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ON THE SOLVABILITY OF A SINGULAR BOUNDARY-VALUE PROBLEM FOR THE EQUATION $f(t, x, x', x'') = 0$

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ABSTRACT. In this work, we consider boundary-value problems of the form

$$f(t, x, x', x'') = 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x'(1) = b, \quad b > 0,$$

where the scalar function $f(t, x, p, q)$ may be singular at $x = 0$. As far as we know, the solvability of the singular boundary-value problems of this form has not been treated yet. Here we try to fill in this gap. Examples illustrating our main result are included.

1. Introduction

In this paper, we deal with the existence of positive solutions to the boundary-value problem (BVP)

$$f(t, x, x', x'') = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \quad x'(1) = b, \quad b > 0, \tag{1.2}$$

where the scalar function $f(t, x, p, q)$ may be singular at $x = 0$, i.e., f may tend to infinity when x tends to zero on the left- and/or on the right-hand side. In fact, we need f to be defined at least for

$$(t, x, p, q) \in [0, 1] \times \{D_x \setminus \{0\}\} \times D_p \times D_q,$$

where the sets $D_x, D_p, D_q \subseteq R$ may be bounded. We also need D_x, D_p, and D_q to be such that $0 \in D_x$, $0 \in D_q$, and the sets $D_q^+ = D_q \cap (0, +\infty)$, $D_q^- = (-\infty, 0) \cap D_q$, and $\{y \in D_p : y > 0\}$ are not empty and also the first derivatives of f are continuous on a suitable subset of the domain of f.

Results on the solvability of various singular BVPs for ordinary differential equations whose main nonlinearity does not depend on the highest derivative can be found, e.g., in [1–5, 7, 10–13, 15, 18–22, 28] and the references therein. Papers [3, 22] deal with higher-order differential equations. In [3, 21, 22], the main nonlinearity satisfies the Carathéodory conditions, while in [21], a differential equation with impulse effects is considered. The results in [2–4, 10, 12, 20, 28] guarantee the existence of positive solutions.

The solvability of various nonsingular BVPs for second-order differential equations whose main nonlinearity depends on x'' has been investigated in [6, 8, 14, 16, 17, 23–27]. The case where the main nonlinearity of the equations is continuous on the set $[0, 1] \times R^3$ is considered in [6, 8, 14, 16, 23–27], while the case where the main nonlinearity is continuous on the set $[0, 1] \times R^n \times R^n \times Y$, where $Y \subseteq R^n$, is considered in [17]. The results in these works guarantee the existence of solutions that may change their own sign.

As far as we know, the solvability of singular BVPs for equations of the form (1.1) has not been studied yet. In this paper, we want to fill in this gap. In order to establish the existence of positive solutions to the BVP (1.1), (1.2), we proceed as follows. For $\lambda \in [0, 1]$ and $n = 1, 2, \ldots$, we construct a family, say $(\Phi)_\lambda$, of regular BVPs. For example, two-parameter families of BVPs have also been used in [4, 5, 19]. As in [8, 13], suitable "barrier strips" yield a priori bounds independent of λ and n for x, x', and x'', where $x \in C^2[0, 1]$ is an eventual solution to the family $(\Phi)_\lambda$. These bounds allow us to apply the topological transversality theorem [9, Chapter I, Theorem 2.6] to prove the solvability of the family $(\Phi)_1$ for each $n = 1, 2, \ldots$. Finally, we establish a bound for x_n''' independent of n in an appropriate domain so that the

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Arzelà–Ascoli theorem yields a solution to the problem (1.1), (1.2) as the limit of a sequence of solutions to the problems $(\Phi)_1,\ n = 1, 2, \ldots$.

2. Basic Hypotheses

In order to obtain our results, we make the following three basic hypotheses.

H1. There are positive constants K, Q, P_i, $i = 1, 2, 3, 4$, and a sufficiently small $\varepsilon > 0$ such that

$$P_3 + \varepsilon \le P_1 \le b \le P_2 \le P_4 - \varepsilon, \qquad P_1 < P_2,$$
$$(0, P_2 + \varepsilon] \subseteq D_x, \qquad [P_3, P_4] \subseteq D_p, \qquad [h_q - \varepsilon, H_q + \varepsilon] \subseteq D_q,$$

where $h_q = -Q + P_1 - b$ and $H_q = Q + P_2 - b$, and the following "barrier strip" conditions are satisfied:

$$f(t, x, p, q) + Kq \ge 0, \qquad (t, x, p, q) \in [0, 1] \times D_x^0 \times [P_2, P_4] \times D_q^-, \tag{2.1}$$
$$f(t, x, p, q) + Kq \le 0, \qquad (t, x, p, q) \in [0, 1] \times D_x^0 \times [P_3, P_1] \times D_q^+, \tag{2.2}$$
$$q\big(f(t, x, p, q) + Kq\big) \le 0, \qquad (t, x, p, q) \in [0, 1] \times (0, P_2 + \varepsilon] \times [P_1, P_2] \times \{D_Q^- \cup D_Q^+\}, \tag{2.3}$$

where $D_x^0 = D_x \setminus \{0\}, D_Q^- = \{z \in D_q : z < -Q\}$ and $D_Q^+ = \{z \in D_q : z > Q\}$.

Remark 2.1. Since $[-Q, Q] \subset [h_q - \varepsilon, H_q + \varepsilon] \subseteq D_q$, it follows that the sets D_Q^- and D_Q^+ are not empty.

