*Journal of Mathematical Sciences, Vol. 147, No. 4, 2007*

# SMALL DEVIATION PROBABILITIES FOR SUMS OF INDEPENDENT POSITIVE RANDOM VARIABLES

**L. V. Rozovsky*** UDC 519.21

*In this note, we give estimates of small deviation probabilities of the sum $\sum_{j\ge1}\lambda_j X_j$, where $\{\lambda_j\}$ are nonnegative numbers and $\{X_j\}$ are i.i.d. positive random variables that satisfy mild assumptions at zero and infinity. Bibliography:* 10 *titles.*

## 1. Introduction

In this paper, we prove results announced in [1].

Let $S=\sum\limits_{j\ge1}\lambda_j X_j$, where $\lambda_1\ge\lambda_2\ge\cdots$ are nonnegative numbers such that

$$1\le n=\#\{j|\lambda_j>0\}\le\infty \tag{1.1}$$

(if $n=\infty$, we assume that the series $S$ converges a.s.) and $\{X_i\}$ are independent copies of a positive random variable $X$ with distribution function $V(x)$ that satisfies the following condition:
there exist constants $b\in(0,1)$, $c_1, c_2>1$, and $r_0>0$ such that

$$c_1 V(br)\le V(r)\le c_2 V(br) \tag{$L$}$$

for any $r\le r_0$. Here and below, $c_1, c_2,\dots$ denote positive constants depending only on the distribution $V$ and on parameters from conditions $(L)$, (1.2), etc which are connected with $V$.

Condition $(L)$ introduced in [2] is obviously satisfied if the function $V(r)$ is regularly varying at zero. In particular, this is so in the important Gaussian case, i.e., when $X=|\xi|^p$, $p>0$, where $\xi$ is the standard normal variable.

Condition $(L)$ also implies that

$$\begin{gathered} c_1u^\beta\ge V(ur)/V(r)\ge u^\alpha/c_2,\ u\le1,\ r\le r_0,\\ u^\beta/c_1\le V(ur)/V(r)\le c_2u^\alpha,\ u\ge1,\ r\le r_0,\end{gathered} \tag{1.2}$$

$$c_3r^\alpha\le V(r)\le c_4r^\beta,\ r\le r_0, \tag{1.3}$$

for some constants $\alpha,\beta$ such that $\log c_1/|\log b|\le\beta\le\alpha\le\log c_2/|\log b|$.

We are mostly interested in the behavior of the probabilities $\mathbf{P}\,(r-s<S\le r)$ for $r$, $s>0$. Similar problems (mostly, for the Gaussian case) were considered by many authors (see references in [2] and [3]). The most general and exact statements (in the context of the present work) were obtained in [2]. Let us formulate the basic result of this note. For this purpose, we introduce some notation.

For $\gamma\ge0$, set

$$\begin{gathered}\Lambda(\gamma)=\mathbf{E}\,e^{-\gamma S},\quad m(\gamma)=-\left(\log\Lambda(\gamma)\right)',\quad \sigma^2(\gamma)=\left(\log\Lambda(\gamma)\right)'',\\ Q(\gamma)=-\gamma m(\gamma)-\log\Lambda(\gamma).\end{gathered} \tag{1.4}$$

**Theorem 1.1.** *Assume that a random variable $X$ has*
*(a) finite variance*
*and*
*(b) absolutely continuous distribution*
*and satisfies condition $(L)$. If $\{\lambda_j\}$ is a nonincreasing positive sequence such that $\sum\limits_{j\ge1}\lambda_j<\infty$, then*

$$\mathbf{P}\,(S\le r)=(2\pi)^{-1/2}\,(\gamma\sigma(\gamma))^{-1}\,e^{-Q(\gamma)}\left(1+o(1)\right),\ r\searrow0,$$

*Department of Mathematics, Chemical-Pharmaceutical Academy, St.Petersburg, Russia, e-mail: l_rozovsky@mail.ru.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 341, 2007, pp. 151–167. Original article submitted November 1, 2006.

 

*where the parameter $\gamma = \gamma(r)$ is a solution of the equation $m(\gamma) = r$.*
*Moreover, if $\mathbf{E}\, X^3 < \infty$ and $\alpha$ satisfies condition* (1.3), *then*

$$\mathbf{P}\,(S \le r) = (2\pi)^{-1/2}\,(\gamma\sigma(\gamma))^{-1}\, e^{-Q(\gamma)}\,\Big(1 + O\big((\gamma\sigma(\gamma))^{-1} + (\gamma\sigma(\gamma))^{-(2-\kappa)/\alpha}\big)\Big),\ r \searrow 0, \tag{1.5}$$

for any $\kappa \in (0, 2)$.

Note that assumptions (a) and (b) of Theorem 1.1 are essentially used in the proofs.

Our main purpose is to show that relations of the type (1.5) are still valid without assumption (b) and under an essential relaxation of condition (a).

## 2. Results

First we state an assertion which holds under condition $(L)$ only.

**Theorem 2.1.** *Assume that the distribution function $V$ satisfies condition $(L)$. Then*

$$e^{-Q(\gamma)} \ge\ \mathbf{P}\,(S \le r) \ge e^{-Q(\gamma) - c_5\,\sqrt{1+Q(\gamma)}},\quad 0 < r <\ \mathbf{E}\, S, \tag{2.1}$$

*where $\gamma$ is the unique solution of the equation $m(\gamma) = r$.*

**Remark 1.** Since the function $m(u)$ monotonically decreases on $(0, \infty)$, $m(0) = \mathbf{E}\, S \le \infty$, and $m(\infty) = 0$, relation (2.1) is equivalent to the following relations:

$$e^{-Q(\gamma)} \ge\ \mathbf{P}\,(S \le m(\gamma)) \ge e^{-Q(\gamma) - c_5\,\sqrt{1+Q(\gamma)}},\quad 0 < \gamma < \infty.$$

Relation (2.1) allows us to find asymptotics of the logarithm of the probability $\mathbf{P}\,\big(S \le r\big)$.

