Journal of Mathematical Sciences, Vol. 135, No. 3, 2006

THE METHOD OF CHARACTERISTICS FOR HAMILTON–JACOBI EQUATIONS AND APPLICATIONS TO DYNAMICAL OPTIMIZATION

N. N. Subbotina UDC 517.93

Dedicated to the shiny memory of my parents
Nikolay Maximovich and Zoya Nikolaevna Barabanovs

CONTENTS

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 20, Differential Equations, 2004.

INTRODUCTION

This monograph deals with generalized solutions of first-order partial differential equations and quasilinear parabolic equations.

Partial differential equations arise in many theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, economics, engineering, control, navigation, etc. For example, there are the following well-known equations: the Hamilton–Jacobi equation in mechanics [8], the Bellman equation in optimal control problems [29], the Isaacs equation in differential games [108], the eikonal equation in geometrical optics [62], the inviscid Burgers—Hopf equation in gas and hydrodynamics [73, 104, 161, 222], etc.

A classical method of solving boundary-value problems for partial differential equations is the *method of characteristics* suggested by Cauchy in the nineteenth century. This method reduces the integration of a partial differential equation to the integration of a system of ordinary differential equations. Solutions of the system are called *characteristics*. The Cauchy method for a boundary-value problem for a first-order partial differential equation (see, e.g., [62, 207, 222, 233]) provides a construction of the classical solution, and uses the *invariance* of the graph of the classical solution with respect to the characteristics. However, this method is restricted in applications since boundary-value problems for nonlinear partial differential equations have local classical solutions, at the best.

At the same time, there are *nonsmooth* (not everywhere differentiable or discontinuous) functions vital for the considered problems, for example, optimal time of capture in pursuit-evasion games; optimal distance to a target set at a given terminal time moment in antagonistic differential games; nonsmooth wave front in inhomogeneous aggregate medium, etc. These nonsmooth functions are defined globally, satisfy the given boundary conditions everywhere, and the corresponding partial differential equations at points of smoothness. These functions can be understood as global generalized solutions of boundary-value problems.

The need for an improved concept of generalized solutions of the Hamilton–Jacobi equation and other types of partial differential equations stimulated active research in the 50s–70s. Problems connected with notions of *weak* solutions of partial differential equations were investigated by N. S. Bakhvalov, L. C. Evans, W. H. Fleming, I. M. Gel'fand, S. K. Godunov, E. Hopf, O. A. Ladyzhenskaya, P. Lax, O. A. Oleinik, B. L. Rozhdestvenskii, A. A. Samarskii, S. L. Sobolev, A. N. Tikhonov, and many other famous mathematicians. The researches used mostly integration methods and the integral properties of weak solutions.

Among the research, we should note the results of S. N. Kruzhkov obtained for the Hamilton–Jacobi equations with convex Hamiltonians (see, e.g., [140]). He incorporated tools of convex analysis, subdifferentials, to studies of nonsmooth solutions of Hamilton–Jacobi equations. Also, another new tool of nonsmooth analysis, namely, generalized directional derivatives, was suggested by F. H. Clarke [51] to investigate generalized solutions of the Bellman equation.

In the early 80s, M. G. Crandall and P. L. Lions introduced the concept of viscosity solutions [59, 60, 167], where the development and applications of subdifferential and superdifferential tools of nonsmooth analysis play a key role. In the first papers, the existence of a viscosity solution was proved by using the method of vanishing viscosity. Within the theory of viscosity solutions, many researchers proved the existence and uniqueness theorems for various types of first-order partial differential equations, elliptic and parabolic equations, and various types of boundary-value problems. Reviews of results in the theory of viscosity solutions can be found in [21, 61, 80].

Now, the focus of research is concentrated on the development of analytic, constructive, and numerical methods in the theory. Also, application of the theoretical results to solving different problems in chemistry, economics, biology, and so on has attracted much attention (see, e.g., [23, 24, 73, 109, 232]). Important contributions to the research were made by O. Alvarez, Z. Artstein, M. Bardi, G. Barles, E. N. Barron, I. Capuzzo-Dolcetta, P. D. Christofides, M. G. Crandall, L. C. Evans, M. Falcone, W. H. Fleming, V. G. Gaitsgory, H. Ishii, R. Jensen, S. Koike, P. L. Lions, B. Perthame, H. M. Soner, P. E. Souganidis, X. Y. Zhou, and others.

Another known concept of a generalized solution based on the idempotent analysis was suggested by V. P. Maslov [122, 176]. He and his disciples studied first-order partial differential equations with convex Hamiltonians and applications to problems of mathematical physics, using this approach for linearized convex problems.

The results presented in this book are obtained in the framework of the concept of generalized *minimax* solution introduced by A. I. Subbotin [235, 236, 238]. Based on this approach, there are minimax estimates and operations. The concept of minimax solutions has sources in the theory of positional differential games [128, 133–135] developed in the Ekaterinburg (Sverdlovsk) school headed by N. N. Krasovskii. Fundamental contributions to the development of the theory of positional guaranteed control, supervision, estimate, and dynamical reconstruction were given in works by N. N. Krasovskii [126, 128, 134], A. B. Kurzhanskii [96, 150, 152], Yu. S. Osipov [143, 144, 195], and A. I. Subbotin [133, 240, 250] (see, e.g., references at the end of this monograph). Leading positions in the research are occupied by A. V. Kryazhimskii, V. E. Tretyakov, and A. G. Chentsov. Active researchers in this school are also E. H. Al'brekht, B. I. Anan'ev, V. D. Batukhtin, Yu. I. Berdyshev, S. A. Brykalov, V. L. Gasilov, M. I. Gusev, Kh. G. Guseinov, S. N. Zavalishchin, I. Ya. Katz, A. V. Kim, A. F. Kleimenov, A. I. Korotkii, A. N. Krasovskii, V. I. Maksimov, O. I. Nikonov, B. G. Pimenov, A. N. Sesekin, I. F. Sivergina, A. M. Tarasyev, V. N. Ushakov, T. F. Filippova, G. I. Shishkin, A. F. Shorikov, V. S. and N. L. Patsko, V. M. and T. N. Reshetov, S. I. Kumkov, N. Yu. Lukoyanov, and their disciples.

In the theory of positional differential games the property of the epigraph and hypograph of the value function of a differential game to be invariant relative to special differential inclusions was established. N. N. Krasovskii and A. I. Subbotin included the property into the definition of the *u-stability* and *v-stability* properties for real functions. The properties that define the value function are kernel stones in the theory of positional differential games. Since the value function of a differential game is a generalized solution for the corresponding Isaacs equation, the properties can be considered as prototypes of the concept of minimax solutions of first-order partial differential equations. The concept is defined in various equivalent ways, including infinitesimal forms, with the help of different tools of nonsmooth analysis: directional derivatives, tangent or contingent cones, subdifferentials and superdifferentials, etc. The definitions and the proof of their equivalence are given, for example, in [236, 238]. All the definitions describe the same property of *weak invariance* of the minimax solution relative to *generalized characteristics*, which are solutions of so-called *characteristic* differential inclusions.

Theorems on the existence, uniqueness, correctness, and vitality of the minimax solutions are proved for various types of boundary-value problems for first-order partial differential equations [1, 40, 98, 159, 160, 239, 241, 274, 276]. Within the theory, research on constructive and numerical methods, including grid methods, play an active role [95, 99, 242, 249, 259, 279, 280]. The important result in the theory of minimax solutions to first-order partial differential equations is the nontrivial proof of equivalence of the concepts of minimax and viscosity solutions [238, 250]. Note that methods of the theory of differential games, dynamical optimization, and nonsmooth analysis find numerous applications in the theory of minimax solutions. Also, research on minimax solutions stimulated the development of these new branches of mathematics (see, e.g., [53, 54]).

It is proved that the minimax solution to a Hamilton–Jacobi–Isaacs–Bellman equation coincides with the value function of an optimal control problem or of a differential game. Therefore, the theory of minimax solutions has many applications connected with control problems (see, e.g., [135, 149, 235, 236, 288–291, 293]). The value function defines equal values of optimal guaranteed results of two antagonistic players (or a control and a disturbance) that are achieved from a given initial state (initial position). In addition, this function plays a key role in the construction of optimal and almost optimal feedbacks (see [29, 34, 52, 72, 76, 82, 108, 127, 163, 213]). The theory of optimal guaranteed controls is based on the pioneer works of R. Isaacs, L. S. Pontryagin, R. Bellman, N. N. Krasovskii, W. H. Fleming, R. J. Elliott, N. J. Kalton, and A. Fridman. Important contributions to the research were made also by the works of V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, E. O. Roxin, G. Leitmann, L. D. Berkovitz, A. E. Bryson, Y.-C. Ho, R. Olsder, J. Warga, N. N. Moiseev, B. N. Pshenichnyi, V. I. Arnold, D. V. Anosov, F. L. Chernous'ko, V. A. Yakubovich, V. I. Zubov, A. A. Chikrii, A. A. Melikyan, L. A. Akulenko, G. K. Pozharitzkii, V. I. Blagodatskikh, N. L. Grigorenko, P. B. Gusyatnikov, M. I. Zelikin, Yu. S. Ledyaev, M. S. Nikol'skii, A. A. Agrachev, A. V. Arutyunov, S. M. Aseev, S. A. Vakhrameev, A. Ya. Dubovitzkii, A. A. Milyutin, V. M. Tikhomirov, A. D. Ioffe, E. S. Polovinkin, V. I. Ukhobotov, V. I. Zhukovskii, N. N. Petrov, L. A. Petrosyan, etc., the above-mentioned works by the Ekaterinburg (Sverdlovsk) school, and by many other schools and researchers.

The present monograph deals with materials on the further development of the theory of minimax solutions for new types of equations, namely, for singularly perturbed Hamilton–Jacobi equations and quasi-linear parabolic Isaacs equations. Also, there are applications of the theory of minimax solutions for the mentioned partial differential equations and for the Bellman equation to problems of optimal control and differential games. The *method of characteristics* is key to the research, and, therefore, it is mentioned in the title of the monograph.

It should be mentioned that there is now the increasing interest in the various generalizations of the method of characteristics in modern research on dynamical optimization and boundary problems for the corresponding partial differential equations. New approaches are suggested in works of J. P. Aubin, F. H. Clarke, H. Frankowska, G. Haddad, A. B. Kurzhanskii, Yu. S. Ledyaev, A. A. Melikyan, D. B. Silin, and many others.