H2. The functions $f(t, x, p, q)$ and $f_q(t, x, p, q)$ are continuous on the set $[0, 1] \times (0, P_2 + \varepsilon] \times [P_1 - \varepsilon, P_2 + \varepsilon] \times [h_q - \varepsilon, H_q + \varepsilon]$ and there is a constant $K_q > K$ such that

$$f_q(t, x, p, q) \le -K_q, \quad (t, x, p, q) \in [0, 1] \times (0, P_2 + \varepsilon] \times [P_1 - \varepsilon, P_2 + \varepsilon] \times [h_q - \varepsilon, H_q + \varepsilon],$$

where K, Q, P_1, P_2, h_q, H_q, and ε are as in H1.

H3. The functions $f_t(t, x, p, q)$, $f_x(t, x, p, q)$, and $f_p(t, x, p, q)$ are continuous for $(t, x, p, q) \in [0, 1] \times (0, P_2 + \varepsilon] \times [P_1, P_2] \times [h_q, H_q]$.

3. An Auxiliary Result

For $\lambda \in [0, 1]$ and $n \in \mathbb{N}$, we construct the family of BVPs

$$\begin{cases} K\big(x'' - (1 - \lambda)(x' - b)\big) = \lambda\Big(K\big(x'' - (1 - \lambda)(x' - b)\big) + f\big(t, x, x', x'' - (1 - \lambda)(x' - b)\big)\Big), \\ x(0) = \dfrac{1}{n}, \quad x'(1) = b, \end{cases} \tag{3.1_λ}$$

which for $\lambda = 1$ includes the BVP (1.1), (1.2) and where the constant $K > 0$ is as in H1, when it is satisfied. Relatively, the following proposition is fulfilled.

Lemma 3.1. *Let* H1 *be satisfied and let* $x(t) \in C^2[0, 1]$ *be a solution to the family* $(3.1)_\lambda$. *Then*

$$0 < \frac{1}{n} \le x(t) \le P_2 + \frac{1}{n}, \quad P_1 \le x'(t) \le P_2, \quad h_q \le x''(t) \le H_q$$

for $t \in [0, 1]$, $n \in \mathbb{N}$, *and* $n > 1/\varepsilon$.

Proof. Let the number $n \in \mathbb{N}$, $n > 1/\varepsilon$, be fixed and suppose that the set

$$S = \big\{t \in [0, 1] : P_2 < x'(t) \le P_4\big\}$$

is not empty. The continuity of $x'(t)$ and the boundary condition at $t = 1$ imply that there is an interval $[\alpha, \beta] \subseteq S$ such that

$$x'(\alpha) > x'(\beta). \tag{3.2}$$

Then there is a $\gamma \in [\alpha, \beta]$ such that

$$x''(\gamma) < 0.$$

Without loss of generality, assume that $x(\gamma) \neq 0$. Since $x(t)$ is a solution to $(3.1)_\lambda$, we have

$$\Big(\gamma, x(\gamma), x'(\gamma), x''(\gamma) - (1-\lambda)(x'(\gamma) - b)\Big) \in [0,1] \times D_x^0 \times D_p \times D_q.$$

But $x'(\gamma) \in (P_2, P_4]$ and $x''(\gamma) - (1-\lambda)(x'(\gamma) - b) < 0$. So,

$$\Big(\gamma, x(\gamma), x'(\gamma), x''(\gamma) - (1-\lambda)(x'(\gamma) - b)\Big) \in [0,1] \times D_x^0 \times (P_2, P_4] \times D_q^-$$

and we obtain by H1 that

$$\begin{aligned} 0 &> K\Big(x''(\gamma) - (1-\lambda)(x'(\gamma) - b)\Big) \\ &= \lambda\Big(K\Big(x''(\gamma) - (1-\lambda)(x'(\gamma) - b)\Big) + f\Big(\gamma, x(\gamma), x'(\gamma), x''(\gamma) - (1-\lambda)(x'(\gamma) - b)\Big)\Big) \geq 0, \end{aligned}$$

which is impossible. Therefore,

$$x'(t) \leq P_2 \ \text{ for } \ t \in [0,1].$$

Similarly, the assumption that the set

$$S_0 = \big\{t \in [0,1] : P_3 \leq x'(t) < P_1\big\}$$

is not empty leads to a contradiction, and from this we conclude that

$$0 < P_1 \leq x'(t) \ \text{ for } \ t \in [0,1].$$

But the fact that $x'(t) > 0$ on $[0,1]$ means that $x(t) \geq 1/n$ for $t \in [0,1]$ and for fixed $n \in N$. On the other hand, by the mean-value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x(t) - x(0) = x'(\xi)t,$$

from which it follows that

$$x(t) \leq P_2 + 1/n < P_2 + \varepsilon \ \text{ for } \ t \in [0,1].$$

Suppose now that there is $(t_0, \lambda_0) \in [0,1] \times [0,1]$ such that

$$x''(t_0) - (1-\lambda_0)(x'(t_0) - b) < -Q.$$

Then, using the fact that $\big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)(x'(t_0) - b)\big) \in [0,1] \times (0, P_2+\varepsilon] \times [P_1, P_2] \times D_Q^-$ and having in mind (2.3), we find that

$$\begin{aligned} 0 &> K\Big(x''(t_0) - (1-\lambda_0)(x'(t_0) - b)\Big) \\ &= \lambda_0\Big(K\Big(x''(t_0) - (1-\lambda_0)(x'(t_0) - b)\Big) + f\Big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)(x'(t_0) - b)\Big)\Big) \geq 0. \end{aligned}$$

The obtained contradiction shows that

$$-Q \leq x''(t) - (1-\lambda)(x'(t) - b)$$

for each $(t, \lambda) \in [0,1] \times [0,1]$. In a similar way, assuming that there exists $(t_1, \lambda_1) \in [0,1] \times [0,1]$ such that

$$x''(t_1) - (1-\lambda_1)(x'(t_1) - b) > Q$$

and using (2.1), we again lead to a contradiction. So, we see that

$$-Q \leq x''(t) - (1-\lambda)(x'(t) - b) \leq Q, \quad (t, \lambda) \in [0,1] \times [0,1],$$

which yields

$$h_q = -Q + P_1 - b \leq x''(t) \leq Q + P_2 - b = H_q, \quad t \in [0,1].$$

□

4. An Appropriate Extension of the Main Nonlinearity

In order to prove our main result, it is necessary to extend the function f on the set $[0,1] \times R^3$ in a suitable way. With that end in view, we proceed as follows.

