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VANISHING THEOREMS IN AFFINE, RIEMANNIAN, AND LORENTZ GEOMETRIES

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ABSTRACT. In this survey, we consider one aspect of the Bochner technique, the proof of vanishing theorems by using the Weitzenbock integral formulas, which allows us to extend the technique to pseudo-Riemannian manifolds and equiaffine connection manifolds.

CONTENTS

Introduction

1. This paper is devoted to one of the most important analytic methods of global differential geometry, which historically was common for proving so-called *vanishing theorems*. These theorems state the vanishing of some topological or geometric invariants (such as Betti numbers or the dimension of the Killing vector space) of a closed Riemannian manifold (M, g) under certain restrictions on its curvature.

This method is based on the Weitzenbock formulas comparing rough Laplacians on sections of vector bundles over a Riemannian manifold (M, g) since their difference is expressed through the curvature of the manifold (see [16, pp. 77–83] and [13, pp. 261–279]).

Such formulas were initially obtained by Bochner for the study of harmonic vector fields and exterior differential forms (see [23, 24]).

As was noted in [86], this idea traces back to Bernstein (about 1900); he found that for a harmonic function $f : \mathbb{R}^n \to \mathbb{R}$, the Laplacian satisfies the condition

$$\Delta \frac{1}{2} \|\nabla f\|^2 = \|\nabla f\|^2 \geq 0. \tag{1}$$

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This implies that the the energy $\frac{1}{2}\|\nabla f\|^2$ of the harmonic function f is a subharmonic function.

Arguing in a similar way, Bochner found (see [23]) that for a harmonic vector field $\xi : M \to TM$ on an n-dimensional Riemannian manifold (M, g), the Hodge–de Rham Laplacian satisfies the condition

$$\Delta \frac{1}{2}\|\xi\|^2 = \|\nabla\xi\|^2 + \mathrm{Ric}(\xi, \xi). \tag{2}$$

Then Eq. (2) implies that $\Delta\frac{1}{2}\|\xi\|^2 \geq 0$ only if the Ricci tensor of the manifold (M, g) satisfies the inequality $\mathrm{Ric} \geq 0$ (see [42, pp. 68–72 and 97-99]).

The Bochner formula (2) immediately implies that

$$\Delta = \nabla^*\nabla - \mathrm{Ric} \tag{3}$$

for the rough Laplacian $\nabla^*\nabla$ (see [16, p. 77]) and the Levi-Civita connection ∇ on (M, g). At present, formula (3) is called the *Weitzenbock formula* (see [16, pp. 77-83] and [13, pp. 261–279]). This name dates back to 1923 when the book [163] was published; this book contains a similar formula for p-forms. In contemporary notation, this formula has the form (see [16, p. 77] and [42, p. 98])

$$\Delta = \nabla^*\nabla - F_p, \tag{4}$$

where Δ is the Hodge–de Rham Laplacian on p-forms, $\nabla^*\nabla$ is the rough Bochner Laplacian, ∇ is the Levi-Civita connection on (M, g) extended to the bundle $\bigwedge^p M$ of p-forms on (M, g), and F_p is a symmetric endomorphism of $\bigwedge^p M$ linearly depending on the curvature tensor (M, g).

Using the Stokes theorem, we obtain from (2) the following integral formula for a closed Riemannian manifold (M, g) for the volume element $\eta = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$ of the manifold (M, g) in a local coordinate system $x^1, \ldots, x^n$:

$$\int\limits_M \|\nabla\xi\|^2 \eta = -\int\limits_M \mathrm{Ric}(\xi, \xi)\eta. \tag{5}$$

Formula (5) is called the *integral Weitzenbock formula*.

Assume that everywhere on (M, g), the Ricci tensor satisfies the inequality $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} > 0$ at at least one point. Then Eq. (3) implies that any harmonic vector field vanishes, $\xi = 0$. In concordance with the Hodge theory, this means that the first Betti number $b_1(M)$ of the manifold (M, g) vanishes; this is the first Bochner vanishing theorem.

2. In works of Lichnerowicz, Nomizu, Chen, Yano, and others, the analytic Bochner method was substantially developed and successfully applied to complex, complete Riemannian, and Lorentzian manifolds.

At present, there are at least five surveys [14, 86, 137, 162, 166] in which results obtained by using this method (the Bochner technique) are discussed. The monographs [24, 105, 167, 170] are devoted to the Bochner technique. The Bochner technique is described and/or used in almost all monographs on differential geometry published in the last three decades.

In 1982, Wu, a famous American mathematician, wrote: "By now, this technique has achieved the status of being part of the basic vocabulary of every geometer" (see [166]).

Studies using the Bochner technique are constantly performed. At present there are more than hundreds papers in which the Bochner technique was applied. Along with the classical fields of its applications, theory of differential forms and geometry of submanifolds, in recent years, the technique has been actively applied in global affine differential geometry, theory of differential-geometric structures, and theory of mappings of Riemannian manifolds. Also, ways for its applications in relativistic physics have been opened. It is important to note that a considerable part of these results was not reflected in books, and only a certain part of the results was mentioned in surveys.

In Russia, the Bochner technique was purposefully applied starting from the later 1970s by members of the Odessa school headed by professor N. S. Sinyukov for the study of geodesic mappings. Later on, the author and his two collaborates started to study the subject. The results of their studies are contained in two surveys published outside Russia; the first of them [137] is devoted to the global geometry of Riemannian structures of almost product and submersions, whereas the second [140] describes new methods in the Bochner technique and their numerous applications.

It is worth mentioning that to the present, in Russian scientific literature, there are no surveys of works carried out by using the Bochner technique. We think that this paper is necessary and hope that it will be useful not only to geometers but to physicists studying the general relativity theory.

3. In the present survey, we consider only one aspect of the Bochner technique, the proof of vanishing theorems by using the Weitzenbock integral formulas, which allow us to extend the technique to pseudo-Riemannian manifolds and equiaffine connection manifolds. Simultaneously with this, we note that within the framework of this technique, there exists another approach based on the application of the maximum principle, which will be discussed below in more detail.

The present paper does not include results obtained by using the Bochner techniques in the theory of differential-geometric structures on Riemannian manifolds and the geometry of mappings of Riemannian manifolds (see [137]) and also in the global geometry of submanifolds of Riemannian and pseudo-Riemannian manifolds to which many works are devoted for now. The latter direction will be partially considered in Secs 2.1 and 3.1.

Chapter 1

VANISHING THEOREMS IN RIEMANNIAN GEOMETRY

1.1. Analysis of Main Problems

1.1.1. P. Berard characterized the Bochner technique as a method of proving vanishing theorems as follows (see [14]): "As a matter of fact, the word technique might be misleading. On the one hand it is not so easy to explain the technical details of the proofs in which S. Bochner's ideas are used, and this is not our purpose here; on the other hand, the ideas are quite simple. Indeed, the gist is to show that some object (a harmonic form in the case of Betti numbers, a Killing vector field, . . .) satisfies an elliptic inequality provided that some curvature assumption is satisfied. The proofs then reduce to applying the maximum principle or to integrating over the manifold."

Applications of the *maximum principle* in the Bochner technique are discussed in the papers mentioned above (see also [5]). Investigations in this direction are extensively performed in foreign countries; several results were also obtain in Russia (see, e.g., [91, 114]). Note that there exist some versions of the maximum principle (see, e.g., [55]) that still await applications; this matter requires a separate paper. In this paper, we are interested in the method of proving the theorems by using integral Weitzenbock formulas, or, in the figurative words of Berard, by "integration over a manifold." We start with tho classical examples.

We present the first vanishing theorem, which can be stated by using the Gauss–Bonnet formula (see [60, p. 325])

$$\int_M K ds = 2\pi\chi(M), \tag{1.1.1}$$

where M is a closed, oriented surface in $\mathbb{R}^3$ with Gaussian curvature K, Euler characteristic $\chi(M)$, and area element ds. Formula (1.1.1) implies the following vanishing theorem.

Theorem 1.1.1. *Let M be a closed, oriented surface. If $K \geq 0$ and $K \neq 0$, then $\chi(M) > 0$ and, therefore, M is homeomorphic to the sphere S.*

For the proof, it suffices to recall that a closed surface M is homeomorphic to the sphere S with p handles and its topological invariant, the Euler characteristic, is $\chi(M) = 2(1 - p)$.

The second example of the vanishing theorem can be stated by using the Stokes formula

$$\int_M \operatorname{div} X\eta = \int_{\partial M} g(X,\mathcal{N})\eta', \tag{1.1.2}$$

where X is a smooth vector field on a compact, oriented manifold (M,g) with oriented boundary ∂M equipped with a field of outward, unit, normal vectors $\mathcal{N}$ and η and η' are the volume elements of the manifold (M,g) and its boundary ∂M, respectively.

We denote by Ric, Q, and H the Ricci tensor of the manifold (M,g), the second fundamental form, and the mean curvature of its boundary ∂M, respectively, and by $b_1(M)$, $b_{n-1}(M)$, and $b_1(M,\partial M)$, $b_{n-1}(M,\partial M)$ the absolute and relative Betti numbers, which are equal to the dimensions of one-dimensional and $(n-1)$-dimensional absolute $H^1(M)$, $H^{n-1}(M)$ and relative $H^1(M,\partial M)$, $H^{n-1}(M,\partial M)$ homology groups of the compact manifold (M,g) with boundary ∂M (see, e.g., [101, p. 51–60]). Now we can formulate the following vanishing theorem (see [170, Chap. 7, Theorems 1.9 and 1.10]).

Theorem 1.1.2 (see [170])**.** *Let (M,g) be a compact, oriented, Riemannian manifold with boundary ∂M. Then the following assertions hold*:

(1) *if* $\operatorname{Ric} > 0$ *and* $Q \leq 0$*, then* $b_1(M) = b_{n-1}(M,\partial M) = 0$;
(2) *if* $\operatorname{Ric} > 0$ *and* $H \leq 0$*, then* $b_{n-1}(M) = b_1(M,\partial M) = 0$.

For the proof, one must, first, use the Yano integral formulas

$$\int_M [\operatorname{Ric}(\xi,\xi) + g(\nabla\xi,\nabla\xi)]\eta = \int_{\partial M} Q(\xi,\xi)\eta',$$

$$\int_M [\operatorname{Ric}(\xi,\xi) + g(\nabla\xi,\nabla\xi)]\eta = (n-1)\int_{\partial M} g(\xi,\xi)H\eta',$$

which are versions of the Stokes formula (1.1.2) for harmonic vector fields ξ on the manifold (M,g) that are tangent and orthogonal to ∂M, respectively (see [170, Chap. 7, Sec. 1, formula (1.17)]).

Next, one must apply the Duff–Spencer theorem (see [33]), which states that the first absolute Betti number $b_1(M)$ is equal to the number of linearly independent harmonic vector fields on the manifold (M,g) that are tangent to its boundary ∂M and the first relative Betti number $b_1(M,\partial M)$ of the manifold M modulo ∂M is equal to the number of linearly independent harmonic vector field on the manifold (M,g) that are orthogonal to its boundary ∂M.

Recall that a vector field ξ on a closed manifold (M,g) is said to be *harmonic* (see [35, pp. 34–35]) if the 1-form ω dual to it is harmonic; by definition, this form satisfies the equation $\Delta\omega = 0$ with the Hodge–de Rham Laplacian Δ on the Riemannian manifold (M,g) (see, e.g., [16, p. 54]).

If $\partial M = 0$, then either of the Weitzenbock formulas allows one to obtain the following vanishing theorem, which is the cradle of the Bocher technique.

Theorem 1.1.3 (see [23])**.** *Let (M,g) be a closed, oriented, Riemannian manifold. If* $\operatorname{Ric} \geq 0$*, then* $b_1(M) \leq \dim M$*. If* $\operatorname{Ric} > 0$*, then* $b_1(M) = 0$.

1.1.2. A natural generalization of the class of harmonic 1-forms is the class of *exterior differential harmonic p-forms* ω, or, in another terminology, covariant, skew-symmetric, harmonic tensor fields (see, e.g., [5, pp. 240–241] and [35, pp. 54–55]). These forms are also defined by the condition $\Delta\omega = 0$. An important role played by these forms in geometry and topology is emphasized by the classical Hodge theorem for closed oriented manifold (M,g) and its generalization, the Duff–Spencer theorem for compact oriented manifolds (M,g) with boundaries ∂M. These theorems state the relationships between the Betti numbers of a manifold (M,g) and the dimension of the vector space of harmonic forms on it. Theorem 1.1.2 admits a generalization (see [170, Chap. 8, Theorems 3.4 and 3.5]) in which certain conditions imposed on the curvature of the manifold (M,g) and the second fundamental form of the boundary ∂M guarantee the vanishing of the absolute $b_p(M)$ and relative $b_{n-p}(M,\partial M)$ Betti numbers for all $0 < p < n$. A key role in the proof of this vanishing theorem is again played by integral Weitzenbock formulas for harmonic p-forms on (M, g) that are tangent or normal to ∂M.

Investigations of harmonic forms, including investigations using the Bochner techniques, are constantly carried out (see [18, 39, 84, 99, 173]).

The Bochner technique is also used for the investigation of *Killing vector fields* (see [171, pp. 35–36] and [5, pp. 243–244]) and their generalizations, so-called *exterior differential Killing p-forms*, $1 \le p \le n-1$; in another terminology, *Killing–Yano tensors* (see [18, 19, 127, 145, 152, 171]). As is known, Killing vector fields generate local one-parameter motion groups in (M, g) and hence have numerous applications (see [7]). The interest in Killing forms was motivated by the needs of relativistic physics (see [9, 31, 32, 81]). For example, Killing–Yano tensors of valence 2 are used for the description and classification of symmetries of the Dirac and Klein–Gordon–Fock equations (see [57, 102]).

The sphere of applications of the Bochner technique is steady broadening. For example, as generalizations of Killing vector fields and Killing forms, *affine Killing vector and tensor fields* were introduced and studied using the Bochner technique (see [113]).

Conformal Killing vector fields generating one-parameter groups of conformal infinitesimal transformations in (M, g) were studied by using the Bochner technique in the 1950s (see [171]). Their generalization, *conformal Killing p-forms*, introduced in the late 1960s (see [54, 146]) and studied by using the Bochner technique, have aroused the interest of geometers (see [50, 63, 131, 143]). These forms and conformal Killing vector fields (see [66, 67]) are also applied in relativistic physics (see [128, 136]).

The Bochner technique was successfully applied for studying closed conformal Killing p-forms (called also *flat p-forms*, see [126, 128] and used in relativistic physics, see [128, 136]), projective Killing (see [51, 147, 148]), special projective Killing (see [149]), and other p-forms and tensor fields (see, e.g., [30, 115, 168]).

