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MINIMAX DETECTION OF A SIGNAL IN THE HETEROSCEDASTIC GAUSSIAN WHITE NOISE

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We consider the problem of signal detection in the heteroscedastic Gaussian white noise when the set of alternatives is essentially nonparametric. In this setting, we find a family of asymptotically minimax tests. The results are extended to the case of testing a parametric hypothesis against nonparametric sets of alternatives. Bibliography: 8 titles.

1. Introduction and main results

Assume that we observe a realization of the random process $Y(t)$, $t \in (0,1)$, defined by the stochastic differential equation

$$dY(t) = S(t)dt + \epsilon q(t)dw(t), \quad \epsilon > 0, \tag{1}$$

where $dw(t)$ is the Gaussian white noise and $q(t)$ is a weight function. The noise $q(t)dw(t)$ is usually called the *heteroscedastic Gaussian white noise* (see [2, 6]). This model naturally arises in the theory of statistical inferences. One can show the local asymptotic equivalence of this model and the models of statistical inferences about the density and regression (see [1, 8]). Problems of estimation in the heteroscedastic Gaussian white noise were analyzed in [2, 6]. Problems of nonparametric hypothesis testing for this model were not studied, though similar models were investigated in another setting (see [4, 5]). The goal of this paper is to show that the results do not essentially differ from the case $q(t) = 1$ studied in [3].

Assume that

$$S(t) = \sum_{j=1}^{\infty} s_j \phi_j(t), \qquad S = \{s_j\}_1^\infty \in U = \Big\{S = \{s_j\}_1^\infty,\ \sum_{j=1}^{\infty} a_j s_j^2 \le P_0,\ s_j \in R^1\Big\}, \tag{2}$$

where $\phi_j(t)$ is an orthonormal system of functions in $L_2(0,1)$ and $P_0 > 0$ and $a_j > 0$ are given numbers. In the case of the trigonometric system of functions and $a_{2j} = a_{2j+1} = 1 + (2\pi j)^{2\beta}$, $\beta > 0$, this information corresponds to the assumption that the signal belongs to a ball in the Sobolev space.

The problem is to test the hypothesis $H_0\colon S(t) = 0$, $t \in (0,1)$, against the alternatives

$$S \in V_\epsilon = \{S : ||S||_2 > \rho_\epsilon, S \in U\}, \tag{3}$$

where $||S||_2^2 = \sum\limits_{j=1}^{\infty} \theta_j^2$. We suppose that $\rho_\epsilon \to 0$ as $\epsilon \to 0$.

For any test M_ϵ, denote by $\alpha_\epsilon(M_\epsilon)$ its type I error probability and by $\beta_\epsilon(M_\epsilon, \theta)$ its type II error probability for an alternative $\theta \in V_\epsilon$. Set $\beta_\epsilon(M_\epsilon) = \sup\limits_{S \in V_\epsilon} \beta_\epsilon(M_\epsilon, S)$.

We say that a family of tests K_ϵ with type I error probabilities $\alpha(K_\epsilon) = \alpha$, $0 < \alpha < 1$, is *asymptotically minimax* if for any family of tests M_ϵ with $\alpha(M_\epsilon) \le \alpha$,

$$\limsup_{\epsilon \to 0} \{\beta_\epsilon(K_\epsilon) - \beta_\epsilon(M_\epsilon)) \le 0.$$

Our results are obtained under the following assumptions (cf. [4]).

(A1) The sequence a_j is increasing, and there exists $\delta > 0$ such that $a_j / j^\delta \to \infty$ as $j \to \infty$.

(A2) There exists c, $0 < c < 1$, such that

$$\lim\inf_{t \to \infty} \frac{Z(ct)}{Z(t)} > 0. \tag{4}$$

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Here $Z(t)$ stands for the number of elements $\{a_j : a_j < t,\, 1 \le j < \infty\}$.

It follows from (A2) that there exists $\gamma > 0$ such that $a_j < Cj^\gamma$.

(A3) $\sup\{a_{t+1}/a_t : t > j\} \to 1$ as $j \to \infty$.

(A4) There exists $C > 0$ such that for all j

$$\sup\{|\phi_j(x)| : x \in (0,1)\} \le C < \infty.$$

For any $p, h \in L_2(0,1)$ and any j, j_1, denote

$$(p,h) = \int_0^1 ph\, dx, \qquad p_j = (p)_j = (p, \phi_j), \qquad p_{jj_1} = (p\phi_j)_{j_1}.$$

(A5) $0 < c < q(x) < C < \infty$ for all $x \in (0,1)$.

(A6) There exist $\tau > \frac{1}{2}$ and $C > 0$ such that for all j

$$\sum_{r=1}^{\infty} |j-r|^{2\tau}(q)_{jr}^2 < C, \tag{5}$$

$$\sum_{r=1}^{\infty} |j-r|^{2\tau}(q^2)_{jr}^2 < C. \tag{6}$$

In [4], we pointed out that (6) can be replaced by simpler sufficient assumptions.

