(u)$ in Ω and using the Fatou lemma, from inequality (4.12) we deduce that the function $|\delta u|^p h'(u)/(1+|u|)^{p_1}$ is summable in Ω and inequality (4.7) is true. □

Corollary 4.1. *Let $p_2 = 0$ and $g_1 = 0$ in Ω. Let u be an entropy solution of problem* (3.4), (3.5). *Then the following statements are true*:

(1) *if $h \in C(\mathbb{R})$, $h \ge 0$ in $\mathbb{R}$, and $\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} h(t)\, dt < +\infty$, then the function $|\delta u|^p h(u)$ is summable in Ω*;

(2) *if $h \in C^1(\mathbb{R})$, $h(0) = 0$, and $\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} |h'(t)|^p\, dt < +\infty$, then $h(u) \in L^{p^*}(\Omega)$.*

Proof. Let $h \in C(\mathbb{R})$, $h \geq 0$ in $\mathbb{R}$, and $\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} h(t)\,dt < +\infty$. Let h_1 be a function over $\mathbb{R}$ such that

$$h_1(s) = \begin{cases} \int\limits_0^s (1+|t|)^{p_1} h(t)\,dt & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -\int\limits_s^0 (1+|t|)^{p_1} h(t)\,dt & \text{if } s < 0. \end{cases}$$

Then $h_1 \in C^1(\mathbb{R})$, $h_1(0) = 0$, the function h_1 is bounded on $\mathbb{R}$, and $h_1'(s) = (1+|s|)^{p_1} h(s)$ for any $s \in \mathbb{R}$. Hence, by Proposition 4.1, we deduce that the function $|\delta u|^p h(u)$ is summable over Ω. Therefore, statement (1) is proved.

Now let $h \in C^1(\mathbb{R})$, $h(0) = 0$, and $\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} |h'(t)|^p\,dt < +\infty$. Choose some $k \in \mathbb{N}$. By Lemma 4.1, we have $h(T_k(u)) \in \overset{\circ}{W}{}^{1,p}(\Omega)$ and $|\nabla h(T_k(u))| = |h'(T_k(u))|\,|\nabla T_k(u)|$ almost everywhere in Ω. Then, using (2.12) and Proposition 2.1, we obtain

$$\int\limits_\Omega |h(T_k(u))|^{p^*}\,dx \leq c_{n,p}^{p^*} \biggl\{ \int\limits_\Omega |h'(T_k(u))|^p |\nabla T_k(u)|^p\,dx \biggr\}^{p^*/p} = c_{n,p}^{p^*} \biggl\{ \int\limits_{\{|u|<k\}} |h'(u)|^p |\delta u|^p\,dx \biggr\}^{p^*/p}. \quad (4.13)$$

Due to the assumptions about h and statement (1), the function $|h'(u)|^p |\delta u|^p$ is summable over Ω. Then, applying the Fatou lemma, from (4.13) we derive that $h(u) \in L^{p^*}(\Omega)$. Therefore, statement (2) is proved. ☐

Proposition 4.2. *Let $p_2 < n(p-1-p_1)/(n-p)$ and let u be an entropy solution of problem* (3.4), (3.5). *Let $h \in C^1(\mathbb{R})$, $h(0) = 0$, h and h' be bounded in $\mathbb{R}$, and $h' \geq 0$ in $\mathbb{R}$. Then the function $|\delta u|^p h'(u)/(1+|u|)^{p_1}$ is summable over Ω and the following inequality holds*:

$$\int\limits_\Omega \frac{|\delta u|^p}{(1+|u|)^{p_1}}\, h'(u)\,dx \leq \frac{1}{c_1} \int\limits_\Omega [\, f - a_0(x,u,\delta u)]\, h(u)\,dx + \frac{1}{c_1} \int\limits_\Omega [\, c_2 p_2 (1+|u|)^{p_2} + g_1]\, h'(u)\,dx. \quad (4.14)$$

Proof. Choose some $k > \sup\limits_{\mathbb{R}} |h|$ and define the function v_m and the set E_m by (4.8) and (4.9) for any $m \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Reasoning analogously to the proof of Proposition 4.1, we see that $h(T_m(u)) \in \overset{\circ}{W}{}^{1,p}(\Omega)$, $v_m \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$, and relations (4.10) and (4.11) hold.