For a fixed $n \in \mathbb{N}$, we construct the functions

$$\varphi = \begin{cases} f(t,(2n)^{-1},p,q), & (t,x,p,q) \in [0,1] \times (-\infty,(2n)^{-1}) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \\ f(t,x,p,q), & (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \\ f(t,P_2+\varepsilon,p,q), & (t,x,p,q) \in [0,1] \times (P_2+\varepsilon,\infty) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \end{cases}$$

where h_p, H_p, ε, and $P_i, i = 1, 2$, are the constants of H1.

Remark 4.1. Observe that any other function considered below, which involves the function φ, depends on this fixed value of $n \in \mathbb{N}$. But, for the sake of simplicity, in the sequel we will omit all n-indexes.

Some properties of the function φ are described by the following two lemmas.

Lemma 4.1. *Let* H2 *be satisfied. Then* $\varphi(t,x,p,q)$ *and its derivative* $\varphi_q(t,x,p,q)$ *are continuous on* $\Omega_x \equiv [0,1] \times R \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon]$ *and* $\varphi_q(t,x,p,q) \le -K_q$ *for* $(t,x,p,q) \in \Omega_x$.

Proof. Clearly, $\varphi(t,x,p,q)$ and

$$\varphi_q = \begin{cases} f_q(t,(2n)^{-1},p,q), & (t,x,p,q) \in [0,1] \times (-\infty,(2n)^{-1}) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \\ f_q(t,x,p,q), & (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \\ f_q(t,P_2+\varepsilon,p,q), & (t,x,p,q) \in [0,1] \times (P_2+\varepsilon,\infty) \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon], \end{cases}$$

are continuous on Ω_x. Moreover, in view of H2,

$$f_q(t,x,p,q) \le -K_q, \quad (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon].$$

In particular, for $(t,p,q) \in [0,1] \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon]$, we have

$$f_q(t,(2n)^{-1},p,q) \le -K_q, \quad f_q(t,P_2+\varepsilon,p,q) \le -K_q.$$

Consequently,

$$\varphi_q(t,x,p,q) \le -K_q, \quad (t,x,p,q) \in [0,1] \times R \times [P_1-\varepsilon, P_2+\varepsilon] \times [h_q-\varepsilon, H_q+\varepsilon].$$

□

Lemma 4.2. *Let* H1 *be satisfied. Then the function* $\varphi(t,x,p,q)$ *has the following "barrier strip" properties*

$$\varphi(t,x,p,q) + Kq \ge 0, \quad (t,x,p,q) \in [0,1] \times R \times \{P_2\} \times [h_q-\varepsilon, 0), \tag{4.1}$$

$$\varphi(t,x,p,q) + Kq \ge 0, \quad (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times [h_q-\varepsilon, -Q], \tag{4.2}$$

$$\varphi(t,x,p,q) + Kq \le 0, \quad (t,x,p,q) \in [0,1] \times R \times \{P_1\} \times [0, H_q+\varepsilon], \tag{4.3}$$

$$\varphi(t,x,p,q) + Kq \le 0, \quad (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times [Q, H_q+\varepsilon]. \tag{4.4}$$

Proof. In particular, by the definition of φ, we see that

$$\varphi(t,x,p,q) = f(t,x,p,q), \quad (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

Now, since $[(2n)^{-1}, P_2+\varepsilon] \subseteq D_x^0$, $[P_2, P_2+\varepsilon] \subseteq [P_2, P_4]$, and $[h_q-\varepsilon, 0) \subseteq D_q^-$, in view of H1, we get

$$f(t,x,p,q) + Kq \ge 0, \quad (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

Therefore,

$$\varphi(t,x,p,q) + Kq \ge 0, \quad (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0). \tag{4.5}$$

Next, having in mind H1 and the fact that $(2n)^{-1} \in D_x^0$, $[P_2, P_2+\varepsilon] \subseteq [P_2, P_4]$, and $[h_q-\varepsilon, 0) \subseteq D_q^-$, we see that

$$f(t,(2n)^{-1},p,q) + Kq \ge 0, \quad (t,p,q) \in [0,1] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

But, since the definition of φ implies

$$\varphi(t,x,p,q)=f(t,(2n)^{-1},p,q),\quad (t,x,p,q)\in[0,1]\times(-\infty,(2n)^{-1})\times[P_2,P_2+\varepsilon]\times[h_q-\varepsilon,0),$$

we conclude that

$$\varphi(t,x,p,q)+Kq\geq 0,\quad (t,x,p,q)\in[0,1]\times(-\infty,(2n)^{-1})\times[P_2,P_2+\varepsilon]\times[h_q-\varepsilon,0). \tag{4.6}$$

In a similar way, we obtain

$$\varphi(t,x,p,q)+Kq\geq 0,\quad (t,x,p,q)\in[0,1]\times(P_2+\varepsilon,\infty)\times[P_2,P_2+\varepsilon]\times[h_q-\varepsilon,0),$$

which, together with (4.5) and (4.6), gives (4.1). Note that the same reasoning as above yields (4.3).

