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SUFFICIENT CONDITION OF LOCAL REGULARITY FOR THE NAVIER–STOKES EQUATIONS

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We prove a local regularity result for the non-stationary three-dimensional Navier–Stokes equations. Bibliography: 10 titles.

1. Introduction

In the papers [1–3], interesting point-wise sufficient conditions for local regularity of weak solutions to the nonstationary three-dimensional Navier–Stokes equations are proved. These estimates are deduced from a new version of the so-called Ladyzhenskaya–Prodi–Serrin (LPS) condition, and the regularity follows from smallness of some mixed Lorentz norm of the velocity field. We should point out that the classical LPS condition is formulated in terms of mixed Lebesgue spaces (see [4] and [5]).

In the present paper, we wish to obtain such point-wise conditions with the help of the local regularity theory based on the notion of suitable weak solutions. To this end, we prove a sufficient condition for regularity in terms of smallness of some functionals involving norms in mixed Lebesgue spaces only, see Theorem 1.2. As a consequence of this theorem, we deduce the same point-wise conditions which cover all the cases of [1–3].

To formulate our result, we recall the definition of suitable weak solutions following F.-H.Lin [6]. For a more general definition, we refer the reader to the celebrated paper of Caffarelli–Kohn–Nirenberg [7].

Definition 1.1. *A pair of functions v and p is called a suitable weak solution to the Navier–Stokes equations in a domain $Q(z_0,R)=B(x_0,R)\times]t_0-R^2,t_0[$ (where $z_0=(x_0,t_0)$ and $B(x_0,R)=\{x:|x-x_0|<R\}$) if the following conditions hold:*

$$v\in L_{2,\infty}(Q(z_0,R))\cap W^{1,0}_2(Q(z_0,R))\quad\text{and}\quad p\in L_{\frac32}(Q(z_0,R)); \tag{1.1}$$

the Navier–Stokes equations:

$$\partial_t v+v\cdot\nabla v-\Delta v+\nabla p=0,\quad \operatorname{div} v=0, \tag{1.2}$$

hold in $Q(z_0,R)$ in the sense of distributions;

v and p satisfy the local energy inequality:

$$\int\limits_{B(x_0,R)}\varphi(x,t)|v(x,t)|^2dx+2\int\limits_{t_0-R^2}^{t}\int\limits_{B(x_0,R)}\varphi|\nabla v|^2dxdt'$$

$$\leq\int\limits_{t_0-R^2}^{t}\int\limits_{B(x_0,R)}\Big[|v|^2(\Delta\varphi+\partial_t\varphi)+v\cdot\nabla\varphi(|v|^2+2p)\Big]dxdt' \tag{1.3}$$

for a.a. $t\in[t_0-R^2,t_0]$ and for all nonnegative smooth cut-off functions φ vanishing in a neighborhood of the parabolic boundary of the cylinder $Q(z_0,R)$.

In the above definition, we use the following mixed Lebesgue and Sobolev spaces:

$$L_{m,n}(Q(z_0,R))=L_n(t_0-R^2,t_0;L_m(B(x_0,R))),$$

where $L_m=L_{m,m}$, $m,n\in[1,\infty]$, and

$$W^{1,0}_2(Q(z_0,R))=\{u:u,\nabla u\in L_2(Q(z_0,R))\}.$$

Now we formulate the main result.

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Theorem 1.2. *Let v and p be a suitable weak solution to the Navier–Stokes equations in $Q(z_0,R)$. Assume that real numbers $l\geq 1$ and $s\geq 1$ satisfy the condition*

$$\frac{1}{2}\geq \frac{3}{s}+\frac{2}{l}-\frac{3}{2}>\max\Big\{\frac{1}{2l},\frac{1}{2}-\frac{1}{s},\frac{1}{s}-\frac{1}{6}\Big\}. \tag{1.4}$$

There exists a number $\varepsilon>0$ that depends on s and l only and such that if the condition

$$M^{s,l}(z_0,R)=\frac{1}{R^\kappa}\int\limits_{t_0-R^2}^{t_0}dt\Big(\int\limits_{B(x_0,R)}|v|^s dx\Big)^{\frac{l}{s}}<\varepsilon,\qquad \kappa=l\Big(\frac{3}{s}+\frac{2}{l}-1\Big), \tag{1.5}$$

is satisfied, then z_0 is a regular point of v, i.e., there exists a number $r\in]0,R]$ such that v is Hölder continuous in the closure of the cylinder $Q(z_0,r)$.

Corollary 1.3. *Let v and p be a suitable weak solution to the Navier–Stokes equations in $Q(z_0,R)$. Let $\vartheta\in[0,1]$ be fixed.*

There is a positive number ε_1 that depends on ϑ only and such that if

$$|v(x,t)|\leq \frac{\varepsilon_1}{|x-x_0|^{1-\vartheta}|t_0-t|^{\frac{\vartheta}{2}}} \tag{1.6}$$

for a.a. $z=(x,t)\in Q(z_0,R)$, then z_0 is a regular point of v.