It follows from the assumption about p_2 and from Proposition 3.2 that the function $(1+|u|)^{p_2}$ is summable over Ω. Taking this into account and using Proposition 2.1, relations (3.2) and (4.10), the inclusion $\{|u| < m\} \subset E_m$, and the fact that h' is nonnegative and bounded in $\mathbb{R}$, we obtain

$$\int\limits_{E_m} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u)(\delta_i u - \delta_i v_m) \right\} dx$$
$$= \int\limits_{E_m} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u)(\delta_i u - D_i T_m(u)) \right\} dx + \int\limits_{E_m} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u) D_i h(T_m(u)) \right\} dx$$
$$= \int\limits_{E_m \cap \{|u| \geq m\}} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u)\delta_i u \right\} dx + \int\limits_{\{|u|<m\}} \left\{ \sum_{i=1}^{n} a_i(x,u,\delta u)\delta_i u \right\} h'(u)\, dx$$
$$\geq c_1 \int\limits_{\{|u|<m\}} \frac{|\delta u|^p}{(1+|u|)^{p_1}} h'(u)\, dx - \int\limits_{\{|u|<m\}} [\, c_2 p_2 (1+|u|)^{p_2} + g_1]\, h'(u)\, dx$$
$$- \int\limits_{\{|u|\geq m\}} [\, c_2 p_2 (1+|u|)^{p_2} + g_1]\, dx.$$

Combining this with (4.11), we get (for any $m \in \mathbb{N}$)

$$\int\limits_{\{|u|<m\}} \frac{|\delta u|^p}{(1+|u|)^{p_1}} h'(u)\, dx \leq \frac{1}{c_1} \int\limits_{\Omega} [\, f - a_0(x,u,\delta u)]\, T_k(u - v_m)\, dx$$
$$+ \frac{1}{c_1} \int\limits_{\{|u|<m\}} [\, c_2 p_2 (1+|u|)^{p_2} + g_1]\, h'(u)\, dx + \frac{1}{c_1} \int\limits_{\{|u|\geq m\}} [\, c_2 p_2 (1+|u|)^{p_2} + g_1]\, dx. \quad (4.15)$$

Observe that statement (1) and Proposition 3.2 imply that $\operatorname{meas}\{|u| \geq m\} \to 0$ as $m \to \infty$. Then, taking into account that $T_k(u - v_m) \to h(u)$ in Ω and using the Fatou lemma, from (4.15) we derive that the function $|\delta u|^p h'(u)/(1+|u|)^{p_1}$ is summable over Ω and inequality (4.14) holds. ☐

Corollary 4.2. *Let $p_2 < n(p-1-p_1)/(n-p)$ and let u be an entropy solution of problem* (3.4), (3.5). *Then the following statements are true*:

(1) *if $h \in C(\mathbb{R})$, $h \geq 0$ in $\mathbb{R}$, and*

$$\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} h(t)\, dt < +\infty, \qquad \sup_{t\in\mathbb{R}} (1+|t|)^{p_1} h(t) < +\infty, \quad (4.16)$$

then the function $|\delta u|^p h(u)$ is summable over Ω;

(2) *if $h \in C^1(\mathbb{R})$, $h(0) = 0$, and*

$$\int\limits_{-\infty}^{+\infty} (1+|t|)^{p_1} |h'(t)|^p\, dt < +\infty, \qquad \sup_{t\in\mathbb{R}} (1+|t|)^{p_1} |h'(t)|^p < +\infty, \quad (4.17)$$

then $h(u) \in L^{p^*}(\Omega)$.

Proof. Let $h \in C(\mathbb{R})$, $h \geq 0$ in $\mathbb{R}$, and inequalities (4.16) hold. We define the function h_1 in the same way as in the proof of Corollary 4.1. We have $h_1 \in C^1(\mathbb{R})$, $h_1(0) = 0$, h_1 and h_1' are bounded in $\mathbb{R}$, and $h_1' \geq 0$ in $\mathbb{R}$. Then the function $|\delta u|^p h_1'(u)/(1+|u|)^{p_1}$ is summable over Ω by Proposition 4.2. Hence, taking into account that $h_1'(s) = (1+|s|)^{p_1} h(s)$ for any $s \in \mathbb{R}$, we deduce that $|\delta u|^p h(u)$ is summable over Ω. Thus, statement (1) is proved.