Denote by $\Phi(x)$ the distribution function of the standard normal distribution. For $z \in R^1$, set $(z)_+ = \max(z,0)$ and denote by $[z]$ the integer part of z. For an event D, denote by $\chi(D)$ the indicator of this event. Let C, c denote positive constants.

Consider the sequence $\kappa_j^2 = \kappa_{j\epsilon}^2 = (\lambda_1 - \lambda_2 a_j)_+$, where λ_1 and λ_2 are determined by the equations

$$\sum_{j=1}^{\infty} \kappa_j^2 ||q^2\phi_j||^2 = \rho_\epsilon, \tag{7}$$

$$\sum_{j=1}^{\infty} a_j \kappa_j^2 ||q^2\phi_j||^2 = P_0. \tag{8}$$

Set $k = k_\epsilon = \max\{j : \kappa_j \ne 0\}$ and

$$A_\epsilon = \epsilon^{-4} \sum_1^k \kappa_j^4 ||q^2\phi_j||^2. \tag{9}$$

Consider the test statistics

$$T_\epsilon(Y_\epsilon) = \epsilon^{-2} \sum_1^k \kappa_j^2 \left(\left(\int_0^1 \phi_j(t)dY_\epsilon(t) \right)^2 - \epsilon^2 (q^2)_{jj} \right).$$

Define the tests

$$K_\epsilon(Y_\epsilon) = \chi(\epsilon^{-2} T_\epsilon(Y_\epsilon) > (2A_\epsilon)^{1/2} x_\alpha),$$

where x_α is the critical value of K_ϵ.

Theorem 1. *Assume that conditions* (A1)–(A6) *hold. Then the family of tests* K_ϵ, $0 < C_1 < \alpha_\epsilon = \alpha(K_\epsilon) < C_2 < 1$, *is asymptotically minimax. Let*

$$0 < \liminf_{\epsilon\to 0} A_\epsilon \le \limsup_{\epsilon\to 0} A_\epsilon < \infty. \tag{10}$$

Then x_{α_ϵ} *can be defined by the equation* $\alpha_\epsilon = 1 - \Phi(x_{\alpha_\epsilon})$ *and*

$$\beta_\epsilon(K_\epsilon) = \Phi(x_{\alpha_\epsilon} - (A_\epsilon/2)^{1/2})(1 + o(1)) \tag{11}$$

as $\epsilon \to 0$.

Remark 1. In the case of the trigonometric system of functions, the multipliers $||q^2\phi_j||^2$ in (7)–(9) vanish and are replaced by $\frac{1}{2}||q^2||^2$. This is due to the identity $\sin^2(2\pi jt) + \cos^2(2\pi jt) = 1$.

Remark 2. A similar setting was considered in [4] for the problem of nonparametric testing of hypotheses on the density. Assume that we have a sample $X_1, \dots, X_n$ of independent identically distributed random variables with density $f(x)$. It is known that

$$f(x) = p(x) + q(x)\sum_{j=1}^{\infty} \theta_j \phi_j(x).$$

One needs to test the hypothesis $H_0 : f = p$ against the alternatives $H_\epsilon \theta \in V_\epsilon$, where V_ϵ is defined by (3). In this setting, the result is similar. The main difference is that $||q^2\phi_j||^2$ in (7)–(9) is replaced in [4] by $||pq^{-2}\phi_j||^2$.

In fact, Theorem 1 holds for a much wider set of alternatives

$$\overline{V}_\epsilon = \left\{ S : \sum_{j=1}^{\infty} \kappa_{j\epsilon}^2 s_j^2 > \rho_\epsilon,\ S = \{s_j\}_1^\infty \right\} \tag{12}$$

with $\{\kappa_{j\epsilon}\}_1^\infty$ satisfying rather weak assumptions.

Assume that for each $\epsilon > 0$, we are given a sequence $\{\kappa_{j\epsilon}\}_1^\infty$ satisfying the following assumptions.

(B1) For each $\epsilon > 0$, the sequence $\kappa_{j\epsilon}^2$ is decreasing.

(B2) Relation (10) holds, and

$$\lim_{\epsilon\to 0} \sum_{j=1}^{\infty} \kappa_{j\epsilon}^2 = 0.$$

The definition of A_ϵ coincides with (9).