To formulate a more sharp result, we require an additional information on the behavior of $V$ at infinity.

Introduce the following condition:

$$\limsup_{r\to\infty} \frac{r^2\ \mathbf{P}\,(X \ge r)}{\mathbf{E}\, X^2\, \mathbf{I}[X < r]} < \infty. \tag{$F$}$$

This condition, the so-called condition of Feller's stochastic compactness, is satisfied if $X$ belongs to the domain of attraction of a stable law, including the normal case. It is also possible to show that condition $(F)$ is equivalent to the following condition: there exists $\omega \in (0, 1)$ such that

$$r^{-\omega}\, \mathbf{E}\,\big(1 \wedge r\, X^2\big) \nearrow, r > 0, \tag{2.2}$$

which implies, in particular, that there exists $\delta > 0$ such that $\mathbf{E}\, X^\delta < \infty$.

In fact, take $\xi \in (0, 2)$ and $g(x) = x^{-\xi}\, \mathbf{E}\,\big(x^2 \wedge X^2\big)$, $x \ge x_0$. In this case, $g(x) = x^{-\xi} \int_0^x \bar{V}(u) du^2$, where $\bar{V}(u) = 1 - V(u)$. Next,

$$g'(x) = -\xi\, g(x)/x + 2x^{1-\xi}\bar{V}(x) = x^{-1-\xi}(2\, x^2\bar{V}(x) - \xi\, x^2\bar{V}(x) - \xi\ \mathbf{E}\, X^2\, \mathbf{I}[X \le x]).$$

We note that $g'(x) < 0$ if and only if $\frac{x^2\,\bar{V}(x)}{\mathbf{E}\, X^2\,\mathbf{I}[X<x]} < \xi/(2-\xi)$, which proves the equivalence of conditions $(F)$ and (2.2).

**Theorem 2.2.** *Let the distribution function $V$ satisfy conditions $(L)$ and $(F)$. Then*

$$\mathbf{P}\,(r - s < S \le r) = e^{-Q(\gamma)}\,\frac{1 - e^{-\gamma\, s}}{\tau\sqrt{2\pi}}\,\left(1 + \theta\,\left(\tau^{-1} + \left(\frac{\log\,(1+\tau)}{\tau^2}\right)^{1/\alpha}\,\big(1 + (\gamma s)^{-1}\big)\right)\right) \tag{2.3}$$

*for all $r$, $0 < r <\ \mathbf{E}\, S$, and all positive $s$, where $\gamma$ is a solution of the equation $m(\gamma) = r$, $\tau = \gamma\,\sigma(\gamma)$(see* (1.4)), *the constant $\alpha$ is taken from conditions* (1.3), *and $|\theta| \le c_6$.*

Theorem 2.2 is a corollary of the following more general statement.

**Theorem 2.3.** *Let conditions* $(L)$ *and* $(F)$ *be satisfied. Then*

$$\mathbf{P}\,(r - s < S \le r) = \Lambda(\gamma)\, e^{\gamma\, r}\, \frac{1 - e^{-\gamma\, s}}{\tau\sqrt{2\pi}} \tag{2.4}$$

$$\left(e^{-(r-m(\gamma))^2/2\sigma^2(\gamma)} + \theta\left(\tau^{-1} + \left(\frac{\log(1+\tau)}{\tau^2}\right)^{1/\alpha}\left(1 + (\gamma s)^{-1}\right)\right)\right)$$

*for all positive* $r$, $s$ *and* $\gamma$, *where* $\tau = \gamma\,\sigma(\gamma)$ *and* $|\theta| \le c_7$.

**Remark 2.** If $n < \infty$ (see (1.1)), then, under the conditions of Theorems 2.2 and 2.3, relations (2.3) for $0 < r < m(\delta/\lambda_n)$ and (2.4) for $\gamma > \delta/\lambda_n$ remain valid for $|\theta| \le c(V, \delta)$ and for any positive $\delta$ without condition $(F)$.

Next, we formulate a local version of Theorem 2.3.

By analogy with [2, (3.6) and (3.7)], let us assume that a random variable $X$ has an absolutely continuous distribution with density $f$ such that

$$|df(x)| \le C\, V(x)\, x^{-2}\, dx,\ \ 0 < x \le x_0,\ \ dx > 0, \tag{2.5}$$

and

$$\int_{x_0}^{\infty} \left(e^{-\delta x} f(x)\right)^p dx < \infty \tag{2.6}$$

for some positive $C, x_0, \delta$, and $p > 1$.