Now let $h \in C^1(\mathbb{R})$, $h(0) = 0$, and inequalities (4.17) hold. Then inequality (4.13) is true for any $k \in \mathbb{N}$. Moreover, by statement (1), the function $|h'(u)|^p |\delta u|^p$ is summable over Ω. Hence, using the Fatou lemma, we obtain that $h(u) \in L^{p^*}(\Omega)$. Thus, statement (2) is proved. □

Corollary 4.3. *Let $p_2 < n(p-1-p_1)/(n-p)$ and let u be an entropy solution of problem* (3.4), (3.5). *Let $\beta > 1$. Then the function* $\dfrac{|\delta u|^p}{(1+|u|)^{p_1+1}[\ln(2+|u|)]\,[\ln\ln(3+|u|)]^\beta}$ *is summable over Ω.*

Proof. Let h be a function over $\mathbb{R}$ such that

$$h(s) = \frac{1}{(1+|s|)^{p_1+1}[\ln(2+|s|)]\,[\ln\ln(3+|s|)]^\beta}$$

for any $s \in \mathbb{R}$. We have $h \in C(\mathbb{R})$, $h > 0$ in $\mathbb{R}$, and inequalities (4.16) hold. Then $|\delta u|^p h(u)$ is summable over Ω by statement (1) of Corollary 4.2. Therefore, the corollary is proved. □

Corollary 4.4. *Let $p_2 < n(p-1-p_1)/(n-p)$ and let u be an entropy solution of problem* (3.4), (3.5). *Let $h \in C(\mathbb{R})$ and the following conditions be fulfilled: h is even, $h \geq 0$ in $\mathbb{R}$, h is nonincreasing in the set $[0, +\infty)$, and*

$$\int_1^{+\infty} \frac{1}{t}\,[h(t)]^{(n-p)/n}\,dt < +\infty. \tag{4.18}$$

Then the functions $|u|^{n(p-1-p_1)/(n-p)} h(u)$ and $|\delta u|^{n(p-1-p_1)/(n-1-p_1)}[h(u)]^{(n-p)/(n-1-p_1)}$ are summable over Ω.

Proof. Let h_1 be a function over $\mathbb{R}$ such that

$$h_1(s) = \begin{cases} \displaystyle\int_0^s \frac{[h(t)]^{1/p^*}}{(1+|t|)^{(p_1+1)/p}}\,dt & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \displaystyle -\int_s^0 \frac{[h(t)]^{1/p^*}}{(1+|t|)^{(p_1+1)/p}}\,dt & \text{if } s < 0. \end{cases}$$

We have

$$h_1 \in C^1(\mathbb{R}), \qquad h_1(0) = 0, \tag{4.19}$$

$$\forall\, t \in \mathbb{R} \qquad (1+|t|)^{p_1}|h_1'(t)|^p = \frac{[h(t)]^{(n-p)/n}}{1+|t|}. \tag{4.20}$$

It follows from relations (4.18) and (4.20) that

$$\int_{-\infty}^{+\infty} (1+|t|)^{p_1}|h_1'(t)|^p\,dt < +\infty. \tag{4.21}$$

Since h is even and nonincreasing in $[0, +\infty)$, we have

$$\forall\, t \in \mathbb{R} \qquad h(t) \leq h(0). \tag{4.22}$$

Combining the latter inequality and (4.20), we obtain

$$\sup_{t\in\mathbb{R}} (1+|t|)^{p_1}|h_1'(t)|^p < +\infty. \tag{4.23}$$

Using relations (4.19), (4.21), and (4.23), by statement (2) of Corollary 4.2 we derive

$$h_1(u) \in L^{p^*}(\Omega). \tag{4.24}$$

Let $s \in \mathbb{R}$. Since h is even and nonincreasing in the set $[0, +\infty)$, we have