1.1.3. Together with the theory of exterior differential forms, the theory of symmetric tensor fields (or, in another terminology, *symmetric differential forms*) was developed. The results of these studies are contained in a number of monographs (see, e.g., [13, 16, 35, 81, 103]). The most developed theory is constructed for Killing and Codazzi symmetric tensors.

Killing symmetric p-tensors φ or *Killing symmetric differential p-forms* are defined as fields of covariant symmetric tensors of valence p $(1 \le p \le n)$ satisfying the equation $\delta^*\varphi := \operatorname{sym}(\nabla\varphi) = 0$. The local geometry of these tensors is elaborated in a large amount of literature (see [16, 29, 35, 73, 100, 144, 150]). These fields are extensively used in geometry and physics (see, e.g., [81, 104]). For example, coordinates admitting the separation of variables for the Hamilton–Jacobi equation in a Lorentz manifold are connected with eigenvectors of symmetric Killing tensors of second order (see [165]).

A series of vanishing theorems for symmetric Killing tensors is known (see [122, 126, 138]).

Codazzi tensors of valence p or *Codazzi symmetric differential p-forms* are no less popular than Killing tensors (see, e.g., [16]). An example of such tensors is the second fundamental form of a hypersurface in a space of constant curvature, which satisfies the Codazzi equations (this is the reason for the name of these tensors).

Some vanishing theorems for symmetric Codazzi tensors are known (see [16, 122]).

The theory of *harmonic symmetric p-forms* is developed (see [117, 119, 121]). These forms satisfy the equation $\square\varphi = 0$, where $\square = \delta\delta^* - \delta^*\delta$ is the Laplacian and δ is the operator formally conjugate to δ^*. Components of harmonic p-forms on a locally flat Riemannian manifold (M, g) are harmonic functions; this property is also typical for harmonic exterior differential p-forms. The Weitzenbock formula $\nabla^*\nabla\varphi - \square\varphi = B_p(\varphi)$ holds. It is proved that for $p = 1$, the kernel of the operator $\square$ consists of infinitesimal harmonic transformations (see [121, 141]). The corresponding vanishing theorem is proved.

Conformal symmetric Killing tensors and relative harmonic tensors also occur in the literature (see [26, 64]); they can also be studied using the Bochner technique.

1.1.4. The above list of definitions and theorems of local and global theory of vector fields, differential exterior and symmetric p-forms, and their applications in physics can lead to disappointment in the Bochner technique and this branch of geometry.

For example, this happened in the geometry of almost Hermitian manifolds up to the moment when the accumulations of facts in the theory had been accomplished by the Gray–Hervela classification of manifolds of this type (see [40]). For this, an irreducible decomposition of the covariant derivative $\nabla\Omega$ of the fundamental form Ω of an almost Hermitian manifold was found and then sixteen classes of almost Hermitian manifolds were segregated by stepwise vanishing of components of irreducible representations. All known manifolds were included into this scheme and a series of new manifolds was obtained.

This approach is used in Sec. 1.2 for presentation of the local theory of exterior differential forms (see [131, 136, 143]).

In Sec. 1.3, we present some universal integral Weitzenbock formulas that allow one to prove a series of vanishing theorems for exterior and symmetric forms.

In Sec. 1.4, we describe eventual applications for the geometry of compact, almost Hermitian manifolds.

In concluding this section, we note that the conditions of vanishing theorems usually contain the requirement of sign-definiteness of the sectional, Ricci, or scalar curvature of a Riemannian manifold. In this case, some problems on the existence and geometric structure of such manifolds appear; the answers can be found, for example, in [151].

1.2. Local Geometry of Exterior Differential and Symmetric Forms

1.2.1. Let M be a C^∞-manifold with linear connection ∇ without torsion. We consider the $C^\infty M$-module $\mathrm{Diff}(\Lambda^p M, T^*M \otimes \Lambda^p M)$ of first-order linear differential operators on the space $C^\infty\Lambda^p M$ of $C^\infty M$-sections of the bundle $\Lambda^p M$ of exterior differential p-forms.

A first-order linear differential operator D on the space $C^\infty\Lambda^p M$ of $C^\infty M$-sections is said to be *fundamental* (see [120]) if its principal symbol (with respect to the connection ∇ extended to the bundle $\Lambda^p M$ of p-forms on M) is a projector to a pointwise $GL(n,\mathbb{R})$-irreducible subbundle of the tensor bundle $T^*M \otimes \Lambda^p M$. For the operator of exterior differentiation $d : C^\infty\Lambda^p M \to C^\infty\Lambda^{p+1} M$, these operators are $D_1 = \dfrac{1}{p+1} d$ and $D_2 = \nabla - \dfrac{1}{p+1} d$.

For any exterior differential p-form ω, the $GL(m,\mathbb{R})$-irreducible decomposition $\nabla\omega = D_1\omega + D_2\omega$ pointwise holds. This implies that D_1 and D_2 are generalized $GL(n,\mathbb{R})$-gradients (see [52, 53]). Each of them is $\mathbb{R}$-linear but is not linear with respect to multiplication by functions in the ring $C^\infty M$. The kernel of D_1 consists of closed forms and the kernel of D_2 consists of *Killing* p-forms (see [130]). These forms compose two vector spaces $\mathbf{D}^p(M,\mathbb{R})$ and $\mathbf{K}^p(M,\mathbb{R})$ over the field $\mathbb{R}$ or, in other words, $\mathbb{R}$-modules, which, in their turn, are submodules of the $\mathbb{R}$-module $\mathbf{\Omega}^p(M,\mathbb{R})$ of exterior forms on the manifold M.

In [130], it is proved that

$$\dim \mathbf{K}^p(M,\mathbb{R}) \geq \frac{n!}{p!(n-p)!}$$

on an n-dimensional manifold M with an equiprojective $GL(n,\mathbb{R})$-structure (see also Chap. 3).

In a local coordinate system $x^1, \ldots, x^n$ on a locally flat manifold M, the components $\omega_{i_1\ldots i_p}$ of a Killing p-form ω have the form (see [48, 120])

$$\omega_{i_1\ldots i_p} = K_{ki_1\ldots i_p} x^k + K_{i_1\ldots i_p}$$

for arbitrary constants $K_{ki_1\ldots i_p}$ and $K_{i_1\ldots i_p}$ that are skew-symmetric with respect to all their subscripts. Therefore, on a locally flat manifold M, the following relation holds:

$$\dim \mathbf{K}^p(M,\mathbb{R}) = \frac{(n+1)!}{(p+1)!(n-p)!}.$$

Similar expressions for components of Killing 2-forms were obtained in [145] for Euclidean spaces and in [58] for Minkowski spaces.

1.2.2. Let $C^\infty\Lambda^p M$ be the space of C^∞-sections of the bundle $\Lambda^p M$ of exterior differential p-forms over an n-dimensional $(1 \le p \le n-1)$ Riemannian manifold (M, g). The problem of the search for fundamental differential operators (in the sense of Sec. 1.2.1) such that their principal symbols (with respect to the Levi-Civita connection ∇ extended to the bundle $\Lambda^p M$ of p-forms on M) are projectors on a pointwise $O(n,\mathbb{R})$-irreducible subbundle of the tensor bundle $T^*M \otimes \Lambda^p M$ is equivalent (see [13]) to the problem of the search for a basis of the space of natural first-order Riemannian differential operators defined on the space $C^\infty\Lambda^p M$ and taking their values in the space of homogeneous tensors.

Such a basis (see [13, 128, 136]) consists of three operators of the form

$$D_1 = \frac{1}{(p+1)}d, \quad D_2 = \frac{1}{(n-p+1)}g \wedge d^*, \quad D_3 = \nabla - \frac{1}{p+1}d - \frac{1}{n-p+1}g \wedge d^*,$$

where $d^* : C^\infty\Lambda^p M \to C^\infty\Lambda^{p-1}M$ is the codifferentiation operator formally conjugate to the operator d and

$$(g \wedge d^*\omega)(X_0, X_1, \ldots, X_p) = \sum_{a=2}^{p}(-1)^a g(X_0, X_a)(d^*\omega)(X_1, \ldots, X_{a-1}, X_{a+1}, \ldots, X_p)$$

for arbitrary $\omega \in C^\infty\Lambda^p M$ and $X_0, X_1, \ldots, X_p \in C^\infty TM$. Each of these operators is $\mathbb{R}$-linear but is not linear with respect to multiplication by functions of the ring $C^\infty M$. Therefore, the kernels of the operators D_2 and D_3 form $\mathbb{R}$-modules $\mathbf{F}^p(M,\mathbb{R})$ and $\mathbf{T}^p(M,\mathbb{R})$, which are submodules of the $\mathbb{R}$-module $\mathbf{\Omega}^p(M,\mathbb{R})$. They consist of co-closed and *conformal Killing* p-forms, respectively (see [54, 146]).

For an arbitrary exterior differential p-form ω, the following $O(n,\mathbb{R})$-irreducible decomposition holds:

$$\nabla\omega = D_1\omega + D_2\omega + D_3\omega; \tag{1.2.1}$$

this implies that D_1, D_2, and D_3 are generalized $O(m,\mathbb{R})$-gradients (see [52, 53]).

The condition $\omega \in \ker D_1 \cap \ker D_3$ characterizes a p-form ω as a *flat* form (see [126, 128]); therefore, the $\mathbb{R}$-module of flat p-forms is

$$\mathbf{P}^p(M,\mathbb{R}) = \mathbf{T}^p(M,\mathbb{R}) \cap \mathbf{D}^p(M,\mathbb{R}).$$

In [128], it is proved that

$$\mathbf{P}^*(M,\mathbb{R}) = \bigoplus_{p\ge 0}\mathbf{P}^p(M,\mathbb{R})$$

is a skew-commutative, associative algebra, which is a subalgebra in $\mathbf{D}^*(M,\mathbb{R}) = \bigoplus_{p\ge 0}\mathbf{D}^p(M,\mathbb{R})$.

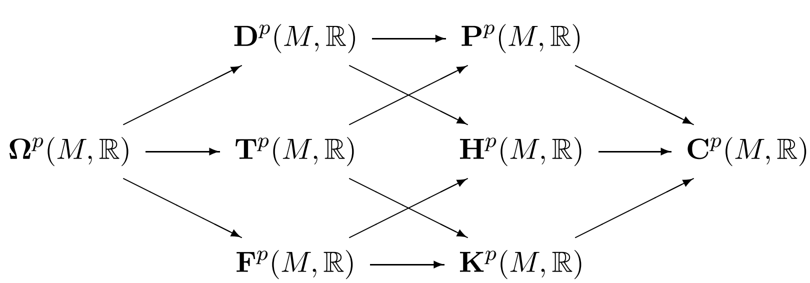
The condition $\omega \in \ker D_3 \cap \ker D_2$ characterizes a p-form ω as a *Killing* form (see [35]); therefore, the $\mathbb{R}$-module of Killing p-forms is

$$\mathbf{K}^p(M,\mathbb{R}) = \mathbf{T}^p(M,\mathbb{R}) \cap \mathbf{F}^p(M,\mathbb{R}).$$

The condition $\omega \in \ker D_1 \cap \ker D_2$ characterizes a p-form ω as a *harmonic* form (see [90, 171]); therefore,

$$\mathbf{H}^p(M,\mathbb{R}) = \mathbf{D}^p(M,\mathbb{R}) \cap \mathbf{F}^p(M,\mathbb{R}).$$

We denote by $\mathbf{C}^p(M,\mathbb{R})$ the vector space of covariantly constant (or, in other words, parallel) p-forms. The following diagram holds (see [131, 136, 143]):



In this diagram, the writing $\mathbf{F}^p(M,\mathbb{R}) \to \mathbf{K}^p(M,\mathbb{R})$ means that the $\mathbb{R}$-module $\mathbf{K}^p(M,\mathbb{R})$ is a submodule of the module $\mathbf{F}^p(M,\mathbb{R})$.

Fix a local orientation of a manifold (M,g) and consider the action of the Hodge operator (see [85, 90]) such that $* : C^\infty\Lambda^p M \cong C^\infty\Lambda^{n-p}M$ is an isomorphism and

$$*^2 = (-1)^{p(n-p)}\,\mathrm{id}_{\Lambda^p M}\,. \tag{1.2.2}$$

The following isomorphisms are obvious:

$$* : \mathbf{\Omega}^p(M,\mathbb{R}) \cong \mathbf{\Omega}^{n-p}(M,\mathbb{R}), \quad * : \mathbf{C}^p(M,\mathbb{R}) \cong \mathbf{C}^{n-p}(M,\mathbb{R}).$$

The codifferentiation operator $d^* : C^\infty\Lambda^p M \to C^\infty\Lambda^{p-1}M$ is defined by the relation (see [85, 90])

$$d^* = (-1)^{np+n+1} * d*; \tag{1.2.3}$$

therefore, by (1.2.2), we obtain the isomorphism

$$* : \mathbf{F}^p(M,\mathbb{R}) \cong \mathbf{D}^{n-p}(M,\mathbb{R}). \tag{1.2.4}$$

From this isomorphism, we obtain the well-known isomorphism of the $\mathbb{R}$-modules of harmonic forms:

$$* : \mathbf{H}^p(M,\mathbb{R}) \cong \mathbf{H}^{n-p}(M,\mathbb{R}). \tag{1.2.5}$$

The following isomorphism of the $\mathbb{R}$-modules of conformal Killing forms is also well known:

$$* : \mathbf{T}^p(M,\mathbb{R}) \cong \mathbf{T}^{n-p}(M,\mathbb{R}) \tag{1.2.6}$$

(see [63, 132]). Finally, (1.2.3) and (1.2.5) imply the isomorphism

$$* : \mathbf{P}^p(M,\mathbb{R}) \cong \mathbf{K}^{n-p}(M,\mathbb{R}) \tag{1.2.7}$$

of the $\mathbb{R}$-modules of flat and Killing forms (see [136]).

On a manifold with constant curvature $C \neq 0$, the (pointwise) sum (see [54])

$$\mathbf{T}^p(M,\mathbb{R}) = \mathbf{P}^p(M,\mathbb{R}) \oplus \mathbf{K}^p(M,\mathbb{R})$$

holds. Therefore, taking isomorphism (1.2.7) into account, we obtain the inequality

$$\dim \mathbf{T}^p(M,\mathbb{R}) \geq \frac{2n!}{p!(n-p)!}.$$

On a locally flat, n-dimensional Riemannian manifold (M,g), components of flat and conformal Killing p-forms ω in a local orthonormal coordinate system $x^1,\dots,x^n$ of the manifold have the following expressions:

$$\omega_{i_1\dots i_p} = x_{[i_1}P_{i_2\dots i_p]} + P_{i_1\dots i_p},$$
$$\omega_{i_1\dots i_p} = A_{kji_1\dots i_p}x^k x^j + B_{ji_1\dots i_p}x^j + C_{i_1\dots i_p},$$

where $P_{i_2\dots i_p}$, $P_{i_1\dots i_p}$, and $C_{i_1\dots i_p}$ are arbitrary constants skew-symmetric with respect to all their subscripts and $A_{kji_1\dots i_p}$ and $B_{ji_1\dots i_p}$ are arbitrary constants skew-symmetric with respect to the subscripts $i_1,\dots,i_p$ and satisfying some additional symmetry conditions (see [48, 128, 136]).