**Theorem 2.4.** *Let the distribution function* $V$ *satisfy conditions* $(L)$ *and* $(F)$ *and let its density* $f$ *satisfy conditions* (2.5) *and* (2.6). *Assume also that* $n \ge n_0$, *where* $n_0$ *is an integer such that* $n_0 > (1+\alpha)(1 \vee 1/\beta)$ *and* $n_0 \ge 2 \vee p/(p-1)$ *(see the notation in* (1.1), (1.2), *and* (2.6)*). Then*

$$\mathbf{P}\,(r - s < S \le r) = \Lambda(\gamma)\, e^{\gamma\, r}\, \frac{1 - e^{-\gamma\, s}}{\gamma\,\sigma(\gamma)\sqrt{2\pi}} \left(e^{-(r-m(\gamma))^2/2\sigma^2(\gamma)} + \theta\,(\gamma\,\sigma(\gamma))^{-1}\right) \tag{2.7}$$

*with*

$$|\theta| \le c_8 \left(1 + \lambda_{n_0}^{-1} \sum_{j \ge 1} \mathbf{E}\,(1 \wedge \lambda_j\, X)\right) \tag{2.8}$$

*for all positive* $r$ *and* $s$ *and all* $\gamma \ge \delta/\lambda_{n_0}$. *In particular, if* $0 < r \le m(\delta/\lambda_{n_0})$ *and* $m(\gamma) = r$, *then*

$$\mathbf{P}\,(r - s < S \le r) = e^{-Q(\gamma)}\, \frac{1 - e^{-\gamma\, s}}{\tau\sqrt{2\pi}} \left(1 + \theta\,(\gamma\,\sigma(\gamma))^{-1}\right).$$

We note that, in constrast to (2.4), equality (2.7) is nontrivial for arbitrarily small values of parameter $s$ as well. Thus, one can divide both parts of (2.7) by $s$ and, letting $s$ tend to zero, find an appropriate asymptotic of the density $q(r)$ of the random variable $S$.

Thus, under the conditions of Theorem 2.4,

$$q(r) = \left(\sigma(\gamma)\sqrt{2\pi}\right)^{-1} \Lambda(\gamma)\, e^{\gamma\, r} \left(e^{-(r-m(\gamma))^2/2\sigma^2(\gamma)} + \theta\,(\gamma\,\sigma(\gamma))^{-1}\right);$$

in addition, if $0 < r \le m(\delta/\lambda_{n_0})$ and $m(\gamma) = r$, then

$$q(r) = \left(\sigma(\gamma)\sqrt{2\pi}\right)^{-1} e^{-Q(\gamma)} \left(1 + \theta\,(\gamma\,\sigma(\gamma))^{-1}\right),$$

where $\theta$ satisfies condition (2.8).

**Remark 3.** If we replace condition (2.6) in the conditions of Theorem 2.4 by a more restrictive assumption that

$$\int_{x_0}^{\infty} e^{-\delta x} |d(x)) < \infty \tag{2.9}$$

(and omit the assumption that $n_0 \ge 2 \vee p/(p-1)$), then relation (2.7) holds for $|\theta| \le c_8$.

Recall that the constants $c_6, \ldots c_8, \ldots$ depend only on the distribution $V$ and on parameters which are connected with the distribution (including conditions (2.5) and (2.6)).

**Remark 4.** If $n_0 = n < \infty$ in Theorem 2.4, then its statement remains valid without condition $(F)$.

## 3. Corollaries

Let $n = \infty$ in (1.1), i.e., let $\{\lambda_j\}$ be a nonincreasing sequence of *positive* numbers such that the series $S$ converges, or (by the three series theorem)

$$\sum_{j\geq 1} \mathbf{E}\,(1 \wedge \lambda_j X) < \infty. \tag{3.1}$$

Note that if $\mathbf{E}\,X < \infty$, then condition (3.1) implies the convergence of the series $\sum_j \lambda_j$ and vice versa. In addition, condition (3.1) imposes some moment restrictions on the random variable $X$. For example, if $\lambda_j = j^{-\omega}$, $\omega > 1$, then (3.1) $\iff$ $\mathbf{E}\,X^{1/\omega} < \infty$, and if $\lambda_j = q^j$, $0 < q < 1$, then (3.1) $\iff$ $\mathbf{E}\,\log(1+X) < \infty$, and the last condition follows from $(F)$.

**Theorem 3.1.** *If condition* $(L)$ *holds, then*

$$-\log \mathbf{P}\,(S \leq r) \sim Q(\gamma),\ r \to 0, \tag{3.2}$$

*where* $m(\gamma) = r$, *or, equivalently,*

$$-\log \mathbf{P}\,(S \leq m(\gamma)) \sim Q(\gamma),\ \gamma \to \infty.$$

*If conditions* $(L)$ *and* $(F)$ *are satisfied, then*

$$\mathbf{P}\,(S \leq r) = \Lambda(\gamma) e^{\gamma\, r}\, (\tau\sqrt{2\pi})^{-1} \left(e^{-((r-m(\gamma))/\sigma(\gamma))^2/2} + O\,(1/\tau + (1/\tau)^{2/\alpha} \log o f^{1/\alpha} \tau)\right),\ \gamma \to \infty, \tag{3.3}$$

*uniformly in* $r$. *In particular, if* $m(\gamma) = r$, *then*

$$\mathbf{P}\,(S \leq r) = e^{-Q(\gamma)}\, (\tau\sqrt{2\pi})^{-1} \left(1 + O\,(1/\tau + (1/\tau)^{2/\alpha} \log^{1/\alpha} \tau\right),\ r \to 0. \tag{3.4}$$

Here we use the same notation as in Theorems 2.1–2.3, and the case $s = \infty$ is considered for simplicity.

**Theorem 3.2.** *Let the distribution function* $V$ *satisfy conditions* $(L)$ *and* $(F)$ *and let its density* $f$ *satisfy conditions* (2.5) *and* (2.6). *Then*

$$Q(r) = \left(\sigma(\gamma)\sqrt{2\pi}\right)^{-1} e^{-Q(\gamma)} \left(1 + O\left((\gamma\,\sigma(\gamma))^{-1}\right)\right),\ r \to 0, \tag{3.5}$$

*where* $m(\gamma) = r$.

