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NONLOCAL WELL-POSEDNESS OF THE MIXED PROBLEM FOR THE ZAKHAROV–KUZNETSOV EQUATION

A. V. Faminskii UDC 517.956

ABSTRACT. The nonlocal well-posedness of the mixed problem for the Zakharov–Kuznetsov equation is considered.

We consider the mixed problem in the domain

$$\Pi_T^+ = (0,T) \times \mathbb{R}_+^2,$$

where $T > 0$ and $\mathbb{R}_+^2 = \{(x,y) : x > 0\} = \mathbb{R}_+ \times \mathbb{R}$, for the Zakharov–Kuznetsov equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = f(t,x,y), \tag{1}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}_+^2, \tag{2}$$

$$u(t,0,y) = u_1(t,y), \quad (t,y) \in S_T = (0,T) \times \mathbb{R}. \tag{3}$$

Equation (1) is one of the generalizations of the Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = f,$$

which describes the propagation of nonlinear waves in a two-dimensional dispersive medium (see [12]). The purpose of this paper is to prove the well-posedness of problem (1)–(3) for any $T > 0$. We set

$$L_p = L_p(\mathbb{R}^2), \quad H^k = H^k(\mathbb{R}^2) = W_2^k(\mathbb{R}^2), \quad L_{p,+} = L_p(\mathbb{R}_+^2), \quad H_+^k = H^k(\mathbb{R}_+^2).$$

To describe the properties of the boundary function u_1, we also use the anisotropic Sobolev spaces of fractional order

$$H^{s_1,s_2} = H_{t,y}^{s_1,s_2}(\mathbb{R}^2) = \big\{\mu(t,y) : (1 + |\lambda|^{s_1} + |\eta|^{s_2})\widehat{\mu}(\lambda,\eta) \in L_2\big\},$$

where $\widehat{\mu} = \mathcal{F}[\mu]$ denotes the Fourier transform of the function μ:

$$\widehat{\mu}(\lambda,\eta) = \iint\limits_{\mathbb{R}^2} e^{-i(\lambda t + \eta y)}\mu(t,y)\,dt\,dy, \quad \mu \in L_1;$$

the symbol $\mathcal{F}^{-1}[\mu]$ is used for the inverse Fourier transform.

For a domain $\Omega \subset \mathbb{R}^2$, $H^{s_1,s_2}(\Omega)$ denotes the space of restrictions to Ω of the functions from H^{s_1,s_2} with the natural norm.

For an integer $k \geq 0$, by $C_b^k(\overline{\Omega})$ we denote the space of bounded continuous functions on $\overline{\Omega}$ having all partial derivatives up to order k. We set

$$C_b^k = C_b^k(\mathbb{R}^2), \quad C_{b,+}^k = C_b^k(\overline{\mathbb{R}}_+^2), \quad C_b(\overline{\Omega}) = C_b^0(\overline{\Omega}).$$

As usual, for a Banach space $\mathcal{X}$ and an interval I of the real line, $L_p(I;\mathcal{X})$ denotes the space of Bochner-measurable, pth-power integrable (essentially bounded at $p = +\infty$) mappings from I to $\mathcal{X}$. By $C_b(\overline{I};\mathcal{X})$ we denote the space of bounded continuous mappings from $\overline{I}$ to $\mathcal{X}$ (if I is a bounded interval, then we omit the subscript b).

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Definition 1. A function $u(t,x,y) \in L_2(\Pi_T^+)$ is called a generalized solution to problem (1)–(3) if, for any function $\phi(t,x,y)$ such that

$$\phi \in L_\infty(0,T;H_+^3), \quad \phi_t \in L_\infty(0,T;L_{2,+}), \quad \phi\big|_{t=T}=0, \quad \phi\big|_{x=0}=\phi_x\big|_{x=0}=0,$$

the following equality is valid:

$$\iiint\limits_{\Pi_T^+} \left[u(\phi_t+\phi_{xxx}+\phi_{xyy})+\frac{1}{2}u^2\phi_x+f\phi\right]dx\,dy\,dt + \iint\limits_{\mathbb{R}_+^2} u_0\phi\big|_{t=0}\,dx\,dy + \iint\limits_{S_T} u_1\phi_{xx}\big|_{x=0}\,dy\,dt = 0. \tag{4}$$

We prove the nonlocal well-posedness of the problem in a class $Z(\Pi_T^+)$ of generalized solutions with higher order of smoothness.

Definition 2. We say that a function $u(t,x,y)$ belongs to $Z(\Pi_T^+)$ if

$$u \in C([0,T];H_+^1) \cap L_2(0,T;C_{b,+}^1) \cap L_2(\mathbb{R}_+^x;C_b(\overline{S}_T)),$$
$$D_x^k u \in C_b(\overline{\mathbb{R}}_+^x;H^{(2-k)/3,2-k}(S_T)), \qquad k=0,1,2.$$

The main result of this paper is the following theorem.

Theorem 1. *Suppose that $u_0 \in H_+^1$, $u_1 \in H^{2/3,2}(S_T)$, and $f \in L_2(0,T;H_+^1)$ for some $T>0$, and $u_1(0,y) \equiv u_0(0,y)$. Then problem (1)–(3) has a unique solution $u(t,x,y)$ from the space $Z(\Pi_T^+)$, and the mapping $(u_0,u_1,f) \mapsto u$ is Lipschitz continuous on any ball in the norm of mappings from $H_+^1 \times H^{2/3,2}(S_T) \times L_2(0,T;H_+^1)$ to $Z(\Pi_T^+)$.*

The conditions imposed on the boundary data in this theorem can be considered natural, because they are determined by the properties of the differential operator $D_t + D_x^3 + D_x D_y^2$. Indeed, let $v(t,x,y)$ be a solution from the space $C_b(\mathbb{R}^t;H^1)$ to the Cauchy problem for the equation

$$v_t + v_{xxx} + v_{xyy} = 0 \tag{5}$$

with initial function $v_0 \in H^1$. Then

$$\|v(\cdot,x,\cdot)\|_{\dot H^{2/3,2}} = \|(|\lambda|^{2/3}+|\eta|^2)\widehat{v}(\lambda,x,\eta)\|_{L_2^{\lambda,\eta}} \sim \|\nabla v_0\|_{L_2}$$

uniformly in $x \in \mathbb{R}$ (see [5]).

The Zakharov–Kuznetsov equation is a special case of the equation

$$u_t - P(D_X)u + \operatorname{div}_X g(u) = f(t,X), \tag{6}$$

where

$$X=(x_1,\dots,x_n), \quad P(D_X)=\sum_{|\alpha|=3} a_\alpha D_{x_1}^{\alpha_1}\dots D_{x_n}^{\alpha_n}, \quad \alpha=(\alpha_1,\dots,\alpha_n), \quad |\alpha|=\alpha_1+\dots+\alpha_n.$$

A mixed problem for Eq. (6), which is similar to (1)–(3) for $x_1>0$, was considered in [6]. The main condition imposed on the operator P in [6] is as follows. If $P(\xi)=\sum\limits_{|\alpha|=3} a_\alpha \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$ is the symbol of the operator P (here $\xi=(\xi_1,\dots,\xi_n)$), then the operator $Q(D_X)$ with symbol $Q(\xi)=\partial P(\xi)/\partial\xi_1$ is elliptic. Obviously, this condition holds for the operator $-(D_x^3+D_xD_y^2)$, where $X=(x,y)$. In [6], nonlocal existence and uniqueness results for generalized solutions to the mixed problem for Eq. (6) were obtained. As applied to problem (1)–(3), these solutions belong to weaker classes in comparison with the solutions constructed in this paper.