Let us briefly describe the contents of the book. The monograph consists of five chapters divided into twenty sections.

The known results used in the monograph are called *assertions*. The results obtained by the author are called *lemmas* for the auxiliary results and *theorems* for the basic results. All *assertions*, *lemmas*, and *theorems* have a double number: the Roman number is the number of the *chapter* and the Arabian number means the current number of the *assertion*, *lemma*, or *theorem* in this chapter. Sections are numbered continuously. Subsections and formulas have also a double number, where the first number means the number of the section and the second number means the number of the subsection or the number of a formula in this section.

In Chap. I, there are some basic notions and results in the classical theory of Hamilton–Jacobi equations; the tools of nonsmooth analysis are applied to the research presented in this monograph, and also notions and facts of the theories of minimax and viscosity solutions useful in the research. In Chap. II, there is a detailed investigation of the value function (the Bellman function) in the nonlinear problem of optimal control with a cost functional of the Bolza type. The coincidence of the value function and the minimax solution of the corresponding Cauchy problem for the Bellman equation is essential to the research. The theoretical and applied aspects of this problem have gained the interest of many researchers. Among the investigations of the problem using the Bellman function that are closest to the materials of this monograph, let us mention the works of F. M. Kirillova, R. F. Gabasov, S. N. Kruzhkov, L. I. Rozonoer, M. M. Khrustalev, V. A. Vyazgin, V. F. Krotov, V. F. Dem'yanov, V. I. Gurman, M. Bardi, E. N. Barron, L. D. Berkovitz, P. Cannarsa, F. H. Clarke, H. Frankowska, R. Jensen, G. Leitman, P. L. Lions, S. Mirică, R. T. Rockafellar, and R. Vinter.

Applications of the theory of minimax solutions for the Bellman equation to the optimal control problem considered in Chap. III provide the following results:

- the representative formula for minimax solutions in the form of the lower envelope of a family of smooth functions;
- a new proof of the Pontryagin maximum principle via differentiability of solutions of ordinary differential equations with respect to the parameters;
- the coincidence of extremals and conjugate variables satisfying the conditions of the Pontryagin maximum principle with classical characteristics for the Bellman equation;
- the necessary and simultaneously sufficient optimality conditions of first order supplementing the Pontryagin maximum principle;
- the justification of the dynamical programming method for Lipschitz continuous Bellman functions, and the justification of the structure of optimal feedbacks (optimal synthesis);
- the formula of the minimax solution to the Cauchy problem for a first-order nonlinear partial differential equation with concave Hamiltonian in terms of the classical characteristics.

In Chaps. III and IV, there are results on singular approximations of the minimax solutions to the unperturbed Hamilton–Jacobi equations. The approximations use minimax solutions for the Hamilton–Jacobi equations considered in the extended phase space. The equations are singularly perturbed with respect to a part of the impulse variables. Sufficient conditions for the singular approximation are obtained. Also, there is a construction of the limit unperturbed equations, called asymptotics, to the given singularly perturbed Hamilton–Jacobi equations. Applications of the results to singularly perturbed differential games demonstrate the importance of the approximations. The origin of the research lies in works of A. N. Tikhonov. It should be noted that the presented results are also close to the results of E. F. Mishchenko and L. S. Pontryagin connected with mathematical models of dynamical systems with fast and slow motions. This area in the theory and applications of generalized solutions have attracted the interest of many researchers. Among the works most relevant to the material of this monograph we mention the works of A. B. Vasil'eva, V. F. Butuzov, V. G. Gaitsgory, M. G. Dmitriev, A. B. Kurzhanskii, A. A. Pervozvanskii, E. L. Tonkov, N. H. Rosov, L. D. Akulenko, T. F. Filippova, A. M. Fradkov, Z. Artstein, M. Bardi, E. N. Barron, A. Bensoussan, L. C. Evans, P. Donchev, R. Jensen, P. V. Kokotovitc, and V. Veliov.

In Chap. III, there is the concept of the minimax solution to singularly perturbed Hamilton–Jacobi equations that contain a small parameter of singularity in the denominators of coefficients at a part of the impulse variables. The occurrence of fast and slow components (variables) is a new feature of the generalized characteristics defining the minimax solutions. Sufficient conditions for minimax solutions to singularly perturbed Hamilton–Jacobi equations are suggested to provide convergence, as the parameter of singularity tends to zero. The structure of the limit unperturbed guaranteed control problems is described. A key requirement in the sufficient conditions is the existence of compact attractors in the subspace of fast variables. One can consider the conditions as a development and generalization of the Tikhonov reduction technique, which was suggested to investigate singularly perturbed systems of ordinary differential equations. The results presented apply to research on convergence of the value functions and constructions of asymptotics to differential games with fast and slow motions. The applications and model examples are presented in Chap. IV.

Chapter V deals with the development of the theory of minimax solutions to quasi-linear parabolic partial differential equations of the Hamilton–Jacobi–Isaacs type. The interest in the generalized solution of a quasi-linear parabolic Isaacs equation stems from the same reasons as in the above-mentioned deterministic case. The solution coincides with the value function to a diffusion differential game with noise degenerate in the whole or in a part of the variables. In the works of N. N. Krasovskii, V. E. Tretyakov, and A. N. Krasovskii [126, 136], the existence of the value function of the game was proved. Also, notions and justifications of stochastic *u-stability* and *v-stability* properties of the value function are suggested.

The concept of generalized *viscosity* solution to a quasi-linear parabolic partial differential equation of the Hamilton–Jacobi–Isaacs type is introduced by P. L. Lions using the notions of sub- and superjets involving tools of sub- and superdifferentials and constant matrices, which approximate the corresponding Laplace operator of the second derivatives [168]. Substantial contributions to investigations of generalized solutions to quasi-linear parabolic partial differential equations and of the value functions to the corresponding diffusion differential were made by O. A. Oleinik, O. A. Ladyzhenskaya, S. N. Kruzhkov, A. M. Il'in, H. Ishii, W. H. Fleming, A. Fridman, M. G. Crandall, L. C. Evans, P. L. Lions, H. M. Soner, etc.

Chapter V deals with diffusion differential games, where the value function is Lipschitz continuous. For the class of Lipschitz continuous real functions, the concept of generalized stochastic derivatives relative to the set of the drift and the diffusion matrix defining the diffusion control process, is presented. The concept of the generalized minimax solution to the corresponding quasi-linear parabolic Isaacs equation is introduced. The definition uses a pair of differential inequalities in terms of the generalized stochastic derivatives for the minimax solution. Since the minimax solution coincides with the value function for a diffusion differential game, the concept is one way of applying the method of generalized characteristics to investigations of the value function. Note that the definition is an infinitesimal form of the mentioned properties of stochastic *u-stability* and *v-stability.* In Chap. V, formulas for stochastic derivatives are also obtained for a class of functions differentiable in a part of the variables. There are applications of the formulas to stochastic differential games, where controls, disturbances, and noise act in a part of the variables. There is presented a corrected form of the generalized quasi-linear parabolic Isaacs equation.

The results presented provide perspectives to develop the theory of minimax solutions for new types of partial differential equations and boundary problems. The results can also be used for the analysis of applied control problems and the construction of feedbacks solving these problems.

This work i dedicated to the shiny memory of A. I. Subbotin who piqued her scientific interests and provided constructive criticism and helpful advise.

The author thanks her teacher, Academician N. N. Krasovskii, for kind attention, valuable advices, support, and his proposal to prepare the presented results for publication.

The author expresses her sincere gratitude to Academician R. V. Gamkrelidze for the opportunity to publish this monograph.

Taking advantage of the opportunity, the author expresses her gratitude to Yu. S. Osipov, A. B. Kurzhanskii, V. E. Tretyakov, E. G. Al'brekht, G. S. Shelement'ev, A. V. Kryazhimskii, A. A. Melikyan, A. G. Chentsov, V. N. Ushakov, M. I. Gusev, and to all colleagues in the Departments of Dynamical

Systems, Optimal Controls, Control Systems, and Differential Equations in the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, and also in the Chair of Applied Mathematics in Ural State University for constant attention and fruitful discussions of the results presented at seminars and personal meetings.

This work was supported by the Russian Foundation for Basic Research (project Nos. 02-01-00769 and 03-01-00599) and the grant of Scientific Schools NS-791.2003.1.

General Notation

In this monograph, we use the following notation.

$\mathbb{R}^n$ — n-dimensional Euclidean space;

$^\top$ — transposition operation;

$\langle x, y\rangle$ — inner product of vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^n$;

$\|x\|$ — Euclidean norm $x \in \mathbb{R}^n$, $\|x\| = \langle x, x\rangle^{1/2}$;

$\operatorname{dist}(x, X)$ — distance between a point $x \in \mathbb{R}^n$ and a set $X \subseteq \mathbb{R}^n$, $\operatorname{dist}(x, X) = \inf\limits_{x' \in X} \|x - x'\|$;

$\operatorname{int} X$ — the set of interior points of the set $X \subseteq \mathbb{R}^n$;

$\operatorname{cl} X$ — closure of $X \subseteq \mathbb{R}^n$;

∂X — the set of boundary points of $X \subseteq \mathbb{R}^n$;

$\operatorname{co} X$ — convex hull of $X \subseteq \mathbb{R}^n$;

$\operatorname{comp} \mathbb{R}^n$ — the set of all compact subsets of $\mathbb{R}^n$;

$\operatorname{diam} X$ — diameter of the set $X \subset \operatorname{comp} \mathbb{R}^n$, $\operatorname{diam} X = \max\limits_{x \in X, x' \in X} \|x - x'\|$;

$\operatorname{dist}(X_1, X_2)$ — Hausdorff distance between sets $X_1 \subset \operatorname{comp} \mathbb{R}^n$ $X_2 \subset \operatorname{comp} \mathbb{R}^n$:

$$\operatorname{dist}(X_1, X_2) = \max\{\max_{x_1 \in X_1} \operatorname{dist}(x_1, X_2), \max_{x_2 \in X_2} \operatorname{dist}(x_2, X_1)\};$$

B_n — closed unit ball in $\mathbb{R}^n$ centered at the origin;