Theorems 3.1 and 3.2 follow from Theorems 2.1–2.4 (see (4.4a), (4.4b), and the proof of relation (4.8) in [2]) since $\tau = \gamma\sigma(\gamma) \to \infty$ as $\gamma$ grows, and, in addition, $Q(\gamma) \nearrow \infty$ and $m(\gamma) \searrow 0$ if and only if $\gamma \nearrow \infty$.

Note that relations (3.3)–(3.5) generalize and refine the coresponding results of [2, Theorems 2 and 3 and (6.19)].

Concerning the case of finite $n$, we only point out that Theorems 1 and 2 of [6] follow from Theorems 2.2 and 2.3 and Remark 2 of the present paper.

**Remark.** In [7] and [8], conditions for the validity of relation (3.2) were independently obtained under much weaker, compared to $(L)$, restrictions on the behavior of the distribution $V$ at zero. Moreover, the right-hand side of asymptotic (3.2) in these papers is given in an explicit form. But the authors of the above-mentioned papers examined cases of *polynomial* (and some other) weights, while our result holds for *any* admissible $\{\lambda_j\}$. Within the framework of the present paper, we prove the following statement.

**Theorem 3.3.** *Assume that*

$$\lim_{\gamma\to\infty} \sup_{u\geq\gamma} u\,\sigma(u)/Q(u) = 0. \tag{3.6}$$

*Then relation* (3.2) *is valid.*

Note that condition (3.6) follows from condition $(L)$; the former condition is also carried out if the coefficients $\lambda_j$ are roughly equivalent to $\lambda(j)$ as $j \to \infty$, where a function $\lambda(x)$ is regularly varying at infinity with index not exceeding $-1$ (see (3.1)), and $\lim_{u\to\infty} |\log V(\lambda(u))|/u = 0$.

## 4. Lemmas

**Lemma 1.** *Let $\gamma$ be the unique solution of the equation $m(\gamma) = r$, $0 < r < \mathbf{E}\, S$. Then*

$$e^{-Q(\gamma)} \geq \mathbf{P}\,(S \leq r) \geq \tfrac{1}{2}\, e^{-2a\,(1+\sqrt{1+2Q(\gamma)/a})}\; e^{-Q(\gamma)}$$

*and*

$$\mathbf{P}\,(S \leq r) \geq \tfrac{1}{2}\, e^{-Q(\gamma)(1+b\sqrt{2})/(1-b\sqrt{2})^{+}}, \tag{4.1}$$

*where* $a = \sup\limits_{u \geq \gamma} \frac{u^2\,\sigma^2(u)}{Q(u)}$ *and* $b = \sup\limits_{u \geq \gamma} \frac{u\,\sigma(u)}{Q(u)}$.

Lemma 1 can be proved similarly to Lemma 1 of [4] (see also [5]).

Assume that a random variable $S(h)$, $h \geq 0$, has distribution

$$\mathbf{P}\,(S(h) \leq r) = \int_0^r e^{-hy}\, d\,\mathbf{P}\,(S \leq y)/\Lambda(h),\; r \geq 0.$$

Note (see (1.4)) that $m(h) = \mathbf{E}\, S(h)$ and $\sigma^2(h) = \mathbf{Var}\, S(h)$.

Denote

$$G_h(t) = \mathbf{E}\, \exp\left(it\, \frac{S(h) - m(h)}{\sigma(h)}\right),\; \tau = h\sigma(h)\; (h > 0),$$

$$\delta_\varepsilon(h) = \int\limits_0^{1/\varepsilon} |g_h(t) - e^{-t^2/2}|\, dt\; (\varepsilon > 0). \tag{4.2}$$

**Lemma 2.** *For any positive $r$, $h$, $s$, and $\varepsilon$,*

$$\mathbf{P}\,(r - s < S \leq r) = \Lambda(h) e^{h\,r}\, \frac{1 - e^{-h\,s}}{\tau\sqrt{2\pi}}\, \left(e^{-\beta^2/2} + \theta\,(\beta\, e^{-\beta^2/2}/\tau + 1/\tau^2 + \rho_\varepsilon(h,s))\right).$$

*Here*

$$\beta = \frac{r - m(h)}{\sigma(h)},\; \rho_\varepsilon(h,s) = \delta_\varepsilon(h) + (1 + \delta_\varepsilon(h))(1 + \tfrac{1}{h\,s})\,\tau\varepsilon,$$

*and $|\theta| \leq c$, where $c$ is an absolute constant. In particular, if $m(h) = r$, then $\beta = 0$ and*

$$\mathbf{P}\,(r - s < S \leq r) = e^{-Q(h)}\, \frac{1 - e^{-h\,s}}{\tau\sqrt{2\pi}}\, \left(1 + \theta\,(1/\tau^2 + \rho_\varepsilon(h,s))\right).$$

Lemma 2 is a special case of Lemma 2 of [4]. We point out that we do not assume conditions $(L)$ or $(F)$ to be fulfilled in Lemmas 1 and 2.

Let a random variable $X(h)$, $h \geq 0$, have distribution $E^{-hr}\, V(dr)/L(h)$, where $L(h) = \mathbf{E}\, e^{-hX}$. Denote $G(h) = \mathbf{E}\left(1 \wedge (h\,X)^2\right)$ and $G_1(h) = \mathbf{E}\,(1 \wedge h\,X)$.