The Cauchy problem for the Zakharov–Kuznetsov equation was studied in [4, 5, 8]. In [4, 8], more general quasilinear equations of odd order were considered. In [5], classes in which the Cauchy problem for Eq. (1) with an initial function from the space H^k is nonlocally well-posed were constructed for any positive integer k.

The nonlocal solvability theory of the Cauchy problem for the Zakharov–Kuznetsov equation is based on *a priori* estimates implied by conservation laws. If $u(t,x,y)$ is a sufficiently smooth solution to Eq. (1) with $f \equiv 0$, which decreases together with its derivatives at infinity, then

$$\iint\limits_{\mathbb{R}^2} u^2\,dx\,dy = \text{const}, \qquad \iint\limits_{\mathbb{R}^2} \Big(u_x^2 + u_y^2 - \frac{1}{3}u^3\Big)dx\,dy = \text{const}. \tag{7}$$

Similarly, for the mixed problem, multiplying Eq. (1) by $2u(t,x,y)$ and integrating the result over $\mathbb{R}^2_+$, we easily obtain the equality

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2_+} u^2\,dx\,dy + \int\limits_{\mathbb{R}} \Big(u_x^2 - 2uu_{xx} + u_y^2 - \frac{2}{3}u^3\Big)\Big|_{x=0} dy = 2\iint\limits_{\mathbb{R}^2_+} fu\,dx\,dy. \tag{8}$$

Clearly, if $u_1 \equiv 0$, then equality (8) readily implies an *a priori* estimate for the solution in $L_{2,+}$. In the case of inhomogeneous boundary conditions, obtaining such an estimate is obstructed by the term $uu_{xx}|_{x=0}$. In this case, it is natural to pass to the new function $U(t,x,y) \equiv u(t,x,y) - \psi(t,x,y)$, where the auxiliary function ψ is chosen so that $\psi|_{x=0} = u_1$. The function U satisfies the homogeneous boundary conditions (3), but the equation becomes more complicated; it acquires variable coefficients depending on ψ. Certainly, this approach supposes that the properties of the function u_1 must ensure the possibility of extending it over Π_T^+ in such a way that the corresponding analogue of equality (8) gives an estimate for the solution u in the space $L_{2,+}$.

In [6], a solution to the linear equation (5) of boundary potential type was constructed (in fact, it was constructed in a more general case for a linear analogue of Eq. (6)). This solution was used as the auxiliary function ψ. A similar approach was earlier applied to study mixed problems for the Korteweg–de Vries equation (see, e.g., [7] and the references therein).

In [6], to construct the boundary potential, the function

$$A(x,y) \equiv \mathcal{F}^{-1}_{x,y}\left[e^{i(\xi^3+\xi\eta^2)}\right](x,y)$$

was introduced and studied. It was proved that A belongs to the space $\mathcal{S}(\overline{\mathbb{R}}^2_+)$ of restrictions to $\overline{\mathbb{R}}^2_+$ of the functions from the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of rapidly decreasing functions; at any point $(x,y) \in \overline{\mathbb{R}}^2_+$, it satisfies the equation

$$3A_{xx} + A_{yy} = xA; \tag{9}$$

moreover,

$$\iint\limits_{\mathbb{R}^2_+} A(x,y)\,dx\,dy = \frac{1}{3}, \qquad \int\limits_{\mathbb{R}} A(0,y)y\,dy = 0. \tag{10}$$

Definition 3. For a function $\mu(t,y)$, we set

$$J(t,x,y;\mu) \equiv \int\limits_{-\infty}^{t}\int\limits_{\mathbb{R}} (3D_x^2 + D_y^2)G(t-\tau,x,y-z)\mu(\tau,z)\,dz\,d\tau, \tag{11}$$

where $x > 0$ and

$$G(t,x,y) \equiv \frac{1}{t^{2/3}}A\Big(\frac{x}{t^{1/3}},\frac{y}{t^{1/3}}\Big) = \mathcal{F}^{-1}_{x,y}\Big[e^{it(\xi^3+\xi\eta^2)}\Big](x,y). \tag{12}$$

In the definition of the boundary potential J suggested in [6], the integral with respect to τ was from 0 to t (unlike in (11)). However, in this paper, we use the potential J only for functions μ which are compactly supported as $t \to -\infty$ uniformly in y.

If $\mu \in L_2$ and $\mu(t,y)=0$ for $t<0$, then, according to [6], the function J is defined and infinitely differentiable, and it satisfies Eq. (5) for $x>0$. Moreover, it was shown in [6] that, for any $T>0$, $x_0>0$, and $\beta \geq 0$ and any integers $k,l,m \geq 0$, the estimate

$$(1+x)^{\beta}\|D_t^l D_x^k J(t,x,\cdot;\mu)\|_{H^m(\mathbb{R})} \leq c(T,x_0,\beta,k,l,m)\|\mu\|_{L_2} \tag{13}$$

holds at $t \leq T$ and $x \geq x_0$. Moreover, if, e.g., $\mu \in C_b^1$, then the limit $J(t,0+0,y)=\mu(t,y)$ exists. These properties justify the use of the term boundary potential for the function J.

Let us find an alternative representation for J.

Lemma 1. *Let $\mu \in L_2$. Then, for any $x>0$,*

$$J(t,x,y;\mu) = \mathcal{F}_{t,y}^{-1}\Big[e^{r(\lambda,\eta)x}\widehat{\mu}(\lambda,\eta)\Big](t,y), \tag{14}$$

where $r(\lambda,\eta)$ is the unique root of the equation

$$r^3 - r\eta^2 + i\lambda = 0$$

such that $\operatorname{Re} r < 0$ *for $(\lambda,\eta) \neq (0,0)$ (it is assumed that $r(0,0)=0$).*

Proof. Let $\xi = \varphi(\lambda,\eta)$ be the function inverse to $\lambda = \xi^3 + \xi\eta^2$ for fixed η. Making a suitable change of variable on the right-hand side of (12), we obtain

$$G(t,x,y) = \mathcal{F}_{t,y}^{-1}\Big[\varphi_\lambda(\lambda,\eta)e^{ix\varphi(\lambda,\eta)}\Big](t,y). \tag{15}$$

Applying the formula for the Fourier transform of a convolution and using the Fourier transform of the Heaviside function θ (see, e.g., [10]), we see that (11) implies

$$\begin{aligned}\mathcal{F}_{t,y}[J](\lambda,\eta) &= \mathcal{F}_{t,y}\big[(3D_x^2+D_y^2)G(t,x,y)\theta(t)\big](\lambda,\eta)\widehat{\mu}(\lambda,\eta)\\ &= -\frac{1}{4\pi^2}\Big(e^{ix\varphi(\lambda,\eta)} * \big(\mathcal{F}[\theta](\lambda)\times\delta(\eta)\big)\Big)\widehat{\mu}(\lambda,\eta)\\ &= -\left(\frac{1}{2}e^{ix\varphi(\lambda,\eta)} + \frac{i}{2\pi}\,\text{v. p.}\int_{\mathbb{R}}\frac{e^{ix\varphi(\zeta,\eta)}}{\zeta-\lambda}d\zeta\right)\widehat{\mu}(\lambda,\eta),\end{aligned}$$

because $(3\varphi^2(\lambda,\eta)+\eta^2)\varphi_\lambda(\lambda,\eta) \equiv 1$. To find the last integral, we make the change $\zeta = z^3 + z\eta^2$ and apply Jordan's lemma. Since the equation $z^3+z\eta^2-\lambda=0$ has one real root $\varphi(\lambda,\eta)$ and two complex conjugate roots, one of which $(-ir(\lambda,\eta))$ belongs to the upper half-plane, it follows that

$$\text{v. p.}\int_{\mathbb{R}}\frac{e^{ixz}(3z^2+\eta^2)}{z^3+z\eta^2-\lambda}dz = 2\pi i e^{xr(\lambda,\eta)} + \pi i e^{ix\varphi(\lambda,\eta)}$$

for $x>0$. This completes the proof of the lemma. □

Lemma 2. *Suppose that $\mu \in H^{2/3,2}$ and $\mu(t,y)=0$ for $t \leq 0$. Then, for any $T>0$,*

$$\|J(\cdot,\cdot,\cdot;\mu)\|_{Z(\Pi_T^+)} \leq c(T)\|\mu\|_{H^{2/3,2}}. \tag{16}$$