$$|h_1(s)| \geq \frac{[h(s)]^{1/p^*}|s|}{(1+|s|)^{(p_1+1)/p}}. \tag{4.25}$$

If $|s| \geq 1$, then, using (4.25), we get

$$|s|^{1-(p_1+1)/p}[h(s)]^{1/p^*} \leq 2\,|h_1(s)|. \tag{4.26}$$

If $|s| < 1$, then, by virtue of (4.22), we obtain

$$|s|^{1-(p_1+1)/p}[h(s)]^{1/p^*} \leq [h(0)]^{1/p^*}. \tag{4.27}$$

Inequalities (4.26) and (4.27) imply that

$$|s|^{1-(p_1+1)/p}[h(s)]^{1/p^*} \leq 2\,|h_1(s)| + [h(0)]^{1/p^*}$$

for any $s \in \mathbb{R}$. Then

$$|u|^{n(p-1-p_1)/(n-p)}h(u) \leq 4^{p^*}|h_1(u)|^{p^*} + 2^{p^*}h(0) \quad \text{in } \Omega.$$

Hence, taking inclusion (4.24) into account, we conclude that $|u|^{n(p-1-p_1)/(n-p)}h(u)$ is summable over Ω.

Next, set

$$q = \frac{n(p-1-p_1)}{n-1-p_1}, \qquad \lambda = \frac{n-p}{n-1-p_1}.$$

Observe that $q < p$,

$$\frac{q(1+p_1)}{p-q} = \frac{n(p-1-p_1)}{n-p}, \tag{4.28}$$

$$(\lambda - 1)\frac{p}{q} + 1 = \frac{n-p}{n}. \tag{4.29}$$

Let $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Using the Young inequality with parameters p/q and $p/(p-q)$ and equations (4.28) and (4.29), we obtain

$$|\xi|^q[h(s)]^\lambda = \frac{|\xi|^q[h(s)]^{\lambda-1+q/p}}{(1+|s|)^{(p_1+1)q/p}}(1+|s|)^{(p_1+1)q/p}\,[h(s)]^{(p-q)/p}$$
$$\leq \frac{|\xi|^p[h(s)]^{(n-p)/n}}{(1+|s|)^{p_1+1}} + (1+|s|)^{n(p-1-p_1)/(n-p)}h(s).$$

Then

$$|\delta u|^{n(p-1-p_1)/(n-1-p_1)}[h(u)]^{(n-p)/(n-1-p_1)}$$
$$\leq \frac{|\delta u|^p[h(u)]^{(n-p)/n}}{(1+|u|)^{p_1+1}} + (1+|u|)^{n(p-1-p_1)/(n-p)}h(u) \quad \text{in } \Omega. \tag{4.30}$$

It follows from the properties of the function h and from statement (1) of Corollary 4.2 that the function $\frac{|\delta u|^p[h(u)]^{(n-p)/n}}{(1+|u|)^{p_1+1}}$ is summable over Ω. Hence, taking into account that $|u|^{n(p-1-p_1)/(n-p)}h(u)$ is summable (as was proved above) and using inequalities (4.22) and (4.30), we deduce that the function

$$|\delta u|^{n(p-1-p_1)/(n-1-p_1)}[h(u)]^{(n-p)/(n-1-p_1)}$$

is summable over Ω. □

Corollary 4.5. *Let $p_2 < n(p-1-p_1)/(n-p)$, the function u be an entropy solution of problem* (3.4), (3.5), *and $\beta > n/(n-p)$. Then the functions*

$$\frac{|u|^{n(p-1-p_1)/(n-p)}}{[\ln(2+|u|)]^{n/(n-p)}[\ln\ln(3+|u|)]^\beta}, \qquad \frac{|\delta u|^{n(p-1-p_1)/(n-1-p_1)}}{[\ln(2+|u|)]^{n/(n-1-p_1)}[\ln\ln(3+|u|)]^{\beta(n-p)/(n-1-p_1)}}$$

are summable over Ω.

Proof. Let h be a function over $\mathbb{R}$ such that

$$h(s) = \frac{1}{[\ln(2+|s|)]^{n/(n-p)}[\ln\ln(3+|s|)]^{\beta}}$$

for any $s \in \mathbb{R}$. The function h satisfies the conditions of Corollary 4.4, which completes the proof. $\square$

From Corollary 4.5, we derive the following result.