It is known (see [138]) that on an n-dimansional Riemannian manifold (M,g) with nonzero constant curvature C, an arbitrary conformal Killing p-form ω admits the representation

$$\omega = \omega' - \frac{1}{pC}\omega'',$$

where ω' and ω'' are Killing p-form and $(p-1)$-form, respectively. In [139], it is proved by using this representation that there exists a local coordinate system $x^1,\dots,x^n$ on (M,g) in which components of ω have the form

$$\omega_{i_1\dots i_p} = e^{(p+1)\psi}\big(A'_{ki_1\dots i_p}x^k + B'_{i_1\dots i_p}\big) - \frac{1}{C}e^{p\psi}\left(\psi_{[i_1}A''_{|k|i_2\dots i_p]}x^k + \psi_{[i_1}B''_{i_2\dots i_p]}\right) + \frac{1}{p}A_{i_1 i_2\dots i_p}$$

for $\psi = \dfrac{1}{2(n+1)} \ln|\det g|$, $\psi_i = \partial_i \psi$, and arbitrary constants $A'_{ki_1\dots i_p}$, $A''_{i_1\dots i_p}$, $B'_{i_1\dots i_p}$, and $B''_{i_2\dots i_p}$ skew-symmetric with respect to all their subscripts.

1.2.3. Let M be a C^∞-manifold with linear connection ∇ without torsion. Consider a $C^\infty M$-module $\mathrm{Diff}(S^pM, T^*M \otimes S^pM)$ of first-order, linear differential operators on the space $C^\infty S^pM$ of $C^\infty M$-sections of the bundle S^pM of differential p-forms.

A first-order, linear differential operator D on the space $C^\infty S^pM$ of $C^\infty M$-sections is said to be *fundamental* if its principal symbol (with respect to the connection ∇ extended to the bundle S^pM) is a projector on a pointwise $GL(n,\mathbb{R})$-irreducible subbundle of the tensor bundle $T^*M \otimes S^pM$. The operators $D_1 = \dfrac{1}{p+1}\delta^*$ and $D_2 = \nabla - \dfrac{1}{p+1}\delta^*$ are fundamental operators (see [120]).

For any symmetric p-form φ, the following pointwise $GL(n,\mathbb{R})$-irreducible decomposition holds:

$$\nabla\varphi = D_1\varphi + D_2\varphi.$$

This implies that D_1 and D_2 are generalized $GL(n,\mathbb{R})$-gradients (see [52, 53]). The kernels of $D-1$ and D_2 consist of Killing (see [130]) and Codazzi (see [69, p. 169], [16, p. 589], [122]) symmetric p-forms that compose two submodules $\mathbf{W}^p(M,\mathbb{R})$ and $\mathbf{V}^p(M,\mathbb{R})$ of the $\mathbb{R}$-module $\mathbf{S}^p(M,\mathbb{R})$ of symmetric differential p-forms on the manifold M.

The symmetric multiplication turns the direct sum

$$\mathbf{W}^*(M,\mathbb{R}) = \bigoplus_{p\ge 0} \mathbf{W}^p(M,\mathbb{R})$$

into the subalgebra

$$\mathbf{S}^*(M,\mathbb{R}) = \bigoplus_{p\ge 0} \mathbf{S}^p(M,\mathbb{R})$$

of the graded, commutative, associative algebra of symmetric forms on M (see [144]).

On an n-dimensional manifold M with an equiprojective $SL(n,\mathbb{R})$-structure, we have

$$\dim \mathbf{W}^p(M,\mathbb{R}) \ge \frac{(n+p-1)!}{p!(n-p)!}$$

(see [130, 138]).

In a local coordinate system $x^1,\dots,x^n$ of a locally affine manifold M, components $\varphi_{i_1\dots i_p}$ of a symmetric, Killing p-form φ have the form

$$\varphi_{i_1\dots i_p} = \sum_{q=0}^{p} A_{i_1\dots i_p j_1\dots j_q} x^{j_1}\dots x^{j_q},$$

where $A_{i_1\dots i_p j_1\dots j_q}$ are arbitrary constants symmetric with respect to the groups of subscripts $i_1,\dots,i_p$ and $j_1,\dots,j_q$; the symmetrization with respect to the subscripts $i_1,\dots,i_p,j_1,\dots,j_{q-1}$ for $q=1,\dots,p$ yields zero (see [73]). Then on the manifold M with an equiprojective $SL(n,\mathbb{R})$-structure, we have

$$\dim \mathbf{W}^p(M,\mathbb{R}) = \frac{p(p+1)^2(p+2)^2\dots(n+p-1)^2(n+p)^2}{p!(p+1)!}$$

(see [120]).

On a locally affine manifold M in a local coordinate system $x^1,\dots,x^n$, a p-form φ with components $\varphi_{i_1i_2\dots i_p} = \dfrac{\partial^p f}{\partial x^{i_1}\partial x^{i_2}\dots\partial x^{i_p}}$ is a Codazzi form for any $f \in C^\infty M$. It is known (see [69, p. 169]) that a Codazzi 2-form φ on a manifold M with an equiprojective $SL(N,\mathbb{R})$-structure has the form

$$\varphi = \nabla^2 f + \frac{1}{n-1} f\,\mathrm{Ric}$$

for any $f \in C^\infty M$.

On a (pseudo) Riemannian manifold (M, g), there exists an interrelation between symmetric Killing 2-forms and exterior, differential Killing flat p-forms; it is useful for physics (see [128, 136] and [81, pp. 339–340]). Let ω be an exterior differential p-form; we set

$$\varphi(X, Y) = \sum_{1 \le i_2 < \cdots < i_p \le n} \omega(X, e_{i_2}, \ldots, e_{i_p}) \omega(Y, e_{i_2}, \ldots, e_{i_p}),$$

where $X, Y \in T_x M$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the space $T_x M$ at an arbitrary point $x \in M$. If ω is a Killing p-form, then the associated 2-form φ is a symmetric Killing form; if ω is a flat p-form, then the form

$$\varphi' = \varphi - \frac{1}{p}(\operatorname{trace}_g \varphi) g$$

is also a symmetric Killing form.

1.2.4. An analogue of the Hodge–de Rham Laplacian on the bundle $S^p M$ of symmetric forms is the Laplacian $\Box : C^\infty S^p M \to C^\infty S^p M$ defined by the rule

$$\Box = \delta\delta^* - \delta^*\delta$$

(see [119, 144]). Compare it with the rough Laplacian $\nabla^*\nabla$.

It is easy to see that these two operators coincide if (M, g) is a locally Euclidean space.

Further (see [119, 144]), the differential operator $\Box - \nabla^*\nabla$ has order zero and can be defined by a symmetric endomorphism B_p of the bundle $S^p M$, where B_p can be algebraically (even linearly) expressed through the curvature tensor on (M, g). This can be expressed by the Weitzenbock formula $\Box = \nabla^*\nabla + B_p$.

For $p \ge 2$, the explicit expression for $\Box$ is sufficiently complicate but for $p = 1$, it has the form $\Box = \Delta - 2Q$ (see [121]), where, in local coordinates, $Q^k_j = g^{ki} R_{ij}$ (R_{ij} are local coordinates of the Ricci tensor Ric). This form of the operator $\Box$ was used by Yano (see [170, p. 40]) for the investigation of local isometries of the manifold (M, g) preserving the metric g.

Recall that in a neighborhood U of an arbitrary point $x \in M$, a vector field ξ generates a one-parametric group of local automorphisms $f_t : U \to M$, $t \in (-\varepsilon; \varepsilon) \subset \mathbb{R}$. In this case, we can define the Lie derivative

$$(L_\varepsilon \nabla)t = \nabla'(x) - \nabla(x),$$

where $\nabla'(x) = f_t^*(\nabla(f_t(x)))$.

An automorphism $f : (M, g) \to (M, g)$ is harmonic if and only if $\operatorname{trace}_g(\nabla'(x) - \nabla(x)) = 0$ for all $x \in M$. Therefore, $\operatorname{trace}_g(L_\xi \nabla) = \Delta\xi - 2Q\xi$. The following theorem holds.

Theorem 1.2.1 (see [121, 141])**.** *The Yano differential operator $\Box = \Delta - 2Q$ acting on the space of sections of the tangent bundle of a manifold (M, g) is a Laplacian of the form $\Box = \delta\delta^* - \delta^*\delta$ and its kernel is a finite-dimensional vector space of local harmonic automorphisms of the manifold (M, g).*

1.3. Vanishing Theorems in Geometry of Differential and Symmetric Forms

1.3.1. We consider the Hodge–de Rham Laplacian $\Delta = dd^* + d^*d$. It is known that on a closed, oriented Riemannian manifold, the relation

$$\ker \Delta = \mathbf{H}^p(M, \mathbb{R})$$

holds and, in addition,

$$\dim \mathbf{H}^p(M, \mathbb{R}) = b_p(M) < \infty$$

(see [85, pp. 79 and 178]). We obtain that the Poincaré duality theorem holds:

$$b_p(M) = b_{n-p}(M), \quad 1 \le p \le n - 1.$$

Now we consider a second-order differential operator

$$D_3^* D_3 : C^\infty \Lambda^p M \to C^\infty \Lambda^p M,$$

where D_3^* is the operator formally conjugate to D_3. The operator $D_3^* D_3$ is strongly elliptic for all $n \le 2p$; it coincides with the Laplacian for $n = 2p$. Moreover,

$$\ker D_3^* D_3 = \mathbf{T}^p(M, \mathbb{R})$$

on closed, oriented manifolds (see [135, 160]). Therefore, by using (1.2.5), we obtain the following analogue of the Poincaré duality theorem for Betti numbers on an n-dimensional, closed, oriented Riemannian manifold (M, g):

$$t_p = t_{n-p}, \quad t_p = \dim \mathbf{T}^p(M, \mathbb{R}).$$

By isomorphism (1.2.6), we obtain

$$\dim \mathbf{P}^{n-p}(M, \mathbb{R}) = \dim \mathbf{K}^p(M, \mathbb{R}) < \infty, \quad 1 \le p \le n-1.$$

1.3.2. We consider the Bochner $\nabla^*\nabla$ (see [42, pp. 91 and 97]) and Hodge–de Rham $\Delta = d^*d + dd^* = (d + d^*)(d^* + d)$ Laplacians related by the classical Weitzenbock formula

$$\Delta\omega = \nabla^*\nabla\omega + F_p(\omega) \tag{1.3.1}$$

(see [171, p. 52] and [163, pp. 393–397]), where

$$F_p(\omega)(X_1, \ldots, X_p) = \sum_{a=1}^{p} \mathrm{Ric}(e_j, X_a)\omega(X_1, \ldots, X_{a-1}, e_j, X_{a+1}, \ldots, X_p)$$
$$- \sum_{a<b}^{1\ldots p} R(e_j, e_k, X_a, X_b)\omega(X_1, \ldots, X_{a-1}, e_j, X_{a+1}, \ldots, X_{b-1}, e_k, X_{b+1}, \ldots, X_p)$$

for arbitrary $\omega \in C^\infty\Lambda^p M$, $X_1, \ldots, X_p \in C^\infty TM$, an orthonormal basis $\{e_1, \ldots, e_n\}$, and the curvature R and Ricci Ric tensors of the manifold (M, g).

Using the identity

$$\int_M g(\nabla\omega, \nabla\omega)\eta = \int_M g(\nabla^*\nabla\omega, \omega)\eta,$$

where (M, g) is a closed, oriented manifold, and the Weitzenbock formula (1.3.1), we obtain the following integral formula:

$$\int_M \left\{ \mathcal{F}_p(\omega, \omega) + \frac{1}{p}\|\nabla\omega\|^2 - \frac{1}{p(p+1)}\|d\omega\|^2 - \|d^*\omega\|^2 \right\} \eta = 0 \tag{1.3.2}$$

(see [171, p. 58]), where $\mathcal{F}_p(\omega, \omega) = g(F_p(\omega), \omega)$ is a quadratic form $\mathcal{F}_p : C^\infty\Lambda^p M \otimes C^\infty\Lambda^p M \to C^\infty M$ whose coordinates are the components of the curvature R and Ricci Ric tensors of the manifold (M, g).

Formula (1.3.2) implies that the inequality

$$\mathcal{F}_p(\omega, \omega) > 0$$

is an obstruction for the existence of harmonic forms ω on a closed Riemannian manifold.

Meyer proved (see [68]) that the quadratic form $\mathcal{F}_p(\omega, \omega)$ is positive definite if the symmetric curvature operator $\bar{R} : C^\infty\Lambda^2 M \to C^\infty\Lambda^2 M$ is positive definite; this operator is defined by the formula

$$\bar{\omega}_{ij} = R_{ijkl}\omega^{kl},$$

where ω^{kl} are the local contravariant components of the form $\omega \in C^\infty\Lambda^2 M$ and R_{ijkl} are the components of the curvature tensor R (see [42, pp. 39 and 94–97] and [61, p. 333]). Therefore, the following theorem holds.

Theorem 1.3.1 (see [166, 167]). *Let (M, g) be a closed, oriented Riemannian manifold of dimension n with positive definite curvature operator. Then for all $1 \le p \le n-1$, the following relation holds*:

$$b_p(M) = \dim \mathbf{H}(M, \mathbb{R}) = 0.$$

Note (see [44]) that a compact, four-dimensional Riemannian manifold (M, g) with positive definite curvature operator is diffeomorphic to a sphere or a real projective space. Similar result for other dimensions can be found in [42, pp. 39–52], [151], and [41, pp. 283–285].

It is easy to prove (see [61, p. 333]) that the positive definiteness of the curvature operator $\bar{R}$ of (M, g) implies the positive definiteness of the section curvature. Generally speaking, the inverse assertion is invalid. A valid inverse assertion is proved in [25] for an n-dimensional Riemannian manifold (M, g) isometrically embedded in $\mathbb{R}^{n+2}$.

Corollary 1.3.1 (see [25]). *Let (M, g) be a closed, oriented, n-dimensional Riemannian manifold with positive section curvature isometrically embedded in $\mathbb{R}^{n+2}$. Then for all $1 \le p \le n-1$, the following relation holds*:

$$b_p(M) = \dim \mathbf{H}^p(M, \mathbb{R}) = 0.$$

A vanishing theorem for Killing forms similar to Theorem 1.3.1 can be proved (see [171, p. 59]) since for a Riemannian manifold (M, g) isometrically embedded in $\mathbb{R}^{n+2}$, the negative definiteness of the section curvature implies the negative definiteness of the curvature operator $\bar{R}$ (see [25]).