Proof. First, let us prove that

$$\|J(t,\cdot,\cdot;\mu)\|_{H_+^1} \leq c\|\mu\|_{H^{2/3,2}} \tag{17}$$

uniformly in $t \in \mathbb{R}$. We use representation (14). For any nonnegative integers k and m, we have

$$D_x^k D_y^m J(t,x,y;\mu) = \mathcal{F}_{t,y}^{-1}\Big[(i\eta)^m r^k(\lambda,\eta)e^{r(\lambda,\eta)x}\widehat{\mu}(\lambda,\eta)\Big](t,y). \tag{18}$$

Let us make the change $\lambda = \vartheta^3$. The Parseval equality gives

$$\|D_x^k D_y^m J\|_{L_{2,+}^{x,y}} = \left\|\frac{3}{2\pi}\int_{\mathbb{R}} e^{i\vartheta^3 t}(i\eta)^m r^k(\vartheta^3,\eta)e^{r(\vartheta^3,\eta)x}\widehat{\mu}(\vartheta^3,\eta)\vartheta^2 d\vartheta\right\|_{L_{2,+}^{x,\eta}}. \tag{19}$$

In [2], it was proved that if some continuous function $\gamma(\vartheta)$ satisfies the condition $\operatorname{Re}\gamma(\vartheta) \le -\varepsilon|\vartheta|$ for some $\varepsilon > 0$ and any $\vartheta \in \mathbb{R}$, then

$$\Bigl\| \int\limits_{\mathbb{R}} e^{\gamma(\vartheta)x} f(\vartheta)\,d\vartheta \Bigr\|_{L_2(\mathbb{R}_+^x)} \le c(\varepsilon)\|f\|_{L_2(\mathbb{R}^\vartheta)}. \tag{20}$$

It is easy to see that

$$\operatorname{Re} r(\lambda,\eta) \le -\frac{\sqrt{3}}{2}|\lambda|^{1/3}, \qquad |r(\lambda,\eta)| \le c(|\lambda|^{1/3} + |\eta|). \tag{21}$$

Therefore, extending equality (19), we obtain

$$\|D_x^k D_y^m J\|_{L_{2,+}^{x,y}} \le c\bigl\|\eta^m(|\vartheta|^k + |\eta|^k)\vartheta^2 \widehat{\mu}(\vartheta^3,\eta)\bigr\|_{L_2^{\vartheta,\eta}} \le c_1\bigl\|\eta^m(|\lambda|^{k/3} + |\eta|^k)\lambda^{1/3}\widehat{\mu}(\lambda,\eta)\bigr\|_{L_2^{\lambda,\eta}} \le c_2\|\mu\|_{H^{2/3,2}}.$$

Next, by analogy with (17), we obtain the estimate

$$\|J(\cdot,\cdot,\cdot;\mu)\|_{L_2(\mathbb{R}_+^x; C_b^{t,y})} \le c\|\mu\|_{H^{2/3,2}}. \tag{22}$$

Indeed, representation (14) and inequality (20) imply

$$\begin{aligned}\bigl\|\sup_{t,y}|J|\bigr\|_{L_2(\mathbb{R}_+^x)} &\le \int\limits_{\mathbb{R}} \Bigl\| \int\limits_{\mathbb{R}} e^{\operatorname{Re} r(\lambda,\eta)x}|\widehat{\mu}(\lambda,\eta)|\,d\lambda \Bigr\|_{L_2(\mathbb{R}_+^x)} d\eta \\ &= 3\int\limits_{\mathbb{R}} \Bigl\| \int\limits_{\mathbb{R}} e^{\operatorname{Re} r(\vartheta^3,\eta)x}\vartheta^2|\widehat{\mu}(\vartheta^3,\eta)|\,d\vartheta \Bigr\|_{L_2(\mathbb{R}_+^x)} d\eta \le c\int\limits_{\mathbb{R}} \|\vartheta^2\widehat{\mu}(\vartheta^3,\eta)\|_{L_2(\mathbb{R}^\vartheta)}\,d\eta \\ &\le c_1\|\lambda^{1/3}(1+|\eta|)\widehat{\mu}(\lambda,\eta)\|_{L_2^{\lambda,\eta}} \le c_2\|\mu\|_{H^{2/3,2}}.\end{aligned}$$

Moreover, it follows directly from the Parseval equality, equality (18), and inequalities (21) that

$$\|D_x^k J(\cdot,x,\cdot;\mu)\|_{H^{(2-k)/3,2-k}}\bigl\|(1+|\lambda|^{1/3}+|\eta|)^{2-k} r^k(\lambda,\eta)\widehat{\mu}(\lambda,\eta)\bigr\|_{L_2^{\lambda,\eta}} \le c\|\mu\|_{H^{2/3,2}} \tag{23}$$

uniformly in $x > 0$ for $k \le 2$.

Finally, let us prove that

$$\|J\|_{L_2(0,T;C_{b,+}^1)} \le c(T)\|\mu\|_{H^{2/3,2}}. \tag{24}$$

We set

$$B(x,y) \equiv xA(x,y), \qquad R(t,x,y) \equiv \frac{1}{t^{4/3}} B\Bigl(\frac{x}{t^{1/3}}, \frac{y}{t^{1/3}}\Bigr).$$

By virtue of (9), equality (11) implies

$$\begin{aligned} J(t,x,y;\mu) &= \int\limits_{-\infty}^{t}\int\limits_{\mathbb{R}} R(t-\tau,x,y-z)\mu(\tau,z)\,dz\,d\tau \\ &= \int\limits_{0}^{+\infty}\int\limits_{\mathbb{R}} R(\tau,x,z)\bigl(\mu(t-\tau,y-z)-\mu(t,y-z)\bigr)\,dz\,d\tau \\ &+ \int\limits_{0}^{+\infty}\int\limits_{\mathbb{R}} R(\tau,x,z)\mu(t,y-z)\,dz\,d\tau \equiv J_1(t,x,y;\mu) + J_2(t,x,y;\mu). \end{aligned} \tag{25}$$