To prove (4.2), observe first that, by the definition of φ,

$$\varphi(t,x,p,q)=f(t,x,p,q),\quad (t,x,p,q)\in[0,1]\times[(2n)^{-1},P_2+\varepsilon]\times[P_1,P_2]\times[h_q-\varepsilon,-Q),$$

and then, using (2.1), we obtain

$$\varphi(t,x,p,q)+Kq\geq 0,\quad (t,x,p,q)\in[0,1]\times[(2n)^{-1},P_2+\varepsilon]\times[P_1,P_2]\times[h_q-\varepsilon,-Q).$$

Moreover, (2.1) implies that

$$f(t,(2n)^{-1},p,q)+Kq\geq 0,\quad f(t,P_2+\varepsilon,p,q)+Kq\geq 0$$

for $(t,p,q)\in[0,1]\times[P_1,P_2]\times[h_q-\varepsilon,-Q)$ and, by the definition of φ, we derive

$$\varphi(t,x,p,q)+Kq\geq 0,\quad (t,x,p,q)\in[0,1]\times\Big\{(-\infty,(2n)^{-1})\cup(P_2+\varepsilon,\infty)\Big\}\times[P_1,P_2]\times[h_q-\varepsilon,-Q).$$

Thus, we see that

$$\varphi(t,x,p,q)+Kq\geq 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,-Q).$$

Finally, by the same arguments, we conclude that

$$\varphi(t,x,p,q)+Kq\leq 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(Q,H_q+\varepsilon].$$

□

Now, using the function φ, we introduce the function

$$\Phi(t,x,p,q)=\begin{cases}\varphi(t,x,P_1,q), & (t,x,p,q)\in[0,1]\times R\times(-\infty,P_1)\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi(t,x,P_2,q), & (t,x,p,q)\in[0,1]\times R\times(P_2,\infty)\times[h_q-\varepsilon,H_q+\varepsilon],\end{cases}$$

whose properties are described by the following proposition.

Lemma 4.3. *Let* H2 *be satisfied. Then* $\Phi(t,x,p,q)$ *and its derivative* $\Phi_q(t,x,p,q)$ *are continuous on* $\Omega_p:\equiv[0,1]\times R\times R\times[h_q-\varepsilon,H_q+\varepsilon]$ *and* $\Phi_q(t,x,p,q)\leq -K_q$ *for* $(t,x,p,q)\in\Omega_p$.

Proof. Clearly, $\Phi(t,x,p,q)$ and

$$\Phi_q(t,x,p,q)=\begin{cases}\varphi_q(t,x,P_2,q), & (t,x,p,q)\in[0,1]\times R\times(P_2,\infty)\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi_q(t,x,P_1,q), & (t,x,p,q)\in[0,1]\times R\times(-\infty,P_1)\times[h_q-\varepsilon,H_q+\varepsilon],\end{cases}$$

are continuous on Ω_p. Moreover, by Lemma 4.1,

$$\varphi_q(t,x,p,q)\leq -K_q,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],$$

and, hence, it follows that

$$\Phi_q(t,x,p,q)\leq -K_q,\quad (t,x,p,q)\in\Omega_p.$$

□

In order to extend the main nonlinearity appropriately, we suppose that the condition H2 is satisfied. We also assume that ψ is a function with the properties

$$\Psi(t,x,p,q) \text{ and } \Psi_q(t,x,p,q) \text{ are continuous on } [0,1]\times R^2\times[H_q+\varepsilon,\infty),$$

$$\Psi(t,x,p,H_q+\varepsilon)=\Phi(t,x,p,H_q+\varepsilon) \text{ and } \Psi_q(t,x,p,H_q+\varepsilon)=\Phi_q(t,x,p,H_q+\varepsilon) \text{ for } (t,x,p)\in[0,1]\times R^2,$$

and

$$\Psi_q(t,x,p,q)\le -K_q,\quad (t,x,p,q)\in[0,1]\times R^2\times[H_q+\varepsilon,\infty),$$

which is possible because, by Lemma 4.3, $\Phi_q(t,x,p,H_q+\varepsilon)\le -K_q$ for $(t,x,p)\in[0,1]\times R^2$.

Finally, suppose that Ψ is a function with the properties

$$\psi(t,x,p,q) \text{ and } \psi_q(t,x,p,q) \text{ are continuous on } [0,1]\times R^2\times(-\infty,h_q-\varepsilon],$$

$$\psi(t,x,p,h_q-\varepsilon)=\Phi(t,x,p,h_q-\varepsilon) \text{ and } \psi_q(t,x,p,h_q-\varepsilon)=\Phi_q(t,x,p,h_q-\varepsilon) \text{ for } (t,x,p)\in[0,1]\times R^2,$$

and

$$\psi_q(t,x,p,q)\le -K_q,\quad (t,x,p,q)\in[0,1]\times R^2\times(-\infty,h_q-\varepsilon],$$

which is possible since, by Lemma 4.3, $\Phi_q(t,x,p,h_p-\varepsilon)\le -K_q$ for $(t,x,p)\in[0,1]\times R^2$.