**Lemma 3.** (1) *If condition $(F)$ holds, then*

$$c_9 \leq h\,\mathbf{E}\, X(h)/G_1(h) \leq c_{10}, \tag{4.3a}$$

$$c_9 \leq h^2 \mathbf{Var} X(h)/G(h) \leq c_{10}, \tag{4.3b}$$

*and*

$$h\,\mathbf{E}\, X^3(h)/\mathbf{Var} X(h) \leq c_{11} \tag{4.3c}$$

*for all $0 < h \leq 1$.*

(2) *If condition $(L)$ holds, then*

$$c_9 \leq h\,\mathbf{E}\, X(h) \leq c_{10}, \tag{4.4a}$$

$$c_9 \leq h^2 \mathbf{Var} X(h) \leq c_{10}, \tag{4.4b}$$

*and*

$$h^3\,\mathbf{E}\, X^3(h) \leq c_{11} \tag{4.4c}$$

*for all $h \geq 1$.*

The first statement of Lemma 3 was established in [9, Lemma 4.1] and can be proved by the same reasoning; the second statement coincides with [6, Lemma 1].

**Lemma 4.** *Denote* $f_h(v) = \mathbf{E}\, e^{iv\,X(h)}$. *If condition* $(L)$ *holds, then*

$$1 - |f_h(v)) \ge c_{15} e^{-(2\pi+b)\,h/v}\, V(1/v)/V(1/h),\ V \ge (2\pi+b)/r_0,\ h > 0. \tag{4.5}$$

*If condition* $(F)$ *holds, then*

$$|f_h(v)| \le e^{-\delta G(h)\,(v/h)^\omega},\ \varepsilon h \le v \le v_0, \tag{4.6}$$

*for any sufficiently small positive* $v_0$ *and for any* $\varepsilon \in (0,1)$*, where* $\delta = \delta(V, \varepsilon, v_0) > 0$ *and* $\omega = \omega(V) \in (0,2)$.

*Proof of Lemma 4.* Relation (4.5) was obtained in [6, Lemma 1]. Let us check estimate (4.6). We assume that $\mathbf{E}\, X^2 = \infty$ (the opposite case is considered similarly with simplifications). Obviously,

$$L(h)\,|f_h(v) - 1| \le |\int\limits_{1/v}^{\infty} e^{-hy}(e^{ivy} - 1)\, V(dy)| + |\int\limits_{0}^{1/v} e^{-hy}(e^{ivy} - 1)\, V(dy)| = I + J,$$

$$I \le 2\int_{1/v}^{\infty} e^{-hy}\, V(dy) \le 2\,(1 - V(1/v)),$$

and

$$J = |\int_{0\le y<1/v} (e^{ivy} - 1 - ivy)\, V(dy) + \int\limits_{0\le y<1/v} ivy\, V(dy)$$

$$+ \int\limits_{0\le y<1/v} (e^{hy} - 1)(e^{ivy} - 1)\, V(dy)|$$

$$\le 0.5\, v^2 \int_{0\le y<1/v} y^2\, V(dy) + v\int_{0\le y<1/v} y\, V(dy) + vh\, e^{h/v}\int_{0\le y<1/v} y^2\, V(dy).$$

The reasoning above implies that

$$|f_h(v) - 1| \le c_{12} e^{1/\varepsilon}/\varepsilon\,(G(v) + v\,\mathbf{E}\, X\mathbf{I}[0 \le X \le 1/v]) \tag{4.7}$$

for any positive $v_0$ that is small enough and for all $h$ and $v$ such that $\varepsilon h \le v \le v_0$. Now, if condition $(F)$ holds, then

$$L(h)\,\mathbf{Re}(f_h(v) - 1) = \int\limits_{0}^{\infty}(\cos vy - 1)e^{-hy}\, V(dy) \le \int\limits_{0\le y<1/v} \frac{\cos vy - 1}{(vy)^2}(vy)^2 e^{-hy}\, V(dy)$$

$$\le (\cos 1 - 1)e^{-h/v} v^2 \int\limits_{0\le y<1/v} y^2\, V(dy) \le -c_{13} e^{-1/\varepsilon} G(v) \tag{4.8}$$

under the same restrictions on $h$ and $v$ (we use condition $(F)$ in the last inequality only).

Since

$$\mathbf{E}^2 X\mathbf{I}[|X| \le z] = o(z^2 G(1/z)),\ z \to \infty,$$

if $\mathbf{E}\, X^2 = \infty$ (see [9, (4.9)]), we deduce from inequalities (4.7) and (4.8) and the estimate

$$L(h) \ge \int_0^{1/h} e^{-th} V(dt) \ge e^{-1} V(1/h)$$

that

$$|f_h(v)| \le e^{\mathbf{Re}(f_h(v)-1) + |f_h(v)-1|^2} \le e^{-\delta_1 G(v)},\ \varepsilon h \le v \le v_0, \tag{4.9}$$

for $v_0$ small enough.

Further, under condition $(F)$ (see (2.2)) there exists an $\omega \in (0,2)$ such that $v^{-\omega} G(v) \nearrow$ for all sufficiently small $v$. Thus, if $\varepsilon h \le v$ and $\varepsilon \in (0,1)$, then

$$G(v) = v^\omega v^{-\omega} G(v) \ge v^\omega (\varepsilon h)^{-\omega} G(\varepsilon h) \ge \varepsilon^{2-\omega} v^\omega h^{-\omega} G(h).$$

These inequalities and (4.9) imply estimate (4.6).

Lemma 4 is proved.

**Lemma 5.** *If condition* $(L)$ *holds, then (see the notation in* (1.4)*)*

$$Q(u) \ge c_{14} u^2 \sigma^2(u)$$

*and*

$$u m(u) \ge c_{15} u^2 \sigma^2(u) \tag{4.10}$$

*for any positive* $u$.