First, we estimate J_1. We have

$$|J_1(t,x,y;\mu)| + |J_{1x}(t,x,y;\mu)| + |J_{1y}(t,x,y;\mu)| \le \int_{\mathbb{R}_+} \big\| |R(\tau,x,\cdot)| + |R_x| + |R_y| \big\|_{L_2(\mathbb{R})} \|\mu(t-\tau,\cdot) - \mu(t,\cdot)\|_{L_2(\mathbb{R})} d\tau$$

uniformly in $y \in \mathbb{R}$; here

$$\|R(\tau,x,\cdot)\|_{L_2(\mathbb{R})} \le \frac{1}{\tau^{7/6}} \sup_{\xi\ge 0} \|B(\xi,\cdot)\|_{L_2(\mathbb{R})} \le \frac{c}{\tau^{7/6}},$$

$$\big\| |R_x(\tau,x,\cdot)| + |R_y(\tau,x,\cdot)| \big\|_{L_2(\mathbb{R})} \le \frac{1}{\tau^{3/2}} \sup_{\xi\ge 0} \|B(\xi,\cdot)\|_{H^1(\mathbb{R})} \le \frac{c}{\tau^{3/2}}$$

uniformly in $x \ge 0$. Since $\mu(t-\tau,\cdot) = 0$ for $\tau \ge t$, it follows that

$$\begin{aligned} \|J_1\|_{L_2(0,T;C^1_{b,+})} &\le c(T) \left(\int_0^T \Big(\int_0^t \frac{1}{\tau^{3/2}} \|\mu(t-\tau,\cdot) - \mu(t,\cdot)\|_{L_2(\mathbb{R})} d\tau \Big)^2 dt \right)^{1/2} \\ &+ c(T) \Big(\int_0^T \frac{1}{t} \|\mu(t,\cdot)\|^2_{L_2(\mathbb{R})} dt \Big)^{1/2} \\ &\le c_1(T) \Big(\int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{\tau^{7/3}} \int_{\mathbb{R}_+} \big(\mu(t-\tau,y) - \mu(t,y)\big)^2 \, dt \, d\tau \, dy \Big)^{1/2} \\ &+ c_1(T) \Big(\int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{\mu^2(t,y)}{t^{4/3}} \, dt \, dy \Big)^{1/2} \le c_2(T) \|\mu\|_{L_2(\mathbb{R}^y; W_2^{2/3}(\mathbb{R}^t_+))} \le c_3(T) \|\mu\|_{H^{2/3,0}}, \end{aligned} \tag{26}$$

because $W_2^s(\mathbb{R}_+) = H^s(\mathbb{R}_+)$ (see, e.g., [1]) and

$$\int_{\mathbb{R}_+} \frac{1}{t^{ps}} |f(t) - f(0)|^p dt \le c(p,s) \|f\|^p_{W_p^s(\mathbb{R}_+)}$$

for $1/p < s < 1$ (see [9]); we here use the Slobodetskii space

$$W_p^s(I) = \left\{ f \in L_p(I) : \iint_{I\times I} \frac{|f(t) - f(\tau)|^p}{|t-\tau|^{1+ps}} \, dt \, d\tau < +\infty \right\}$$

for $s \in (0,1)$ and $p \in [1,+\infty)$.

We estimate the term J_2 separately for the function itself and its derivative with respect to y and for the derivative with respect to x. First, we can represent J_2 in the form

$$\begin{aligned} J_2(t,x,y;\mu) &= \int_0^{+\infty} \int_{|z|<1} R(\tau,x,z)\big(\mu(t,y-z) - \mu(t,y)\big) \, dz \, d\tau \\ &+ \int_0^{+\infty} \int_{|z|>1} R(\tau,x,z)\big(\mu(t,y-z) - \mu(t,y)\big) \, dz \, d\tau + \mu(t,y) \\ &\equiv J_3(t,x,y;\mu) + J_4(t,x,y;\mu) + \mu(t,y), \end{aligned} \tag{27}$$

because the first equality in (10) implies

$$\int_0^{+\infty}\int_{\mathbb{R}} R(\tau,x,z)\,dz\,d\tau = 3\iint_{\mathbb{R}_+^2} A(\xi,\eta)\,d\xi\,d\eta = 1 \tag{28}$$

for any $x>0$. We have

$$\begin{aligned}\big\||J_3|+|J_{3y}|\big\|_{L_2(0,T;C_{b,+})} &\le c\sup_{x\ge 0}\int_{|z|<1}\int_{\mathbb{R}_+}\frac{|z|^{1/2}}{\tau^{4/3}}\Big|B\Big(\frac{x}{\tau^{1/3}},\frac{z}{\tau^{1/3}}\Big)\Big|\,d\tau\,dz\,\|\mu\|_{L_2(\mathbb{R}^t;H^2(\mathbb{R}^y))}\\ &\le c_1\int_{\mathbb{R}}\sup_{\xi\ge 0}|B(\xi,\eta)|\,d\eta\int_{|z|<1}|z|^{-1/2}\,dz\,\|\mu\|_{H^{0,2}}\le c_2\|\mu\|_{H^{0,2}},\end{aligned} \tag{29}$$

$$\begin{aligned}\big\||J_4|+|J_{4y}|\big\|_{L_2(0,T;C_{b,+})} &\le c\left(\sup_{x\ge 0}\int_{|z|>1}\Big(\int_{\mathbb{R}_+}\frac{1}{\tau^{4/3}}B\Big(\frac{x}{\tau^{1/3}},\frac{z}{\tau^{1/3}}\Big)d\tau\Big)^2dz\right)^{1/2}\|\mu\|_{L_2(\mathbb{R}^t;H^1(\mathbb{R}^y))}\\ &\le c_1\int_{\mathbb{R}}\sup_{\xi\ge 0}|B(\xi,\eta)|\,d\eta\Big(\int_{|z|>1}|z|^{-2}\,dz\Big)^{1/2}\|\mu\|_{H^{0,1}}\le c_2\|\mu\|_{H^{0,1}}.\end{aligned} \tag{30}$$

To estimate J_{2x}, we again apply (28) and represent J_2 as

$$\begin{aligned}J_2(t,x,y;\mu) &= \int_1^{+\infty}\int_{\mathbb{R}} R(\tau,x,z)\big(\mu(t,y-z)-\mu(t,y)\big)\,dz\,d\tau\\ &+\int_0^1\int_{|z|>1} R(\tau,x,z)\big(\mu(t,y-z)-\mu(t,y)\big)\,dz\,d\tau\\ &+\int_0^1\int_{|z|<1} R(\tau,x,z)\big(\mu(t,y-z)-\mu(t,y)+z\mu_y(t,y)\big)\,dz\,d\tau\\ &-\mu_y(t,y)\int_0^1\int_{\mathbb{R}} R(\tau,x,z)z\,dz\,d\tau+\mu_y(t,y)\int_0^1\int_{|z|>1} R(\tau,x,z)z\,dz\,d\tau+\mu(t,y)\\ &\equiv J_5(t,x,y;\mu)+J_6(t,x,y;\mu)+J_7(t,x,y;\mu)+\mu_y(t,y)\big(J_8(x)+J_9(x)\big)+\mu(t,y).\end{aligned} \tag{31}$$