Now we are ready to extend the function f to the function defined in $[0,1]\times\mathbb{R}^3$ by

$$\overline{f}_n(t,x,p,q)=\begin{cases}\psi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times(-\infty,h_q-\varepsilon),\\ \Phi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times[h_q-\varepsilon,H_q+\varepsilon],\\ \Psi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times(H_q+\varepsilon,\infty).\end{cases}$$

The next two lemmas establish some useful properties of the functions $\overline{f}_n$ and its derivative

$$(\overline{f}_n)_q(t,x,p,q)=\begin{cases}\psi_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times R\times(-\infty,h_q-\varepsilon),\\ \Phi_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times R\times[h_q-\varepsilon,H_q+\varepsilon],\\ \Psi_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times R\times(H_q+\varepsilon,\infty).\end{cases}$$

Lemma 4.4. *Let* H2 *be satisfied. Then*

$$\overline{f}_n(t,x,p,q) \textit{ and } (\overline{f}_n)_q(t,x,p,q) \textit{ are continuous on } [0,1]\times R^3$$

and

$$(\overline{f}_n)_q(t,x,p,q)\le -K_q,\quad (t,x,p,q)\in[0,1]\times R^3.$$

Proof. Since the conclusion of this lemma follows from the properties of the functions ψ and Ψ and from Lemma 4.3, the details of the proof are omitted. ☐

Lemma 4.5. *Let* H1 *and* H2 *be satisfied. Then the function* $\overline{f}_n$ *has the following "barrier strip" properties*:

$$\begin{aligned}\overline{f}_n(t,x,p,q)+Kq\ge 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_2,P_2+\varepsilon]\times(-\infty,0),\\ \overline{f}_n(t,x,p,q)+Kq\le 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1-\varepsilon,P_1]\times(0,\infty),\end{aligned}\tag{4.7}$$

and

$$q\Big(\overline{f}_n(t,x,p,q)+Kq\Big)\le 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times\Big\{R\setminus[-Q,Q]\Big\}.$$

Proof. The definitions of the functions Φ and $\overline{f}_n$ imply that

$$\overline{f}_n(t,x,p,q)=\varphi(t,x,p,q),\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon].\tag{4.8}$$

On the other hand, by Lemma 4.2,

$$\varphi(t,x,P_2,q)+Kq\ge 0,\quad (t,x,q)\in[0,1]\times R\times[h_q-\varepsilon,0).$$

So, from the fact that

$$\overline{f}_n(t,x,p,q)=\Phi(t,x,p,q)=\varphi(t,x,P_2,q),\quad p\ge P_2,\quad q\in[h_q-\varepsilon,0),$$

it follows that

$$\overline{f}_n(t,x,p,q)+Kq\ge 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_2,P_2+\varepsilon]\times[h_q-\varepsilon,0). \tag{4.9}$$

Observe that, by Lemma 4.4, for each $(t,x,p,q)\in[0,1]\times R\times[P_2,P_2+\varepsilon]\times(-\infty,0)$, we have

$$\Big(\overline{f}_n(t,x,p,q)+Kq\Big)_q=(\overline{f}_n)_q(t,x,p,q)+K<(\overline{f}_n)_q(t,x,p,q)+K_q\le 0,$$

which, together with (4.9), yields

$$\overline{f}_n(t,x,p,q)+Kq\ge 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_2,P_2+\varepsilon]\times(-\infty,0).$$

Now note that the same reasoning as above yields (4.7).

Note also that, in particular, it follows from (4.8) that

$$\overline{f}_n(t,x,p,q)=\varphi(t,x,p,q),\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(Q,H_q+\varepsilon],$$

from which, according to (4.3), we get

$$\overline{f}_n(t,x,p,q)+Kq\le 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(Q,H_q+\varepsilon]. \tag{4.10}$$

In view of Lemma 4.4, for each $(t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(0,\infty)$ it follows that

$$\Big(\overline{f}_n(t,x,p,q)+Kq\Big)_q=(\overline{f}_n)_q(t,x,p,q)+K<(\overline{f}_n)_q(t,x,p,q)+K_q\le 0.$$

So, by (4.10), we conclude that

$$\overline{f}_n(t,x,p,q)+Kq\le 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(Q,\infty). \tag{4.11}$$

Finally, observe that the inequality

$$\overline{f}_n(t,x,p,q)+Kq\ge 0,\quad (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times(-\infty,-Q),$$

can be obtained in a similar manner. □

Now, for $\lambda\in[0,1]$ and $n\in\mathbb{N}$, $n>1/\varepsilon$, consider the family of regular problems

$$\begin{cases}K\big(x''-(1-\lambda)(x'-b)\big)=\lambda\Big(K\big(x''-(1-\lambda)(x'-b)\big)+\overline{f}_n\big(t,x,x',x''-(1-\lambda)(x'-b)\big)\Big),\\ x(0)=\dfrac{1}{n},\quad x'(1)=b.\end{cases} \tag*{$(4.12)_\lambda$}$$

The following two lemmas establish some useful properties of solutions to the family $(4.12)_\lambda$.

Lemma 4.6. *Let* H1 *and* H2 *be satisfied and let* $x(t)\in C^2[0,1]$ *be a solution to the family* $(4.12)_\lambda$. *Then*

$$\frac{1}{n}\le x(t)\le P_2+\varepsilon,\quad P_1\le x'(t)\le P_2,\quad h_q\le x''(t)\le H_q$$

for $t\in[0,1]$.

Proof. Since the conclusions of Lemma 4.5 hold, the proof of this lemma is similar to that of Lemma 3.1. □

The next result is a direct consequence of Lemma 4.6 and the definition of the function $\overline{f}_n$.

Lemma 4.7. *Let* H1 *and* H2 *be satisfied. Then each* $C^2[0,1]$*-solution to the family* $(4.12)_\lambda$ *is also a solution to the family* $(3.1)_\lambda$, $\lambda\in[0,1]$.