Define

$$k_\epsilon = \sup\left\{ k : \sum_{j\le k_\epsilon} \kappa_{j\epsilon}^2 \le \frac{1}{2}\sum_{j=1}^{\infty} \kappa_{j\epsilon}^2 \right\}.$$

(B3) For any $\delta > 0$,

$$\lim_{\epsilon\to 0} \sup_{\delta k_\epsilon < j < \delta^{-1}k_\epsilon} \left| \frac{\kappa_{j\epsilon}^2}{\kappa_{j+1,\epsilon}^2} - 1 \right| = 0.$$

(B4)

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \frac{\sum\limits_{\delta k_\epsilon < j < \delta^{-1}k_\epsilon} \kappa_{j\epsilon}^2 ||q^2\phi_j||^2}{\sum\limits_{j=1}^{\infty} \kappa_{j,\epsilon}^2 ||q^2\phi_j||^2} = 1, \tag{13}$$

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \epsilon^{-4} A_\epsilon^{-1} \sum_{\delta k_\epsilon < j < \delta^{-1}k_\epsilon} \kappa_{j\epsilon}^4 ||q^2\varphi_j|| = 1. \tag{14}$$

Theorem 2. *Assume that conditions* (B1)–(B4), (A4)–(A6) *hold. Then the family of tests* K_ϵ *is asymptotically minimax in the problem of testing the hypothesis* $H_0 : S = 0$ *against the alternatives* $H_\epsilon : S \in \overline{V}_\epsilon$. *If* $0 < \alpha(K_\epsilon)(1 + o(1)) = \alpha < 1$, *then the critical value* x_α *is determined by the equation* $\alpha = 1 - \Phi(x_\alpha)$ *and*

$$\beta_\epsilon(K_\epsilon) = \Phi(x_\alpha - (A_\epsilon/2)^{1/2})(1 + o(1))$$

as $\epsilon \to 0$.

The proof of Theorem 2 is omitted, since it is essentially a slight modification of the proof of Theorem 1.

Similar results hold for the problem of testing parametric hypotheses against nonparametric sets of alternatives.

2. The problem of testing a parametric hypothesis against nonparametric sets of alternatives

Assume that we need to test the hypothesis $H_0 : S(t) = S(t,\theta)$, $\theta \in \Theta$, agaist the nonparametric sets of alternatives

$$S \in \widetilde{V}_\epsilon = \left\{ S : \inf_{\theta\in\Theta} ||S - S(\theta)|| > \rho_\epsilon,\ S \in \widetilde{U}_\epsilon \right\}, \tag{15}$$

where

$$\widetilde{U}_\epsilon = \left\{ S : S(t) = S(t,\theta) + \sum_{j=1}^{\infty} s_j \phi_j(t), \sum_{j=1}^{\infty} a_j s_j^2 \le P_0 \right\}.$$

Assume that Θ is an open bounded set in R^d.

We say that a family of tests M_ϵ, where $\alpha_\theta(M_\epsilon) = E_\theta M_\epsilon \le \alpha$, $0 < \alpha < 1$, $\theta \in \Theta$, is *uniformly asymptotically minimax* for the set of alternatives $\widetilde{V}_\epsilon$ if the family of tests M_ϵ is asymptotically minimax for each fixed $\theta \in \Theta$ in the problem of testing the simple hypothesis $S = S(s,\theta)$ against the alternatives $S \in \widetilde{V}_\epsilon$.

Denote $\beta(K_\epsilon, \theta) = \sup\{\beta(K_\epsilon, S),\ S \in \widetilde{V}_\epsilon\}$.

We make the following assumptions.

(D1) For all θ_1, $\theta_2 \in \Theta$, $\theta_1 \ne \theta_2$, the inequality $||S(\theta_1) - S(\theta_2)|| \ne 0$ holds.

Assume that the signal $S(t,\theta)$ is continuously differentiable with respect to $\theta \in \Theta$ and denote by $S_{\theta_i}(t,\theta) = \frac{\partial S(t,\theta)}{\partial \theta_i}$, $1 \le i \le d$, its partial derivatives. Set $S_\theta(t,\theta) = \{S_{\theta_i}(t,\theta)\}_1^d$.

Given vectors u, $v \in R^d$, denote their inner product by $u'v$.

(D2) There exists $\omega > 0$ such that for all θ_1, $\theta_2 \in \Theta$

$$||S(\theta_2) - S(\theta_1) - S'_\theta(\theta_1)(\theta_2 - \theta_1)||^2 < C|\theta_1 - \theta_2|^{2+\omega}.$$

(D3) There exists $C > 0$ such that $||S_{\theta_i}(\cdot,\theta)||^2 < C$ for all $1 \le i \le d$ and $\theta \in \Theta$. The relation

$$(S_{\theta_i}(\cdot,\theta)q^2)_j \to 0 \quad \text{as} \quad j \to \infty \tag{16}$$

holds uniformly in $\theta \in \Theta$.

(D4) There exists a functional $\overline{\theta} : L_2(0,1) \to \Theta$ such that $\overline{\theta}(S) \to \theta$ as $||S - S(\theta)|| \to 0$ uniformly in $\theta \in \Theta$.