*Proof of Lemma 5.* Set $\bar{Q}(u) = -u\,\mathbf{E}\,X(u) - \log L(u)$ and $\sigma_j^2(u) = \lambda_j^2\,\mathbf{Var}\,X(u\lambda_j)$, $j \ge 1$.

Let $k$ satisfy the condition $u\lambda_k > 1 \ge u\lambda_{k+1}$. Then

$$Q(u) = \sum_{j=1}^{k} \bar{Q}(u\lambda_j) + \sum_{j>k} \bar{Q}(u\lambda_j) = Q_1 + Q_2 \tag{4.11}$$

and (see (4.4b))

$$Q_1 \ge k\,\bar{Q}(u\lambda_k) \ge k\,\bar{Q}(1) \ge c_{16} \sum_{j=1}^{k} \sigma_j^2(u). \tag{4.12}$$

Further, by [9, Lemma 2.2] (without assumption $(L)$),

$$\bar{Q}(\gamma) \ge c_{17}\gamma^2 G(1/\gamma) \ge c_{18}\gamma^2\,\mathbf{Var}\,X(\gamma),\ \ 0 < \gamma \le 1;$$

hence,

$$Q_2 \ge c_{19} u^2 \sum_{j>k} \sigma_j^2(u). \tag{4.13}$$

The first statement of Lemma 5 follows from inequalities (4.11)–(4.13). Relation (4.10) is checked similarly. It is enough to note (see Lemma 3) that $u\,G_1(1/u) \ge u^2\,G(1/u)$.

Lemma 5 is proved.

**Lemma 6.** *Let conditions* (L) *and* (F) *be satisfied. Then (see* (4.2)*)*

$$\int_0^{\tau\rho} |g_h(t) - e^{-t^2/2}|\,dt \le c_{20}\,(1 \wedge 1/\tau) \tag{4.14}$$

*for any* $h > 0$*, where* $\rho = \delta_0\,(\tau/\sqrt{\log(1+\tau)}\,)^{2/\alpha}$*,* $\delta_0 = \delta_0(V)$ *is a sufficiently small positive constant depending on* $V$ *only, and* $\alpha$ *is defined in* (1.2) *and* (1.3).

**Remark.** Assume that $n < \infty$ in condition (3.1). If $h > \delta/\lambda_n$ for some positive $\delta$, then inequality (4.14) is valid without assumption $(F)$ with constants $c_{20}$ and $\delta_0$ depending, in addition, on $\delta$.

*Proof of Lemma 6.* Estimate (4.14) for $\tau \le \tau_0$ is obvious. Thus, in what follows, we assume that $\tau$ is large enough and $\rho > 1$.

It is known (see [10, Chap. 5, Lemma 1]) that

$$|g_h(t) - e^{-t^2/2}| \le 16\mu e^{-t^2/3},\ \ |t| \le 1/4\mu,$$

where

$$\mu = \frac{1}{\sigma^3(h)} \sum_{j\ge 1} \lambda_j^3\,\mathbf{E}\,|X(h\lambda_j) - \mathbf{E}\,X(h\lambda_j)|^3.$$

By Lemma 3 (see (4.3c) and (4.4c)),

$$\mu \le \frac{8}{h\sigma^3(h)} \sum_{j\ge 1} \sigma_j^2(h) \frac{h\lambda_j\,\mathbf{E}\,X^3(h\lambda_j)}{\mathbf{Var}\,X(h\lambda_j)} \le \frac{8}{\tau} \sup_{\gamma \ge h\lambda_n} \frac{\gamma\,\mathbf{E}\,X^3(\gamma)}{\mathbf{Var}\,X(\gamma)} \le 8c_{11}/\tau. \tag{4.15}$$

For $\varepsilon = (32\,c_{11})^{-1}$, inequalities (4.15) imply that

$$\int\limits_0^{\varepsilon\,\tau} |g_h(t) - e^{-t^2/2}|\,dt \le c_{21}/\tau. \tag{4.16}$$

If $\rho \le \varepsilon$, estimate (4.14) follows from (4.16). Let $\rho > \varepsilon$. Consider $I(h) = \int\limits_{\varepsilon\,\tau}^{\tau\,\rho} |g_h(t)|\,dt$. In the notation of (4.2) and Lemma 4 (see (4.5)), we have the equality

$$|g_h(t)| = \prod_j |f_{h\lambda_j}(t/\sigma(h))|. \tag{4.17}$$

Let $k$ be chosen from the condition that

$$\sum_{j=1}^{k} \sigma_j^2(h) \ge \sigma^2(h)/2 > \sum_{j=1}^{k-1} \sigma_j^2(h). \tag{4.18}$$

Set $\nu = v_0/\lambda_k$, where the constant $v_0$ is the same as in (4.6). If $\rho \le \nu/h$, then

$$\varepsilon h\lambda_j \le t\lambda_j/\sigma(h) \le \tau\,\rho\lambda_j/\sigma(h) \le \tau\,\nu\,\lambda_j/(h\,\sigma(h)) \le v_0\,\lambda_j/\lambda_k \le v_0$$

for any $j \ge k$ and all $t \in [\varepsilon\,\tau, \tau\,\rho]$. These inequalities combined with (4.17), (4.18), and (4.6) imply by virtue of (4.3b) that

$$I(h) \le \int\limits_{\varepsilon\,\tau}^{\tau\,\nu/h} \prod_{j\ge k} |f_{h\lambda_j}(t\lambda_j/\sigma(h))|\,dt \le \tau \int\limits_{\varepsilon}^{\nu/h} \prod_{j\ge k} \exp\left(-\delta_2\,\sigma_j^2(h)\,h^2\,t^{\omega}\right)dt \le \tau \int\limits_{\varepsilon}^{\infty} \exp\left(-\delta_2\,\tau^2\,t^{\omega}\right)dt \le \delta_3/\tau, \tag{4.19}$$

where $\delta_i = \delta_i(V, v_0)$. The statement of Lemma 6 for $\varepsilon < \rho \le \nu/h$ follows from (4.16) and (4.19).