Formula (1.3.2) was obtained more than fifty years ago. It has been used for the study of harmonic and Killing forms until now (see, e.g., [19, 20, 115]).

For example, by using the representation of the quadratic form $F_p(\omega, \omega)$ in the form

$$\mathcal{F}_p(\omega, \omega) = \frac{n-2p}{n-2} R_{ij} \omega^{ii_2 \dots i_p} \omega^j_{i_2 \dots i_p} + \frac{p-1}{(n-1)(n-2)} S \omega^{i_1 \dots i_p} \omega_{i_1 \dots i_p} - \frac{p-1}{2} W_{ijkl} \omega^{iji_3 \dots i_p} \omega^{kl}_{i_3 \dots i_p}$$

(see [171, p. 72]), where S is the scalar curvature and W_{ijkl} is the Weyl tensor, the following assertions are proved in [34].

Corollary 1.3.2 (see [34]). *Let (M, g) be a compact, locally irreducible Riemannian manifold with nonnegative Ricci curvature. If $n = 4$ and $S - 6W_2 \ge 0$ or $n > 4$ and $S - (n-2)(n-1)W_2 \ge 0$ for the operator W_2 on $\Lambda^2 M$ induced by the Weyl tensor, then either the Betti numbers $b_p(M)$ vanish for $1 \le p \le n-1$ or the covering space for (M, g) is a compact symmetric space.*

Corollary 1.3.3 (see [34]). *Let (M, g) be a compact, locally irreducible Riemannian manifold such that the sum of two minimal eigenvalues of its Ricci tensor* Ric *is nonnegative. If $[\mathrm{Ric} \wedge \mathrm{Id}, W_2] = 0$ and $(n-2)(n-1)W_2 - S \ge 0$, then either the Betti numbers $b_p(M)$ vanish for $2 \le p \le n-2$ or the covering space for (M, g) is symmetric.*

There exist different analogues of formula (1.3.2) for p-forms on a compact, oriented manifold (M, g) with boundary ∂M (see [170]). For example, for an arbitrary p-form ω, which touches the boundary of the manifold, the corresponding integral Weitzenbock formula has the form

$$\int_M \Big[\mathcal{U}_p(\omega, \omega) + \mathcal{Z}_p(\omega, \omega) + \mathcal{W}_p(\omega, \omega) - p(p+1)\|D_1\omega\|^2 - (n-p)(p+1)\|D_2\omega\|^2 + (p+1)\|D_3\omega\|^2 \Big] \eta$$

$$= \int_{\partial M} Q_p(t\omega, t\omega)\eta' \quad (1.3.3)$$

(see [143]) for a quadratic form $Q_p : C^\infty \Lambda^p \partial M \otimes C^\infty \Lambda^p \partial M \to C^\infty M$ whose coefficients are components of the second fundamental form Q of the boundary ∂M of the manifold (M, g) and a decomposition $F_p = U_p + Z_p + W_p$ irreducible with respect to actions of $O(n, \mathbb{R})$.

Recall that a p-form ω is said to be tangent to the boundary ∂M if

$$\omega(\mathcal{N}, X_2, \dots, X_p) = 0 \forall X_2, \dots, X_p \in C^\infty T\partial M$$

(see [170, p. 126]).

Integral Weitzenbock formula (1.3.3), in contrast to (1.3.2), can be used for the study of four classes of forms: harmonic, conformal Killing, Killing, and flat forms on compact, oriented Riemannian manifolds,

manifolds of zero scalar curvature, Einstein manifolds, and conformally flat manifolds under one restriction: the form must be tangent to the boundary of the manifold or $\partial M = \emptyset$. For example, the following vanishing theorem holds.

Theorem 1.3.2 (see [135])**.** *On an n-dimensional $(n > 3)$, compact, oriented, conformally flat Riemannian manifold (M, g) with convex boundary ∂M, there is no conformal Killing p-forms $(1 \le p \le n-1)$ tangent to ∂M if the Ricci curvature of the manifold is negative.*

For the proof, we note that for $2p \le n$, the following inequality holds:

$$\mathcal{U}_p(\omega, \omega) + \mathcal{Z}_p(\omega, \omega) \le -r_{\max} \|\omega\|^2 < 0, \tag{1.3.4}$$

where $-r_{\max}$ is the maximal negative value of the Ricci curvature on (M, g). We also note that if the boundary of the manifold (M, g) is convex, then the inequality $Q_p(t_\omega, t_\omega) \ge 0$ holds. This inequality, (1.3.4), and the condition $\mathcal{W}_p(\omega, \omega) = 0$ contradict the integral Weitzenbock formula (1.3.3). This implies that $\dim \mathbf{T}(M, \mathbb{R}) = t_p = 0$ for $2p \le n$. If we apply the relation $t_p = t_{n-p}$, then we obtain the required result.

Similarly, by using the integral Weitzenbock formula (1.3.3), one can prove vanishing theorems for harmonic, Killing, and flat p-forms on a compact, oriented manifold with boundary.

1.3.3. Let (M, g) be a closed, oriented Riemannian manifold and S^pM be the bundle of symmetric p-forms over (M, g). It is easy to prove that the fundamental operators D_1 and D_2 on the space of $C^\infty S^pM$-section of the bundle S^pM have the form

$$D_1^* D_1 = \frac{1}{(p+1)^2} \delta\delta^*, \quad D_2^* D_2 = \frac{p}{p+1} \left(\nabla^* \nabla - \frac{1}{p+1} \delta\delta^* \right).$$

The operator $D_1^* D_1$ is elliptic (see [103, p. 53]); hence, the operator $D_2^* D_2$ is also elliptic. Therefore (see [85, pp. 176–178] and [16, p. 632]), we have

$$\dim \mathbf{W}^p(M, \mathbb{R}) = w_p < \infty, \quad \dim \mathbf{V}^p(M, \mathbb{R}) = v_p < \infty.$$

For symmetric differential forms, the following integral Weitzenbock formula holds (see [122, 126]):

$$\int_M \left\{ B_p(\varphi, \varphi) + \frac{1}{p(p+1)} \|\delta^* \varphi\|^2 - \frac{1}{p} \|\nabla \varphi\|^2 - \|\delta \varphi\|^2 \right\} \eta = 0, \tag{1.3.5}$$

where $B_p(\omega, \omega)$ is a quadratic form $B_p : C^\infty S^pM \otimes C^\infty S^pM \to C^\infty M$ whose coefficients are the components of the curvature R and Ricci Ric tensors of the manifold (M, g). It is proved in [122] that the sign of the quadratic form $B_p(\omega, \omega)$ is opposite to the sign of the second-type symmetric curvature operator $\overset{\circ}{R} : C^\infty S^2M \to C^\infty S^2M$, which is defined in local coordinates by the formula

$$\bar{\varphi}_{ij} = R_{ikjl} \varphi^{kl}, \quad \varphi^{kl} = g^{ki} g^{lj} g \varphi_{ij},$$

where $g^{kl} = (g_{kl})^{-1}$ (see [42] and [13, p. 278]). The following vanishing theorem holds.

Theorem 1.3.3 (see [122])**.** *Let (M, g) be a closed, oriented, n-dimensional Riemannian manifold with positive-definite curvature operator of second type. Then the relation*

$$w_p = \dim \mathbf{W}^p(M, \mathbb{R}) = 0$$

holds.

A similar assertion for Codazzi p-forms can be stated only in particular cases, for example, for forms $\varphi \in \ker D_2 \cap \ker \delta$ or for traceless Codazzi p-forms (see [122]).

Note that for $p = 2$, the form

$$B_2(\varphi, \varphi) = \frac{1}{2} \sum_{i<j} K(e_i, e_j)(\lambda_i - \lambda_j)^2,$$

where $\varphi(e_i, e_j) = \lambda_i \delta_{ij}$, δ_{ij} is the Kronecker symbol, and $\{e_1, \dots, e_n\}$ is an orthonormal basis of the space $T_x M$ at an arbitrary point $x \in M$ (see [16, p. 529]).

Corollary 1.3.4 (see [126]). *Let φ be a symmetric Killing 2-form on a closed, oriented Riemannian manifold (M, g) with nonpositive section curvature K. Then $\nabla\varphi = 0$. If, in addition, $K < 0$ at some point, then $\varphi = \lambda g$, where $\lambda = \text{const}$.*

In the considered case where $p = 2$, the co-closedness of a Codazzi 2-form is equivalent to the condition that its trace is constant. Therefore, the following assertion holds.

Theorem 1.3.4 (see [16]). *Any Codazzi 2-form with constant trace on a closed, oriented Riemannian manifold (M, g) with nonnegative section curvature K is parallel. If, in addition, $K > 0$ at some point, then $\varphi = \lambda g$, where $\lambda = \text{const}$.*

Let (M, g) be a compact, oriented Riemannian manifold with boundary ∂M such that at any point $x \in \partial M$, a unit, orthogonal to ∂M, and outward directed vector $\mathcal{N}_x$ is given. If for a symmetric 2-form φ, the condition $\varphi(\mathcal{N}, X) = 0$ holds for all $X \in T\partial M$, we say that φ is tangent to the boundary ∂M of the manifold (M, g). In this case, φ satisfies the integral formula

$$\int\limits_M \left\{ \frac{1}{2} \sum_{i<j} K(e_i, e_j)(\lambda_i - \lambda_j)^2 + \frac{1}{p(p+1)} \|\delta^* \varphi\|^2 - \frac{1}{p} \|\nabla\varphi\|^2 - \|\delta\varphi\|^2 \right\} \eta = \int\limits_{\partial M} Q_2(t\varphi, t\varphi)\eta'$$

(see [138]). This formula allows one to prove the following vanishing theorem.

Theorem 1.3.5 (see [127, 138]). *Let (M, g) be an n-dimensional, compact, oriented Riemannian manifold with boundary ∂M equipped with a Killing 2-form φ, which is tangent to the boundary ∂M.*

(1) *If the manifold (M, g) has convex boundary and $K \le 0$, then either (M, g) is locally a Riemannian-product manifold or $\varphi = \lambda g$ for $\lambda = \text{const}$.*
(2) *If the manifold (M, g) has convex boundary, $K \le 0$, and the inequality $K < 0$ holds at least at one point, then $\varphi = \lambda g$ for $\lambda = \text{const}$.*
(3) *If the manifold (M, g) has strongly convex boundary and $K \le 0$, then $\varphi = 0$.*

Note that the *strongly convex boundary* is the boundary ∂M of the manifold (M, g) whose second fundamental form is strongly positive-definite.

We can easily state a similar theorem for Codazzi 2-forms with constant trace, which generalizes Theorem 1.3.4.

Taking into account the relationship between symmetric and exterior Killing differential forms, we obtain the following assertion.

Corollary (see [126, 138]). *Let an n-dimensional, compact, oriented Riemannian manifold (M, g) with boundary ∂M be equipped with an exterion Killing p-form, which is tangent for the boundary ∂M, $1 \le p \le m - 1$.*

(1) *If the manifold (M, g) has convex boundary and $K \le 0$, then $\|\omega\| = \text{const}$ and either (M, g) is locally a Riemannian-product manifold or ω satisfies the relation*

$$\sum_{1 \le i_2 < \dots < i_p \le n} \omega(X, e_{i_2}, \dots, e_{i_p}) \omega(Y, e_{i_2}, \dots, e_{i_p}) = \frac{p!}{n} \|\omega\|^2 g(X, Y), \tag{1.3.6}$$

where $X, Y \in T_x M$ and $\{e_1, \dots, e_n\}$ is an orthonormal basis of the space $T_x M$ at an arbitrary point $x \in M$.
(2) *If the manifold (M, g) has convex boundary, $K \le 0$, and the inequality $K < 0$ holds at least at one point, then ω satisfies Eq. (1.3.6).*
(3) *If the manifold (M, g) has strongly convex boundary and $K \le 0$, then $\omega = 0$.*

Note that there exists a similar assertion for flat p-forms.

1.3.4. Recall that the Laplacian $\Box$ on the space $C^\infty S^p M$ is defined by the formula (see [117, 119]) $\Box = \delta\delta^* - \delta^*\delta$. The Weitzenbock formula $\Box = \nabla^*\nabla + F_p$ has the form

$$(\nabla^*\nabla\varphi)_{i_1\dots i_p} - (\Box(\varphi))_{i_1\dots i_p} = \sum_{a=1}^{p} R^k_{i_a}\varphi_{i_1\dots i_{a-1}ki_{a+1}\dots i_p} - g^{kj}g^{lm}\sum_{a<b}^{1\dots p} R_{i_a j i_b m}\varphi_{i_1\dots i_{a-1}ki_{a+1}\dots i_{b-1}ji_{b+1}\dots i_p}.$$

On a locally flat manifold (M, g), the equation $\Box\varphi = 0$ becomes

$$\sum_{k=1}^{n}\frac{\partial^2\varphi_{i_1\dots i_p}}{(\partial x^k)^2} = 0;$$

therefore, all components of a symmetric p-form are harmonic functions and symmetric forms belonging to the kernel of the Laplacian $\Box$, in analogy with exterior harmonic differential forms, are said to be *harmonic* (see [117]). Denote the vector space of symmetric harmonic p-forms by $\mathbf{U}^p(M,\mathbb{R})$. The following vanishing theorem can be proved by using the Weitzenbock formula.

Theorem 1.3.6 (see [117, 119])**.** *Let (M, g) be a closed, oriented, n-dimensional Riemannian manifold whose second-type curvature operator is positive definitie. Then the relation*

$$u_p = \dim \mathbf{U}^p(M,\mathbb{R}) = 0$$

holds.

Theorem 1.3.7 (see [121, 141])**.** *Let (M, g) be a closed Riemannian manifold. The vector space of local, harmonic automorphisms of this manifold is finite-dimensional. If* $\mathrm{Ric} < 0$*, then it is zero-dimensional.*

1.4. Applications to Hermitian Geometry

1.4.1. Consider exterior Killing differential forms and flat 2-forms of maximal rank on a $2n$-dimensional, closed Riemannian manifold (M, g). Note that a 2-form ω of maximal rank on such a manifold exists if the manifold is oriented.

Recall that an almost Hermitian manifold (M, J, g) with Killing fundamental form $\Omega = (\sqrt{2n}\|\omega\|)^{-1}\omega$ is said to be *approximately Kählerian* (see, e.g., [40]).

Theorem 1.4.1 (see [126])**.** *Let a $2n$-dimensional, closed Riemannian manifold (M, g) be equipped with a Killing 2-form ω of maximal rank. Let one of the following two conditions hold*:

(1) *either $K < 0$,*
(2) *or $K > 0$ and the manifold (M, g) is not a Riemannian-product manifold.*

Then (M, g) is an approximately Kählerian manifold with fundamental form $\Omega = (\sqrt{2n}\|\omega\|)^{-1}\omega$.