We would like to note that Corollary 1.3 includes the above-mentioned point-wise conditions of local regularity as particular cases. For $\vartheta=1$, see (1.7) in [1], and for $0\leq\vartheta<1$, see Theorem 1.2 in [2, 3].

2. Proof of Theorem 1.2

Due to invariance with respect to the natural scaling, we may assume that $z_0=0$ and $R=1$.

Let us introduce auxiliary functionals:

$$C(\varrho)=\frac{1}{\varrho^2}\int\limits_{Q(\varrho)}|v|^3dz,\quad A(\varrho)=\operatorname{ess}\sup_{-\varrho^2<t<0}\frac{1}{\varrho}\int\limits_{B(\varrho)}|v(x,t)|^2dx,$$

$$E(\varrho)=\frac{1}{\varrho}\int\limits_{Q(\varrho)}|\nabla v|^2dz,\ H(\varrho)=\frac{1}{\varrho^3}\int\limits_{Q(\varrho)}|v|^2dz,\quad\text{and}\quad D(\varrho)=\frac{1}{\varrho^2}\int\limits_{Q(\varrho)}|p|^{\frac{3}{2}}dz,$$

where $Q(\varrho)=Q(0,\varrho)$, $B(\varrho)=B(0,\varrho)$, $B=B(1)$, and $Q=Q(1)$. In this notation,

$$M^{s,l}(\varrho)=\frac{1}{\varrho^\kappa}\int\limits_{-\varrho^2}^{0}dt\Big(\int\limits_{B(\varrho)}|v|^sdx\Big)^{\frac{l}{s}}.$$

The following lemma is but a consequence of interpolation and imbedding theorems.

Lemma 2.1. *Assume that a function v has finite energy in Q, i.e.,*

$$v\in L_{2,\infty}(Q)\cap W^{1,0}_2(Q). \tag{2.1}$$

Let numbers s and l satisfy condition (1.4). *Then the inequality*

$$C(\varrho)\leq cA^\mu(\varrho)(M^{s,l}(\varrho))^{\frac{1}{q}}(E(\varrho)+H(\varrho))^{\frac{1}{q'}} \tag{2.2}$$

holds with

$$\mu=\frac{\frac{3}{s}+\frac{2}{l}-2}{2(\frac{3}{s}+\frac{2}{l}-\frac{3}{2})},\quad q=2l\Big(\frac{3}{s}+\frac{2}{l}-\frac{3}{2}\Big),\quad\text{and}\quad q'=\frac{q}{q-1}.$$

In (2.2), *the positive constant c depends on s and l only.*

Remark 2.2. By condition (1.4), $q > 1$ and $\mu > 0$.

Proof of Lemma 2.1. We are going to use the interpolation in Lebesgue spaces in the following way. By setting

$$\lambda = \frac{1}{2s(\frac{3}{s} + \frac{2}{l} - \frac{3}{2})} \quad \text{and} \quad \gamma = \frac{\frac{2}{s} + \frac{1}{l} - 1}{2(\frac{3}{s} + \frac{2}{l} - \frac{3}{2})}, \tag{2.3}$$

we observe that the numbers λ and γ are positive and obey the following two identities:

$$\lambda s + 2\mu + 6\gamma = 3 \quad \text{and} \quad \lambda + \gamma + \mu = 1. \tag{2.4}$$

Applying identities (2.4) and the Hölder inequality, we see that

$$\int\limits_{B(\varrho)} |v|^3 dx = \int\limits_{B(\varrho)} |v|^{\lambda s} |v|^{2\mu} |v|^{6\gamma} dx \leq \Big(\int\limits_{B(\varrho)} |v|^s dx \Big)^{\lambda} \Big(\int\limits_{B(\varrho)} |v|^2 dx \Big)^{\mu} \Big(\int\limits_{B(\varrho)} |v|^6 dx \Big)^{\gamma}.$$

Next, the imbedding theorem with the limit exponent is used:

$$\Big(\int\limits_{B(\varrho)} |v|^6 dx \Big)^{\frac{1}{6}} \leq c \Big(\int\limits_{B(\varrho)} (|\nabla v|^2 + \frac{1}{\varrho^2} |v|^2) dx \Big)^{\frac{1}{2}}.$$

Taking into account the definition of A, we derive from the latter relation the estimate

$$\int\limits_{B(\varrho)} |v|^3 dx \leq c \varrho^{\mu} A^{\mu}(\varrho) \Big(\int\limits_{B(\varrho)} |v|^s dx \Big)^{\lambda} \Big(\int\limits_{B(\varrho)} (|\nabla v|^2 + \frac{1}{\varrho^2} |v|^2) dx \Big)^{3\gamma}.$$