B_n^ε — closed ball of radius ε in $\mathbb{R}^n$ centered at the origin;

$\operatorname{proj}_x G$ — projection of the set G in the space with the coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$ on the subspace $\mathbb{R}^n$ with the coordinates x;

$\varphi : W \to Q$ — mapping with domain W and range Q,

$\varphi(\cdot)$ — mapping φ (note that $\varphi(w)$ is the value of the mapping φ at a point $w \in W$);

$\operatorname{gr} \varphi$ — the graph of a function $\varphi(\cdot) : W \to Q \subseteq \mathbb{R}^1$:

$$\operatorname{gr} \varphi = \{(w, r) \in W \times \mathbb{R}^1 : w \in W, r = \varphi(w)\};$$

$\operatorname{epi} \varphi$ — the epigraph of $\varphi(\cdot) : W \to Q \subseteq \mathbb{R}^1$:

$$\operatorname{epi} \varphi = \{(w, r) \in W \times \mathbb{R}^1 : w \in W, r \geq \varphi(w)\};$$

$\operatorname{hypo} \varphi$ — the hypograph of $\varphi(\cdot) : W \to Q \subseteq \mathbb{R}^1$:

$$\operatorname{hypo} \varphi = \{(w, r) \in W \times \mathbb{R}^1 : w \in W, r \leq \varphi(w)\};$$

$D_x\omega$ — gradient with respect to $x = (x_1, \ldots, x_n)$ of a function $\mathbb{R}^n \times \mathbb{R}^k \ni (x, y) \mapsto \omega(x, y) \in \mathbb{R}^1$:

$$D_x\omega = \left(\frac{\partial \omega}{\partial x_1}, \ldots, \frac{\partial \omega}{\partial x_n}\right);$$

$D\omega$ — gradient of a function $\omega(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$:

$$D\omega = \left(\frac{\partial \omega}{\partial x_1}, \ldots, \frac{\partial \omega}{\partial x_n}\right);$$

$\operatorname{rpm} W$ — the set of all regular probability measures defined on $W \subset \operatorname{comp} \mathbb{R}^m$;

$C_n(W)$ — the space of continuous real functions $\phi(\cdot) : W \to \mathbb{R}^n$, $W \subset \operatorname{comp} \mathbb{R}^m$;

$C_n(W)^*$ the space conjugate to the space $C_n(W)$ of continuous real functions;

$L_n([t_0,T],C_n(W))$ the space of real functions $\xi(\cdot,\cdot):[t_0,T]\times W\to\mathbb{R}^n$, $t_0\in[0,T]$, such that for any $w\in W$, the functions $\xi(\cdot,w):[t_0,T]\to\mathbb{R}^n$ are Borel-measurable and summable, i.e., $\xi(\cdot,w)\in L_n([t_0,T])$, and for almost $t\in[t_0,T]$, the functions $\xi(t,\cdot):W\to\mathbb{R}^n$ are continuous, i.e., $\xi(t,\cdot)\in C_n(W)$;

$(\Omega,\mathcal{F},\mathbb{P})$ probability space, where Ω is a nonempty set, $\mathcal{F}$ is the σ-algebra of subsets Ω, and $\mathbb{P}$ is the probability measure;

$\mathbb{P}$-a.e. means that the corresponding condition holds almost everywhere in Ω with respect to the measure $\mathbb{P}$;

$\{\mathcal{F}_s\}$, $s\ge 0$ nondecreasing family of σ-algebras of subsets of Ω;

$\operatorname{tr}A$ the trace of the matrix $A=\{a_{ij}\}$, $i\in\overline{1,n}$, $j\in\overline{1,n}$, $\operatorname{tr}A=\sum\limits_{i=1}^{n}a_{ij}$;

$\mathcal{L}[\mathbb{R}^m,\mathbb{R}^n]$ the set of all constant $(n\times m)$-matrices $A=\{a_{ij}\}$, $i\in\overline{1,n}$, $j\in\overline{1,m}$.

Chapter I

CAUCHY PROBLEM FOR HAMILTON–JACOBI EQUATIONS

1. Hamilton–Jacobi Equations. Basic Notions

We consider first-order partial differential equations of the Hamilton–Jacobi type:

$$\begin{gathered}\frac{\partial V'(t,x)}{\partial t}+H\big(t,x,V'(t,x),D_xV'(t,x)\big)=0,\quad (t,x)\in\Pi_T=(0,T)\times\mathbb{R}^n,\\ D_xV'(t,x)=\left(\frac{\partial V'(t,x)}{\partial x_1},\dots,\frac{\partial V'(t,x)}{\partial x_n}\right)\in\mathbb{R}^n.\end{gathered}\tag{1.1}$$

Such equations arise in many applied and theoretical problems of engineering, control, navigation, economics, chemistry, biology, and so on.

This monograph deals with the following boundary-value Cauchy problem for Eq. (1.1):

$$V'(T,x)=\sigma(x),\quad x\in\mathbb{R}^n.\tag{1.2}$$

Recall basic notions of the theory of Hamilton–Jacobi equations (see, e.g., [62, 112, 207]).

1.1. Classical solutions of the Cauchy problem and the classical Cauchy method of characteristics for the Hamilton–Jacobi equation. The function $H(t,x,z,p):[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ in Eq. (1.1) is called the *Hamiltonian.* We study Eq. (1.1), where the Hamiltonian is independent of the variable z. The function $\sigma(\cdot):\mathbb{R}^n\to\mathbb{R}$ in condition (1.2) is called the *boundary function.* The Hamiltonian and the boundary function are called the *input data* of the problem (1.1), (1.2).

A continuous function $V'(\cdot):\operatorname{cl}\Pi_T=[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is called a *classical solution* of the boundary-value Cauchy problem (1.1), (1.2) if it is differentiable in the open strip $\Pi_T=(0,T)\times\mathbb{R}^n$, satisfies Eq. (1.1) *everywhere* in the strip, and also satisfies the boundary condition (1.2).

In the first half of the nineteenth century, Cauchy suggested a basic method for solving such problems, now called the Cauchy method of characteristics. In this method, the classical solution of the first-order partial differential equation (1.1) under condition (1.2) can be found by using the following system of ordinary differential equations on the interval $[0,T]$:

$$\begin{aligned}
\frac{d\hat{x}}{dt} &= D_{\hat{p}}H(t,\hat{x},\hat{p}),\\
\frac{d\hat{p}}{dt} &= -D_{\hat{x}}H(t,\hat{x},\hat{p}),\\
\frac{d\hat{z}}{dt} &= \langle \hat{p}, D_{\hat{p}}H(t,\hat{x},\hat{p})\rangle - H(t,\hat{x},\hat{p}),
\end{aligned} \tag{1.3}$$

and the following boundary conditions corresponding to (1.2):

$$\hat{x}(T,y) = y, \quad \hat{p}(T,y) = D_y\sigma(y), \quad \hat{z}(T,y) = \sigma(y), \quad y \in \mathbb{R}^n, \tag{1.4}$$

where

$$D_{\hat{p}}H(t,\hat{x},\hat{p}) = \left(\frac{\partial H}{\partial \hat{p}_1},\dots,\frac{\partial H}{\partial \hat{p}_n}\right), \quad D_{\hat{x}}H(t,\hat{x},\hat{p}) = \left(\frac{\partial H}{\partial \hat{x}_1},\dots,\frac{\partial H}{\partial \hat{x}_n}\right)$$

and the symbol $\langle p,q\rangle$ denotes the inner product of vectors p and q.

The family of solutions of the *characteristic* ordinary differential equations (1.3) forms the graph of the classical smooth solution ω of the Hamilton–Jacobi equation:

$$\operatorname{gr}\omega = \big\{(t,x,z) : (t,x) \in \operatorname{cl}\Pi_T = [0,T]\times\mathbb{R}^n,\ z = \omega(t,x)\big\}$$

and defines the vector field of gradients of the solution. In other words, the graph of the classical solution $\omega(\cdot) \in C^1(\operatorname{cl}\Pi_T)$ of the Hamilton–Jacobi equation (1.1) *is invariant* with respect to solutions of the characteristic equations (1.3). The solutions

$$(\hat{x}(\cdot,y), \hat{p}(\cdot,y), \hat{z}(\cdot,y)) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \tag{1.5}$$

depend on the parameter $y \in \mathbb{R}^n$; they are called *characteristics* (see, e.g., [62, 207, 222]).

The Cauchy method of characteristics is applicable in cases where the phase-space projections $\hat{x}(\cdot,y)$ of characteristics (1.3) do not intersect, in other words, if for all $(t,x) \in \Pi_T$, the set of parameters

$$Y(t,x) = \{\forall y \in \mathbb{R}^n : \hat{x}(t,y) = x\} = \{y(t,x)\} \tag{1.6}$$

is a singleton. In this case, the classical solution can be represented in the form

$$V'(t,x) = \hat{z}(t, y(t,x)), \quad (t,x) \in \operatorname{cl}\Pi_T. \tag{1.7}$$

As is well known, the Cauchy method of characteristics is a local method for nonlinear input data of the boundary-value Cauchy problem (1.1), (1.2). As a rule, the characteristics do not intersect in a small neighborhood of the smooth boundary manifold. The classical solution of the boundary-value problem does not exist outside this neighborhood. Consider the following well-known example illustrating the influence of smooth but nonlinear Hamiltonians (for more details, see [238]). The self-intersecting integral surface is covered with the linear characteristics of Eq. (1.1). This type of singularity is called a "swallow's tail."