If (M, g) is a manifold satisfying the conditions above, then a flat 2-form ω turns this manifold into an almost Hermitian manifold with fundamental form $\Omega = (\sqrt{2n}\|\omega\|)^{-1}\omega$. However, a flat closed 2-form ω annihilates the Nijenhuis tensor

$$N(X, Y) = [wX, wY] + w^2[X, Y] - w[X, wY] + w[wX, Y]$$

for $g(wX, Y) = \omega(X, Y)$ and all $X, Y \in C^\infty TM$ (see [60]). Therefore, an almost Hermitian manifold becomes a Kählerian manifold (see [61]).

Theorem 1.4.2 (see [126])**.** *Let a $2n$-dimensional, closed Riemannian manifold (M, g) be equipped with a flat 2-form ω of maximal rank. Let one of the following two conditions hold*:

(1) *either $K < 0$,*
(2) *or $K \le 0$ and the manifold (M, g) is not a Riemannian-product manifold.*

Then (M, g) is a Kählerian manifold with fundamental form $\Omega = (\sqrt{2n}\|\omega\|)^{-1}\omega$.

If this 2-form ω is recurrent, i.e., $\nabla\omega = \theta \otimes \omega$ for $\theta \in C^\infty T^*M$, then the following assertion holds.

Theorem 1.4.3 (see [116, 118])**.** *Any* $2n$*-dimensional Riemannian manifold* (M, g) *with sign-definite sectional curvature equipped with a recurrent* 2*-form of maximal rank is a Kählerian manifold.*

All these assertions are generalizations of the following classical result of Shirokov [106]: an even-dimensional Riemannian manifold equipped with a covariantly constant exterior differential 2-form is a Kählerian manifold.

Chapter 2

VANISHING THEOREMS IN LORENTZ GEOMETRY

2.1. Analysis of Main Problems

2.1.1. The progress in causality and singularity theories and in the study of black holes in general relativity achieved in the last decades generated great interest in global Lorentz geometry among physicists and mathematicians. As a rule, their attention was attracted by the well-known monograph of Beem and Ehrlich [10], which bridges modern differential geometry "in the whole" and mathematical physics of general relativity. This monograph and numerous papers on this matter study various synthetic methods used by the authors, from classical and elementary to modern topological. For this reason, each chapter of this monograph contains a comprehensive introduction, in which the authors explain the choice of the methods used.

The endeavor to obtain a universal method for solving such problems led scientists to turn their attention to the Bochner technique, which was very productive in global Riemannian geometry. However, the following obstruction appeared (see [92]): the Laplacian $\Delta f = \operatorname{div}(\operatorname{grad} f)$ for a function $f \in C^\infty M$ on a Lorentz manifold (M, g) is not an elliptic operator and, therefore, the maximum principle underlying the classical Bochner technique in inapplicable. In support of this, it suffices to recall that "harmonic functions" on a compact Lorentz manifold, in contrast to the Riemannian case, are not constant. For this reason, much effort was made to prove integral formulas for sections of different vector bundles over Lorentz manifolds, which would serve as a tool for obtaining results similar to results in the Riemannian case. Conventionally, we can indicate two main directions of investigations.

The representatives of the first direction study, using the traditional Bochner technique, the geometry of $(n-1)$-dimensional, closed, space-like submanifolds (M', g') of a Lorentz manifold (M, g), whose metric g' is, by definition, positive definite (see [3, 4, 6, 8, 10, 12, 70, 82, 169]). As a rule, submanifolds of constant or even zero mean curvature (so-called maximal submanifolds) are considered since this theme for the Riemannian case is well developed. For example, in Yano's monograph [170], the whole chapter is devoted to the study of the geometry of such submanifolds. Investigations in this direction are important since similar problems appear in general relativity and cosmology (see [28, 36, 45, 65]).

In their turn, the representatives of the second direction study vector fields and infinitesimal transformations of closed Lorentz manifolds by using integral formulas obtained by Bochner more than fifty years ago (see [22, 23, 72, 92–94]). In this chapter, we focus our attention on this second direction.

2.1.2. Describe some peculiar properties of the version of the Bochner technique used in the present chapter.

The applicability of the classical Bochner technique to Lorentz manifolds satisfying the closedness and sign-definiteness conditions is restricted by a number of circumstances. For example, recall the following well-known result of Galloway [38]: an n-dimensional ($n \geq 3$), closed Lorentz manifold satisfying the condition $\operatorname{Ric}(X, X) > 0$ for all unit time-like vectors $X \in C^\infty TM$ has no closed space-like hypersurfaces. However, manifolds possessing such hypersurfaces are very important in physics (see [69]); therefore, it is desirable to abjure the closedness condition for Lorentz manifolds. As a paradigm, the Hawking vanishing theorem was chosen (see [47, p. 164]); this theorem is yet classical in general relativity.

Theorem 2.1.1. *The following conditions for the space-time* (M, g) *are mutually exclusive*:

(1) *there exists a closed, space-like hypersurface in* M;
(2) $\operatorname{div} \mathcal{N} \geq 0$ *for the unit normal vector field* $\mathcal{N}$ *in* M;
(3) $\operatorname{Ric}(\xi, \xi) \geq 0$ *for any time-like vector* ξ;
(4) (M, g) *is complete with respect to time-like geodesics directed to the past.*

To use the scheme prescribed by this theorem for the study of objects on manifolds with Lorentz metrics "in the whole," it is necessary to apply the Stokes theorem (see [69]), which allows one to ignore the signature of the metric. Moreover, it is necessary to improve the obtained integral formulas by using the group representation theory. As a result, a number of useful theorems appeared (see [47, 123, 133, 134, 142]).

2.1.3. Chapter 2 consists of four sections. Section 2.1 contains the analysis of main problems in Lorentz geometry "in the whole." In Sec. 2.2, we recall necessary facts of local Lorentz geometry. Section 2.3 is devoted to analogues of the Weitzenbock formulas on Lorentz manifolds and vanishing theorems for conformal Killing vector fields, completely umbilic and maximal space-like hyper-distributions on Lorentz manifolds. In Sec. 2.4, we discuss some possible applications in relativistic hydro- and electrodynamics.

2.2. Local Geometry of Time-Oriented Lorentz Manifolds

2.2.1. Consider an n-dimensional Lorentz manifold (M, g) with the metric g of signature $(-++\cdots+)$ and the Levi-Civita connection ∇.

Let a continuous, nonvanishing imaginary-unit vector field ξ be defined in a neighborhood U of the manifold M. If $U = M$, then (M, g) is said to be a *time-oriented Lorentz manifold* or n*-dimensional space-time* (see [10]).

Any tangent vector space $\mathbf{E} = T_xM$ in the domain U of a vector space is the orthogonal sum of the subspaces $\mathbf{V} = \operatorname{span}\{\xi_x\}$ and $\mathbf{H} = \mathbf{V}^{\perp}$. Denote by $v_x : T_xM \to \mathbf{V}$ and $h_x : T_xM \to \mathbf{H}$ the orthogonal projectors; then the metric $q = g_x$ of the space $\mathbf{E} = T_xM$ has the form $q = q^h \oplus q^v$ for $q^h = q(h, h)$ and $q^v = q(v, v)$; moreover, q^h is a positive-definite quadratic form on the $(n-1)$-dimensional vector space $\mathbf{H}$.

The orthogonal group $O(q)$ defined by the form q and acting on the vector space splits into the product $O(q) = O(q^h) \times O(q^v)$ of the orthogonal groups acting on $\mathbf{H}$ and $\mathbf{V}$, respectively.

The imaginary-unit vector field ξ defined on the Lorentz manifold (M, g) a tensor field $A_\xi = -\nabla\xi$ of type $(1, 1)$ such that at any point $x \in M$, the tensor $A_{\xi|x}$ of the field A_ξ belongs to the space $\mathbf{E}^* \otimes \mathbf{H}$. By [123, 129], the following $O(q^h) \times O(q^v)$-irreducible decomposition holds for the space $\mathbf{E}^* \otimes \mathbf{H}$:

$$\mathbf{E}^* \otimes \mathbf{H} = \Lambda^2\mathbf{H} \oplus S_0^2\mathbf{H} \oplus \mathbf{R}q^h \oplus (\mathbf{V}^* \otimes \mathbf{H}^*).$$

Therefore, the tensor field A_ξ can be represented as the following sum of four pointwise $O(q^h) \times O(q^v)$-irreducible tensor components:

$$g(A_\xi X, Y) = \omega(X, Y) + \sigma(X, Y) + \frac{1}{n-1}\theta g(hX, hY) + g(\xi, X)g(A_\xi\xi, Y), \tag{2.2.1}$$

where, in particular, $\omega_x \in \Lambda^2\mathbf{H}$, $\sigma_x \in S_0^2\mathbf{H}$, and $(n-1)^{-1}\theta_x g_x^h \in \mathbf{R}q^h$ at any point $x \in U$. The following relations hold:

$$\omega(X, Y) = \frac{1}{2}\left[g(hX, A_\xi hY) - g(A_\xi hX, hY)\right], \tag{2.2.2}$$

$$\sigma(X, Y) = -\frac{1}{n-1}\theta g^h(X, Y) - \frac{1}{2}\left[g(hX, A_\xi hY) + g(A_\xi hX, hY)\right], \tag{2.2.3}$$

$$\theta = -\operatorname{trace} A_\xi \tag{2.2.4}$$

for any vector fields $X, Y \in C^\infty TM$.

2.2.2. For an imaginary-unit vector field ξ on an n-dimensional Lorentz manifold (M,g), we consider a space-like hyper-distribution $H : x \in U \to \mathbf{H} \subset T_xM$ for any points x of the domain U of the vector field ξ. The integrability tensor A of the distribution H (see [89]) is defined by the formula

$$A(X,Y) = \frac{1}{2}v[\nabla_{hX}hY - \nabla_{hY}hX]$$

for any $X, Y \in C^\infty TM$ (see [124]). It is known (see [123, 134]) that $A = -\omega \otimes \xi$ and, therefore, $\|A\|^2 \le 0$.

The second fundamental form G of the distribution H is defined by the formula

$$G(X,Y) = \frac{1}{2}v\Big[\nabla_{hX}hY + \nabla_{nY}hX\Big]$$

for all $X, Y \in C^\infty TM$ (see [123, 133, 134]). The vector

$$H^h = (n-1)^{-1} \operatorname{trace}_{g^h} G$$

is called the *mean-curvature vector* of the distribution H (see [89]). If everywhere in the domain of the distribution H we have $A = 0$ and $H^h = 0$, then the distribution H is integrable with $(n-1)$-dimensional *maximal submanifolds* of the Lorentz manifold (M,g). The maximality (instead of the minimality) is explained by the fact that the metric of the target space is a Lorentz metric (see, e.g., [28, 70]). Therefore, if everywhere in the domain of the distribution H we have $H^h = 0$, then H is called a *maximal distribution*, but not minimal as in the Riemannian case (see [116]).

A distribution H is said to be *umbilic* (see [116]) if everywhere in its domain, we have $G = g^h \otimes H$. An integrable umbilic distribution H defines on (M,g) a fibration consisting of $(n-1)$-dimensional umbilic integral manifolds. Metrics of such manifolds (M,g) are described, for example, in [104].

Note that the traceless part G_0 of the second fundamental form G of the hyper-distribution H is defined by the formula

$$G_0 = G - \frac{1}{n-1} g^h \otimes \operatorname{trace}_{g^h} G.$$

It is known (see [133, 134]) that $G_0 = -\sigma \otimes \xi$ and, therefore, $\|G_0\|^2 \le 0$. In addition, it is easy to prove that $\theta^2 = -\|\operatorname{trace}_{g^h} G\|^2 \le 0$.

2.3. Vanishing Theorems in Lorentz Geometry

2.3.1. For an imaginary-unit vector field ξ on an n-dimensional Lorentz manifold (M,g), the following equation holds (see [133, 134]):

$$-\operatorname{div}(A_\xi \xi) = \operatorname{Ric}(\xi,\xi) - \|\omega\|^2 + \|\sigma\|^2 + \frac{1}{n-1}\theta^2 + \xi(\theta); \tag{2.3.1}$$

for $n = 4$, it is called the *Landau–Raychaudhuri equation* (see [46]). For the vector field $\theta\xi$, we have

$$\operatorname{div}(\theta\xi) = \xi(\theta) + \theta^2;$$

therefore,

$$\operatorname{div}(A_\xi \xi - \theta\xi) = \operatorname{Ric}(\xi,\xi) - \|\omega\|^2 + \|\sigma\|^2 - \frac{n-2}{n-1}\theta^2. \tag{2.3.2}$$

Consider an oriented domain U of a Lorentz manifold (M,g) with boundary ∂U. The volume element in (M,g) is defined by the formula $\eta = \sqrt{|\det(g)|}$. An outward vector field N transversal to ∂U defines an orientation of the boundary ∂U and a volume element η' (see [130]). For $\Omega = \operatorname{trace}[(A_\xi\xi - \theta\xi) \otimes \eta]$, by (2.3.2), the Stokes theorem

$$\int_U d\Omega = \int_{\partial U} \Omega$$

takes the following form (see [123, 133, 134]):

$$\int\limits_U \left[\mathrm{Ric}(\xi,\xi) - \|\omega\|^2 + \|\sigma\|^2 - \frac{n-2}{n-1}\theta^2\right]\eta = \int\limits_{\partial U} \mathrm{trace}\left[(A_\xi\xi - \theta\xi)\otimes\eta\right]. \tag{2.3.3}$$

If ∂U is an $(n-1)$-dimensional, closed, time-like submanifold (see [10]) with orthogonal vectors of the field N at all points and the imaginary-unit vector field ξ touches the boundary ∂U at any of its points, then formula (2.3.3) can be written as follows (see [123, 133, 134]):

$$\int\limits_U \left[\mathrm{Ric}(\xi,\xi) - \|\omega\|^2 + \|\sigma\|^2 - \frac{n-2}{n-1}\theta^2\right]\eta = \int\limits_{\partial U} Q'(\xi,\xi)\eta', \tag{2.3.4}$$

where Q' is the second fundamental form of the boundary ∂U (see [10]) defined by the Gauss equation

$$Q'(X',Y') = g(\nabla_{X'}Y', N), \quad X',Y' \in C^\infty T\partial U.$$

Now let the boundary ∂U of a domain U of a Lorentz manifold (M,g) be an $(n-1)$-dimensional, space-like submanifold (see [10]). We also assume that a time-like, unit vector field ξ orthogonal to the boundary ∂U at all points is defined in U. Then formula (2.3.3) takes the form

$$\int\limits_U \left[\mathrm{Ric}(\xi,\xi) - \|\omega\|^2 + \|\sigma\|^2 - \frac{n-2}{n-1}\theta^2\right]\eta = (n-1)\int\limits_{\partial U} H'\eta', \tag{2.3.5}$$

where $H' = (n-1)^{-1}\,\mathrm{trace}_{g'}\,Q'$ is the mean curvature of the boundary ∂U (see [123, 133, 134]).