It is easy to see that

$$\begin{aligned}\big\|J_{5x}\big\|_{L_2(0,T;C_{b,+})} &\le 2\sup_{x\ge 0}\int_1^{+\infty}\int_{\mathbb{R}}|R_x(\tau,x,z)|\,dz\,d\tau\,\|\mu\|_{L_2(\mathbb{R}^t;C_b(\mathbb{R}^y))}\\ &\le c\int_{\mathbb{R}}\sup_{\xi\ge 0}|B_\xi(\xi,\eta)|\,d\eta\int_1^{+\infty}\frac{d\tau}{\tau^{4/3}}\|\mu\|_{L_2(\mathbb{R}^t;H^1(\mathbb{R}^y))}\le c_1\|\mu\|_{H^{0,1}},\end{aligned} \tag{32}$$

$$
\begin{aligned}
\left\|J_{6x}\right\|_{L_2(0,T;C_{b,+})} &\leq 2 \int\limits_{|z|>1} \int\limits_0^1 \frac{1}{\tau^{5/3}} \sup_{\xi\geq 0}\left|B_\xi\left(\xi, \frac{z}{\tau^{1/3}}\right)\right| \, d\tau \, dz \, \|\mu\|_{L_2(\mathbb{R}^t; C_b(\mathbb{R}^y))} \\
&\leq c \int\limits_{\mathbb{R}} \sup_{\xi\geq 0} |\eta B_\xi(\xi,\eta)| \, d\eta \int\limits_{|z|>1} |z|^{-2} \, dz \, \|\mu\|_{L_2(\mathbb{R}^t; H^1(\mathbb{R}^y))} \leq c_1 \|\mu\|_{H^{0,1}},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\left\|J_{7x}\right\|_{L_2(0,T;C_{b,+})} &\leq c \int\limits_{|z|<1} \int\limits_0^1 \frac{|z|^{3/2}}{\tau^{5/3}} \sup_{\xi\geq 0}\left|B_\xi\left(\xi, \frac{z}{\tau^{1/3}}\right)\right| \, d\tau \, dz \|\mu\|_{L_2(\mathbb{R}^t; H^2(\mathbb{R}^y))} \\
&\leq c_1 \int\limits_{\mathbb{R}} \sup_{\xi\geq 0} |\eta B_\xi(\xi,\eta)| \, d\eta \int\limits_{|z|<1} |z|^{-1/2} \, dz \, \|\mu\|_{L_2(\mathbb{R}^t; H^2(\mathbb{R}^y))} \leq c_2 \|\mu\|_{H^{0,2}}.
\end{aligned} \tag{34}
$$

Setting $\widetilde{B}(x) \equiv \int\limits_{\mathbb{R}} B_x(x,y) y dy$, we see that $\widetilde{B}(0) = 0$ by the second equality in (10); therefore,

$$
\sup_{x\geq 0} |J_8'(x)| \leq \sup_{x\geq 0} \left| \int\limits_0^1 \frac{1}{\tau} \widetilde{B}\left(\frac{x}{\tau^{1/3}}\right) d\tau \right| \leq c \left| \int\limits_{\mathbb{R}_+} \frac{\widetilde{B}(\eta)}{\eta} d\eta \right| < +\infty, \tag{35}
$$

and, moreover,

$$
\sup_{x\geq 0} |J_9'(x)| \leq \int\limits_0^1 \frac{1}{\tau} \int\limits_{|\eta|>\tau^{-1/3}} \sup_{\xi\geq 0} |B_\xi(\xi,\eta)\eta| \, d\eta \, d\tau < +\infty. \tag{36}
$$

Combining (25)–(27) and (29)–(36), we obtain (24). This completes the proof of the lemma. ☐

Remark 1. In proving Lemmas 1 and 2 and substantiating calculations, we can first assume that $\mu \in C_0^\infty(\mathbb{R}^2)$ and then pass to the limit on the basis of the estimates obtained.

Now let us study the properties of the linear problem on Π_T^+, that is,

$$
v_t + v_{xxx} + v_{xyy} = f(t,x,y), \tag{37}
$$

$$
v(0,x,y) = v_0(x,y), \tag{38}
$$

$$
v(t,0,y) = v_1(t,y). \tag{39}
$$

First, we introduce two potentials related to the Cauchy problem for Eq. (37):

$$
S(t,x,y;v_0) \equiv \mathcal{F}_{x,y}^{-1}\left[e^{it(\xi^3+\xi\eta^2)} \widehat{v}_0(\xi,\eta)\right](x,y), \tag{40}
$$

$$
K(t,x,y;f) \equiv \int\limits_0^t S(t-\tau,x,y;f(\tau,\cdot,\cdot)) d\tau. \tag{41}
$$

In what follows, we assume that if the functions v_0 and f are defined only on $\mathbb{R}_+^2$ and Π_T^+, respectively, then the potentials S and K are constructed for their extensions over the entire plane $\mathbb{R}^2$ and the layer $\Pi_T = (0,T) \times \mathbb{R}^2$ with the same properties.

Lemma 3. *Suppose that* $v_0 \in H_+^1$, $v_1 \in H^{2/3,2}(S_T)$, *and* $f \in L_2(0,T;H_+^1)$ *for some* $T > 0$, *and* $v_1(0,y) \equiv v_0(0,y)$. *Then problem* (37)–(39) *has a unique solution* $v(t,x,y)$ *from the space* $Z(\Pi_T^+)$, *and*

$$
\|v\|_{Z(\Pi_{t_0}^+)} \leq c(T)\left(\|v_0\|_{H_+^1} + \|v_1\|_{H^{2/3,2}(S_T)} + t_0^{1/6}\|f\|_{L_2(0,t_0;H_+^1)}\right) \tag{42}
$$

for any $t_0 \in (0,T]$; here v is the function on Π_T^+ defined by

$$v(t,x,y) = S(t,x,y;v_0) + K(t,x,y;f) + J(t,x,y;\widetilde{v}_1), \tag{43}$$

where

$$\widetilde{v}_1(t,y) \equiv v_1(t,y) - S(t,0,y;v_0) - K(t,0,y;f)$$

for $t \in [0,T]$ and $\widetilde{v}_1(t,y) \equiv 0$ for $t < 0$.

Proof. The uniqueness of a generalized solution to problem (37)–(39) in the larger space $L_\infty(0,T;L_{2,+})$ was proved in [6] by the method of Holmgren.

The Cauchy problem (37), (38) was studied in [5]. In particular, it was shown that if $Z'(\Pi_T)$, where $\Pi_T = (0,T)\times\mathbb{R}^2$, is the space of functions $v(t,x,y)$ such that

$$v \in C([0,T];H^1) \cap L_2(0,T;C_b^1) \cap L_2(\mathbb{R}^x;C_b(\overline{S}_T)),$$
$$D_x^k v \in C_b(\mathbb{R}^x;H^{0,2-k}(S_T)), \qquad k = 0,1,2,$$

then problem (37), (38) has a solution $\widetilde{v}(t,x,y)$ belonging to the space $Z'(\Pi_T)$, and

$$\|\widetilde{v}\|_{Z'(\Pi_{t_0})} \le c(T)\left(\|v_0\|_{H^1} + \|f\|_{L_1(0,t_0;H^1)}\right) \tag{44}$$

for any $t_0 \in (0,T]$ (see [5]); moreover,

$$\widetilde{v}(t,x,y) = S(t,x,y;v_0) + K(t,x,y;f). \tag{45}$$

To determine the smoothness properties of $\widetilde{v}$ with respect to t, we prove the following auxiliary estimate: if $v_0 \in H^s$ for some $s \in \mathbb{R}$, then

$$\|S(\cdot,\cdot,\cdot;v_0)\|_{C_b(\mathbb{R}^x;H^{(s+1)/3,s+1}((-T,T)\times\mathbb{R}))} \le c(T,s)\|v_0\|_{H^s} \tag{46}$$

for any $T > 0$. Indeed, let us represent the function S in the form

$$\begin{aligned} S(t,x,y;v_0) =& \mathcal{F}^{-1}_{x,y}\left[e^{it(\xi^3+\xi\eta^2)}\widehat{v}_0(\xi,\eta)\chi(\xi,\eta)\right](x,y) \\ &+ \mathcal{F}^{-1}_{x,y}\left[e^{it(\xi^3+\xi\eta^2)}\widehat{v}_0(\xi,\eta)(1-\chi(\xi,\eta))\right](x,y) \equiv S_1(t,x,y;v_0) + S_2(t,x,y;v_0), \end{aligned}$$