Let

$$H(t,x,p) = \sqrt{1+p^2}, \quad \sigma(x) = \frac{x^2}{2},$$

where $0 \le t \le T = 2$, $x \in \mathbb{R}$, and $p \in \mathbb{R}$. Consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} + \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} = 0, \qquad u(2,x) = \frac{x^2}{2}. \tag{1.8}$$

The characteristic system (1.3) has the form

$$\dot{x} = \frac{p}{\sqrt{1+p^2}}, \qquad \dot{p} = 0, \qquad \dot{z} = \frac{-1}{\sqrt{1+p^2}}.$$

According to (1.4), solutions of system (1.3) must satisfy the following boundary conditions:

$$\tilde{x}(2,y) = y, \qquad \tilde{p}(2,y) = y, \qquad \tilde{z}(2,y) = \frac{y^2}{2},$$

where $y \in \mathbb{R}$ is a parameter. One can easily calculate the solution:

$$\tilde{x}(t,y) = y + \frac{(t-2)y}{\sqrt{1+y^2}}, \qquad \tilde{p}(t,y) = y, \qquad \tilde{z}(t,y) = \frac{y^2}{2} - \frac{(t-2)}{\sqrt{1+y^2}}.$$

These characteristics do not intersect if $1 < t \le 2$. The equation $x = \tilde{x}(t,y)$ has a unique root $y(t,x)$ in the strip $D = \{(t,x) : 1 < t \le 2,\ x \in \mathbb{R}\}$. The function $u(t,x) = \tilde{z}(t, y(t,x))$ is continuously differentiable in D. According to the Cauchy method of characteristics, this function is a local classical solution of problem (1.8) in the domain D.

One can see that applications of the Cauchy method of characteristics are restrictive. The idea of invariance of the graph of a classical solution of partial differential equations with respect to a system of characteristic ordinary differential equations was very fruitful. The invariance with respect to systems of differential inclusions (generalized characteristics) is a substantial property of generalized (minimax) solutions of Hamilton–Jacobi equations. Below, one can see that classical and generalized characteristics are convenient in the study of nonsmooth solutions and adequate to the problems considered.

Assume that input data $\sigma(x)$ and $H(t,x,p)$ in the Cauchy problem (1.1), (1.2) satisfy the following assumptions, which are standard in the theory of minimax and viscosity solutions (see, e.g., [59, 235]):

(H1) the function $\sigma(x)$ is continuous and locally bounded;

(H2) the Hamiltonian $H(t,x,p)$ is continuous in the domain $\operatorname{cl}\Pi \times \mathbb{R}^n$ and satisfies the estimate

$$\sup_{(t,x)\in \operatorname{cl}\Pi_T} \frac{|H(t,x,0)|}{(1+\|x\|)} < \infty; \tag{1.9}$$

(H3) the Hamiltonian $H(t,x,p)$ satisfies the Lipschitz condition in the variable p:

$$|H(t,x,p') - H(t,x,p'')| \le \lambda(x)\|p' - p''\| \tag{1.10}$$

for any $(t,x) \in \operatorname{cl}\Pi_T$, $p', p'' \in \mathbb{R}^n$, where $\lambda(x) := (1+\|x\|)\mu$ and $\mu > 0$ is a constant;

(H4) the Hamiltonian $H(t,x,p)$ satisfies the local Lipschitz condition in the variable x:

$$\sup_{(t,x',x'',p)} \left\{ \frac{|H(t,x',p) - H(t,x'',p)|}{\|x' - x''\|(1+\|p\|)} \right\} < \infty \tag{1.11}$$

for any $(t,x',x'',p) \in [0,T] \times B \times B \times \mathbb{R}^n$, where $B \subset \mathbb{R}^n$ is a bounded set.

1.2. Viscosity solutions of the Hamilton–Jacobi equation. As is known, there are functions nondifferentiable on a set of zero measure or discontinuous but important for problems of theoretical mechanics, optimal control, fluid dynamics, and many other fields. Nonsmooth wave fronts in inhomogeneous media, the value functions in time-optimal control problems, the minimal distance between a pursuer and an evader, to the corresponding differential games are examples of such functions.

Such functions are defined in sufficiently large domains $\operatorname{cl}\Pi_T$ or even everywhere in the phase space of the problem. Moreover, it is known that they satisfy the corresponding Hamilton–Jacobi equation at all points of differentiability, i.e., almost everywhere. These functions coincide with the classical solution of the Hamilton–Jacobi equation in the domain where the classical solution is defined. Thus, these functions can be interpreted as generalized solutions of the Hamilton–Jacobi equation. However, it is incorrect to define generalized solutions as continuous functions satisfying the Hamilton–Jacobi equation almost everywhere since there exist examples in which many functions satisfy the Hamilton–Jacobi equation almost everywhere (see, e.g., [73]). Therefore, the problem of appropriate definition of generalized solutions arises.

This problem stimulated active research into the Hamilton–Jacobi equations in the 50s–70s. Concepts of weak solutions of partial differential equations of the first and higher orders were suggested on the basis of integral representations and methods.

In the 1970s, development of convex and nonsmooth analysis allowed one to apply new results and methods based on generalizations of differentiability to research into generalized solutions of partial differential equations. In the early 1980s, Crandall and Lions introduced the notion of viscosity solution.

Recall one of the equivalent definitions of a viscosity solution.

Definition I.1. A continuous function $\operatorname{cl}\Pi_T \ni (t,x) \mapsto \omega(t,x) \in \mathbb{R}$ is called a *viscosity supersolution* of Eq. (1.1) if the following condition holds: if the difference $\omega(t,x) - \varphi(t,x)$ achieves a local minimum at a point $(t_0,x_0) \in \Pi_T$ and the function φ is differentiable at this point, then the following inequality holds:

$$\frac{\partial \varphi(t_0,x_0)}{\partial t} + H(t_0,x_0,D_x\varphi(t_0,x_0)) \leq 0. \tag{1.12}$$

A continuous function $\operatorname{cl}\Pi_T \ni (t,x) \mapsto \omega(t,x) \in \mathbb{R}$ is called a *viscosity subsolution* of Eq. (1.1) if the following condition holds: if the difference $\omega(t,x) - \varphi(t,x)$ achieves a local maximum at a point $(t_0,x_0) \in \Pi_T$ and the function φ is differentiable at this point, then the following inequality holds:

$$\frac{\partial \varphi(t_0,x_0)}{\partial t} + H(t_0,x_0,D_x\varphi(t_0,x_0)) \geq 0. \tag{1.13}$$

A continuous function $\operatorname{cl}\Pi_T \ni (t,x) \mapsto \omega(t,x) \in \mathbb{R}$ is called a *viscosity solution* of Eq. (1.1) if it is a supersolution and a subsolution simultaneously.

Further in Sec. 2.5, an equivalent definition of viscosity solutions will be given via subdifferentials and superdifferentials (see also [60]).

2. Generalization and Relaxation of the Classical Method of Characteristics for the Hamilton–Jacobi Equation

2.1. Generalized characteristics and continuous minimax solutions of the Hamilton–Jacobi equation. The research presented in this monograph is carried out within the framework of the concept of minimax solution suggested by Subbotin [235, 236, 238]. The concept of a generalized solution of the Hamilton–Jacobi equation is vital and can be interpreted as a generalization and relaxation of the classical Cauchy method.

Recall two equivalent definitions (see [238]) of the minimax solution of problem (1.1), (1.2) (see Definitions I.3 and I.6).

Let S be a nonempty set and M be a multi-valued mapping

$$[0,T] \times \mathbb{R}^n \times S \ni (t,x,s) \mapsto M(t,x,s) \subset \mathbb{R}^n \times \mathbb{R}. \tag{2.1}$$

Definition I.2. A pair (S,M) is called a *characteristic complex* (or, briefly, a complex) if the following requirements are satisfied.

(1°) The set $M(t,x,s) \subset \mathbb{R}^n \times \mathbb{R}$ is nonempty, convex, and compact for any $(t,x) \in [0,T] \times \mathbb{R}^n$ and $s \in S$. For any $(t,x,s) \in [0,T] \times \mathbb{R}^n \times S$ and $(f,g) \in M(t,x,s)$, the following estimates hold:

$$\|f\| \leq \lambda(x), \quad |g| \leq m(t,s)(1 + \|x\|),$$

where $\lambda(x)$ is defined in conditions (1.10). For any $s \in S$, the function $t \mapsto m(t,s)$ is summable on $[0,T]$ and the multi-valued mapping $(t,x) \mapsto M(t,x,s)$ is upper semicontinuous

(2°a) For any $(t,x) \in [0,T] \times \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the following relation holds:

$$\max_{s\in S} \min_{(f,g)\in M(t,x,s)} [\langle f,p\rangle - g] = H(t,x,p).$$

(2°b) For any $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the following relation holds:

$$\min_{s \in S} \max_{(f,g) \in M(t,x,s)} [\langle f, p \rangle - g] = H(t, x, p).$$

We denote by $\mathcal{C}(H)$ the set of all complexes (S, M).

Remark I.1. Note that a pair (S, M), where $S = \mathbb{R}^n$,

$$M(t, x, s) = \{(f, g) \in \mathbb{R}^n \times \mathbb{R} : \|f\| \leq \lambda(x),\ g = \langle f, s \rangle - H(t, x, s)\}$$

for all $s \in \mathbb{R}^n$ and $(t, x) \in \operatorname{cl} \Pi_T$, and $\lambda(x) = (1 + \|x\|)\mu$ is the Lipschitz constant (see (1.10)), satisfies all the above requirements.

Choose a complex $(S, M) \in \mathcal{C}(H)$ and $s \in S$.

Denote by $\operatorname{Sol}(t_0, x_0, z_0, s)$ the set of all absolutely continuous functions

$$(x(\cdot), z(\cdot)) : \quad [0, T] \to \mathbb{R}^n \times \mathbb{R}$$

satisfying the condition $(x(t_0), z(t_0)) = (x_0, z_0)$ and the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in M(t, x(t), z(t), s), \quad t \in [t_0, T]. \tag{2.2}$$

Definition I.3. A continuous function $[0, T] \times \mathbb{R}^n \ni (t, x) \mapsto V'(t, x) \in \mathbb{R}$ is called a *minimax solution* of Eq. (1.1) if for any $(t_0, x_0, z_0) \in \operatorname{gr} u$, $s \in S$, and $\tau \in [t_0, T]$, there is a trajectory

$$(x(\cdot), z(\cdot)) \in \operatorname{Sol}(t_0, x_0, z_0, s)$$

such that $(\tau, x(\tau), z(\tau)) \in \operatorname{gr} V'$.

The differential inclusion (2.2) is said to be *characteristic* and its solutions are called *generalized characteristics*.

It is known that this definition is independent of the choice of a complex $(S, M) \in \mathcal{C}(H)$ (see [238]). There is a wide variety of characteristic complexes unified with respect to the given Hamiltonian. They play a key role in the study and construction of minimax solutions. In particular, this will be used in Chap. IV to provide effective sufficient conditions for convergence of minimax solutions of singularly perturbed Isaacs equations.