2.3.2. A vector field ζ defined on a pseudo-Riemannian manifold (M,g) is called a *conformal Killing vector field* if the one-parametric diffeomorphism group generated by it consists of local conformal transformations (see [60]). This means that

$$(L_\zeta g)(X,Y) = g(A_\zeta X, Y) + g(X, A_\zeta Y) = 2\lambda g(X,Y),$$

where L_ζ is the Lie derivative along the vector field ζ, $A_\zeta = -\nabla\zeta$ is a tensor field, $\lambda = -n^{-1}\,\mathrm{trace}\,A_\zeta$, and $X, Y \in C^\infty TM$ are arbitrary. If $\lambda = 0$ everywhere in U, then the diffeomorphism group consists of local isometries and ζ is a Killing vector field (see [60]).

Assume that a time-like, conformal Killing vector field ζ does not vanish in a domain U; we denote its unit vector by ξ. We also assume that the domain U has time-like boundary ∂U such that the vector field ξ is tangent to ∂U at all points. The integral formula (2.3.4) allows one to prove the following vanishing theorem.

Theorem 2.3.1 (see [133, 134]). *There is no n-dimensional* $(n \geq 3)$ *Lorentz manifolds* (M,g) *equipped with a time-like, conformal Killing vector field* ζ *satisfying the following conditions*:

(1) *there exists a domain* U *with time-like boundary* ∂U *in* (M,g);
(2) *the vector field* ζ *does not vanish in* U *and touches* ∂U *at all points*;
(3) $\mathrm{Ric}(\zeta,\zeta) \leq 0$ *in* U;
(4) *the second fundamental form of the boundary satisfies the inequality* $Q'(\zeta,\zeta) \geq 0$ *everywhere except for at least one point* $x \in \partial U$ *where* $Q'(\zeta,\zeta) > 0$.

We replace condition (3) by a weaker condition: the second fundamental form of the boundary takes nonnegative values in all time-like directions; then it can be proved that the vector field ζ is recurrent.

Now we assume that a domain U has space-like boundary ∂U such that a time-like, conformal Killing vector field ζ with unit vector ξ is orthogonal to ∂U. Formula (2.3.5) allows one to prove the following theorem.

Theorem 2.3.2 (see [133, 134]). *There is no n-dimensional* $(n \geq 3)$ *Lorentz manifolds* (M,g) *with time-like, conformal Killing vector field* ζ *satisfying the following conditions*:

(1) *there exists a domain U with space-like boundary ∂U in (M, g) in which the vector field does not vanish and is orthogonal to the boundary at all points*;
(2) $\mathrm{Ric}(\zeta, \zeta) \leq 0$ *in* U;
(3) *the mean curvature of the boundary is nonnegative except for at least one point, where it is positive.*

We replace condition (3) by a weaker condition: the mean curvature of the boundary is nonnegative; then the vector field ζ is recurrent (see [17]).

A time-like vector field ζ on a Lorentz manifold (M, g) is said to be *harmonic* (see [174]) if the tensor field A_ζ satisfies the conditions

$$g(A_\zeta X, Y) - g(X, A_\zeta Y) = 0, \quad \mathrm{trace}\, A_\zeta = 0$$

for any vector fields $X, Y \in C^\infty TM$. Assume that a time-like, harmonic vector field ζ does not vanish in a domain U. Arguments similar to that for the case of conformal Killing vector fields allow one to prove the following vanishing theorem.

Theorem 2.3.3 (see [133, 134])**.** *There is no n-dimensional ($n \geq 3$) Lorentz manifolds (M, g) equipped with a time-like, harmonic vector field ζ satisfying the following conditions*:

(1) *there exists a domain U with time-like boundary ∂U in (M, g) such that the vector field ζ does not vanish in U and touches ∂U at all points*;
(2) $\mathrm{Ric}(\zeta, \zeta) \geq 0$ *in* U;
(3) *the second fundamental form of the boundary satisfies the inequality $Q'(\zeta, \zeta) \leq 0$ everywhere except for at least one point $x \in \partial U$ where $Q'(\zeta, \zeta) < 0$.*

2.3.3. Using the relation $G_0 = -\sigma \otimes \xi$, we rewrite formula (2.3.5) in the form

$$\int\limits_U \Big[\mathrm{Ric}(\xi, \xi) + \|A\|^2 - \|g_0\|^2 - (n-1)(n-2)\|H^h\|^2 \Big] \eta = (n-1) \int\limits_{\partial U} H' \eta'$$

for a space-like distribution H defined in a domain U with space-like boundary ∂U such that H touches ∂U at all points. In this case, the boundary ∂U is a closed integral manifold of the distribution H. Assume that H is a completely umbilic distribution. Then the following vanishing theorem holds.

Theorem 2.3.4 (see [123, 133, 134])**.** *In an n-dimensional ($n \geq 3$) Lorentz manifold, there is no domains U with space-like boundaries such that the following conditions hold*:

(1) *the time-like Ricci curvature in the domain U is nonpositive*;
(2) *the boundary of the domain U has nonnegative mean curvature everywhere except for at least one point where this curvature is positive*;
(3) *there exists an umbilic, space-like distribution in U such that the boundary ∂U is an integral manifold of it.*

Consider a maximal integrable distribution H in the domain U. Assume that one of the integral manifolds (M, g) is a closed, simply-connected submanifold (M', h') of the Lorentz manifold (M, g). Then the following equation holds (see [134]):

$$\int\limits_{M'} \big[\mathrm{Ric}(\xi, \xi) - \|G_0\|^2 \big] \eta' = 0.$$

This equation implies the following vanishing theorem.

Theorem 2.3.5 (see [134])**.** *There exists no fibrations consisting of maximal sections and possessing closed fibers in a domain U of an n-dimensional ($n \geq 0$) Lorentz manifold if the time-like Ricci curvature is nonnegative at all points of U except for at least one point, where it is positive.*

2.4. Vanishing Theorems in Relativistic Hydrodynamics

2.4.1. In this section, we describe some results on dynamics of a relativistic fluid in a domain U of the four-dimensional space-time (M, g) assuming that an imaginary-unit vector field ξ defined in this domain is formed by time-like tangent vectors to streamlines of the relativistic fluid. The kinematics of an infinitesimal volume element of the fluid is described by the following quantities (see [64, 69]): the *vorticity* (or *rotation*) *tensor* (2.2.2), the *tensor of* (*in-plane*) *shear* (2.2.3), the *expansion of fluid world lines* (2.2.4), and the vector field $\xi' = -A_\xi \xi$ of 4-*acceleration*. Recall that a motion of a relativistic fluid in a four-dimensional space-time is said to be *rotationless* if $\omega = 0$, *shearless* if $\sigma = 0$, *expansionless* if $\theta = 0$, and *rigid in the Born sense* if $\sigma = \theta = 0$ (see [91]).

By the Einstein equation $\operatorname{Ric} - 2^{-1} s g = T$ for the scalar curvature s of a Lorentz manifold (M, g) and the energy-momentum tensor of matter T, we have

$$\operatorname{Ric}(\xi, \xi) = \rho - 2^{-1} s,$$

where $\rho = T(\xi, \xi)$ is the mass-energy density of the fluid (see [69]). The quantities

$$\rho(U) = \int_U \rho\eta, \quad s(U) = \int_U s\eta$$

are called the *total mass-energy density of the relativistic fluid* in the domain U and the *total scalar curvature* of the domain U of the space-time (M, g).

Let the fluid flow in the domain U and orthogonally pass the space-like boundary ∂U. Then the following integral formula holds (see [134]):

$$\rho(U) - \frac{1}{2} s(U) = \int_U \left[\|\omega\|^2 - \|\sigma\|^2 + \frac{3}{2}\theta^2 \right] \eta + (n-1) \int_{\partial U} H' \eta', \tag{2.4.1}$$

which is equivalent to formula (2.3.5).

If we assume that the domain U has time-like boundary ∂U, then we can easily obtain the following integral formula for the fluid filling the domain U and flowing on ∂U (see [134]):

$$\rho(U) - \frac{1}{2} s(U) = \int_U \left[\|\omega\|^2 - \|\sigma\|^2 + \frac{3}{2}\theta^2 \right] \eta + \int_{\partial U} Q'(\xi, \xi) \eta', \tag{2.4.2}$$

which is equivalent to formula (2.3.4).

2.4.2. The problem on Born-rigid motions of relativistic fluids has attracted the attention of scientists for a long time (see [47]). However, since the system of differential equations is parabolic, this problem is rather difficult. The Bochner technique allows one to find the conditions of existence of shearless motions, including Born-rigid motions.

Consider a shearless flow of a fluid in the domain U, whose streamlines orthogonally intersect the space-like boundary ∂U. In this case, by (2.4.1), the following theorem holds.

Theorem 2.4.1 (see [123, 133, 134]). *For a four-dimensional space-time (M, g), the following conditions are incompatible*:

(1) *there exists a domain U with space-like boundary ∂U having nonnegative mean curvature*;
(2) *there exists a shearless flow of the relativistic fluid filling the domain U and orthogonally intersecting its boundary*;
(3) *the double total mass-energy density of the relativistic fluid in the domain U is greater that the total scalar curvature of this domain.*

The condition of sign-definiteness of the mean curvature H^h of the boundary ∂U can be replaced by the following stronger condition: ∂U is a maximal submanifold in (M, g).

Consider a shearless flow of a relativistic fluid filling a domain U and streaming along its time-like boundary ∂U. By (2.4.1), the following theorem holds.

Theorem 2.4.2 (see [123, 133, 134]). *For a four-dimensional space-time* (M, g), *the following conditions are incompatible*:

(1) *there exists a shearless flow of a relativistic fluid filling a domain* U *and streaming along its time-like boundary* ∂U;
(2) *for the normal vector field* $\mathbf{N}$ *of the boundary* ∂U *and the pressure* p *of the relativistic fluid, the inequality* $\mathbf{N} \geq 0$ *holds*;
(3) *the double total mass-energy density of the relativistic fluid in the domain* U *is greater that the total scalar curvature of this domain.*

Rotationless and expansionless flows of relativistic fluids were also studied. Taking Theorem 2.3.3 into account, we can state the following theorem.

Theorem 2.4.3 (see [134]). *In a four-dimensional space-time, there is no rotationless and expansionless (or compressionless) flows of relativistic fluids which orthogonally intersect a given three-dimensional, simply-connected, space-like section at whose points the time-like Ricci curvature is nonnegative.*

2.4.3. Consider a flow of a relativistic fluid in a domain U of a space-time (M, g) whose streamlines orthogonally intersect a closed, simply-connected, umbilic section (M', g'). We assume that the relativistic fluid is charged and the electromagnetic field is described by the "special Maxwell equations" of the form

$$(\nabla_X F)(Y, Z) = \frac{4\pi}{3}[g(X, Z)g(J, Y) - g(X, Y)g(J, Z)],$$

where F is the tensor of electromagnetic field, J is the current 4-vector, and $X, Y \in C^\infty TM$ (see [125, 129]); therefore, $F^h = F(h, h)$ is the tensor of magnetic field strength. Then the following theorem holds.

Theorem 2.4.4 (see [134]). *Let* U *be a domain of a space-time, where the sectional curvatures in all space-like* 2*-directions are nonpositive. Let a flow of a charged fluid in* U *intersect a simply-connected, completely umbilic section. If the electromagnetic field of the fluid is described by the special Maxwell equations, then the norm of the tensor of magnetic strength is constant along this section.*

If the flow is shearless, then the section (M', g') is necessarily completely unbilic and, therefore, this condition in Theorem 2.4.4 can be omitted.

Chapter 3

VANISHING THEOREMS IN AFFINE DIFFERENTIAL GEOMETRY

3.1. Analysis of Problems of Studies

3.1.1. Affine differential geometry is one of the important fields of geometry. Since the first works of Blaschke in the first quarter of the last century it has attracted the attention of geometers. In Germany (1923), in the Soviet Union (1959, 1960, and 1977), in Japan (1991), and in the USA (1994), books [2, 21, 80, 104, 107, 130] especially devoted to affine differential geometry were published. Starting from the first Oberwolfach conference (1986; see [1]), international conferences devoted to affine differential geometry were conducted.

It should be noted that whereas affine differential geometry was the traditional object of studies for Russian mathematicians, which gradually lost its appeal at the end of the last century, on the contrary, starting from that time, one observes a sharp increase in studies in the field of affine differential geometry outside Russia.

The impetus of this renewed interest was due to the lecture [74] of K. Nomizu, one of the classical geometers, at Münster University (1982); the title of the lecture was "What is affine differential geometry?" and Klingenberg in [56] call this title grandiose.

In his lecture, Nomizu suggested the concept according to which by affine differential geometry one must understand the geometry of an n-dimensional smooth manifold M with n-form η nonvanishing on M and torsion-free connection ∇ such that $\nabla\eta = 0$.

After that, there arose a series of works of Nomizu (see, e.g., [75–79]), which ended with his book [80]. The well-known geometers such as Yau, Calabi, Simon, and others promote this new direction. The first results of the studies carried out after that were summarized in the lecture [108] of Simon (1988), and two years later (1990), this was done by Nomizu himself (see [76]). At present, there are tens of works of the "new wave" devoted to affine differential geometry.

Also, in parallel with local affine differential geometry, differential geometry "in the large" of submanifolds M_n of the affine space A_m was developed outside Russia (see, e.g., [96, 98, 109, 110, 172]). Here, as a rule, hypersurfaces $M_n \subset A_{n+1}$ with a metric, for example, the Berwald–Blaschke metric, were considered and, therefore, their geometry is constructed similarly to that of hypersurfaces of Euclidean space.

One should not allow that this found no response in Russia. As an example, we mention the paper [87] of Pogorelov in which he gave an answer to the following question posed by Calabi in his lecture at the first Oberwolfach conference on affine differential geometry: "What is a complete, strictly convex, affine minimal hypersurface $M_n \subset A_{n+1}$?" Note that minimal surfaces of the 3-dimensional affine space were already studied (see, e.g., [107, pp. 227–231]). The elliptic paraboloid is known as their classical example. It is interesting that the answer to the Calabi question is the same surface, but now for the case $n \geq 2$. Other results devoted to the geometry of surfaces of such a kind can be found in the survey [161].

In this paper, we give a review of results in global differential geometry of a submanifold M equipped with the Nomizu structure (η, ∇). The results considered have one specific feature: they are obtained by using an affine analogue of the Bochner technique, which uses integral formulas that relate the curvature of a manifold with actions of differential operators. Similar formulas were deduced by using the Stokes formula and do not assume the existence of a metric on the manifold considered.