(D5) There exists an estimator $\hat{\theta}_\epsilon$ such that for any $\delta > 0$

$$P_S(|\hat{\theta}_\epsilon - \overline{\theta}(S)| > \delta\epsilon^2/\kappa) = o(1)$$

uniformly in $S \in \widetilde{V}_\epsilon$. Here $\kappa = \max\{\kappa_j,\ 1 \le j < \infty\}$.

By Lemma 3.1 of [4], $\kappa^2 = O(\epsilon^2 k^{-1/2})$. Thus, using (D5), we can obtain estimates of large deviation probabilities for $\hat{\theta}_\epsilon$. Note that these probabilities are considered for S not from the parametric set. In statistical inference, the estimator $\hat{\theta}(Y_\epsilon)$ can be interpreted as a functional $\overline{\theta}(S)$ defined on a set including the realizations of Y_ϵ corresponding to the set of all possible signals S, and we can consider large deviation probabilities of $\hat{\theta}(Y_\epsilon) - \overline{\theta}(S)$ for all possible S. Note that in the similar assumption D4 in [5], the estimates of $|\hat{\theta}_\epsilon - \overline{\theta}(S)|$ can be replaced by the more natural assumption that

$$P_S(|\hat{\theta}_\epsilon - \overline{\theta}(S)| > \delta\epsilon^{2/(2+\omega)}) = o(1)$$

uniformly in $S \in \widetilde{V}_\epsilon$. Here the value of $\omega > 0$ is defined in [5]. One can made a similar change in condition E3 from [5].

(D6) There exists a function γ such that $\gamma(u) \to 0$ as $u \to 0$ and

$$\sum_{j=1}^{\infty} a_j(s_j(\theta_1) - s_j(\theta_2))^2 \le \gamma(|\theta_1 - \theta_2|).$$

Let us introduce the test statistics

$$T(Y_\epsilon, \hat{\theta}_\epsilon) = \epsilon^{-2} \sum_{j=1}^{\infty} \kappa_j^2 \left(\left(\int_0^1 \phi_j(t)(dY_\epsilon(y) - S(t,\hat{\theta}_\epsilon)dt) \right)^2 - \epsilon^2 (q^2)_{jj} \right),$$

where κ_j^2 are defined as in Theorem 1.

We define the family of tests $K_\epsilon = \chi(\epsilon^{-2} T(Y_\epsilon, \hat{\theta}_\epsilon) > x_\alpha (A_\epsilon)^{1/2})$, where x_α is the solution of the equation $\alpha = 1 - \Phi(x_\alpha)$.

Theorem 3. *Assume that conditions* (A1)–(A6), (D1)–(D6), *and* (10) *hold. Then the family of tests* K_ϵ *is asymptotically minimax and*

$$\beta(K_\epsilon) = \Phi(x_\alpha - (A_\epsilon/2)^{1/2})(1 + o(1))$$

as $\epsilon \to 0$.

3. Proof of Theorem 1

Consider the matrix $Q=\{q_{ij}\}_{i,j=1}^\infty$ and the matrix $\Lambda = \Lambda_\epsilon = \{\lambda_{ij}\}_{ij=1}^\infty$, where $\lambda_{ij} = \kappa_j^2$ if $i = j$ and $\lambda_{ij} = 0$ if $i \neq j$. Here $\kappa_j^2 = (\lambda_{1\delta} - \lambda_{2\delta} a_j)_+$, where the parameters $\lambda_{1\delta}$ and $\lambda_{2\delta}$ are defined by the equations

$$\sum_{j=1}^\infty \kappa_j^2 ||q^2\phi_j||^2 = (1+\delta)\rho_\epsilon, \qquad \sum_{j=1}^\infty a_j\kappa_j^2||q^2\phi_j||^2 = (1-\delta)P_0,$$

with $0 < \delta < 1$. Consider the vector $\xi = \{\xi_j\}_{j=1}^\infty$, where ξ_j, $1 \le j < \infty$, are independent Gaussian random variables with $E\xi_j = 0$, $E\xi_j^2 = 1$.

In this notation, Eq. (1) has the following form:

$$Y = S + \epsilon Q\xi,$$

where $S = \{s_j\}_1^\infty$, $Y = \{y_j\}_1^\infty$, and $y_j = \int_0^1 \phi_j dY_\epsilon$.

The proof of the lower bound is based on the well-known fact that the Bayes risk does not exceed the minimax risk. We define a family of Bayes tests and show that the minimax risk is attained at this family.

Consider the Gaussian random vector $\zeta = \{\zeta_j\}_1^\infty$ consisting of independent Gaussian random variables ζ_j with $E\zeta_j = 0$, $E\zeta_j^2 = 1$ and the Gaussian random vector

$$\eta = \eta_{\epsilon\delta} = \{\eta_j\}_1^\infty = Q^2\Lambda^{1/2}\zeta.$$

Denote by $\mu_{\epsilon\delta}$ the probability measure of the random vector $\eta_{\epsilon\delta}$. Define a Bayesian prior probability measure $\nu_{\epsilon,\delta}$ as the conditional distribution of $\eta_{\epsilon\delta}$ given $\eta_{\epsilon\delta} \in Q_\epsilon$.