Definition I.4. A pair (S_+, M_+) is called an *upper characteristic complex* (or, briefly, an upper complex) if conditions **(1°)** and **(2°a)** hold.

A pair (S_-, M_-) is called a *lower characteristic complex* (or, briefly, a lower complex) if conditions **(1°)** and **(2°b)** hold.

Denote by $\operatorname{Sol}_+(t_0, x_0, z_0, s_+)$ the set of all absolutely continuous functions

$$(x^+(\cdot), z^+(\cdot)) : [0, T] \to \mathbb{R}^n \times \mathbb{R}$$

satisfying the condition $(x^+(t_0), z^+(t_0)) = (x_0, z_0)$ and the differential inclusion

$$(\dot{x}^+(t), \dot{z}^+(t)) \in M_+(t, x(t), z(t), s_+), \quad t \in [t_0, T], \tag{2.3}$$

where $s_+ \in S_+$ and (S_+, M_+) is an upper characteristic complex.

Similarly, denote by $\operatorname{Sol}_-(t_0, x_0, z_0, s_-)$ the set of all absolutely continuous functions

$$(x^-(\cdot), z^-(\cdot)) : \quad [0, T] \to \mathbb{R}^n \times \mathbb{R}$$

satisfying the condition $(x^-(t_0), z^-(t_0)) = (x_0, z_0)$ and the differential inclusion

$$(\dot{x}^-(t), \dot{z}^-(t)) \in M_-(t, x(t), z(t), s_-), \quad t \in [t_0, T],$$

where $s_- \in S_-$ and (S_-, M_-) is a lower characteristic complex.

Definition I.5. An *upper minimax solution* of Eq. (1.1) is a lower semicontinuous function $[0,T]\times\mathbb{R}^n \ni (t,x)\mapsto V'(t,x)\in\mathbb{R}$ satisfying the following condition:

(i) for any $(t_0,x_0,z_0^+)\in \operatorname{epi} V'$, $s_+\in S_+$, and $\tau\in[t_0,T]$, there is a trajectory

$$(x^+(\cdot),z^+(\cdot))\in \mathrm{Sol}_+(t_0,x_0,z_0^+,s_+)$$

such that $(\tau,x^+(\tau),z^+(\tau))\in \operatorname{epi} V'$, where

$$\operatorname{epi} V'=\{(t,x,z): t\in[0,t],\ x\in\mathbb{R}^n,\ z\geq V'(t,x)\}.$$

A *lower minimax solution* of Eq. (1.1) is an upper semicontinuous function $[0,T]\times\mathbb{R}^n\ni(t,x)\mapsto V'(t,x)\in\mathbb{R}$ satisfying the following condition:

(ii) for any $(t_0,x_0,z_0^-)\in \operatorname{hypo} V'$, $s_-\in S_-$, and $\tau\in[t_0,T]$, there is a trajectory

$$(x^-(\cdot),z^-(\cdot))\in \mathrm{Sol}_-(t_0,x_0,z_0^-,s_-)$$

such that $(\tau,x^-(\tau),z^-(\tau))\in \operatorname{hypo} V'$, where

$$\operatorname{hypo} V'=\{(t,x,z): t\in[0,t],\ x\in\mathbb{R}^n,\ z\leq V'(t,x)\}.$$

Definition I.6. A continuous function $[0,T]\times\mathbb{R}^n\ni(t,x)\mapsto V'(t,x)\in\mathbb{R}$ is called a *minimax solution* of Eq. (1.1) if it is an upper minimax solution and a lower minimax solution of Eq. (1.1), simultaneously.

2.2. Existence and uniqueness theorems for continuous minimax solutions of the Cauchy problem for the Hamilton–Jacobi equation. Recall some properties of minimax solutions (see [238]).

Assertion I.1. *Let conditions* **(H1)**–**(H4)** *hold. Then any upper solution V^* and any lower solution V_* of the Cauchy problem* (1.1), (1.2) *satisfy the inequality $V^*\geq V_*$.*

Assertion I.2. *Let conditions* **(H1)**–**(H4)** *hold. Then there are an upper solution V^* and a lower solution V_* of the Cauchy problem* (1.1), (1.2) *satisfying the inequality $V^*\leq V_*$.*

Assertion I.3. *Let conditions* **(H1)**–**(H4)** *hold. Then there is a unique minimax solution $V'=V^*=V_*$ of the Cauchy problem* (1.1), (1.2).

2.3. Directional derivatives and generalized differentials of nonsmooth functions. The notion of minimax solution can be also introduced in other equivalent forms by using different tools of nonsmooth analysis: directional derivatives, contingent or tangent cones, subdifferentials and superdifferentials, and so on. Proofs of the equivalence of different definitions can be found in [236, 238]. Some of these definitions are presented in Sec. 2.5 below. In Secs. 2.3 and 2.4, we recall some notions of nonsmooth analysis used in this paper (see, e.g., [51, 55, 59, 60, 66, 186, 215, 218]).

Definition I.7. The *Clarke generalized differential* of a function $\mathbb{R}^{n+1}\supset\tilde{\Pi}_T\ni(t,x)\mapsto V'(t,x)\in\mathbb{R}$ at a point $(t,x)\in\Pi_T$ is the set

$$\partial_C V'(t,x)=\operatorname{co}\left\{\forall(\rho,p)\in\mathbb{R}\times\mathbb{R}^n:(\rho,p)=\lim_{(t',x')\to(t,x)}\frac{\partial V(t',x')}{\partial(t,x)}\right\},\tag{2.4}$$

where (t',x') are regular points, where the function $V'(\cdot)$ is differentiable, and

$$\frac{\partial V'(t',x')}{\partial(t,x)}:=\left(\frac{\partial V'(t',x')}{\partial t},\frac{\partial V'(t',x')}{\partial x_1},\ldots,\frac{\partial V'(t',x')}{\partial x_n}\right).$$

Definition I.8. The *lower Dini semiderivative* of a function $\mathbb{R}^{n+1}\supset\tilde{\Pi}_T\ni(t,x)\mapsto V'(t,x)\in\mathbb{R}$ at a point $(t,x)\in\Pi_T$ in a direction $(\eta,h)\in\mathbb{R}\times\mathbb{R}^n$ is defined as follows:

$$\frac{d^-V'(t,x)}{(\eta,h)}=\liminf_{\substack{\delta\downarrow 0\\(\eta',h')\to(\eta,h)}}\frac{V'(t+\delta\eta',x+\delta h')-V'(t,x)}{\delta}.\tag{2.5}$$

Similarly, the *upper Dini semiderivative* of a function $\mathbb{R}^{n+1} \supset \tilde{\Pi}_T \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ at a point $(t,x) \in \Pi_T$ in a direction $(\eta,h) \in \mathbb{R} \times \mathbb{R}^n$ is defined as follows:

$$\frac{d^+V'(t,x)}{(\eta,h)} = \limsup_{\substack{\delta\downarrow 0\\(\eta',h')\to(\eta,h)}} \frac{V'(t+\delta\eta', x+\delta h') - V'(t,x)}{\delta}. \tag{2.6}$$

Definition I.9. The (*regular*) *subdifferential* of a function $\mathbb{R}^{n+1} \supset \tilde{\Pi}_T \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ at a point $(t,x) \in \Pi_T$ is the set

$$\partial^- V'(t,x) := \left\{(\rho,p) \in \mathbb{R}\times\mathbb{R}^n : \forall(\eta,h) \in \mathbb{R}\times\mathbb{R}^n, \langle(\rho,p),\ (\eta,h)\rangle - \frac{d^-V'(t,x)}{(\rho,p)} \le 0\right\}. \tag{2.7}$$

Similarly, the (*regular*) *superdifferential* of a function $\mathbb{R}^{n+1} \supset \tilde{\Pi}_T \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ at a point $(t,x) \in \Pi_T$ is the set

$$\partial^+ V'(t,x) := \left\{(\rho,p) \in \mathbb{R}\times\mathbb{R}^n : \forall(\eta,h) \in \mathbb{R}\times\mathbb{R}^n, \langle(\rho,p),\ (\eta,h)\rangle - \frac{d^+V'(t,x)}{(\eta,h)} \ge 0\right\}. \tag{2.8}$$

Remark I.2. The definitions of generalized differentials imply that the following relations hold at points $(t_*,x_*) \in (0,T)\times\mathbb{R}^n$, where regular subdifferentials and superdifferentials are nonempty:

$$\partial_C V'(t_*,x_*) \supset \partial^- V'(t_*,x_*), \quad \partial_C V'(t_*,x_*) \supset \partial^+ V'(t_*,x_*). \tag{2.9}$$

Assertion I.4 (see [51, 218]). *Let $\mathbb{R}^{n+1} \supset \operatorname{cl}\Pi_T \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ be a locally Lipschitz-continuous function. Then the set $\partial_C V'(t,x)$ is nonempty, convex, and compact for any point $(t,x) \in \Pi_T$. The Dini semiderivatives $\dfrac{d^\pm V'(t,x)}{(\eta,h)}$ in any direction $(\eta,h) \in \mathbb{R}\times\mathbb{R}^n$ exist and satisfy the inequalities*

$$\frac{d^+V'(t,x)}{(\eta,h)} \le \max_{(\rho,p)\in\partial_C V'(t,x)} \langle(\rho,p),(\eta,h)\rangle, \tag{2.10}$$

$$\frac{d^-V'(t,x)}{(\eta,h)} \ge \min_{(\rho,p)\in\partial_C V'(t,x)} \langle(\rho,p),(\eta,h)\rangle. \tag{2.11}$$

2.4. Invariance of sets with respect to differential inclusions. The main notion of the theory of minimax solutions is the concept of weak invariance (see, e.g., [13, 58, 102, 218]).

Let S be a nonempty, closed set in $\mathbb{R}\times\mathbb{R}^n$. Denote by S_t its section at a moment t. Let $(t,x) \mapsto F(t,x)$ be a multi-valued mapping, which transforms points of the strip $[0,T]\times\mathbb{R}^n$ to compact subsets of the space $\mathbb{R}^n$. Consider the differential inclusion

$$\dot{x}(t) \in F(t,x(t)), \quad t \in [t_0,T], \quad x(t_0) = x_0. \tag{2.12}$$

Definition I.10. A set S is said to be *weakly tnvariant* with respect to the differential inclusion (2.12) if there exists a trajectory $x(\cdot)$ of the differential inclusion (2.12) starting at a point $x_0 \in S_{t_0}$ and defined on an interval $[t_0,T]$, $t_0 \in [0,T]$, which remains in the set S for all $t \in [t_0,T]$, i.e., $x(t) \in S_t$.