Taking into consideration equality (4.17), we see that to complete the proof of Lemma 6, it is enough to derive the estimate

$$J(h) = \tau \int\limits_{\varepsilon\vee\nu/h}^{\rho} \prod_{1\le j\le k} |f_{h\lambda_j}(t\,h\,\lambda_j)|\,dt \le c_{22}/\tau. \tag{4.20}$$

For this purpose, we show that

$$|f_{h\lambda_j}(t\,h\,\lambda_j)| \le e^{-c_{23}\,t^{-\alpha}},\ \ t \ge \varepsilon \vee \nu/h,\ \ j \le k. \tag{4.21}$$

For brevity, denote $h\lambda_j$ $(1 \le j \le k)$ by $\gamma$ and set (see (4.5)) $\xi(t) = |f_\gamma(t\gamma)|$, $H_0 = (2\pi + b)/r_0$, and $\bar\gamma = H_0/\varepsilon$.

At first let $\varepsilon \le \nu/h$. If $\gamma < \bar\gamma$, then for $t \ge \varepsilon$ we derive from (4.5) and (1.2) that

$$\xi(t) \le 1 - c_{24}\,V(1/t\gamma) \le 1 - c_{25}\,t^{-\alpha},\ \ t\gamma \ge H_0, \tag{4.22}$$

for $t \ge \varepsilon$, and, moreover,

$$\xi(t) \le e^{-\delta},\ \ \delta = \delta(V, v_0),\ \ t\gamma \in [v_0, H_0], \tag{4.23}$$

since the random variable $X$ is nonlattice and the mapping between the distributions $X$ and $X(h)$ is continuous in $h$.

Since $t\gamma \ge v_0\,\lambda_j/\lambda_k \ge v_0$ for $t \ge \nu/h$, estimates (4.22) and (4.23) imply (4.21).

Let now $\gamma \ge \bar\gamma$. Then we deduce from (1.2) that

$$V(1/t\gamma)/V(1/\gamma) \ge \frac{1}{c_2}\,(t^{-\alpha} \wedge 1) \ge (\frac{1}{c_2} \wedge \varepsilon^{\alpha})\,t^{-\alpha},\ \ t \ge \varepsilon,\ \ \gamma \ge 1/r_0, \tag{4.24}$$

for $t \geq \varepsilon$, where $t\gamma \geq \varepsilon\gamma \geq \varepsilon\bar{\gamma} = H_0$, i.e., inequalities (4.24) and (4.5) imply estimate (4.21). Thus, the case $\varepsilon \leq \nu/h$ is analyzed completely.

Let $\varepsilon > \nu/h$ and $t \geq \varepsilon$. In this case, $\gamma \geq \tilde{\gamma} = v_0/\varepsilon$ and $t\gamma \geq \varepsilon\tilde{\gamma} = v_0$. If $t\gamma \geq H_0$ and $\gamma < \bar{\gamma}$, we use estimate (4.22); if $t\gamma < H_0$, we apply (4.23). If $\gamma \geq \bar{\gamma}$, then $t\gamma \geq H_0$, and one can apply (4.24) to prove relation (4.21).

Now we use estimate (4.21) to obtain the required estimate (4.20). It follows from (4.3b), (4.4b) and (4.18) that

$$k \geq c_{26} \sum_{j=1}^{k} h^2 \sigma_j^2(h) \geq c_{24} h^2 \sigma^2(h)/2 = c_{27}\tau^2;$$

therefore,

$$J(h) \leq \tau \int\limits_{\varepsilon\vee\nu/h}^{\rho} e^{-c_{23}\,k\,t^{-\alpha}}\,dt \leq \tau\rho e^{-c_{28}\,\tau^2\,\rho^{-\alpha}} \leq \delta_0\,\tau^{1+2/\alpha} e^{-c_{29}\,\delta_0^{-\alpha}\,\log(1+\tau)}.$$

It remains to make the constant $\delta_0$ small enough. Lemma 6 is proved.

The remark to Lemma 6 is proved similarly; we refer to the fact that in the case $\varepsilon > \nu/h$, condition $(F)$ is not applied (see also (4.15)).

## 5. Proofs

Theorem 2.1 follows from Lemmas 1 and 5.

Theorem 2.3 for $\tau < 1$ is obvious; for $\tau \geq 1$, this theorem follows from Lemmas 6 and 2 with $\varepsilon = 1/\tau\rho$.

To establish Theorem 2.4, one can use Lemma 2 with $\varepsilon = \infty$; we only have to prove that

$$\int\limits_{\tau\rho}^{\infty} |g_h(t)|\,dt \leq c_{30}/\tau \tag{5.1}$$

for all $\tau$ large enough. For any integer $m \geq 1$,

$$\int\limits_{\tau\rho}^{\infty} |g_h(t)|\,dt \leq \int\limits_{\rho}^{\infty} \prod_{j=1}^{m} |f_{h\lambda_j}(t\,h\lambda_j)|\,dt \leq 2^{m-1} \max_{1\leq j\leq m} (\int\limits_{\rho}^{\infty} |f_{1j}|^m\,dt + 2^{-1} \int\limits_{-\infty}^{\infty} |f_{2j}|^m\,dt), \tag{5.2}$$

where

$$f_{1j} = \frac{1}{L(s)} \int\limits_{0}^{x_0} e^{(it-1)sx} f(x)\,dx,\ \ f_{2j} = \frac{1}{L(s)} \int\limits_{x_0}^{\infty} e^{(it-1)sx} f(x)\,dx,\ \ s = h\lambda_j. \tag{5.3}$$