Lemma 1. *For any* $\delta > 0$,

$$\lim_{\epsilon\to 0} P(\eta_\epsilon \in Q_\epsilon) = 1. \tag{17}$$

Proof. The lemma follows from the Chebyshev inequality and the following relations:

$$E\biggl(\sum_{j=1}^\infty \eta_j^2\biggr) = \sum_{j=1}^\infty\sum_{k=1}^\infty (q^2)_{jk}\kappa_j^2(q^2)_{kj} = \sum_{j=1}^\infty \kappa_j^2||q^2\phi_j||^2(1+o(1)), \tag{18}$$

$$E\left(\sum_{j=1}^\infty a_j\eta_j^2\right) = \sum_{j=1}^\infty a_j\kappa_j^2||q^2\phi_j||^2(1+o(1)), \tag{19}$$

$$\mathrm{Var}\left(\sum_{j=1}^\infty \eta_j^2\right) = 2\sum_{j,t=1}^\infty\left(\sum_{k=1}^\infty (q^2)_{jk}\kappa_j^2(q^2)_{kt}\right)^2 = 2\sum_{j,t=1}^\infty ((q^4)_{jt})^2\kappa_t^4(1+o(1)) = 2\sum_{j=1}^\infty \kappa_j^4||q^4\phi_j||^2(1+o(1)), \tag{20}$$

and

$$\mathrm{Var}\left(\sum_{j=1}^\infty a_j\eta_j^2\right) = 2\sum_{j,t=1}^\infty\left(\sum_{k=1}^\infty (q^2)_{jk}a_j\kappa_j^2(q^2)_{kt}\right)^2$$

$$= 2\sum_{j,t=1}^\infty ((q^4)_{jt})^2a_t^2\kappa_t^4(1+o(1)) = 2\sum_{j=1}^\infty a_j^2\kappa_j^4||q^4\phi_j||^2(1+o(1)). \tag{21}$$

The detailed proof of (18)–(21) is similar to the estimates of Lemma 2.2 from [4] and is omitted. □

It follows from Lemma 1 that in further arguments, the Bayesian a priori probability measures ν_ϵ can be replaced by the probability measures μ_ϵ (see [3, 5]). Thus it suffices to solve the problem for μ_ϵ.

By (A5), the matrix Q has the inverse matrix Q^{-1}. Set $\Lambda^{-1} = \left\{\lambda_{ij}^{-1}\right\}_{i,j=1}^{\infty}$, where we assume that $0^{-1} = 0$.

Denote by $I : L_2(0,1) \to L_2(0,1)$ the identity operator.

The Bayesian a posteriori likelihoood ratio for the a priori distribution $\mu_{\epsilon\delta}$ equals

$$\begin{aligned}
&C \int \exp\left\{-\frac{1}{2\epsilon^2}(Y-S)'Q^{-2}(Y-S) - \frac{1}{2}S'Q^{-2}\Lambda^{-1}Q^{-2}S + \frac{1}{2}Y'Q^{-2}Y\right\} d\mu_{\epsilon\delta} \\
&= C \int \exp\left\{\frac{1}{2\epsilon^2}Y'Q^{-2}S - \frac{1}{2}S'(\epsilon^{-2}Q^{-2} + Q^{-2}\Lambda^{-1}Q^{-2}S\right\} d\mu_{\epsilon\delta} \\
&= C \int \exp\left\{-\frac{1}{2\epsilon^2}||(I+\epsilon^2Q^{-1}\Lambda^{-1}Q^{-1})^{-1/2}Q^{-1}Y - (I+\epsilon^2Q^{-1}\Lambda^{-1}Q^{-1})^{1/2}Q^{-1}S||^2\right\} d\mu_{\epsilon\delta} \\
&\times \exp\left\{-\frac{1}{2\epsilon^2}Y'Q^{-1}(I+\epsilon^2Q^{-1}\Lambda^{-1}Q^{-1})^{-1}Q^{-1}Y\right\} \\
&= C\exp\left\{-\frac{1}{2\epsilon^2}Y'Q^{-1}(I+\epsilon^2Q^{-1}\Lambda^{-1}Q^{-1})^{-1}Q^{-1}Y\right\}.
\end{aligned} \tag{22}$$

Therefore, as a Bayesian statistics we can take

$$T_\epsilon^a(Y) = Y'Q^{-1}(I+\epsilon^2Q^{-1}\Lambda^{-1}Q^{-1})^{-1}Q^{-1}Y.$$

Let us show that the difference between $T_\epsilon^a(Y)$ and $T_\epsilon(Y)$ is negligible. Denote $W_\epsilon = \epsilon^{-4}Y'\Lambda Q\Lambda QY$. We have

$$|T_\epsilon^a(Y) - T_\epsilon(Y) - E_0[W_\epsilon]| < |W_\epsilon - E_0[W_\epsilon]|. \tag{23}$$