Corollary 4.6. *Let* $p_2 < n(p-1-p_1)/(n-p)$, *function* u *be an entropy solution of problem* (3.4), (3.5), *and* $\beta > 1/(p-1-p_1)$. *Then*

$$\frac{|u|}{[\ln(2+|u|)]^{1/(p-1-p_1)}[\ln\ln(3+|u|)]^{\beta}} \in L^{n(p-1-p_1)/(n-p)}(\Omega),$$
$$\frac{|\delta u|}{[\ln(2+|u|)]^{1/(p-1-p_1)}[\ln\ln(3+|u|)]^{\beta}} \in L^{n(p-1-p_1)/(n-1-p_1)}(\Omega).$$

5. Existence Theorem

Theorem 5.1. *Let* $p_1 < (p-1)/(n-p+1)$, $p_2 = 0$, *and* $g_1 = 0$ *in* Ω. *Let* $c \geq 0$, $0 < \sigma < p-1-p_1$, $g \in L^1(\Omega)$, $g \geq 0$ *in* Ω, $\varphi \in C(\mathbb{R})$, $\varphi \geq 0$ *in* $\mathbb{R}$, *and*

$$\int_{-\infty}^{+\infty} (1+|t|)^{p_1}\varphi(t)\,dt < +\infty.$$

Suppose that the following inequality holds for almost all $x \in \Omega$ *and all* $s \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^n$:

$$|a_0(x,s,\xi)| \leq c\big[\,|s|^{\sigma} + |\xi|^{\sigma}\big] + |\xi|^p\,\varphi(s) + g(x). \tag{5.1}$$

Then there exists an entropy solution of problem (3.4), (3.5).

The proof of this theorem contains a lot of technical details, which cannot be fully described within the framework of this paper. Detailed proofs of this result and of the analogous result in the case where $p_2 \neq 0$ and φ satisfies the additional condition $\sup_{t\in\mathbb{R}}(1+|t|)^{p_1}\varphi(t) < +\infty$ will be published soon. Here we just describe the main steps of the proof of Theorem 5.1. They are as follows.

First we consider the sequence of generalized solutions $u_j \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ of approximating problems

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^{(j)}(x,u,\nabla u) + A_0^{(j)}(x,u,\nabla u) = f_j \quad \text{in } \Omega, \tag{5.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{5.3}$$

where $f_j = T_j(f)$, while $A_i^{(j)}$ and $A_0^{(j)}$ are functions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ such that

$$A_i^{(j)}(x,s,\xi) = a_i(x,T_j(s),\xi), \qquad A_0^{(j)}(x,s,\xi) = T_j(a_0(x,s,\xi))$$

for any triple $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Substituting $h(u_j)$ for u in the integral identity corresponding to the approximating problem with the right-hand side f_j, where $h \in C^1(\mathbb{R})$ is any bounded function such that $h(0) = 0$ and $h' \geq 0$ in $\mathbb{R}$, we obtain (using Lemma 4.1 and inequalities (3.2) and (5.1)) the integral inequality, from which we derive a series of uniform integral estimates for functions u_j.

In particular, this allows us to prove the existence of the function $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and the increasing sequence $\{j_l\} \subset \mathbb{N}$ such that

$$u_{j_l} \to u \quad \text{almost everywhere in } \Omega, \tag{5.4}$$

$$\forall\, k > 0 \quad T_k(u_{j_l}) \to T_k(u) \quad \text{weakly in } \overset{\circ}{W}{}^{1,p}(\Omega). \tag{5.5}$$

Then, using the integral identities corresponding to approximating problems (5.2), (5.3), inequalities (3.1)–(3.3), and properties (5.4) and (5.5), we derive that

$$\forall\, i \in \{1,\dots,n\} \quad D_i u_{j_l} \to \delta_i u \quad \text{in measure,} \tag{5.6}$$

$$\forall\, k > 0 \quad T_k(u_{j_l}) \to T_k(u) \quad \text{strongly in } W^{1,p}(\Omega). \tag{5.7}$$

From the above-mentioned uniform integral estimates for the functions u_j (including a uniform estimate for the integrals of $|\nabla u_j|^p\,\varphi(u_j)$ over Ω), inequality (5.1), and properties (5.4) and (5.6), we deduce that $a_0(x,u,\delta u) \in L^1(\Omega)$.