Integrating by parts for $tsx_0 > 1$, we see that

$$-itsL(s)\,f_{1j} = (1 - e^{itsx_0})\,e^{-sx_0} f(x_0) + (-s) \int\limits_{0}^{x_0} (e^{itsx} - 1)e^{-sx} f(x)\,dx$$

$$+(\int\limits_{0}^{1/ts} + \int\limits_{1/ts}^{x_0})(e^{itsx} - 1)e^{-sx}\,df(x) = I_1 + I_2 + I_3.$$

In this case, we deduce from (1.2) and (2.5) that

$$|I_1| \le 2sL(s), \quad |I_2| \le C\,ts \int\limits_0^{1/ts} V(x)/x\,dx \le C\,ts\,c_1 V(1/s)t^{-\beta}/\beta,$$

$$|I_3| \le 2(\int\limits_{1/ts}^{1/s} e^{-sx}\,|df(x)| + \int\limits_{1/s}^{x_0})e^{-sx}\,|df(x)|)$$

$$\le 2C\,s(\int\limits_{1/t}^{1} V(u/s)du/u^2 + \int\limits_{1}^{sx_0} V(u/s)e^{-u}du/u^2) = 2Cs\,(J_1 + J_2),$$

$$J_2 \le V(1/s)c_2 \int\limits_1^\infty u^{\alpha-2}e^{-u}du, \quad \text{and} \quad J_1 \le V(1/s)c_1\,\kappa(t),$$

where $\kappa(t) = (\beta-1)^{-1}$ if $\beta > 1$, $\kappa(t) = \log t$ if $\beta = 1$, and $\kappa(t) = t^{1-\beta}/(1-\beta)$ if $\beta < 1$. Since

$$L(s) \ge e^{-1}V(1/s) \le e^{-1}c_3 s^{-\alpha}, \tag{5.4}$$

$$|f_{1j}| \le c_{31}\ (t^{-\beta} + \log(1+t)/t),\ s \ge \delta,\ ts \ge 1/x_0. \tag{5.5}$$

Thus,

$$\int\limits_\rho^\infty |f_{1j}|^m\,dt \le c_{32}/\tau^2,\ m > (1+\alpha)/\min(1,\beta),\ h\lambda_m \ge \delta. \tag{5.6}$$

Assume that condition (2.6) holds with some $p > 1$ and $m \ge \max(2, p/(p-1))$. Then (see also (5.3) and (5.4))

$$\int\limits_{-\infty}^{\infty} |f_{2j}|^m\,dt \le c_{33}\left(\int\limits_{x_0}^{\infty}(s^\alpha e^{-sx}f(x))^{m/m-1}dx\right)^{m-1}$$

$$\le s^{\alpha m/m-1}\left(\int\limits_{x_0}^{\infty} e^{-sx}f(x)\,dx + \int\limits_{x_0}^{\infty}(e^{-sx}f(x))^p dx\right).$$

The above estimate, inequality (5.6), and the estimate $\tau^2 = h^2\sigma^2(h) \le c_{34}\max(1,h)\sum\limits_{j\ge1}\mathbf{E}\,(1 \wedge \lambda_j X)$ (which is valid by virtue of Lemma 3) imply (5.1).

To check Remark 3, we use an inequality similar to (5.2) and the fact that an estimate similar to (5.5) holds for $|f_{h\lambda_j}(t)|$.

Finally, Theorem 3.3 follows from Lemma 1 (see (4.1)).

This research was supported by the RFBR (project 06-01-00179-a) and by the Program "Leading Scientific Schools" (project 4222.2006.1).

Translated by L. V. Rozovsky.

## REFERENCES

1. L. V. Rozovsky, "On small deviation probabilities for some random sums," *Obozrenie Prikl. Prom. Mat.*, **12**, 865–866 (2005).
2. M. A. Lifshits, "On the lower tail probabilities of some random series," *Ann. Probab.*, **25**, 424–442 (1997).

3. R. Davis and S. Resnick, "Extremes of moving averages of random variables with finite endpoint," *Ann. Probab.*, **19**, 312–328 (1991).
4. L. V. Rozovsky, "Small deviation probabilities for positive random variables," *Zap. Nauch. Semin. POMI*, **320**, 150–159 (2004).
5. L. V. Rozovsky, "On a lower bound for probabilities of large deviations for a sample mean under the Cramér condition," *Zap. Nauchn. Semin. POMI*, **278**, 208–224 (2001).
6. L. V. Rozovsky, "Small deviation probabilities for a class of distributions with polinomial decay at zero," *Zap. Nauchn. Semin. POMI*, **328**, 182–190 (2005).
7. F. Aurzada, "On the lower tail probabilities of some random sequences in $l_p$," Unpublished manuscript (2006).
8. A. A. Borovkov and P. S. Ruzankin, "On small deviations of series of weighted i.i.d. random variables," in: IV *Intern. Conf. Limit Theorems in Probability Theory and Their Applications*, Novosibirsk, Absracts (2006), p. 14.
9. N. C. Jain and W. E. Pruitt, "Lower tail probability estimates for subordinators and nondecreasing random walks," *Ann. Probab.*, **15**, 76–101 (1987).
10. V. V. Petrov, *Limit Theorems for Sums of Independent Random Variables* [in Russian], Moscow (1987).