Straightforward calculations yield

$$E_0[W_\epsilon] = \epsilon^{-4}\sum_{j,i,l=1}^{\infty}(q^2)_{ji}\kappa_j^2 q_{jl}\kappa_l^2 q_{li} = \epsilon^{-4}\sum_{j,l=1}^{\infty}(q^3)_{jl}\kappa_j^2 q_{jl}\kappa_l^2 = \epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^4(q^4)_{jj}(1+o(1)), \tag{24}$$

$$\begin{aligned}
\mathrm{Var}_0[W_\epsilon] &= \epsilon^{-8}\sum_{j,i,l,j_1,l_1,i_1=1}^{\infty}(q^2)_{jj_1}\kappa_j^2 q_{jl}\kappa_l^2 q_{li}\kappa_{j_1}^2 q_{j_1l_1}\kappa_{l_1}^2 q_{l_1i_1}(q^2)_{ii_1} \\
&+\epsilon^{-8}\sum_{j,i,l,j_1,l_1,i_1=1}^{\infty}(q^2)_{ji_1}\kappa_j^2 q_{jl}\kappa_l^2 q_{li}\kappa_{j_1}^2 q_{j_1l_1}\kappa_{l_1}^2 q_{l_1i_1}(q^2)_{j_1i} = \epsilon^{-8}\sum_{j,l,j_1,l_1,1=1}^{\infty}(q^2)_{jj_1}\kappa_j^2 q_{jl}\kappa_l^2\kappa_{j_1}^2 q_{j_1l_1}\kappa_{l_1}^2(q^4)_{ll_1} \\
&+\epsilon^{-8}\sum_{j,l,j_1,l_1,1=1}^{\infty}(q^3)_{jl_1}\kappa_j^2 q_{jl}\kappa_l^2\kappa_{j_1}^2 q_{j_1l_1}\kappa_{l_1}^2(q^3)_{j_1l} = 2\epsilon^{-8}\sum_{l,l_1=1}^{\infty}(q^4)_{ll_1}^2\kappa_l^4\kappa_{l_1}^4(1+o(1)) \\
&\le 2\epsilon^{-8}\kappa^4\sum_{l=1}^{\infty}||q^4\phi_l||^2\kappa_l^4(1+o(1)) = o(1),
\end{aligned} \tag{25}$$

$$\begin{aligned}
E_\eta[W_\epsilon] &= E_0[W_\epsilon] + \epsilon^{-4}\sum_{i,j,l,m=1}^{\infty}(q^2)_{ij}\kappa_j^4 q_{jl}\kappa_l^2 q_{lm}(q^2)_{mi} = E_0[W_\epsilon] \\
&+\epsilon^{-4}\sum_{j,l=1}^{\infty}(q^5)_{lj}\kappa_j^4 q_{jl}\kappa_l^2 = E_0[W_\epsilon] + O\left(\epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^6\right) = E_0V_\epsilon + o(1).
\end{aligned} \tag{26}$$

$$\begin{aligned}
\mathrm{Var}_\eta[W_\epsilon] &= \mathrm{Var}_0[W_\epsilon] + 2\epsilon^{-8}\mathrm{Sp}\,[Q^2\Lambda Q^2\Lambda Q\Lambda QQ^2\Lambda Q^2\Lambda Q\Lambda Q] + 2\epsilon^{-8}\mathrm{Sp}\,[Q^2\Lambda Q\Lambda QQ^2\Lambda Q^2\Lambda Q\Lambda Q] \\
&\le \mathrm{Var}_0[W_\epsilon] + 2\epsilon^{-8}(\sup_t q^{12}(t)\mathrm{Sp}\,[\Lambda^6] + \sup_t q^{10}(t)\mathrm{Sp}\,[\Lambda^5]) = o(1).
\end{aligned} \tag{27}$$

Here $\mathrm{Sp}\,[A]$ denotes the trace of an operator A.

By (24)–(27) and the Chebyshev inequality, we obtain

$$|T^a_\epsilon(Y) - T_\epsilon(Y) - E_0V_\epsilon| < |V_\epsilon - E_0V_\epsilon| = o_P(1), \tag{28}$$

both in the case of the hypothesis and a Bayesian alternative.