Using the same uniform integral estimates for functions u_j and properties (5.4), (5.6), and (5.7), we pass to the limit in the integral identities corresponding to the approximating problems (5.2), (5.3). This implies that u is an entropy solution of problem (3.4), (3.5).

To conclude, we consider an example of functions a_i, $i = 1,\dots,n$, satisfying inequalities (3.1)–(3.3), and an example of a function a_0, satisfying inequality (5.1) with φ complying the conditions of Theorem 5.1.

Example 5.1. Let $p > 2$, $\alpha \in [0, p-1)$, $\beta \in \mathbb{R}$, $\gamma \in (0, p-1-\alpha)$, ψ be a nonnegative continuous function over $\mathbb{R}$, $g_0 \in L^{p/(p-2)}(\Omega)$, and $g_0 \geq 0$ in Ω. For any $i \in \{1,\dots,n\}$, let a_i be a function over $\Omega \times \mathbb{R} \times \mathbb{R}^n$ such that $a_i(x,s,\xi) = \dfrac{|\xi|^{p-2}\xi_i}{(1+|s|)^\alpha} + \beta|s|^\gamma + g_0(x)\psi(s)\xi_i$ for any triple $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then inequalities (3.1)–(3.3) hold and

$$\overline{c}_k = 6^{p/(p-1)} n, \qquad \overline{g}_k = (3|\beta| k^\gamma)^{p/(p-1)} n + 3^{p/(p-1)} n \Big(\max_{[-k,k]} \psi \Big)^{p/(p-2)} g_0^{p/(p-2)},$$

$$p_1 = \alpha, \qquad p_2 = \Big(\gamma + \frac{\alpha}{p}\Big)\frac{p}{p-1}, \qquad g_1 = 0 \quad \text{in } \Omega.$$

Example 5.2. Let $\beta > 1$, $\beta_1, \beta_2 \in \mathbb{R}$, $0 < \sigma < p - 1 - p_1$, $g \in L^1(\Omega)$, $g \geq 0$ in Ω, and φ be a function over $\mathbb{R}$ such that

$$\varphi(s) = \frac{1}{(1+|s|)^{p_1+1}[\ln(2+|s|)]^\beta}$$

for any $s \in \mathbb{R}$. Let a_0 be a function over $\Omega \times \mathbb{R} \times \mathbb{R}^n$ such that

$$a_0(x,s,\xi) = \beta_1|s|^\sigma + \beta_2|\xi|^\sigma + |\xi|^p\,\varphi(s) + g(x)$$

for any triple $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then φ complies with the conditions of Theorem 5.1 and a_0 satisfies inequality (5.1).

6. Bibliographic Comments

The set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ and some wider sets of functions were introduced in [3]. Equation (2.4) is used in [3] for the definition of a gradient for elements of a function class containing the set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$. The direct definition of functions $\delta_i u$ for $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ by (2.3) and the proof for Proposition 2.1 are given in [14]. Proposition 2.2 can be found in [12, 14]; the particular case for $\lambda = p$ is given by [3, Lemma 2.2]. A statement similar to statement (1) of Proposition 2.3 is mentioned without a proof in [3, Sec. 2]. Lemma 2.1 is essentially proved in [11]. For the analogous results using more qualified estimates instead of (2.10), see [12, 13]. For $\theta = 1$, inequalities (2.14) and (2.15) under condition (2.13) were established earlier in [3]. The proof of Lemma 2.2 is analogous to the one described in [3]. An analog of the set $\overset{\circ}{\mathcal{T}}{}^{1,p}(\Omega)$ suitable for investigation of the solvability of fourth-order nonlinear elliptic equations with right-hand sides belonging to L^1 is introduced in [11].