Integrating in time and applying the Hölder inequality, we deduce that

$$C(\varrho) \leq c \varrho^{\mu - 2} A^{\mu}(\varrho) \Big(\int\limits_{-\varrho^2}^{0} \Big(\int\limits_{B(\varrho)} |v|^s dx \Big)^{\lambda q} dt \Big)^{\frac{1}{q}} \times \Big(\int\limits_{-\varrho^2}^{0} \Big(\int\limits_{B(\varrho)} (|\nabla v|^2 + \frac{1}{\varrho^2} |v|^2) dx \Big)^{3\gamma q'} dt \Big)^{\frac{1}{q'}}. \tag{2.5}$$

By the choice of the numbers λ, γ, and q (see the assumptions of the lemma and relations (2.3) and (2.4)),

$$\lambda q = l/s \quad \text{and} \quad 3\gamma q' = 1.$$

Hence, from (2.5) it follows that

$$C(\varrho) \leq c \varrho^{\mu - 2} A^{\mu}(\varrho) \varrho^{\frac{1}{q'}} (E(\varrho) + H(\varrho))^{\frac{1}{q'}} \Big(\int\limits_{-\varrho^2}^{0} \Big(\int\limits_{B(\varrho)} |v|^s dx \Big)^{\frac{l}{s}} dt \Big)^{\frac{1}{q}}$$

$$\leq c \varrho^{\mu - 2} A^{\mu}(\varrho) \varrho^{\frac{1}{q'}} (E(\varrho) + H(\varrho))^{\frac{1}{q'}} (\varrho^{\kappa} M^{s,l}(\varrho))^{\frac{1}{q}}.$$

Observing that

$$\mu - 2 + \frac{1}{q'} + \frac{\kappa}{q} = 0,$$

we complete the proof. Lemma 2.1 is proved.

Now, we proceed with the proof of Theorem 1.2. Let us assume that conditions (1.4) and (1.5) are fulfilled. We are going to show that if ε is sufficiently small, then z_0 is a regular point of v. Estimates (2.2) and (1.5) and Young's inequality imply that

$$C(\varrho) \leq c A^{\mu}(\varrho) \varepsilon^q (E(\varrho) + H(\varrho))^{\frac{1}{q'}} \leq c \varepsilon^q (E(\varrho) + H(\varrho) + A^{\mu q}(\varrho)).$$

It is easy to check that

$$\mu q \le 1 \quad \text{and} \quad H(\varrho) \le A(\varrho).$$

Hence,

$$C(\varrho) \le c\varepsilon^q (E(\varrho) + A(\varrho) + 1) \tag{2.6}$$

for any $0 < \varrho \le 1$.

For an appropriate choice of the cut-off function φ in the local energy inequality (1.3), we see that

$$A(R/2) + E(R/2) \le c(C^{\frac{2}{3}}(R) + C(R) + D(R)) \tag{2.7}$$

for any $0 < R \le 1$.

The last main estimate is the so-called decay estimate for pressure:

$$D(\varrho) \le c\Big[\frac{\varrho}{r} D(r) + \Big(\frac{r}{\varrho}\Big)^2 C(r)\Big] \tag{2.8}$$

for any $0 < \varrho \le r \le 1$. The reader can find a proof of inequality (2.8) in [8].

Set

$$\mathcal{E}(\varrho) = A(\varrho) + E(\varrho) + D(\varrho).$$

Taking $R = 2\theta\varrho$ in (2.7), replacing ϱ by $\theta\varrho$ and r by ϱ in (2.8), and summing the results, we arrive at the estimate

$$\mathcal{E}(\theta\varrho) \le c(C^{\frac{2}{3}}(2\theta\varrho) + C(2\theta\varrho) + D(2\theta\varrho) + \theta D(\varrho) + \frac{1}{\theta^2} C(\varrho)). \tag{2.9}$$

For $0 < \theta < 1/2$, we derive from the pressure estimate (2.8) the relation

$$D(2\theta\varrho) \le c(2\theta D(\varrho) + \Big(\frac{1}{2\theta}\Big)^2 C(\varrho)). \tag{2.10}$$

In addition, we observe that

$$C(2\theta\varrho) \le \frac{1}{4\theta^2} C(\varrho). \tag{2.11}$$

Now from (2.9)–(2.11) it follows that

$$\mathcal{E}(\theta\varrho) \le c(\theta D(\varrho) + \frac{1}{\theta^2} C(\varrho) + \frac{1}{\theta^{\frac{4}{3}}} C^{\frac{2}{3}}(\varrho)). \tag{2.12}$$