where $\chi(\xi,\eta)$ is the characteristic function of the unit disk $\{\xi^2+\eta^2 < 1\}$. The function S_1, which is infinitely smooth, is estimated in an obvious way; to estimate S_2, we make the change $\lambda = \xi^3 + \xi\eta^2$ and obtain

$$S_2(t,x,y;v_0) = \mathcal{F}^{-1}_{t,y}\left[\varphi_\lambda(\lambda,\eta)e^{ix\varphi(\lambda,\eta)}\widehat{v}_0\big(\varphi(\lambda,\eta),\eta\big)\big(1-\chi(\varphi(\lambda,\eta),\eta)\big)\right](t,y)$$

by analogy with (15). As a result, we have

$$\|S_2\|^2_{H^{(s+1)/3,s+1}} = \iint\limits_{\xi^2+\eta^2>1} \left(1+|\xi^3+\xi\eta^2|+|\eta|^3\right)^{2(s+1)/3}|\widehat{v}_0(\xi,\eta)|^2\frac{d\xi\,d\eta}{3\xi^2+\eta^2} \le c(s)\|v_0\|^2_{H^s}$$

for any x.

To estimate the potential K, we use an idea from [3]. First, applying (46) with $s=-1$, we obtain the uniform (in x) estimate

$$\|K(\cdot,x,\cdot;f)\|_{L_2(S_{t_0})} \le c(T)t_0^{1/2}\|f\|_{L_2(0,t_0;H^{-1})}. \tag{47}$$

Moreover,

$$K_t(t,x,y;f) = f(t,x,y) - \int\limits_0^t S(t-\tau,x,y;(D_x^3 + D_xD_y^2)f(\tau,\cdot,\cdot))d\tau.$$

Again applying (46) with $s=-1$, we see that

$$\|K_t(\cdot,x,\cdot;f)\|_{L_2(S_{t_0})} \le c(T)\|f\|_{L_2(0,t_0;H^2)} \tag{48}$$

uniformly in x. Interpolating (47) and (48), we obtain the uniform (in x) estimate

$$\|K(\cdot,x,\cdot;f)\|_{H^{(s+1)/3,0}(S_{t_0})} \le c(T,s)t_0^{1/3-s/6}\|f\|_{L_2(0,t_0;H^s)} \tag{49}$$

for $s \in [-1,2]$.

Combining (45), (46), and (49), we conclude that

$$\|D_x^k \widetilde{v}\|_{C_b(\mathbb{R}^x;H^{(2-k)/3,0}(S_{t_0}))} \le c(T)\left(\|v_0\|_{H^1} + t_0^{1/6}\|f\|_{L_2(0,t_0;H^1)}\right) \tag{50}$$

for $k = 0$ and $k = 1$.

By virtue of the compatibility condition, we have $\widetilde{v}_1(0,y) \equiv 0$. Therefore, $\widetilde{v}_1 \in H^{2/3,2}((-\infty,T]\times\mathbb{R})$, and, according to (44) and (50),

$$\|\widetilde{v}_1\|_{H^{2/3,2}((-\infty,t_0]\times\mathbb{R})} \le c(T)\left(\|v_0\|_{H^1} + \|v_1\|_{H^{2/3,2}(S_T)} + t_0^{1/6}\|f\|_{L_2(0,t_0;H^1)}\right)$$

for any $t_0 \in (0,T]$. The properties of the potentials S, K, and J (in particular, (16), (44), and (50)) imply that the function v defined by (43) is the required solution to problem (37)–(39). This completes the proof of the lemma. □

In the following lemma, we obtain auxiliary integral inequalities for the solutions to problem (37)–(39), which are analogues of the conservation laws (7).

Lemma 4. *Suppose that the conditions of Lemma* 3 *hold and the function v_1 is extended over the entire plane $\mathbb{R}^2$ so that the extension belongs to the same class and $v_1(t,y) \equiv 0$ for $t \le -1$. Let*

$$V(t,x,y) \equiv v(t,x,y) - J(t,x,y;v_1), \tag{51}$$

where v is the solution to problem (37)–(39) *from the space $Z(\Pi_T^+)$. Then, for any $t \in (0,T]$,*

$$\iint_{\mathbb{R}_+^2} V^2(t,x,y)\,dx\,dy + \iint_{S_t} V_x^2(\tau,0,y)\,dy\,d\tau \le \iint_{\mathbb{R}_+^2} v_0^2\,dx\,dy + 2\iiint_{\Pi_t^+} fV\,dx\,dy\,d\tau + c\|v_1\|_{H^{2/3,2}}^2 \tag{52}$$

and

$$\begin{gathered}\iint_{\mathbb{R}_+^2}\Big(V_x^2 + V_y^2 - \frac{1}{3}V^3\Big)\rho(x)\,dx\,dy + \frac{1}{2}\iiint_{\Pi_t^+}(V_{xx}^2 + V_{xy}^2 + V_{yy}^2)\rho'(x)\,dx\,dy\,d\tau \\ +2\iiint_{\Pi_t^+} VV_x(V_{xx}+V_{yy})\rho\,dx\,dy\,d\tau \le c(\rho,\|v_0\|_{H_+^1},\|v_1\|_{H^{2/3,2}}) \\ +2\iiint_{\Pi_t^+}(f_xV_x + f_yV_y)\rho\,dx\,dy\,d\tau - \iiint_{\Pi_t^+} fV^2\rho\,dx\,dy\,d\tau \\ +c(\rho)\iint_{S_t}(f^2+V_x^2)\Big|_{x=0}dy\,d\tau + c(\rho,\|V\|_{C([0,t];L_{2,+})})\iiint_{\Pi_t^+}(V_x^2+V_y^2)\rho\,dx\,dy\,d\tau,\end{gathered} \tag{53}$$

where $\rho(x) \in C_b^3(\overline{\mathbb{R}}_+)$ is a positive nondecreasing function on $\overline{\mathbb{R}}_+$.

Proof. The function V is a solution to the problem of type (37)–(39) with the same right-hand side f, initial function $V_0(x,y) \equiv v_0(x,y) - J(0,x,y;v_1) \in H_0^1(\mathbb{R}_+^2)$, and zero boundary function. Moreover, by virtue of (17), we have

$$\|V_0\|_{H_+^k} \le \|v_0\|_{H_+^k} + c\|v_1\|_{H^{2/3,2}}$$

for $k = 0$ and $k = 1$. According to [11], if the functions V_0 and f are smooth, then the corresponding mixed problem has a smooth solution. In this case, inequality (52) is obtained similarly to (8) by multiplying the equality (37) for V by $2V(t,x,y)$ and integrating the result.