In the case of alternative, we have

$$Y'\Lambda Y = \sum_{j,k,l=1}^{\infty} (s_j + q_{jk}\zeta_k)\kappa_j^2(s_j + q_{jl}\zeta_l) = \sum_{j=1}^{\infty} s_j^2\kappa_j^2 + 2\sum_{j=1}^{\infty}\kappa_j^2 s_j \sum_{j=1}^{\infty} q_{jk}\zeta_k$$

$$+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_{jk}\zeta_k\kappa_j^2 \sum_{i=1}^{\infty} q_{ji}\zeta_i = J_1(S) + J_2(S) + J_3 = J_1 + J_2 + J_3. \tag{29}$$

Straightforward calculations yield

$$E_\theta[J_2] = 0, \quad E_0[J_2] = 0,$$

$$E_0[J_3] = \epsilon^2 \sum_{j,l=1}^{\infty} \kappa^2 q_{jl}q_{lj} = \epsilon^2 \sum_{j=1}^{\infty} \kappa_j^2 (q^2)_{jj},$$

$$\mathrm{Var}_\theta[J_2] = 4 \sum_{j,i,l=1}^{\infty} \kappa_j^2\theta_j q_{ji}q_{il}\theta_l\kappa_l^2 = 4 \sum_{j,l=1}^{\infty} \kappa_j^2\theta_j (q^2)_{jl}\theta_l\kappa_l^2$$

$$\le 4\sum_{j=1}^{\infty}\kappa_j^2|\theta_j| \left(\sum_{l=1}^{\infty}\kappa_l^4\theta_l^2\right)^{1/2} ||q^2\phi_j|| \le 4C\kappa^2 k^{1/2}\sum_{j=1}^{\infty}\kappa_j^2\theta_j^2||q^2\phi_j|| = o\left(\sum_{j=1}^{\infty}\kappa^2\theta_j^2||q^2\phi_j||\right), \tag{30}$$

$$\mathrm{Var}_0[J_3] = \mathrm{Var}_0 Y'\Lambda Y = 2 \sum_{j,i,l,m=1}^{\infty} \kappa_j^2 q_{ji}q_{il}\kappa_l^2(q^2)_{lm}q_{mj}$$

$$= 2\sum_{j,l=1}^{\infty}\kappa_j^2\kappa_l^2(q^2)_{jl}^2 = 2\sum_{j=1}^{\infty}\kappa_j^4||q^2\phi_j||^2(1+o(1)) = 2A_\epsilon(1+o(1)). \tag{31}$$

In the case of a Bayesian alternative, we obtain

$$\epsilon^{-2}E_\eta[Y'\Lambda Y] = A_\epsilon(1+o(1)) + \sum_{j=1}^{\infty}\kappa_j^2(q^2)_{jj},$$

$$\epsilon^{-4}\mathrm{Var}_\eta[Y'\Lambda Y] = 2A_\epsilon + 4\epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^6(q^2)_{jj} = 2A_\epsilon(1+o(1)).$$

Note that

$$\epsilon^{-4}\mathrm{Var}_\eta[J_2] = 4\epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^6(q^2)_{jj}.$$

Since

$$\mathrm{Var}_\theta[J_2] = o(J_1^2), \qquad \mathrm{Var}\,[J_3] = 2A_\epsilon(1+o(1)) = O(J_1),$$

it follows from the Chebyshev inequality that $\beta(K_\epsilon, S_\epsilon) \to 0$ as $\epsilon \to 0$ provided that $J_1(S_\epsilon) \to \infty$ as $\epsilon \to 0$.

Thus, to prove Theorem 1, it suffices to show that the distribution of the statistics J_3 is asymptotically normal. This proof is similar to that of Lemma 3.7 in [4] and is omitted. □

Proof of Theorem 3. We begin with the proof of the lower bound. We have

$$T_\epsilon^{1/2}(Y_\epsilon - S(\hat\theta)) \le T_\epsilon^{1/2}(Y_\epsilon - S) + T_\epsilon^{1/2}(S - S(\theta(S))) + T_\epsilon^{1/2}(S(\theta(S)) - S(\hat\theta)). \tag{32}$$

The distribution of the statistics $T_\epsilon(Y_\epsilon - S)$ coincides with the distribution of the statistics $T_\epsilon(Y_\epsilon)$ in the case of the zero hypothesis in the setting of Theorem 1.

We have

$$T_\epsilon(S - S(\theta(S))) = \epsilon^{-2}\sum_{j=1}^{\infty}\kappa_j^2(s_j - s_j(\theta(S)))^2$$

$$\geq \epsilon^{-2}\lambda_1\sum_{j=1}^{k}(s_j - s(\overline{\theta}(S)))^2 - \epsilon^{-2}\lambda_2\sum_{j=1}^{k}a_j(s_j - s(\overline{\theta}(S)))^2 = \Gamma_1 - \Gamma_2 = \Gamma, \tag{33}$$

where $s_j(\theta) = (S(\theta))_j$. Now let us obtain an upper estimate for Γ_2. Assume that

$$||S - S(\overline{\theta}(S))|| = o(1). \tag{34}$$

Otherwise $A_\epsilon = o(\epsilon^{-2}T_\epsilon(S - S(\overline{\theta}(S))))$ and $\beta_\epsilon(K_\epsilon, S) \to 0$ as $\epsilon \to 0$.