Proof. Observe that, in view of Lemma 4.6, for each solution $x(t)\in C^2[0,1]$ to $(4.12)_\lambda$, we have

$$\Big(t,x(t),x'(t),x''(t)\Big)\in[0,1]\times[n^{-1},P_2+\varepsilon]\times[P_1,P_2]\times[h_q,H_q].$$

On the other hand, the definition of $\overline{f}_n$ implies that

$$\overline{f}_n(t,x,p,q)=f(t,x,p,q),\quad (t,x,p,q)\in[0,1]\times[n^{-1},P_2+\varepsilon]\times[P_1,P_2]\times[h_q,H_q],$$

from which the assertion of the lemma follows immediately. □

We conclude this section by proving the following important assertion.

Lemma 4.8. *Let* H1 *and* H2 *be satisfied. Then for each* $n \in \mathbb{N}$, $n > 1/\varepsilon$ *the problem* $(3.1)_\lambda$ *with* $\lambda = 1$ *has at least one solution in* $C^2[0,1]$.

Proof. Let n be fixed. Then, using Lemma 4.4, we conclude that the functions

$$F(\lambda, t, x, p, q) := \lambda \overline{f}_n(t, x, p, q) + (\lambda - 1)Kq, \qquad F_q(\lambda, t, x, p, q) = \lambda(\overline{f}_n)_q(t, x, p) + (\lambda - 1)K$$

are continuous for $(\lambda, t, x, p, q) \in [0,1]^2 \times R^3$ and that

$$F_q(\lambda, t, x, p, q) < 0, \quad (\lambda, t, x, p) \in [0,1]^2 \times R^3.$$

On the other hand, according to Lemma 4.5, we have

$$\overline{f}_n(t, x, p, H_q) + KH_q \le 0, \quad (t, x, p) \in [0,1] \times R^2,$$

and

$$\overline{f}_n(t, x, p, h_q) + Kh_q \ge 0, \quad (t, x, p) \in [0,1] \times R^2.$$

Thus, we see that $F < 0$ for $q = H_q$ and $F > 0$ for $q = h_q$. Thus, there is a unique function $V(\lambda, t, x, p) \in (h_q, H_q)$, which is continuous on the set $[0,1]^2 \times R^2$ and such that the equations

$$q = V(\lambda, t, x, p), \quad (\lambda, t, x, p) \in [0,1]^2 \times R^2,$$

and

$$F(\lambda, t, x, p, q) = 0, \quad (\lambda, t, x, p, q) \in [0,1]^2 \times R^3,$$

are equivalent. This means that for any $\lambda \in [0,1]$ the family $(4.12)_\lambda$ is equivalent to the family of BVPs

$$\begin{cases} x'' - (1-\lambda)(x' - b) = V(\lambda, t, x, x'), \quad t \in [0,1], \\ x(0) = \dfrac{1}{n}, \quad x'(1) = b. \end{cases} \tag{4.13$_\lambda$}$$

Note that $F(0, t, x, p, 0) = 0$ yields

$$V(0, t, x, p) = 0, \quad (t, x, p) \in [0,1] \times R^2. \tag{4.14}$$

Denote now $C^2_B[0,1] := \{x(t) \in C^2[0,1] : x(0) = 1/n,\ x'(1) = b\}$ and define the maps

$$j : C^2_B[0,1] \to C^1[0,1], \quad jx = x,$$

$$L_\lambda : C^2_B[0,1] \to C[0,1], \quad L_\lambda x = x'' - (1-\lambda)(x' - b), \quad \lambda \in [0,1],$$

and

$$V_\lambda :\ C^1[0,1] \to C[0,1], \quad (V_\lambda x)(t) = V(\lambda, t, x(t), x'(t)), \quad t \in [0,1], \quad \lambda \in [0,1].$$

Let us introduce the set

$$U = \left\{ x \in C^2_B[0,1] : \frac{1}{2n} < x < P_2 + \varepsilon,\ \ P_1 - \varepsilon < x' < P_2 + \varepsilon,\ \ h_q - \varepsilon < x'' < H_q + \varepsilon \right\},$$

which is a relatively open set in the convex set $C^2_B[0,1]$ of the Banach space $C^2[0,1]$. Since L_λ, $\lambda \in [0,1]$, is a continuous, linear, and one-to-one map of $C^2_B[0,1]$ onto $C[0,1]$, we conclude that L_λ^{-1} exists for each $\lambda \in [0,1]$ and is also a continuous map. In addition, V_λ is a continuous map, while the natural embedding j is a completely continuous map. Therefore, the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1]$$

defined by $H(x, \lambda) \equiv H_\lambda(x) \equiv L_\lambda^{-1} V_\lambda j(x)$ is a compact map. Moreover, the equations

$$L_\lambda^{-1} V_\lambda j(x) = x, \quad L_\lambda x = V_\lambda jx$$

are equivalent, i.e., the fixed points of $H_\lambda(x)$ are solutions to the family $(4.13)_\lambda$. Further, obverse that the solutions to $(4.13)_\lambda$ are not elements of ∂U, which means that $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Moreover, in view of (4.14), $H_0(x) = n^{-1} + bt$. Since $n^{-1} + bt \in U$, we can apply [9, Chapter I, Theorem 2.2] to conclude that H_0 is an essential map. By the topological transversality [9, Chapter I, Theorem 2.6], $H_1 = L_1^{-1} V_1 j$ is also an essential map. Consequently, the problem $(4.13)_\lambda$ with $\lambda = 1$ has $C^2[0,1]$-solutions, which are also solutions to the problem $(4.12)_\lambda$ with $\lambda = 1$. Finally, by Lemma 4.7, the solutions of the problem $(4.12)_\lambda$ with $\lambda = 1$ are also solutions to the problem $(3.1)_\lambda$ with $\lambda = 1$. □

5. Main Result

Using the results of the previous sections, we are ready to prove our main result, which is the following existence theorem.