We also recall the notion of strong invariance (see, e.g., [13]).

Definition I.11. A set S is said to be *strongly invariant* with respect to the differential inclusion (2.12) if *all* trajectories $x(\cdot)$ of the differential inclusion (2.12) starting at a point $x_0 \in S_{t_0}$ and defined on an interval $[t_0,T]$, $t_0 \in [0,T]$, remain in the set S for all $t \in [t_0,T]$, i.e., all $x(t) \in S_t$.

2.5. Equivalent definitions of minimax solutions. As is known, the notions of upper, lower, and minimax solutions of the Hamilton–Jacobi equations can be introduced in different equivalent forms (see, e.g., [238]).

Conditions defining upper solutions. First, consider the following conditions **(U1)**–**(U3)** defining upper solutions of Eq. (1.1). Assume that a function $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto V^*(t,x) \in \mathbb{R}$ is lower semicontinuous.

(U1) For any $s_+ \in S_+$, the epigraph of the function V^* is weakly invariant with respect to the differential inclusion

$$(\dot{x}^+(t), \dot{z}^+(t)) \in M^+(t, x^+(t), z^+(t), s_+). \tag{2.13}$$

Here and in condition **(U2)** below, (S_+, M^+) is an upper characteristic complex satisfying conditions **(1°)** and **(2°a)**.

(U2) For any $(t,x) \in \Pi_T$ and $s_+ \in S_+$, we have

$$\inf_{(f,g)\in M^+(t,x,V^*(t,x),s_+)} \left[\frac{d^-V^*(t,x)}{(1,f)} - g\right] \le 0. \tag{2.14}$$

(U3) For any $(t,x) \in \Pi_T$ and $(\rho,p) \in D^-V^*(t,x)$, we have

$$\rho + H(t,x,p) \le 0. \tag{2.15}$$

Conditions defining lower solutions. Let us consider a lower solution of Eq. (1.1). Assume that the function $V_*(t,x)$ in the conditions **(L1)**–**(L3)** is upper semicontinuous.

(L1) For any s_-inS_-, the hypograph of the function V_* is weakly invariant with respect to the differential inclusion

$$(\dot{x}^-(t), \dot{z}^-(t)) \in M^-(t, x^-(t), z^-(t), s_-). \tag{2.16}$$

Here and in condition **(L2)** below, (S_-, M^-) is a lower characteristic complex satisfying conditions **(1°)** and **(2°b)**.

(L2) For any $(t,x) \in \Pi_T$ and $s_- \in S_-$, we have

$$\sup_{(f,g)\in M^-(t,x,V_*(t,x),s_-)} \left[\frac{d^+V_*(t,x)}{(1,f)} - g\right] \ge 0. \tag{2.17}$$

(L3) For any $(t,x) \in \Pi_T$ and $(\rho,p) \in D^+V_*(t,x)$, we have

$$\rho + H(t,x,p) \ge 0. \tag{2.18}$$

Note that

$$\frac{d^+V'(t,x)}{(\alpha,f)} = -\frac{d^-(-V'(t,x))}{(\alpha,f)}, \quad D^+V'(t,x) = -D^-(-V'(t,x)).$$

Conditions defining minimax solutions. Consider the following conditions **(M1)** and **(M2)** defining minimax solutions of Eq. (1.1). Let a function $V'(t,x)$ be continuous.

(M1) For any $(t_0,x_0,z_0) \in \operatorname{gr} V'$ and $s \in \mathbb{R}^n$, there exists a number $\tau \in (0,T)$ and absolutely continuous functions $(x(\cdot), z(\cdot)) : [t_0,\tau] \mapsto \mathbb{R}^n \times \mathbb{R}$ satisfying the initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$, the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in M(t, x(t), z(t), s), \tag{2.19}$$

and the equation $z(t) = V'(t, x(t))$ for any $t \in [t_0, \tau]$; here and in condition **(M2)** below, (S,M) is an arbitrary characteristic complex satisfying conditions **(1°)**, **(2°a)**, and **(2°b)**.

(M2) The function V' is an upper and lower solution of Eq. (1.1) simultaneously, i.e., V' satisfies a pair of conditions **(Ui)** and **(Lj)** for $i,j = 1,2,3$.

Assertion I.5 (see [238]). *For any lower semicontinuous function $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto V^*(t,x) \in \mathbb{R}$, conditions* **(U1)**–**(U3)** *are equivalent.*

For any upper semicontinuous function $[0,T]\times\mathbb{R} \ni (t,x) \mapsto V_*(t,x) \in \mathbb{R}$, conditions **(L1)**–**(L3)** *are equivalent.*

For any continuous function $[0,T]\times\mathbb{R} \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$, conditions **(M1)** *and* **(M2)** *are equivalent.*

According to Assertion I.5, let us introduce the definitions of the upper, lower, and minimax solutions.

Definition I.12. A lower semicontinuous function $[0, T] \times \mathbb{R} \ni (t, x) \mapsto V^*(t, x) \in \mathbb{R}$ is called an *upper solution* of Eq. (1.1) if it satisfies any of (equivalent) conditions **(U1)**–**(U3)**.

Similarly, an upper semicontinuous function $[0, T] \times \mathbb{R} \ni (t, x) \mapsto V_*(t, x) \in \mathbb{R}$ is called a *lower solution* of Eq. (1.1) if it satisfies any of (equivalent) conditions **(L1)**–**(L3)**.

A continuous function $[0, T] \times \mathbb{R} \ni (t, x) \mapsto V_*(t, x) \in \mathbb{R}$ is called a *minimax solution* of Eq. (1.1) if it satisfies any of (equivalent) conditions **(M1)** or **(M2)**.

Note that conditions **(U3)** and **(L3)** present a definition of a *viscosity solution* of Eq. (1.1) (see [60]). This definition is equivalent to the conditions presented in Sec. 1.2.

Assertions I.1–I.3 and I.5 imply that the conditions **(H1)**–**(H4)** for the Hamiltonian H and the boundary function σ provide the existence of a unique minimax solution of the Cauchy problem (1.1), (1.2). This solution coincides with a unique continuous viscosity solution of the problem. Proofs of these facts can be found in [236, 238].

The above-mentioned notions of generalized solutions of the Hamilton–Jacobi equation are defined in different forms and by using different tools. However, they are equivalent since they determine the same functions. These definitions are based on the *weak invariance* of the graphs of generalized solutions with respect to characteristic complexes. Hence, the notion of a generalized solution of the Hamilton–Jacobi equation is based on ideas having sources in the classical Cauchy method of characteristics.

Chapter II

CLASSICAL AND GENERALIZED METHODS OF CHARACTERISTICS FOR OPTIMAL CONTROL PROBLEMS

3. Statement of Optimal Control Problems

3.1. Optimal open-loop control problem. In this chapter, we consider the following optimal control problem (**OCP**). Let the dynamics of a system be described by the equation

$$\dot{x} = f(t, x, u), \quad u \in P, \quad x(t_0) = x_0, \tag{3.1}$$

where t is time, $t \in [0, T]$, and $x \in \mathbb{R}^n$ is the phase vector of the system. Let values of the control parameters u belong to a given compact set $P \subset \mathbb{R}^n$. Initial conditions for the system are $x(t_0) = x_0 \in \mathbb{R}^n$, $t_0 \in [0, T]$. Assume that the terminal time moment T for the considered control process is fixed. Let the cost functional $I_{t_0,x_0}(x(\cdot), u(\cdot))$ be of the Bolza type:

$$I_{t_0,x_0}(x(\cdot), u(\cdot)) = \sigma(x(T; t_0, x_0, u(\cdot))) + \int_{t_0}^{T} g(t, x(t), u(t))dt, \tag{3.2}$$

where $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot)) : [t_0, T] \to \mathbb{R}^n$ is a trajectory of the dynamical system (3.1) starting at the initial point (t_0, x_0) under a measurable control $u(\cdot) : [t_0, T] \to P$.

Consider the problem **OCP**: how one can guide motions of system (3.1) to provide the optimal cost $V(t_0, x_0)$? The value $V(t_0, x_0)$ is determined as follows:

$$V(t_0, x_0) = \inf_{u(\cdot) \in \mathbf{U}_{t_0}} I_{t_0,x_0}(x(\cdot; t_0, x_0, u(\cdot)), u(\cdot)), \tag{3.3}$$

where $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and $\mathbf{U}_{t_0}$ is the set of all admissible open-loop controls, i.e., measurable functions $u(\cdot) : [t_0, T] \to P$ (so-called *program controls*), $(t_0 \in [0, T])$. We denote by Π_T and $\operatorname{cl} \Pi_T$, respectively, the following sets (strips) in the space $\mathbb{R}^{n+1}$:

$$\Pi_T = (0, T) \times \mathbb{R}^n, \quad \mathrm{cl}\,\Pi_T = [0, T] \times \mathbb{R}^n.$$

3.2. Main assumptions. Assume that input data of the optimal control problem **OCP** satisfy the following conditions.

(A1) The functions $f(t, x, u)$ and $g(t, x, u)$ in (3.1) and (3.2) are continuous on the set $\mathrm{cl}\,\Pi_T \times P$ and are Lipschitz continuous with respect to the variables t and x, i.e.,

$$\|f(t', x', u) - f(t'', x'', u)\| \le L_1(|t' - t''| + \|x' - x''\|),$$
$$\|g(t', x', u) - g(t'', x'', u)\| \le L_1(|t' - t''| + \|x' - x''\|)$$

for any $t', t'' \in [0, T]$, $x', x'' \in \mathbb{R}^n$, and $u \in P$, where $L_1 > 0$ is a constant.

(A2) For all $(t, x, u) \in \mathrm{cl}\,\Pi_T \times P$, the following inequalities hold:

$$\|f(t, x, u)\| \le K_1(1 + \|x\|), \quad \|g(t, x, u)\| \le K_1(1 + \|x\|),$$

where $K_1 > 0$ is a constant.