Let $\widetilde{\theta}$ be such that

$$\sum_{j=1}^{\infty}a_j(s_j - s_j(S(\widetilde{\theta})))^2 \leq P_0. \tag{35}$$

Then

$$\left|\sum_{j=1}^{\infty}a_j(s_j - s_j(\overline{\theta}(S)))^2 - \sum_{j=1}^{\infty}a_j(s_j - s_j(\widetilde{\theta}(S)))^2\right|$$

$$\leq \left(\sum_{j=1}^{\infty}a_j(s_j - s_j(\overline{\theta}(S)) + s_j - s_j(\widetilde{\theta}(S)))^2\right)^{1/2}\left(\sum_{j=1}^{\infty}a_j(s_j(\overline{\theta}(S)) - s_j(\widetilde{\theta}(S)))^2\right)^{1/2}$$

$$\leq 2P_0^{1/2}\left(\sum_{j=1}^{\infty}a_j(s_j(\overline{\theta}(S)) - s_j(\widetilde{\theta}(S)))^2\right)^{1/2} \leq 2P_0^{1/2}\gamma^{1/2}(|\widetilde{\theta} - \overline{\theta}(S)|) = o(1), \tag{36}$$

where the latter equality follows from (34) and (D4), (D6).

It follows from (33), (35), and (36) that

$$\Gamma \geq \epsilon^{-2}\lambda_1\rho_\epsilon - \epsilon^{-2}\lambda_2 P_0(1 + o(1)) \geq \epsilon^2 A_\epsilon. \tag{37}$$

We have

$$T_\epsilon(S(\hat{\theta}) - S(\overline{\theta}(S)) = \epsilon^{-2}\sum_{j=1}^{\infty}\kappa_j^2(s_j(\hat{\theta}) - s_j(\overline{\theta}(S)))^2 \leq \epsilon^{-2}\kappa^2||S(\hat{\theta}) - S(\overline{\theta}(S))||^2$$

$$\leq \epsilon^{-2}\left(\kappa||S(\hat{\theta}) - S(\theta(S)) - (\hat{\theta} - \theta(S))' S_\theta(\overline{\theta}(S))|| + \kappa|\hat{\theta} - \overline{\theta}(S)|||S_\theta(\overline{\theta}(S))||\right)^2 \leq C\epsilon^{-2}\kappa^2|\hat{\theta} - \overline{\theta}(S)|^2 = o_P(\epsilon^2), \tag{38}$$

where the latter equality follows from (D5).

It follows from (32), (33), (37), and (38) that $\beta(K_\epsilon S) \leq \beta_\epsilon(K_\epsilon)(1 + o(1))$ for every $S \in \widetilde{V}_\epsilon$.

The proof of the lower bound uses the same arguments as in [5]. Denote by $\mu_{\epsilon\delta}$ the probability measure of the random process $S(\theta_0) + \eta_\epsilon$. Set

$$\rho_{\eta_\epsilon} = \inf_\theta ||S(\theta_0) + \eta_\epsilon - S(\theta)||^2.$$

Then the lower bound follows from the lemma given below.

Lemma 2.

$$\frac{\rho_{\eta_\epsilon}}{\rho_\epsilon} \to 1 + \delta \tag{39}$$

in probability as $\epsilon \to 0$.

The proof of Lemma 2 is similar to the proof of (4.26) in [5]. The main difference is that the smoothing kernel is replaced by the delta function.

Denote $\widetilde{\theta}_\epsilon = \arg\min_\theta ||S(\theta_0) + \eta_\epsilon - S(\theta)||$. Since $||\eta_\epsilon||^2 = \rho_\epsilon(1 + o_P(1))$ as $\epsilon \to 0$, it follows by (D1)–(D3) that $\widetilde{\theta}_\epsilon - \theta_0 = o_P(1)$. Hence, arguing similarly to (4.50)–(4.59) in [5] and using (D1)–(D3), we obtain

$$\rho_{\eta_\epsilon} = ||\eta_\epsilon||^2(1 + o(1)) + 2\int_0^1 (\widetilde{\theta} - \theta_0)' S_\theta(t, \theta_0)\eta_\epsilon(t)dt + ||(\widetilde{\theta} - \theta_0)' S_\theta(t, \theta_0)||^2(1 + o(1)).$$

Hence we will prove (39) if we show that

$$\int_0^1 S_{\theta_i}(t, \theta_0)\eta_\epsilon(t)dt = o_P(\rho_\epsilon^{1/2}) \tag{40}$$

for $1 \le i \le d$.

By (16), we obtain

$$E\left(\int_0^1 S_{\theta_i}\eta_\epsilon dt\right)^2 = \sum_{j=1}^\infty \kappa_j^2 (S_{\theta_i} q^2)_j^2 = o(\rho_\epsilon).$$

This implies (39) and (40).

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