Theorem 5.1. *Let* H1, H2 *and* H3 *be satisfied. Then problem* (1.1), (1.2) *has at least one solution* $x(t) \in C[0,1] \cap C^2(0,1]$ *with the property* $x(t) > 0$ *on* $(0,1]$.

Proof. Consider the sequence $\{x_n(t)\} \subset C^2[0,1]$, where $x_n(t)$, $n \in \mathbb{N}$, $n > 1/\varepsilon$, is a solution to $(3.1)_\lambda$ with $\lambda = 1$. Note that, by Lemma 4.8, the above sequence exists and, by Lemma 3.1, for $n \in \mathbb{N}$, $n > 1/\varepsilon$, the elements of this sequence satisfy the bounds

$$\frac{1}{n} \le x_n(t) \le P_2 + \varepsilon, \quad P_1 \le x_n'(t) \le P_2, \quad h_q \le x_n''(t) \le H_q, \quad t \in [0,1]. \tag{5.1}$$

Therefore, in view of H2 and H3, we conclude from the differential equation $(3.1)_1$ that for $t \in (0,1)$ and h small enough, we have

$$\begin{aligned} &\left[-f_q(t, x_n(t), x_n'(t), q_{nh}(t))\right] \left[x_n''(t+h) - x_n''(t)\right] \\ &\qquad = h f_t(T_{1h}) + f_x(T_{2h})[x_n(t+h) - x_n(t)] + f_p(T_{3h})[x_n'(t+h) - x_n'(t)] \\ &\qquad\qquad \to f_t(T_n) + f_x(T_n) x_n'(t) + f_p(T_n) x_n''(t) \quad \text{as} \quad h \to 0, \end{aligned} \tag{5.2}$$

where $T_n \equiv T_n(t, x_n(t), x_n'(t), x_n''(t))$ and the points T_{1h}, T_{2h}, T_{3h}, and $(t, x_n(t), x_n'(t), q_{nh}(t))$ tend to T_n. Because of (5.1), (5.2) and in view of H2 and H3, it follows that $x_n'''(t)$ exists for every $t \in [0,1]$, is given by the formula

$$x_n'''(t) = \{f_t(T_n) + f_x(T_n) x_n'(t) + f_p(T_n) x_n''(t)\} / \left[-f_q(T_n)\right], \tag{5.3}$$

and is continuous on $[0,1]$.

Next, integrating the inequality $P_1 \le x_n'(t) \le P_2$ from 0 to t with $t \in (0,1]$, we obtain

$$\frac{1}{n} + P_1 t \le x_n(t) \le \frac{1}{n} + P_2 t, \quad t \in [0,1]. \tag{5.4}$$

Let the constant α belong to $(0,1)$. Then, in view of (5.4),

$$x_n(t) \ge P_1 \alpha > 0, \quad t \in [\alpha, 1].$$

According to H3, using (5.1) and (5.3) we find that

$$|x_n'''(t)| \le \frac{|f_t| + |f_x||x_n'| + |f_p||x_n''|}{K_q} \le C_\alpha, \quad t \in [\alpha, 1],$$

where the constant C_α does not depend on n. Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{x_{n_l}\}_{l=1}^\infty$ converging uniformly on $C^2[\alpha,1]$ to some function $x \in C^2[\alpha,1]$, which is a solution of the differential equation (1.1) for $t \in [\alpha,1]$. The boundary condition $x'(1) = b$ is obviously satisfied. Thus, for $t \in (0,1]$, there exists a solution $x(t) \in C^2(0,1]$ of the differential equation (1.1), which satisfies the boundary condition $x'(1) = b$. Moreover, according to (5.4), we see that

$$0 < P_1 t \le x(t) \le P_2 t, \quad t \in (0,1), \tag{5.5}$$

and thus $x \in C[0,1]$ and $x(0) = 0$, which implies that $x(t)$ is a solution to the boundary-value problem (1.1), (1.2), for which, in view of (5.5), we have $x(t) > 0$ for every $t \in (0,1]$. □

6. Illustrative Examples

We conclude our investigation with the following examples, illustrating our main result.

Example 6.1. Consider the problem

$$\begin{cases} \exp\big((t-2)x''\big) + (x'-5)(x'-10) - 2x'' - \dfrac{x''}{\big(x(30-x)\big)^2} = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = 8. \end{cases}$$

It is easy to check that for $K = 1$, $Q = 15$, $P_1 = 7$, $P_2 = 11$, $P_3 = 6$, $P_4 = 12$, and for a sufficiently small $\varepsilon > 0$, the hypothesis H1 is satisfied. Hence, the hypothesis H2 is satisfied for $K_q = 2$. Moreover, $D_x \equiv D_x^0 \equiv (-\infty, 0) \cup (0, 30) \cup (30, \infty)$, $D_p \equiv D_q \equiv R$, $h_q = -16$, and $H_q = 18$. Obviously, the functions

$$f_t(t,x,p,q) = q\,\exp\big(q(t-2)\big), \quad f_x(t,x,p,q) = \frac{q(60-4x)}{\big(x(30-x)\big)^3}, \quad f_p(t,x,p,q) = 2p - 15$$

are continuous for $(t,x,p,q) \in [0,1] \times (0,12] \times [7,11] \times [-16,18]$. Therefore, the hypothesis H3 is fulfilled and, by Theorem 5.1, the problem considered admits a $C[0,1] \cap C^2(0,1]$-solution.