To derive (53), we multiply the equality (37) for V by

$$-\big(2V_{xx}(t,x,y)+2V_{yy}(t,x,y)+V^2(t,x,y)\big)\rho(x).$$

After integration, we obtain

$$\begin{aligned}&\frac{d}{dt}\iint\limits_{\mathbb{R}^2_+}\big(V_x^2+V_y^2-\frac{1}{3}V^3\big)\rho\,dx\,dy+\iint\limits_{\mathbb{R}^2_+}(3V_{xx}^2+4V_{xy}^2+V_{yy}^2)\rho'\,dx\,dy\\&+\int\limits_{\mathbb{R}}\big(V_{xx}^2\rho+2V_{xx}V_x\rho'-V_x^2\rho''\big)\big|_{x=0}dy-\iint\limits_{\mathbb{R}^2_+}(V_x^2+V_y^2)\rho'''\,dx\,dy\\&+2\iint\limits_{\mathbb{R}^2_+}VV_x(V_{xx}+V_{yy})\rho\,dx\,dy+\iint\limits_{\mathbb{R}^2_+}V^2(V_{xx}+V_{yy})\rho'\,dx\,dy\\&=2\iint\limits_{\mathbb{R}^2_+}(f_xV_x+f_yV_y)\rho\,dx\,dy+2\int\limits_{\mathbb{R}}(fV_x\rho)\big|_{x=0}dy-\iint\limits_{\mathbb{R}^2_+}fV^2\rho\,dx\,dy.\end{aligned}\tag{54}$$

The well-known interpolation inequality (see, e.g., [1])

$$\|g\|_{L_{p,+}}\le c(p)\|\nabla g\|_{L_{2,+}}^{(p-2)/p}\|g\|_{L_{2,+}}^{2/p},\tag{55}$$

where $2\le p<+\infty$, yields

$$\begin{aligned}\left|\iint\limits_{\mathbb{R}^2_+}V^2(V_{xx}+V_{yy})\rho'\,dx\,dy\right|&\le\frac{1}{2}\iint\limits_{\mathbb{R}^2_+}(V_{xx}^2+V_{yy}^2)\rho'\,dx\,dy+c\iint\limits_{\mathbb{R}^2_+}V^4\,dx\,dy\\&\le\frac{1}{2}\iint\limits_{\mathbb{R}^2_+}(V_{xx}^2+V_{yy}^2)\rho'\,dx\,dy+c_1\iint\limits_{\mathbb{R}^2_+}(V_x^2+V_y^2)\rho\,dx\,dy\iint\limits_{\mathbb{R}^2_+}V^2\,dx\,dy;\end{aligned}$$

thus, (54) implies (53) in the smooth case.

In the general case, the required inequalities are obtained by passing to the limit on the basis of (42). This completes the proof of the lemma. □

We return to the initial nonlinear problem. Theorem 1 is implied by the following two lemmas.

Lemma 5. *If the conditions of Theorem 1 hold, then there exists a $t_0>0$ depending on T, $\|u_0\|_{H^1_+}$, $\|u_1\|_{H^{2/3,2}(S_T)}$, and $\|f\|_{L_2(0,T;H^1_+)}$ such that problem (1)–(3) on $\Pi^+_{t_0}$ has a unique generalized solution $u(t,x,y)$ belonging to the space $Z(\Pi^+_{t_0})$. The mapping $(u_0,u_1,f)\mapsto u$ is Lipschitz continuous on any ball in the norm of mappings $H^1_+\times H^{2/3,2}(S_T)\times L_2(0,T;H^1_+)\to Z(\Pi^+_{t_0})$.*

Proof. For an arbitrary $t_0\in(0,T]$, consider the mapping Λ defined on the $Z(\Pi^+_{t_0})$ as follows: $v=\Lambda u$ for $u\in Z(\Pi^+_{t_0})$ if $v\in Z(\Pi^+_{t_0})$ and the function v is a solution to the linear problem

$$v_t+v_{xxx}+v_{xyy}=f-uu_x,\tag{56}$$

$$v\big|_{t=0}=u_0,\qquad v\big|_{x=0}=u_1.\tag{57}$$

Note that

$$\begin{aligned}\|uu_x\|_{L_2(0,t_0;H^1_+)}&\le\|u_x\|_{L_2(0,t_0;C_{b,+})}\|u\|_{C([0,t_0];H^1_+)}\\&+\big\||u_{xx}|+|u_{xy}|\big\|_{C_b(\overline{\mathbb{R}}^x_+;L_2(S_{t_0}))}\|u\|_{L_2(\mathbb{R}^x_+;C_b(\overline{S}_{t_0}))}\le c\|u\|^2_{Z(\Pi^+_{t_0})}\end{aligned}\tag{58}$$

on $\Pi_{t_0}^+$. According to Lemma 3, problem (56), (57) has a solution, and (42) implies

$$\|v\|_{Z(\Pi_{t_0}^+)} \le \widetilde{c}\big(1 + t_0^{1/6}\|u\|^2_{Z(\Pi_{t_0}^+)}\big), \tag{59}$$

where the constant $\widetilde{c}$ depends on the same quantities as t_0 in the statement of the lemma. According to (59), for sufficiently small t_0, the mapping Λ takes each ball of sufficiently large (depending on $\widetilde{c}$) radius in the space $Z(\Pi_{t_0}^+)$ to itself. Moreover, considering the corresponding mixed problem (with homogeneous boundary conditions) for the difference of Λu and $\Lambda\widetilde{u}$, where u and $\widetilde{u}$ belong to such a ball, we see that the mapping Λ is a contraction on this ball for sufficiently small t_0, because, by analogy with (58),

$$\|uu_x - \widetilde{u}\widetilde{u}_x\|_{L_2(0,t_0;H_+^1)} \le c\left(\|u\|_{Z(\Pi_{t_0}^+)} + \|\widetilde{u}\|_{Z(\Pi_{t_0}^+)}\right)\|u-\widetilde{u}\|_{Z(\Pi_{t_0}^+)};$$

this readily implies the existence and uniqueness of a solution. Continuity is proved similarly. This completes the proof of the lemma. □

Lemma 6. *Suppose that the conditions of Theorem* 1 *hold and, for some* $T' \in (0,T]$, *a function* $u(t,x,y)$ *from the space* $Z(\Pi_{T'}^+)$ *is a solution to problem* (1)–(3) *on* $\Pi_{T'}^+$. *Then*

$$\|u\|_{C([0,T'];H_+^1)} \le c(T, \|u_0\|_{H_+^1}, \|u_1\|_{H^{2/3,2}(S_T)}, \|f\|_{L_2(0,T;H_+^1)}). \tag{60}$$

Proof. Let $\widetilde{c}$ denote various constants depending on the same quantities as the constant on the right-hand side of (60). As in the proof of Lemma 4, we extend the function u_1 over the entire plane $\mathbb{R}^2$ so that $\|u_1\|_{H^{2/3,2}} \le c\|u_1\|_{H^{2/3,2}(S_T)}$ and $u_1(t,y) \equiv 0$ for $t \le -1$. We set

$$U(t,x,y) \equiv u(t,x,y) - J(t,x,y;u_1). \tag{61}$$

Writing inequality (52) for the function U, we obtain

$$\iint\limits_{\mathbb{R}_+^2} U^2(t,x,y)\,dx\,dy + \iint\limits_{S_t} U_x^2\Big|_{x=0} dy\,d\tau \le \widetilde{c} + 2\iiint\limits_{\Pi_t^+} (f - uu_x)U\,dx\,dy\,d\tau \tag{62}$$

for $t \in [0,T']$. Since $U\big|_{x=0} = 0$, it follows that

$$2\iint\limits_{\mathbb{R}_+^2} uu_xU\,dx\,dy = \iint\limits_{\mathbb{R}_+^2} (J_xU^2 + 2JJ_xU)\,dx\,dy. \tag{63}$$

Relations (16), (62), and (63) imply the estimate

$$\|u\|_{C([0,T'];L_{2,+})} + \|u_x\big|_{x=0}\|_{L_2(S_{T'})} \le \widetilde{c}. \tag{64}$$