For the first time, this method was used by Grotemeyer (1952) who applied the Stokes formula to a vector field ξ defined in a domain U of a surface $M_2 \subset A_3$ bounded by a smooth curve γ. The integral formulas deduced by using this formula were applied for finding characteristics of two-dimensional spheres (see [43]). Later on, the Stokes formula for a vector field ξ on a compact oriented manifold M with boundary ∂M equipped with an affine connection was used by Ishihara. On the basis of this formula, the formulas used for studying "in the large" infinitesimal affine transformations of a manifold and special concircular vector fields were obtained (see [49]). In the former, as well as in the latter cases, these are effective analogues of the Weitzenbock integral formulas, since their integrands contain curvatures of the surface and the Ricci tensor of the manifold considered, respectively.

Further development of application of the affine analogue of the Bochner technique can be found in the works [109] of Simon and [98] of Schwenk; in the $(n+1)$-dimensional $(n \geq 2)$ affine space A_{n+1}, they studied the global differential geometry of hypersurface M_n with the Blaschke metric by using the Laplacian. After that, there arose a series of works of Russian geometers devoted to the affine analogue of the Bochner formula (see [130, 153, 154, 156–159]), which we review here.

3.1.2. The present chapter consists of four sections. In Sec. 3.1, we analyze the problems of studies in affine differential geometry "in the large." In Sec. 3.2, we present the necessary facts from local affine differential geometry. In Sec. 3.3, we present affine analogues of the Weitzenbock integral formulas, and using them, we prove the vanishing theorems. In Sec. 3.4, we present applications of the obtained results to Lorentz geometry.

3.2. Local Equiaffine Differential Geometry of Vector Fields

3.2.1. Let M be a connected, differentiable C^∞-manifold of dimension n and $L(M)$ be the bundle of linear frames on M with the structural group $GL(n, \mathbb{R})$. We define the $SL(n, \mathbb{R})$-structure on M as the principal $SL(n, \mathbb{R})$-subbundle of the bundle $L(M)$. It is well known that the $SL(n, \mathbb{R})$-structure is exactly the volume element η on M, i.e., an n-form η distinct from zero everywhere (see [59]).

There is the known problem of putting in correspondence to each G-structure on the manifold M a uniquely defined linear connection ∇ reducible to G (see [27]).

A torsion-free linear connection ∇ reducible to $SL(n, \mathbb{R})$ is said to be *equiaffine* (see [81]): it can be characterized by the condition $\nabla\eta = 0$. In this case, the pair (∇, η) is called the *equiaffine structure* on M, and the geometry of M with the equiaffine structure (∇, η) is called *affine differential geometry* (see [74]).

At each point $x \in M$, the curvature tensor R of the equiaffine connection ∇ admits the $SL(n, \mathbb{R})$-invariant decomposition in the form

$$R = \frac{1}{n-1} \operatorname{Ric} \wedge \operatorname{id}_{TM} + W,$$

where Ric is the Ricci tensor and W is the tensor of the Weyl projective curvature of the connection ∇ (see, e.g., [158]).

In accordance with this decomposition, we distinguish between two classes of equiaffine structures: the *Ricci-flat structures* for which $\operatorname{Ric} \equiv 0$ and the *equiprojective structures* for which $W \equiv 0$ (see [81]). In the latter case, as is known (see [112]), the manifold M is projectively flat, i.e., it admits a diffeomorphism onto an n-dimensional affine space which transforms geodesics of M into the corresponding straight lines. Such a diffeomorphism is called a *projective* or *geodesic mapping*. Because of the group property of projective mappings (see [112]), any two projectively flat manifolds admits a projective mapping onto each other. Therefore, all manifolds with equiprojective structure belong to the same projective space class (see [112]).

3.2.2. A self-diffeomorphism of the manifold M is an automorphism of the $SL(n, \mathbb{R})$-structure if and only if it preserves the volume element η. Let X be a vector field on M. The function $\operatorname{div} X$ defined by the equation $(\operatorname{div} X)_\eta = L_X\eta$, where L_X is the Lie derivative in the direction of X, is called the divergence of X with respect to the n-form η (see [60]).

Obviously, X is an infinitesimal automorphism of the $SL(n, \mathbb{R})$-structure if and only if $\operatorname{div} X = 0$. Such a vector field X is said to be *solenoidal*.

For a vector field X, we define the tensor field $A_X = L_X - \nabla_X$ as the field of endomorphisms of the tangent bundle TM (see [60]). The formula $\operatorname{trace} A_X = -\operatorname{div} X$ is directly verified (see [2]).

At each point $x \in M$, we have the $SL(n, \mathbb{R})$-invariant decomposition

$$A_X = \left(-\frac{1}{n} \operatorname{div} X\right) \operatorname{id}_{TM} + \dot{A}_X$$

(see [158]); in accordance with it, we distinguish between two classes of vector fields on M: solenoidal vector fields that compose a subalgebra of the Lie $\mathbb{R}$-algebra of vector fields on M (see [158]) and special concircular vector fields for which, by definition (see, e.g., [97], p. 322), we have

$$A_X = \left(-\frac{1}{n} \operatorname{div} X\right) \operatorname{id}_{TM}.$$

It is directly verified that special concircular vector fields on the manifold M with the equiaffine structure (η, ∇) compose an $\mathbb{R}$-module $\mathbb{S}(M, \mathbb{R})$. Moreover, the following theorem holds.

Theorem 3.2.1 (see [158]). *An equiaffine structure (η, ∇) on an n-dimensional manifold M is equiprojective if and only if on M there exist n linearly independent, special concircular vector fields.*

This assertion generalizes the fact well known for a Riemannian manifold of constant sectional curvature (see [37]).

In conclusion of this subsection, we introduce one concept useful for what follows. The roots $\lambda_1(x), \dots, \lambda_n(x)$ of the characteristic polynomial

$$P[\lambda(x)] = \det[\lambda \operatorname{id}_{TM} - A_X](x)$$

are called the principal curvatures of a vector field X at a point $x \in M$.

In the case of a special concircular vector field X, the following relation holds:

$$\lambda_1(x) = \dots = \lambda_n(x) = -\frac{1}{n} \operatorname{trace} A_X,$$

and for a solenoidal vector field X, we have

$$\lambda_1(x) + \dots + \lambda_n(x) = 0$$

at any point $x \in M$.

Integral curves of the vector field Y which defines the direction of the curvature of the field X at each point $x \in M$, i.e., $A_X Y_x = \lambda(x) Y_x$, are called *affine curvature lines of* X. Moreover, the lines defined by the vector field X and passing through points of the development of each its affine curvature line on the tangent space compose a developed surface (see [155]).

3.2.3. Consider an n-dimensional C^∞-manifold M with an equiaffine structure (η, ∇) and an arbitrary geodesic $\gamma : J \subset R \to M$ on M parameterized by an affine parameter t. In this case,

$$\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0$$

for the tangent vector field $d\gamma/dt$ of the geodesic γ.

Similarly to the Riemannian case (see [130]), a differential p-form ω on M is called a *Killing form* if the $(p-1)$-form

$$i_{\frac{d\gamma}{dt}} \omega = \operatorname{trace}\left(\frac{d\gamma}{dy} \otimes \omega\right)$$

is covariantly constant along any geodesic γ. This means that $d\omega = (p+1)\nabla\omega$; the latter is equivalent to the condition $\nabla\omega \in C^\infty\Lambda^{p+1}M$, where $C^\infty\Lambda^{p+1}M$ is the space of sections of the bundle $\Lambda^{p+1}M$ of differential $(p+1)$-forms over M. Obviously, the set of Killing p-form composes an $\mathbb{R}$-module denoted by $\mathbb{R}^p(M, \mathbb{R})$.

On a manifold M with an $SL(n, \mathbb{R})$-structure, we can define an affine analogue of the Riemannian Hodge operator, the isomorphism $* : C^\infty\Lambda^p TM \to C^\infty\Lambda^{n-p}M$ of the vector bundle $\Lambda^p TM$ of skew-symmetric p-tensors onto the bundle $\Lambda^{n-p}M$ of exterior differential $(n-p)$-forms. In particular, we have

$$\omega = *(\xi_1 \wedge \dots \wedge \xi_p) = *alt(\xi_1 \otimes \dots \otimes \xi_p),$$

that is,

$$\omega = i_{\xi_1 \wedge \dots \wedge \xi_p} \eta.$$

Taking the covariant derivative of the differential p-form ω in direction of an arbitrary vector field, we obtain $\nabla\omega \in C^\infty\Lambda^{n-p+1}M$ whenever all $\xi_1, \dots, \xi_p$ are special concircular vector fields.

Theorem 3.2.2 (see [130])**.** *Let M be an n-dimensional manifold with an equiaffine structure (η, ∇) and $\xi_1, \dots, \xi_p$ be p linearly independent special concircular vector fields on M for $0 < p < n$. Then the $(n-p)$-form ω dual to the tensor field $\xi_1 \wedge \dots \wedge \xi_p$ with respect to the volume n-form η is a Killing form.*

Therefore,

$$\dim \mathbb{R}^p(M, \mathbb{R}) \geq \frac{n!}{p!(n-p)!}$$

on an arbitrary n-dimensional manifold M with an equiprojective structure (η, ∇).

As we already mentioned, if M is an n-dimensional manifold with a flat affine connection ∇, then in a neighborhood of an arbitrary point $x \in M$ of it, there always exists a local coordinate system $x^1, \dots, x^n$ in which the components $\omega_{i_1 \dots i_p}$ of the Killing p-form ω have the form

$$\omega_{i_1 \dots i_p} = K_{ki_1 \dots i_p} x^k + K_{i_1 \dots i_p} \tag{3.2.1}$$

for arbitrary constants $K_{ki_1 \dots i_p}$ and $K_{i_1 \dots i_p}$ are skew-symmetric in all their subscripts (see [114, 119]).

Let $f : M \to \bar{M}$ be a projective diffeomorphism of n-dimensional manifolds with equiprojective $SL(n, \mathbb{R})$-structures and $\bar{\omega}$ be a Killing p-form on $\bar{M}$. Then it is directly verified that the p-form $\omega = e^{-(p+1)\psi}(f^* \bar{\omega})$ is a Killing form for $\psi = (n+1)^{-1} \ln(\bar{\eta}/\eta)$. Taking (3.2.1) into account, we conclude that an arbitrary Killing p-form on a manifold M with an $SL(n, \mathbb{R})$-structure, i.e., a manifold admitting a projective mapping onto the N-dimensional affine space, has the components

$$\omega_{i_1 \dots i_p} = e^{(p+1)\psi}(K_{ki_1 \dots i_p} x^k + K_{i_1 \dots i_p}).$$

This implies the relation

$$\dim \mathbb{R}^p(M, \mathbb{R}) = \frac{(n+1)!}{(p+1)!(n-p)!}.$$

It is easily verified that the operator $*$ defines an isomorphism between the spaces of special concircular vector fields and Killing $(n-1)$-forms.

Theorem 3.2.3 (see [130])**.** *For an n-dimensional manifold M with an equiaffine structure (η, ∇), the spaces $\mathbb{S}(M, \mathbb{R})$ and $\mathbb{R}^{n-1}(M, \mathbb{R})$ are $*$-isomorphic.*

3.3. Vanishing Theorems for Manifolds with an $SL(n, \mathbb{R})$-Structure

3.3.1. Let M be a compact, n-dimensional manifold with boundary. The boundary ∂M is a closed, $(n-1)$-dimensional submanifold of M whose tangent space $T_x \partial M$ is a subspace of $T_x M$ at each point $x \in \partial M$.

Define a vector field $\mathcal{N}$ along ∂M as a section of the tangent bundle TM such that at each point $x \in \partial M$, the vector $\mathcal{N}_x$ is transversal to $T_x \partial M$.

The assignment of an $SL(n, \mathbb{R})$-structure on the *n-dimensional manifold M* implies that of an $SL(n-1, \mathbb{R})$-structure on ∂M. Indeed, let η denote the volume n-form of M. Then we can define the volume form η' on ∂M by setting $\eta'(e_2, \dots, e_n) = \eta(\mathcal{N}_x, e_2, \dots, e_n)$ for oriented adapted frames $\{\mathcal{N}_x, e_2, \dots, e_n\}$ such that $T_x \partial M = \operatorname{span}\{e_2, \dots, e_n\}$ at all points of $x \in \partial M$.

Let ∇ be an equiaffine connection on M. Then for arbitrary vector fields X' and Y' tangent to ∂M, the covariant derivative decomposes into the direct sum

$$\nabla_{X'} Y' = \nabla'_{X'} Y' + Q(X', Y')\mathcal{N},$$

where the mapping $(X', Y') \to \nabla_{X'} y' = \operatorname{Pr}_{T\partial M} \nabla_{X'} Y'$ defines the linear, torsion-free connection ∇' on ∂M (see [75, 81]) and the mapping $(X'Y') \to Q(X', Y')\mathcal{N}$ defines the bilinear symmetric form $Q_x : T_x \partial M \times T_x \partial M \to \mathbb{R}$ at each point $x \in \partial M$. In accordance with the general theory (see [61, 81]), Q_x is called the *second fundamental form of* ∂M at the point x.

Note that the connection ∇' and the form Q depend on the choice of the field $\mathcal{N}$. For example, replacing this field by another vector field $\tilde{\mathcal{N}} = Z + f\mathcal{N}$ framing the boundary of the manifold ∂M, for any nonzero $f \in C^\infty \partial M$ and arbitrary $X', Y', Z' \in C^\infty T\partial M$, we obtain

$$Q = f\tilde{Q}, \quad \nabla'_{X'} Y' = \tilde{\nabla}'_{X'} Y' + \tilde{Q}(X', Y')\tilde{\mathcal{N}}.$$

Moreover, if the boundary ∂M is *nondegenerate* with respect to $\mathcal{N}$, i.e., $\det[Q] \neq 0$, then ∂M is also nondegenerate with respect to the field $\tilde{\mathcal{N}} = Z + fN$. If Q_x is identically equal to zero at each point $x \in \partial M$, then the boundary is a *totally geodesic submanifold of* M (see [61]). As is seen from the above, this property is independent of the choice of the field $\mathcal{N}$. The following theorem is proved by using the Stokes theorem.