(A3) The terminal part $\sigma(\cdot)$ of the cost functional (3.2) satisfies the Lipschitz condition

$$|\sigma(x') - \sigma(x'')| \le L_2\|x' - x''\|$$

for all $x', x'' \in \mathbb{R}^n$, where $L_2 > 0$ is a constant.

(A4) The complete vectograms

$$E(t, x) = (f(t, x, P), g(t, x, P)) \subset \mathbb{R}^n \times \mathbb{R} \tag{3.4}$$

are convex sets for all $(t, x) \in \mathrm{cl}\,\Pi_T$.

It is known (see [207, 210, 233] that assumptions **(A1)**–**(A3)** imply the existence, uniqueness, and extendability of trajectories $x(\cdot; t_0, x_0, u(\cdot)) : [t_0, T] \to \mathbb{R}^n$ of system (3.1) starting at initial points $(t_0, x_0) \in \mathrm{cl}\,\Pi_T$ under open-loop measurable control functions (programs) $u(\cdot) : [t_0, T] \to P$.

For any initial point (t_0, x_0), condition **(A4)** guarantees the existence of optimal control functions (or, briefly, controls) $u^0(\cdot) : [t_0, T] \to P$, $u^0(\cdot) \in \mathbf{U}_{t_0}$, satisfying the relation

$$V(t_0, x_0) = \min_{u(\cdot) \in \mathbf{U}_{t_0}} I_{t_0, x_0}(x(\cdot; t_0, x_0, u(\cdot)), u(\cdot)) = I_{t_0, x_0}(x^0(\cdot; t_0, x_0, u^0(\cdot)), u^0(\cdot)) \tag{3.5}$$

(see, e.g., [3, 38, 87, 127, 163, 213, 300]).

3.3. Generalized controls. Further, in a number of sections of this chapter, assumption **(A4)** will be omitted. It is known that the optimal result $V(t_0, x_0)$ (3.3) to **OCP** can be unattainable on the set $\mathbf{U}_{t_0}$ of programs (open-loop controls). However, the value is attainable on an expansion of $\mathbf{U}_{t_0}$, namely, on the set $\mathbf{M}_{t_0}$ of all *generalized controls*, which are defined (see [300]) as measurable functions $\mu(\cdot|du) : [t_0, T] \to \mathrm{rpm}(P)$, where $\mathrm{rpm}(P)$ is the set of all regular probability Borel measures on P with a topology induced by the weak-$*$ topology on the space $C^*(P)$. The symbol $C^*(P)$ denotes the conjugate space to the space $C(P)$ of all continuous functions defined on the compact set P. Note that another well-known approach to the notion of generalized controls [87, 300] can be also applied to the condidered problems.

Thus, the trajectory $x(\cdot) = x(\cdot, t_0, x_0, \mu(\cdot|du)) : [t_0, T] \to \mathbb{R}^n$ of system (3.1) under a generalized control $\mu := \mu(\cdot|du)$ is understood as a (unique) solution of the equation

$$\dot{x}(t) = \int_P f(t, x(t), u)\mu(t|du), \quad x(t_0) = x_0. \tag{3.6}$$

The corresponding cost functional $I_{t_0,x_0}(x(\cdot),\mu(\cdot du))$ has the form

$$I_{t_0,x_0}(x(\cdot),\mu(\cdot|du)) = \sigma(x(T;t_0,x_0,\mu(\cdot|du))) + \int\limits_{t_0}^{T}\int\limits_{P} g(t,x(t),u)\mu(t|du)dt. \tag{3.7}$$

We denote the set of all trajectories $x(\cdot,t_0,x_0,\mu)$ of (3.6), $\mu \in \mathbf{M}_{t_0}$, by $\mathrm{Sol}(t_0,x_0)$.

It is known that for all $t_0 \in [0,T]$, the set

$$\mathbf{M}_{t_0} = \{\forall \mu(\cdot|du) : [t_0,T] \to \mathrm{rpm}(P) \text{ is measurable}\}$$

is a separable and compact metric set. There is a metric topology induced on $\mathbf{M}_{t_0}$ by the weak-$*$ topology of the space $B^* = L^1([t_0,T],C(P))^*$ (see, e.g., [300, p. 284]).

Moreover, the following inclusion always holds:

$$\mathrm{Sol}(t_0,x_0) \supseteq \{\forall x(\cdot;t_0,x_0,u(\cdot)) : u(\cdot) \in \mathbf{U}_{t_0}\}. \tag{3.8}$$

The main assumptions to the program statement of problem (3.1)–(3.3) imply that the following assertion holds.

Assertion II.1. *If conditions* **(A1)**–**(A4)** *for the problem* ***OCP*** *hold at any point* $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$, *then there is an optimal open-loop control* $u^0(\cdot)$ *in the class of programs* $\mathbf{U}_{t_0}$. *If conditions* **(A1)**–**(A3)** *hold for all* $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$, *then*

$$\begin{aligned} V(t_0,x_0) &= \inf_{u(\cdot)\in\mathbf{U}_{t_0}} I_{t_0,x_0}(x(\cdot;t_0,x_0,u(\cdot)),u(\cdot)) \\ &= \min_{\mu(\cdot|du)\in\mathbf{M}_{t_0}} I_{t_0,x_0}(x(\cdot;t_0,x_0,\mu(\cdot|du)),\mu(\cdot|du)), \end{aligned} \tag{3.9}$$

i.e., the problem ***OCP*** *always has a solution in the class of all generalized controls* $\mu(\cdot du) \in \mathbf{M}_{t_0}$.

4. Value Functions for Optimal Control Problems

4.1. Optimality principle.

The mapping

$$[0,T] \times \mathbb{R}^n \to \mathbb{R} : (t_0,x_0) \mapsto V(t_0,x_0)$$

is called the *value function* (the function of optimal cost or the *Bellman function*) for the problem **OCP**. It is known and it will be also shown below that the value function $V(t,x)$ plays a key role in the study of the problem **OCP** and the corresponding Cauchy problem for the Bellman equation. The Bellman equation is a first-order partial differential equation of the Hamilton–Jacobi type (see, e.g., [21, 29, 30, 55, 79, 80, 83, 127, 137, 150, 213, 301, 307]).

Fix an initial point $(t_0,x_0) \in \mathrm{cl}\,\Pi_T$ and a generalized trajectory $x(\cdot) = x(\cdot,t_0,x_0,\mu)$, $\mu \in M_{t_0}$, of system (3.6). Consider the variation of the value function $V(t,x)$ along this trajectory, i.e., $[t_0,T] \ni t \mapsto V[t] = V(t,x(t))$. Using the definition of $V(t,x)$ (see (3.3)) and Assertion II.1, one can obtain the following properties.

Lemma II.1. *For any trajectory* $x(\cdot) \in \mathrm{Sol}(t_0,x_0)$ (3.6) *and all* $t \in [t_0,T]$ *and* $\delta \in (0,T-t)$, *the following inequality holds*:

$$V(t+\delta,x(t+\delta)) + \int\limits_{t}^{t+\delta} g(\tau,x(\tau),u)\mu(\tau|du) \geq V(t,x(t)).$$

Lemma II.2. *A generalized control* $\mu^0 \in \mathbf{M}_{t_0}$ *and the corresponding trajectory* $x^0(\cdot) \in \mathrm{Sol}(t_0, x_0)$ *are optimal in the sense of* (3.9) *if and only if the following relation holds for any* $t \in [t_0, T]$ *and* $\delta \in (0, T-t)$:

$$V(t+\delta, x^0(t+\delta)) + \int_t^{t+\delta} g(\tau, x^0(\tau), u)\mu^0(\tau|du) = V(t, x^0(t)). \tag{4.1}$$

Condition (4.1) is called the *optimality principle* for the problem **OCP**.

4.2. Representative formula for the value function for an optimal control problem. We prove the following result describing the structure of the value function $V(t,x)$ for the problem **OCP** (3.1)–(3.3).

Theorem II.1. *The value function* $V(t,x)$ *for the problem* ***OCP*** *has the representation*

$$V(t,x) = \min_{\alpha \in \mathbf{A}} \omega(t,x,\alpha) \tag{4.2}$$

for any $(t,x) \in \mathrm{cl}\,\Pi_T$*, where the parameter* α *takes values in the metric compact set* $\mathbf{A}$*. If conditions* **(A1)**–**(A3)** *hold, then the function* $\omega(\cdot) : \mathrm{cl}\,\Pi_T \times A \to \mathbb{R}$ *is continuous. For any fixed* $\alpha \in \mathbf{A}$*, the functions*

$$\mathrm{cl}\,\Pi_T \to \mathbb{R} : (t,x) \mapsto \omega(t,x,\alpha)$$

are Lipschitz continuous on compact sets $G \subset \mathrm{cl}\,\Pi_T$ *with constants* $L = L(G) > 0$*, which are uniform with respect to* $\alpha \in \mathbf{A}$*.*

Proof. Construct the functions $(t,x) \mapsto \omega(t,x,\alpha)$ as follows (see also [245, 247, 255]). Let $\alpha : \tau \mapsto \alpha(\tau|du)$ be a measurable function defined on the standard interval $[0,1]$. Its values are regular probability Borel measures on the set $P \ni u$. The function $\alpha := \alpha(\cdot|du)$ is a standardized generalized control (see Sec. 3.3 above). Introduce the set

$$\mathbf{A} = \{\forall \alpha : \alpha(\cdot|du) : [0,1] \mapsto \mathrm{rpm}(P) \ \text{ is measurable}\}. \tag{4.3}$$

The set $\mathbf{A}$ of all generalized standardized controls α is a metric compact set, as well as the sets $\mathbf{M}_t$ defined above in Sec. 3.3 (see, e.g., [300]).