Next, we apply inequality (53) with

$$\rho(x) \equiv 2 - (1+x)^{-1/2}$$

to the function U. Taking into account the already obtained estimate (64), we see that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}_+^2} (U_x^2 + U_y^2 - \frac{1}{3}U^3)\rho\,dx\,dy + \frac{1}{2}\iiint\limits_{\Pi_t^+} (U_{xx}^2 + U_{xy}^2 + U_{yy}^2)\rho'\,dx\,dy\,d\tau \\
&\le \widetilde{c} + \widetilde{c}\iiint\limits_{\Pi_t^+} (U_x^2 + U_y^2)\rho\,dx\,dy\,d\tau + \iiint\limits_{\Pi_t^+} uu_xU^2\rho\,dx\,dy\,d\tau + c\iint\limits_{S_t} u_1^2u_x^2\big|_{x=0}\,dy\,d\tau \\
&+2\iiint\limits_{\Pi_t^+} uu_xU_x\rho'\,dx\,dy\,d\tau + 2\iiint\limits_{\Pi_t^+} (JU_x + uJ_x)(U_{xx} + U_{yy})\rho\,dx\,dy\,d\tau.
\end{aligned} \tag{65}$$

Inequalities (55), (16), and (64) imply

$$\iiint\limits_{\Pi_t^+} uu_xU^2\rho\,dx\,dy\,d\tau \le \iiint\limits_{\Pi_t^+} \Big(J_xuU^2\rho - \frac{1}{3}(J\rho)_xU^3 - \frac{1}{4}U^4\rho'\Big)\,dx\,dy\,d\tau \le \widetilde{c}\iiint\limits_{\Pi_t^+}(U_x^2+U_y^2)\rho\,dx\,dy\,d\tau + \widetilde{c},$$

$$2\iiint\limits_{\Pi_t^+} uu_xU_x\rho'\,dx\,dy\,d\tau = -\iint\limits_{S_t} u_1^2(U_x\rho')\big|_{x=0}dy\,d\tau - \iiint\limits_{\Pi_t^+} u^2(U_{xx}\rho' + U_x\rho'')\,dx\,dy\,d\tau \le \frac{1}{6}\iiint\limits_{\Pi_t^+} U_{xx}^2\rho'\,dx\,dy\,d\tau + \widetilde{c}\iiint\limits_{\Pi_t^+}(U_x^2+U_y^2)\rho\,dx\,dy\,d\tau + \widetilde{c}.$$

The obvious interpolation inequality

$$u_x^2\big|_{x=0} \le c\Bigg(\int\limits_{\mathbb{R}_+} u_{xx}^2\rho'\,dx\Bigg)^{1/2}\Bigg(\int\limits_{\mathbb{R}_+} u_x^2\rho\,dx\Bigg)^{1/2} + c\int\limits_{\mathbb{R}_+} u_x^2\rho\,dx$$

yields

$$\iint\limits_{S_t} u_1^2u_x^2\big|_{x=0}\,dy\,d\tau \le \varepsilon\iiint\limits_{\Pi_t^+} U_{xx}^2\rho'\,dx\,dy\,d\tau + c(\varepsilon)\int\limits_0^t\Big(1+\sup_{y\in\mathbb{R}}u_1^4\Big)\iint\limits_{\mathbb{R}_+^2} U_x^2\rho\,dx\,dy\,d\tau + \widetilde{c},$$

where $\|u_1\|_{L_4(0,T;C_b(\mathbb{R}))} \le c\|u_1\|_{H^{2/3,2}}$ (see, e.g., [1]) and $\varepsilon > 0$ is arbitrarily small. Finally,

$$2\iiint\limits_{\Pi_t^+}(JU_x + uJ_x)(U_{xx}+U_{yy})\rho\,dx\,dy\,d\tau \le \frac{1}{6}\iiint\limits_{\Pi_t^+}(U_{xx}^2+U_{yy}^2)\rho'\,dx\,dy\,d\tau + \int\limits_0^t \sup_{(x,y)\in\mathbb{R}_+^2}\Big[(J_x^2+J^2)\frac{\rho^2}{\rho'}\Big]\iint\limits_{\mathbb{R}_+^2}(U_x^2\rho + u^2)\,dx\,dy\,d\tau$$

$$\le \frac{1}{6}\iiint\limits_{\Pi_t^+}(U_{xx}^2+U_{yy}^2)\rho'\,dx\,dy\,d\tau + \int\limits_0^t\gamma(\tau)\iint\limits_{\mathbb{R}_+^2}U_x^2\rho\,dx\,dy\,d\tau + \widetilde{c},$$

where $\|\gamma\|_{L_1(0,T)} \le \widetilde{c}$, because $\rho^2(\rho')^{-1} \le c(1+x)^{3/2}$. Thus, inequalities (13) and (16) imply

$$\int\limits_0^T \sup_{(x,y)\in\mathbb{R}_+^2}\big[(1+x)^{3/2}(J_x^2+J^2)\big]d\tau \le \widetilde{c}.$$

Combining the obtained inequalities, we derive estimate (60) from (65). This completes the proof of the lemma. □

Remark 2. In classes of functions of higher order of smoothness, the well-posedness of problem (1)–(3) can be proved for $u_0 \in H_+^s$ and $u_1 \in H^{(s+1)/3,s+1}(S_T)$, where $s = 3k$ or $s = 3k+1$ and $k \in \mathbb{N}$, by similar methods.

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REFERENCES

1. O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, *Integral Representation of Functions and Embedding Theorems* [in Russian], Nauka, Moscow (1996).
2. J. L. Bona, S. Sun, B.-Y. Zhang, "A non-homogeneous boundary-value problem for the Korteweg–de Vries equation in a quarter-plane," *Trans. Amer. Math. Soc.*, **354**, No. 2, 427–490 (2002).
3. J. E. Colliander and C. E. Kenig, "The generalized Korteweg–de Vries equation on the half-line," *Commun. Partial Differ. Equations*, **27**, Nos. 11–12, 2187–2266 (2002).
4. A. V. Faminskii, "The Cauchy problem for odd-order quasilinear equations," *Mat. Sb.*, **180**, No. 9, 1183–1210 (1989).
5. A. V. Faminskii, "The Cauchy problem for the Zakharov–Kuznetsov equation," *Differ. Uravn.*, **31**, No. 6, 1070–1081 (1995).
6. A. V. Faminskii, "On the mixed problem for quasilinear equations of the third order," *J. Math. Sci.*, **110**, No. 2, 2476–2507 (2002).
7. A. V. Faminskii, "An initial-boundary-value problem in a half-strip for the Korteweg–de Vries equation in fractional-order Sobolev spaces," *Commun. Partial Differ. Equations*, **29**, Nos. 11–12, 1653–1695 (2004).
8. J. C. Saut, "Sur quelques généralizations de l'equation de Korteweg–de Vries," *J. Math. Pures Appl.*, **58**, No. 1, 21–61 (1979).
9. V. A. Solonnikov, "A priori estimates for second-order equations of parabolic type," *Tr. Mat. Inst. Steklova*, **70**, 133–212 (1964).
10. V. S. Vladimirov, *Generalized Functions in Mathematical Physics* [in Russian], Nauka, Moscow (1979).
11. L. G. Volevich and S. G. Gindikin, "The mixed problem for $(2b+1)$-hyperbolic equations," *Tr. Mosk. Mat. Obshch*, **43**, 197–259 (1981).
12. V. E. Zakharov and E. A. Kuznetsov, "On three-dimensional solitons," *Zh. Eksp. Teor. Fiz.*, **66**, No. 2, 594–597 (1974).

A. V. Faminskii
E-mail: afaminskii@sci.pfu.edu.ru