Example 6.2. Consider the problem

$$\begin{cases} \sqrt{225-(x')^2}\,\sin x' - \dfrac{x''}{\sqrt{400-(x'')^2}\sqrt{x(625-x^2)}} - (x'')^3 - 0.5x'' = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = 5. \end{cases}$$

Here $D_p = [-15, 15]$ and $D_q = (-20, 20)$. Since $x(0) = 0$, we will investigate this problem only for $D_x^0 = (0, 25)$. Clearly, the function

$$f(t,x,p,q) = \sqrt{225-p^2}\,\sin p - \frac{q}{\sqrt{400-q^2}\sqrt{x(625-x^2)}} - q^3 - 0.5q$$

is singular at $x = 0$ and satisfies the hypothesis H1 for $K = 0.5$, $Q = 10$, $P_1 = 4$, $P_2 = 7$, $P_3 = 3.5$, $P_4 = 7.5$, and a sufficiently small $\varepsilon > 0$. The functions

$$f(t,x,p,q), \quad f_q(t,x,p,q) = -\frac{1}{\sqrt{x(625-x^2)}}\frac{400}{\sqrt{(400-q^2)^2}} - 3q^3 - 0.5$$

are continuous on $\Omega \equiv [0,1] \times (0, 8+\varepsilon] \times [4-\varepsilon, 7+\varepsilon] \times [-11-\varepsilon, 12+\varepsilon]$. Moreover, $f_q(t,x,p,q) < -0.5 - 1/1500$ for $(t,x,p,q) \in \Omega$. Thus, H2 is satisfied for $K_q = 0.5 + 1/1500$. Now observe that the functions

$$f_t(t,x,p,q) = 0, \quad f_x(t,x,p,q) = \frac{q}{2\sqrt{400-q^2}}\frac{625-3x^2}{\sqrt{\big(x(625-x^2)\big)^3}}$$

and

$$f_p(t,x,p,q) = \cos p\sqrt{225-p^2}\,\cos p - \frac{p}{\sqrt{225-p^2}}\sin p$$

are continuous on the set $[0,1] \times (0,8] \times [4,7] \times [-11,12]$. This means that H3 is also satisfied. Consequently, by Theorem 5.1, the problem considered has a $C[0,1] \cap C^2(0,1]$-solution.

Example 6.3. Consider the boundary-value problem

$$\begin{cases} f(t,x,x',x'') = 0, \qquad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = 5, \end{cases}$$

where

$$f(t,x,p,q) = \begin{cases} p + e^{-q} - (2+t)q - 6, & (t,x,p,q) \in [0,1] \times [0,\infty) \times R^2, \\ -q(x^{-2}+1), & (t,x,p,q) \in [0,1] \times (-\infty,0) \times R^2. \end{cases}$$

It is easy to check that for $K = 1,\ Q = 10,\ P_1 = 4,\ P_2 = 7,\ P_3 = 3,\ P_4 = 8$, and a sufficiently small $\varepsilon > 0$, the hypothesis H1 is satisfied. Note also that the functions

$$f(t,x,p,q) = p + e^{-q} - (2+t)q - 6, \qquad f_q(t,x,p,q) = -e^{-q} - (2+t)$$

are continuous on the set $\Omega \equiv [0,1] \times (0, 8+\varepsilon] \times [4-\varepsilon, 7+\varepsilon] \times [-11-\varepsilon, 12+\varepsilon]$ and that $f_q(t,x,p,q) < -2$ for $(t,x,p,q) \in \Omega$. Thus, the hypothesis H2 is fulfilled for $K_q = 2$. Observe now that

$$f_t(t,x,p,q) = -q, \quad f_x(t,x,p,q) = 0, \quad f_p(t,x,p,q) = 1$$

to conclude that H3 is satisfied. Thus, by Theorem 5.1, the above problem has a $C[0,1] \cap C^2(0,1]$-solution.

Example 6.4. Consider the problem

$$\begin{cases} f(t,x,x',x'') = 0, & 0 < t < 1, \\ x(0) = 0, \quad x'(1) = 5, \end{cases}$$

where

$$f(t,x,p,q) = \begin{cases} \sqrt{225-p^2}\sin p - \dfrac{q^3}{\sqrt{400-q^2}}\sqrt{\dfrac{30-x}{x}} - 0.5q, \\ \qquad\qquad (t,x,p,q) \in [0,1] \times (0,30] \times [-15,15] \times (-20,20), \\ \sqrt{225-p^2}\sin p - \dfrac{q}{\sqrt{400-q^2}}\dfrac{1}{\sqrt{x(x^2-900)}} - q, \\ \qquad\qquad (t,x,p,q) \in [0,1] \times [-30,0) \times [-15,15] \times (-20,20). \end{cases}$$

The function $f(t,x,p,q)$ satisfies the hypothesis H1 for $K = 0.4$, $Q = 10$, $P_1 = 4$, $P_2 = 7$, $P_3 = 3.5$, $P_4 = 8$, and some sufficiently small $\varepsilon > 0$. Note that the functions

$$f(t,x,p,q) = \sqrt{225-p^2}\sin p - \frac{q^3}{\sqrt{400-q^2}}\sqrt{\frac{30-x}{x}} - 0.5q$$

and $f_q(t,x,p,q)$ are continuous on the set $\Omega \equiv [0,1] \times (0, 8+\varepsilon] \times [4-\varepsilon, 7+\varepsilon] \times [-11-\varepsilon, 12+\varepsilon]$ and $f_q(t,x,p,q) \le -0.5$ for $(t,x,p,q) \in \Omega$. Thus, the hypothesis H2 is fulfilled for $K_q = 0.5$. Further, observe that the functions

$$f_t(t,x,p,q), \quad f_x(t,x,p,q), \quad f_p(t,x,p,q)$$

are continuous on the set $[0,1] \times (0,8] \times [4,7] \times [-11,12]$. Hence, the hypothesis H3 is also satisfied. Therefore, in view of Theorem 5.1, we see that the above problem has a $C[0,1] \cap C^2(0,1]$-solution.

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