Theorem 3.3.1 (see [130]). *Let M be a compact, n-dimensional manifold with boundary ∂M and an equiaffine structure (η, ∇). Then the following integral equation holds for $\xi \in C^\infty TM$:*

$$\int_M \Big\{ \operatorname{Ric}(\xi, \xi) + \operatorname{trace}(A_\xi)^2 - (\operatorname{trace} A_\xi)^2 \Big\}\eta = \int_{\partial M} i_{(\operatorname{trace} A_\xi)\xi - A_\xi \xi}\eta. \tag{3.3.1}$$

Let ξ be a tangent vector field for ∂M; then we can deduce the following integral equation from (3.3.1):

$$\int_M \Big\{ \operatorname{Ric}(\xi, \xi) + \operatorname{trace}(A_\xi)^2 - (\operatorname{trace} A_\xi)^2 \Big\}\eta = \int_{\partial M} Q(\xi, \xi)\eta'. \tag{3.3.2}$$

In particular, if ξ is a special concircular vector field defined on M and tangent to ∂M, then we have

$$\int_M \left\{ \operatorname{Ric}(\xi, \xi) - \frac{n-1}{n}(\operatorname{div} \xi)^2 \right\}\eta = \int_{\partial M} Q(\xi, \xi)\eta'. \tag{3.3.3}$$

On the other hand, if ξ is a special concircular vector field defined on M and transversal to ∂M at each point ∂M, then we can deduce the following integral equation from (3.3.1):

$$\int_M \left\{ \operatorname{Ric}(\xi, \xi) - \frac{n-1}{n}(\operatorname{div} \xi)^2 \right\}\eta = -\frac{n-1}{n}\int_{\partial M} f(\operatorname{div} \xi)\eta', \tag{3.3.4}$$

where $\xi = Z' + f\mathcal{N}$ for certain $Z' \in C^\infty T\partial M$ and $f \in C^\infty \partial M$.

3.3.2. Let M be a compact, n-dimensional manifold, ω be an $(n-1)$-form on M, and $\mathcal{N}$ be an outward vector field along ∂M. For an arbitrary point $x \in \partial M$ of the boundary, we assume that $X_2, \dots, X_n$ are linearly independent vectors from $T_x M$. Then $X_a = e_a + \lambda_a \mathcal{N}_x$ for $a = 2, \dots, n$. As a result, we have

$$\omega(X_2, \dots, X_n) = t\omega + \sum_{a=2}^{n} (-1)^{a+1} \lambda_a (n\omega)(e_2, \dots, \hat{e}_a, \dots, e_n),$$

where $t\omega = \omega(e_2, \dots, e_n)$ stands for the *tangent component of the form* ω and

$$(n\omega)(e_2, \dots, \hat{e}_a, \dots, e_n) = \omega(\mathcal{N}, e_2, \dots, \hat{e}_a, \dots, e_n)$$

are components of its *normal component*. The form $\omega \in C^\infty \Lambda^{n-1} M$ is said to be normal (tangent) to the boundary ∂M if $t\omega = 0$ (respectively, $n\omega = 0$) at each point ∂M.

Consider a special concircular vector field $\xi = *\omega$ for a Killing $(n-1)$-form ω normal to the boundary ∂M. In this case, the integral equation (3.3.3) holds. Therefore, the following theorem holds.

Theorem 3.3.2 (see [130]). *Let M be a compact, n-dimensional manifold $(n \geq 2)$ with an equiaffine structure (η, ∇) and ω be a Killing $(n-1)$-form defined on M and normal to the boundary ∂M.*

(1) *If for arbitrary $X \in C^\infty TM$ and $X' \in C^\infty T\partial M$, the inequalities* $\operatorname{Ric}(X, X) \leq 0$ *and* $Q(X', X') \leq 0$ *hold, then* $\nabla\omega = 0$.
(2) *If for arbitrary $X \in C^\infty TM$ and $X' \in C^\infty T\partial M$, we have* $\operatorname{Ric}(X, X) \geq 0$ *and* $Q(X', X') \leq 0$ *and at at least one point $x \in M$, the inequality* $\operatorname{Ric}(X, X) < 0$ *holds, then ω is a 0-form.*

The condition $\operatorname{Ric}(X, X) \leq 0$ in item (1) of Theorem 3.3.2 can be replaced by a stronger condition assuming that the structure (η, ∇) is Ricci-flat. In turn, the condition $Q(X', X') \geq 0$ in items (1) and (2) can be omitted assuming that the boundary ∂M is a totally geodesic submanifold in M and even $\partial M = \emptyset$.

Before considering the Killing form $\omega \in C^\infty \Lambda^{n-1} M$ tangent to the boundary of the manifold, we formulate the following lemma.

Lemma (see [130]). *Let M be a compact, n-dimensional manifold $(n \geq 2)$ with an equiaffine structure (η, ∇) and ξ be a special concircular vector field transversal to ∂M. Further, let* $\operatorname{div} \xi \leq 0$ *at all points $x \in \partial M$, where ξ_x is outward directed and* $\operatorname{div} \xi \geq 0$*, where ξ_x is inward directed.*

(1) *If for all $X \in C^\infty TM$, we have* $\operatorname{Ric}(X, X) \leq 0$*, then* $\nabla\xi = 0$.

(2) *If for all* $X \in C^\infty TM$, *we have* $\operatorname{Ric}(X,X) \le 0$, *probably, except for a single point* $x \in M$ *at which* $\operatorname{Ric}(X,X) < 0$, *then* ξ *is zero vector field.*

Let $\omega \in C^\infty \Lambda^{n-1} M$ be a form tangent to the boundary ∂M of the manifold M, i.e., $\omega = t\omega$ along ∂M. Then the following relation holds for the vector field $\xi = *\omega$: $\eta(\xi_x, e_2, \dots, e_n) = \omega(e_2, \dots, e_n)$ at each point $x \in M$. Moreover, if the form ω does not vanish on ∂M, then the vectors ξ are transversal to it at all points. Taking into account all what was said above, we deduce the following assertion from the lemma formulated above.

Theorem 3.3.3 (see [130]). *Let* M *be a compact,* n*-dimensional manifold* ($n \ge 2$) *with an equiaffine structure* (η, ∇) *and* ω *be a Killing* $(n-1)$*-form defined on* M *and tangent to its boundary* ∂M. *Further, let for the vector field* $\xi = *\omega$ *transversal to* ∂M, $\operatorname{div}\xi \le 0$ *at all points* $x \in \partial M$, *where* ξ_x *is outward directed, and* $\operatorname{div}\xi \ge 0$, *where* ξ_x *is inward directed. If for all* $X \in C^\infty TM$, *we have* $\operatorname{Ric}(X,X) \le 0$, *probably, except for a single point* $x \in M$ *at which* $\operatorname{Ric}(X,X) < 0$, *then* ω *is a* 0*-form.*

In conclusion, we formulate one more corollary which is obvious.

Corollary 3.3.1. *Let* M *be a closed,* n*-dimensional manifold* ($n \ge 2$) *with an equiaffine structure* (η, ∇).

(1) *If for all* $X \in C^\infty TM$, *we have* $\operatorname{Ric}(X,X) \le 0$ *on* M, *then every Killing* $(n-1)$*-form on* M *is covariantly constant.*
(2) *If for all* $X \in C^\infty TM$, *we have* $\operatorname{Ric}(X,X) \le 0$ *on* M *and* $\operatorname{Ric}(X,X) < 0$ *at at least one point, then* $\dim \mathbb{R}^{n-1}(M, \mathbb{R}) = 0$.

For a solenoidal vector field ξ on a compact manifold M with an equiaffine structure (η, ∇) tangent to the boundary ∂M of the manifold M, integral formula (3.3.2) becomes

$$\int\limits_M \left\{ \operatorname{Ric}(\xi,\xi) + \sum_{i=1}^n (\lambda)^2 \right\} \eta = \int\limits_{\partial M} Q(\xi,\xi)\eta'. \tag{3.3.5}$$

Using the analysis of formula (3.3.5), we can formulate the following theorems.

Theorem 3.3.4 (see [130]). *Let* M *be a compact manifold with an equiaffine structure* (η, ∇) *and* ξ *be a solenoidal vector field tangent to its boundary* ∂M. *Assume that one of the following two conditions holds*:

(1) $\operatorname{Ric}(\xi,\xi) \ge 0$ *on* M, $Q(\xi,\xi) \le 0$ *along* ∂M, *and the principal curvatures of the field* ξ *are real at each point of the manifold* M;
(2) $\operatorname{Ric}(\xi,\xi) \le 0$ *on* M, $Q(\xi,\xi) \ge 0$ *along* ∂M, *and the principal curvatures of the field* ξ *are imaginary at each point of the manifold* M.

Then $\operatorname{Ric}(\xi,\xi) = 0$ *on* M, $Q(\xi,\xi) = 0$ *along* ∂M, *and* ξ *generates a nilpotent field of endomorphisms* A_ξ *of the tangent bundle* TM.

Theorem 3.3.5 (see [130]). *On a compact manifold* M *with an equiaffine structure* (η, ∇), *there is no solenoidal vector field* ξ *tangent to its boundary* ∂M *such that one of the following two conditions holds*:

(1) *the principal curvatures of the vector field* ξ *are real at each point of* M, $\operatorname{Ric}(\xi,\xi) \ge 0$ *on* M, *except for at least one point at which* $\operatorname{Ric}(\xi,\xi) > 0$ *and, moreover,* $Q(\xi,\xi) \le 0$ *along* ∂M;
(2) *the principal curvatures of the vector field* ξ *are imaginary at each point of* M, $\operatorname{Ric}(\xi,\xi) \le 0$ *on* M, *except for at least one point at which* $\operatorname{Ric}(\xi,\xi) < 0$ *and, moreover,* $Q(\xi,\xi) \ge 0$ *along* ∂M.

The conditions $Q(\xi,\xi) \le 0$ and $Q(\xi,\xi) \ge 0$ in both assertions can be replaced by a stronger condition on the boundary ∂M: to be a totally geodesic submanifold or, moreover, $\partial M = \emptyset$. In turn, the conditions $\operatorname{Ric}(\xi,\xi) \ge 0$ or $\operatorname{Ric}(\xi,\xi) \le 0$ in the first assertion can be replaced by a stronger assumption on the structure (η, ∇) to be Ricci-flat.

3.4. Application to Lorentz Geometry

3.4.1. Consider a pseudo-Riemannian n-dimensional manifold M with metric g of index k. The manifold M has the equiaffine structure (η, ∇) with $\eta = \sqrt{|\det(g)|}dx^1 \wedge \cdots \wedge dx^n$, the volume form with respect to a local coordinate system $x^1, \dots, x^n$ on M, and the Levi-Civita connection ∇, which obviously is an equiaffine connection.

We assume that the manifold M is a Lorentz manifold (see [60, p. 267]) and its metric g has the diagonal form $(-+\dots+)$. We will consider a compact manifold M' which is the closure of an open, n-dimensional submanifold of the Lorentz manifold M. In particular, we will assume that $M' = M$.

Let ξ be a time-like unit vector field on M'. In this case, each tangent space T_xM' is the orthogonal sum of the "vertical space" $V = \operatorname{span}\{\xi_x\}$ and the "horizontal space" H consisting of vectors orthogonal to ξ_x.

Denote by $\nu : T_xM \to V$ and $h : T_xM \to H$ the orthogonal projections. Then $q = q^h + q^\nu$ for $q = g_x$, $q^h = g_x^h = g_x(h,h)$, and $q^v = g_x^\nu = g_x(\nu,\nu)$, where q^h is a positive-definite quadratic form on the $(n-1)$-dimensional space H.

3.4.2. A self-diffeomorphism of a pseudo-Riemannian manifold M is an automorphism of the $O(n,k)$-structure if and only if it preserves the metric g. We have

$$(L_\zeta g)(X,Y) = -[g(A_\zeta X, Y) + g(X, A_\zeta Y)]$$

for any vector fields X and Y on M. Obviously, ζ is an infinitesimal automorphism of the $O(n,k)$-structure if and only if

$$g(A_\zeta X, Y) + g(X, A_\zeta Y) = 0 \qquad (3.4.1)$$

(see [60], p. 223). In this case, ζ is called a *Killing vector field on* M.

Let ζ be a time-like Killing vector field and ξ be its unit vector. Then

$$g(A_\xi X, \xi) = 0, \quad g(A_\xi hX, hY) + g(hX, A_\xi hY) = 0, \quad \operatorname{trace} A_\xi = 0$$

for any $X, Y \in C^\infty TM$. The analysis of integral formula (3.3.1) allows us to conclude that the following theorem holds in this case.

Theorem 3.4.1 (see [130])**.** *An n-dimensional Lorentz manifold M does not admit a time-like Killing vector field ξ if in M, there exists an oriented n-dimensional submanifold M' with space-like boundary $\partial M'$ orthogonal to ξ such that at all its points,* $\operatorname{Ric}(\xi,\xi) \le 0$ *except for at least one point at which* $\operatorname{Ric}(\xi,\xi) < 0$.

Note that the problem to what extent the curvature of a Lorentz manifold prevents the existence of Killing vector fields on it constantly attracts the attention of geometers. So, in [11], it was proved that on a space-time of nonnegative time-like sectional curvature, a time-like Killing vector field ξ is parallel (i.e., $\nabla\xi = 0$) since otherwise the space-time must be time-like and isotropically geodesically incomplete simultaneously. In [89], it is proved that a time-like Killing vector field ξ on a closed manifold with Ricci-flat metric must be parallel.

Now let a time-like vector field ξ on a Lorentz manifold M be harmonic (see [174]). In this case, the following relations hold for the unit vector ξ of the field ξ:

$$g(A_\xi X, \xi) = 0, \quad g(A_\xi hX, hY) - g(hX, A_\xi hY) = 0, \quad \operatorname{trace} A_\xi = 0$$

for any $X, Y \in C^\infty TM$. The analysis of integral formula (3.2.1) allows us to conclude that the following theorem holds in this case.

Theorem 3.4.2 (see [130])**.** *An n-dimensional Lorentz manifold M does not admit a harmonic vector field ξ if in M, there exists an oriented, n-dimensional submanifold M' with space-like boundary $\partial M'$ orthogonal to ξ such that at all its points* $\operatorname{Ric}(\xi,\xi) \ge 0$ *except for at least one point at which* $\operatorname{Ric}(\xi,\xi) > 0$.

Note that among a large number of works devoted to the study of the existence "in the large" of harmonic vector fields on a Riemannian manifold, there are practically no works devoted to the same problem for Lorentz manifolds.

3.4.3. Consider the topological product $M = M_1 \times M_2$ of a one-dimensional Riemannian manifold M_1 with metric tensor g_1 and an $(n-1)$-dimensional Riemannian manifold M_2 with metric tensor g_2. Define the metric tensor g of the manifold M by the formula $g = -g_1 \otimes f g_2$, where $f : M_1 \to (0, \infty)$ is a certain positive function such that $f \in C^\infty M_1$. Such a manifold is called a *curved Lorentz product* (see [10, pp. 22 and 57–58]). Metrics of such a form are studied in the general relativity theory. A particular form of them in which there is no closedness condition for M_1 are the Robertson–Wacker models of "large blow-up" and statictical Einstein model of the Universe (see [10, pp. 117–122]). The following theorem holds.

Theorem 3.4.3 (see [130])**.** *On the topological product $M = M_1 \times M_2$ of a one-dimensional closed manifold M_1 and an $(n-1)$-dimensional closed manifold M_2, there is no metric of curved Lorentz product such that in all time-like directions, the Ricci curvature of M is nonpositive and at least one of its point is strictly negative.*

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