To evaluate the right-hand side of (2.12), we make use of (2.6). As a result, we get the inequality

$$\mathcal{E}(\theta\varrho) \le c\Big[\theta \mathcal{E}(\varrho) + \frac{\varepsilon^q}{\theta^2}(\mathcal{E}(\varrho) + 1) + \frac{\varepsilon^{\frac{2q}{3}}}{\theta^{\frac{4}{3}}}(\mathcal{E}(\varrho) + 1)^{\frac{2}{3}}\Big]. \tag{2.13}$$

Estimating the last term by the Young inequality, we derive from (2.13) the relation

$$\mathcal{E}(\theta\varrho) \le c\Big[\Big(\theta + \frac{\varepsilon^{\frac{q}{2}}}{\theta^2}\Big)\mathcal{E}(\varrho) + \frac{\varepsilon^{\frac{q}{2}}}{\theta^2}\Big] \tag{2.14}$$

for any $0 < \varrho \le 1$ and any $0 < \theta < 1/2$.

Now we fix θ so that

$$c\theta < \frac{1}{4}.$$

Choosing ε sufficiently small, we may assume that

$$c\frac{\varepsilon^{\frac{q}{2}}}{\theta^2} < \frac{1}{4}.$$

Thus, we rewrite (2.14) in the following form:

$$\mathcal{E}(\theta\varrho) \le \frac{1}{2}\mathcal{E}(\varrho) + c\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}. \tag{2.15}$$

Iterating (2.15), we obtain the inequalities

$$\mathcal{E}(\theta^k\varrho) \le \frac{1}{2^k}\mathcal{E}(\varrho) + c\frac{\varepsilon^{\frac{q}{2}}}{\theta^2} \tag{2.16}$$

for any nonnegative integer k. On the other hand, according to (2.6),

$$C(\theta^k\varrho) \le c\varepsilon^q(\mathcal{E}(\theta^k\varrho)+1) \le c\varepsilon^q(\frac{1}{2^k}\mathcal{E}(\varrho)+1+\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}) \le c(\frac{1}{2^k}\mathcal{E}(\varrho)+\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}).$$

The latter inequality with $\varrho = 1$ shows that

$$C(\theta^k) \le c(\frac{1}{2^k}\mathcal{E}(1)+\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}). \tag{2.17}$$

Considering (2.16) with $\varrho = 1$ and taking into account the definition of $\mathcal{E}$, we see that

$$D(\theta^k) \le \frac{1}{2^k}\mathcal{E}(\varrho) + c\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}. \tag{2.18}$$

Adding (2.17) and (2.18), we show that

$$C(\theta^k) + D(\theta^k) \le \frac{c}{2^k}\mathcal{E}(1) + c\frac{\varepsilon^{\frac{q}{2}}}{\theta^2}. \tag{2.19}$$

Let us fix an arbitrary positive number ε_0. We may choose a natural number k_0 so large that

$$\frac{c}{2^{k_0}}\mathcal{E}(1) \le \frac{\varepsilon_0}{2}.$$

It remains to reduce ε so that

$$\frac{c\varepsilon^{\frac{q}{2}}}{\theta^2} < \frac{\varepsilon_0}{2}.$$

From the latter relations and from (2.19) it follows that

$$C(\theta^{k_0}) + D(\theta^{k_0}) < \varepsilon_0.$$

It was shown in [9] and [10], Lemma 2.2, that if we choose ε_0 in an appropriate way, then the velocity v is Hölder continuous in the closure of $Q(\theta^{k_0}/2)$. Theorem 1.2 is proved.

3. Proof of Corollary 1.3

First we consider the case $\theta \in [0,1[$ (see [2] and [3]). Let $l = 2$ and

$$3 < s < \min\Big\{4, \frac{3}{1-\theta}\Big\}.$$

It can be verified that all the conditions of (1.4) are satisfied. According to (1.6), we can calculate the following estimate directly:

$$M^{s,2}(z_0,R) \le \frac{c^{\frac{2}{s}}}{1-\theta}\Big(\frac{1}{3-s(1-\theta)}\Big)^{\frac{2}{s}}\varepsilon_1^2.$$

Choosing ε_1 sufficiently small, we can provide the validity of (1.5). This implies that z_0 is a regular point of v. It remains to treat the case $\theta = 1$ (see [1]). Here we let

$$l = \frac{2}{1+\delta}$$

for some $\delta \in]0, 1/3[$ and choose s satisfying the condition

$$\frac{3}{1-\delta} \leq s < \frac{4}{1-\delta}.$$

One can show that, for such a choice of parameters s and l, all the conditions of (1.4) are fulfilled. Now the estimate of $M^{s,l}$ takes the form

$$M^{s,l}(z_0, R) \leq c^{\frac{l}{s}} \varepsilon^l \frac{1+\delta}{\delta}.$$

Repeating the previous arguments, we complete the proof of the corollary.

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