The idea of using inequality (3.1) allowing an arbitrary growth of the coefficients a_i, $i = 1,\dots,n$, with respect to s was inspired by [5], where an analogous inequality was introduced for coefficients of a parabolic equation with L^1-data. A particular case of the degenerate coercivity (3.2), where $p = 2$, $p_2 = 0$, and $g_1 \equiv 0$, was considered, e.g., in [6] for a linear equation with respect to the gradient of the unknown function. In [1], the Dirichlet problem was considered for Eq. (3.4) with a zero low-order coefficient and the following restrictions regarding a_i, $i = 1, \dots, n$: these coefficients satisfy (3.2) with $p \in (1, n)$, $p_1 \in [0, p-1)$, $p_2 = 0$, and $g_1 \equiv 0$, while their order of growth with respect to s and ξ is not greater than $p-1$. Conditions of the strict monotonicity of the form (3.3) are usual for proofs of existence and (in some cases) uniqueness of entropy solutions (e.g., see [1, 3, 8, 16]). For Definition 3.1, see, e.g., [8], and for Definitions 3.2 and 3.3, see [3]. For $p_1 = 0$, Proposition 3.2 describes the properties of summability actually obtained in [3] for entropy solutions of nonlinear elliptic equations with the standard coercivity condition ($p_1 = 0$, $p_2 = 0$, and $g_1 \equiv 0$). If $p_1 \neq 0$, but $p_2 = 0$ and $g_1 \equiv 0$, then the properties of summability formulated in Proposition 3.2 were essentially established in [1] for entropy solutions which are limits of solutions of the corresponding approximating problems. Here we considered the general case, where the proof of Proposition 3.2 is more complicated than in the above-mentioned particular cases. Propositions 3.3 and 3.4 and Corollary 3.2 generalize the results obtained in [3] for $p_1 = 0$, $p_2 = 0$, and $g_1 \equiv 0$.

As far as we know, Propositions 4.1 and 4.2 together with their corollaries formulated in Sec. 4 are new even in the case where $p_1 = 0$. In particular, even for $p_1 = 0$, $p_2 = 0$, and $g_1 \equiv 0$, Corollary 4.6 gives stronger results of summability than results obtained in [9] for weak solutions.

Note that estimates (4.7) and (4.14) play an important role in the investigation of the properties of summability of entropy solutions under conditions of the form (5.1) for the function a_0 and in the improvement of the summability of f. On the whole, this is a subject of a separate publication.

If the high-order coefficients of Eq. (3.4) do not depend on u, have a growth of order $p-1$ with respect to ∇u, and satisfy the standard coercivity condition ($p_1 = 0$, $p_2 = 0$, and $g_1 \equiv 0$), while the low-order coefficient a_0 does not depend on ∇u, has an arbitrary growth with respect to u, and is nondecreasing with respect to u, then the existence of an entropy solution of problem (3.4), (3.5) is proved in [3]. The proof of Theorem 5.1 mainly follows the method used in [3]. However, in the case under consideration the low-order coefficient without a fixed sign satisfying estimate (5.1) complexifies the necessary uniform estimates of solutions of approximating problems. If the high-order coefficients of Eq. (3.4) have growth of order not greater than $p-1$ with respect to u, have growth of order $p-1$ with respect to ∇u, and satisfy the standard coercivity condition ($p_1 = 0$, $p_2 = 0$, and $g_1 \equiv 0$), the low-order coefficient a_0 satisfies inequality (5.1) with $c = 0$ and $\varphi \in L^1(\mathbb{R})$, and the right-hand side of the equation is a bounded Radon measure on Ω, then the existence of a $\mathcal{T}$-solution of the Dirichlet problem is proved in [16]. This article contains interesting (and similar to ours in some sense) methods of proving integral estimates of solutions of the approximating problems and the ideas used in the proof of Theorem 5.1 concerning the proof of strong convergence in $W^{1,p}(\Omega)$ for cross sections of such solutions. Finally, it should be mentioned that if high-order coefficients of Eq. (3.4) satisfy the standard growth condition with respect to u and ∇u, while $p_1 \in [0, p-1)$, $p_2 = 0$, $g_1 \equiv 0$, and $a_0 \equiv 0$, then the existence of entropy solutions of problem (3.4), (3.5) was proved in [1].

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Alexandr Albertovich Kovalevsky
Institute of Applied Mathematics and Mechanics, NAS of Ukraine
E-mail: `alexkvl@iamm.ac.donetsk.ua`