Define the function

$$\omega(t,x,\alpha) = \sigma(y(1;0,x,\alpha;t)) + z(1;0,0,\alpha;t,x) = \sigma(y(1)) + z(1), \tag{4.4}$$

where the absolutely continuous functions $y(\cdot) = y(\cdot;0,x,\alpha;t) : [0,1] \to \mathbb{R}^n$ and $z(\cdot) = z(\cdot;0,0,\alpha;t,x) : [0,1] \to \mathbb{R}$ are trajectories of the system

$$\dot{y} = (T-t)\int_P f(\xi(t,\tau), y(\tau), u)\alpha(\tau|du), \tag{4.5}$$

$$\dot{z} = (T-t)\int_P g(\xi(t,\tau), y(\tau), u)\alpha(\tau|du) \tag{4.6}$$

and the initial condition is $y(0) = x$, $z(0) = 0$. Here, time t plays the role of a parameter and $\xi(t,\tau)$ is a linear transformation $[0,1] \to [t,T]$ of the form

$$\xi(t,\tau) = t + (T-t)\tau. \tag{4.7}$$

Thus, the functions $(t,x) \mapsto \omega(t,x,\alpha)$ (4.4) are superpositions of the terminal function $\sigma(\cdot)$ of the problem **OCP** and the solutions $y(\cdot;0,x,\alpha;t)$ and $z(\cdot;0,0,\alpha;t,x)$ of the ordinary differential equations (4.5) and (4.6). The properties of functions $\omega(t,x,\alpha)$ declared in Theorem II.1 are consequences of assumptions **(A1)**–**(A3)** and the Lipschitz continuity of solutions of the ordinary differential equations (4.5) and (4.6) with respect to parameters and initial data (see, e.g., [300]).

Calculate the constant $L = L(G) > 0$ [247] by using the Gronwall lemma:

$$L = (1 + K_1 + L_1 \cdot T)\left(1 + L_2 + e^{L_1 T}\right)^2 \cdot [4 + C_1(G)], \tag{4.8}$$

where

$$C_1(G) = e^{K_1 T} \max_{(t_0, x_0) \in G} (1 + \|x_0\|). \tag{4.9}$$

One can see that the set $\mathbf{A}$ becomes the set $\mathbf{M}_t$ and problem (4.4)–(4.6) becomes problem (3.1)–(3.2) under the linear transformation $\xi(t, \cdot) : [0, 1] \to [t, T]$ (4.7).

Hence, Eq. (4.2) follows from the definition of the value function $V(t, x)$ (see (3.3)) and Assertion II.1.

□

4.3. Smoothness of the value function. One can easily obtain the following result using the structure (4.2) of the value function $V(t, x)$.

Theorem II.2. *Let conditions* **(A1)**–**(A3)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Then the value function* $V(t, x)$ *is locally Lipschitz continuous on the strip* $\operatorname{cl} \Pi_T$.

By the Rademacher theorem, a locally Lipschitz continuous function is differentiable almost everywhere (see, e.g., [300]). Hence, the value function satisfies the following relation almost everywhere:

$$\frac{d^{\pm} V(t, x)}{(1, f)} = \left\langle \frac{\partial V(t, x)}{\partial (t, x)}, (1, f) \right\rangle. \tag{4.10}$$

Remark II.1. The definitions of generalized differentials imply that the following relations hold for all $(t, x) \in (0, T) \times \mathbb{R}^n$:

$$\emptyset \neq \partial_C V(t, x) \supset \partial^+ V(t, x), \tag{4.11}$$

$$\emptyset \neq \partial_C V(t, x) \supset \partial^- V(t, x). \tag{4.12}$$

Consider realizations $V[t]$ of the value function $V(t, x)$ along trajectories $x(\cdot)$ of system (3.6). One can prove the following properties by using the absolute continuity of trajectories, Theorem II.2, and Lemmas II.1 and II.2.

Lemma II.3. *For any trajectory* $x(\cdot) = x(\cdot, t_0, x_0, \mu(\cdot)) \in \operatorname{Sol}(t_0, x_0)$ *and any* $t \in [t_0, T]$, *the following relations hold*:

$$\inf_{(\tilde f, \tilde g) \in \tilde E(t, x(t))} \frac{d^{\pm} V(t, x(t))}{(1, \tilde f)} + \tilde g \geq 0, \tag{4.13}$$

i.e., for any $t \in [t_0, T]$, *we have*

$$\frac{d^{\pm} V(t, x(t))}{\left(1, \int\limits_P f(t, x(t), u) \mu(t|du)\right)} \geq -\int\limits_P g(t, x(t), u) \mu(t|du). \tag{4.14}$$

Lemma II.4. *A generalized control* $\mu^0 \in \mathbf{M}_{t_0}$ *and the corresponding trajectory* $x^0(\cdot) \in \operatorname{Sol}(t_0, x_0)$ *are optimal in the sense of* (3.9) *if and only if the equalities*

$$\min_{(\tilde f, \tilde g) \in \tilde E(t, x^0(t))} \frac{d^{\pm} V(t, x^0(t))}{(1, \tilde f)} + \tilde g = \frac{d^{\pm} V(t, x^0(t))}{(1, \tilde f^0)} + \tilde g^0 = \frac{dV(t, x^0(t))}{(1, \tilde f^0)} + \tilde g^0 = 0 \tag{4.15}$$

hold for any $t \in [t_0, T]$, *where*

$$\tilde E(t, x) := \operatorname{co}\{(f(t, x, u), g(t, x, u)) : u \in P\}.$$

Consequently, the relation

$$\frac{dV(t,x^0(t))}{\left(1, \int\limits_P f(t,x^0(t),u)\mu^0(t|du)\right)} = -\int\limits_P g(t,x^0(t),u)\mu^0(t|du) \tag{4.16}$$

holds almost everywhere on $[t_0,T]$.

The following theorem is implied by Lemmas II.3 and II.4.

Theorem II.3. *Let conditions* **(A1)**–**(A3)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. For any point* $(t,x) \in \Pi_T$, *there exist directions* $(1,\tilde{f}^0) \in \mathbb{R}^{n+1}$, *where* $(\tilde{f}^0,\tilde{g}^0) \in \tilde{E}(t,x)$, *such that the locally Lipschitz continuous value function* $V(t,x)$ *is directionally differentiable in* $(1,\tilde{f}^0)$ *and*

$$\frac{dV(t,x)}{(1,\tilde{f}^0)} = -\tilde{g}^0, \quad (t,x) \in \Pi_T, \quad (\tilde{f}^0,\tilde{g}^0) \in \tilde{E}(t,x). \tag{4.17}$$

5. Value Functions and Minimax Solutions of Hamilton–Jacobi–Bellman Equations

5.1. Preliminaries. It is well known [29] that the value function $V(t,x)$ for the problem **OCP** satisfies the following Bellman equation (5.1) at all points of differentiability of the function $V(t,x)$:

$$\frac{\partial V(t,x)}{\partial t} + \min_{u\in P}[\langle D_x V(t,x), f(t,x,u)\rangle + g(t,x,u)] = 0, \tag{5.1}$$

where $D_x V(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1},\ldots,\frac{\partial V(t,x)}{\partial x_n}\right)$ and the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product. The value function also satisfies the following boundary condition in accordance with the definition of $V(t,x)$ (see (3.3)):

$$V(T,x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{5.2}$$

The problem **OCP** is considered under assumptions **(A1)**–**(A3)**. It was shown above that the value function is differentiable and satisfies the Bellman equation almost everywhere in the strip $\Pi_T = (0,T)\times\mathbb{R}^n$.

Obviously, the problem **OCP** can be also interpreted as an antagonistic differential game. One of the players $v \in \mathbb{R}^n$ is fictitious in the game. Its admissible control takes the unique value $\{0\} \in \mathbb{R}^n$. Therefore, in the game, the modified dynamics of system (3.1) has the form

$$\dot{x}(t) = f(t,x,u) + v, \quad u \in P, \quad v \in Q := \{0\} \in \mathbb{R}^n. \tag{5.3}$$

The cost functional is the same as (3.2). It is known [133] that there exists a value of the differential game (5.3) at any initial point $(t_0,x_0) \in \operatorname{cl}\Pi_T$. Comparing dynamics (3.1) and (5.3), one can obtain that the value coincides with the optimal cost $V(t_0,x_0)$ (3.3) of the control problem **OCP** in classes of programs (open-loop controls) and feedbacks (closed-loop controls).

The interpretation is useful when applying the following fact to the problem **OCP**. A result relative to the value function of a differential game was obtained in the theory of differential games [133, 135].

Assertion II.2. *A locally Lipschitz continuous function* $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ *coincides with the value function to the problem* ***OCP*** *if and only if the inequalities*

$$\min_{u\in P}\left[\frac{d^+V'(t,x)}{(1,f(t,x,u))} + g(t,x,u)\right] \geq 0 \geq \min_{(f,g)\in\tilde{E}(t,x)}\left[\frac{d^-V'(t,x)}{(1,f)} + g\right] \tag{5.4}$$

hold for any $(t,x) \in (0,T)\times\mathbb{R}^n$, *where*

$$\tilde{E}(t,x) := \operatorname{co}\{(f(t,x,u), g(t,x,u) : u \in P\} \tag{5.5}$$

and the boundary condition

$$V'(T,x) = \sigma(x), \quad x \in \mathbb{R}^n, \tag{5.6}$$

holds.

5.2. Generalized Bellman equation and minimax solutions. As is known in the theory of generalized solutions of Hamilton–Jacobi equations [59, 80, 238], there are connections between the value function $V(t,x)$ (see (3.3)) of the optimal control problem **OCP** and the generalized (minimax and/or viscosity) solution $V'(t,x)$ of the Cauchy problem (5.2) for the Hamilton–Jacobi equation (5.1).

Assertion II.3. *Let conditions* **(A1)**–**(A3)** *for the problem* ***OCP*** *hold. There exists a minimax* (*and/or viscosity*) *solution* $\operatorname{cl}\Pi_T = [0,T] \times \mathbb{R}^n \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ *of the corresponding Cauchy problem* (5.2) *for the Hamilton–Jacobi equation* (5.1)*. It is unique and coincides with the value function* $V(t,x)$ *of the considered control problem* ***OCP****.*

We also prove the following theorem (see, e.g., [247, 273]).

Theorem II.4. *A locally Lipschitz continuous function* $\operatorname{cl}\Pi_T \ni (t,x) \mapsto V'(t,x) \in \mathbb{R}$ *coincides with the value function* $V(t,x)$ *of the problem* ***OCP*** *if and only if the following conditions hold*:

- *the pair of equalities*

$$\min_{(\tilde f,\tilde g)\in \tilde E(t,x)} \frac{d^{\pm}V'(t,x)}{(1,\tilde f)} + \tilde g = 0 \tag{5.7}$$

  holds for any point $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$*, where*

$$\tilde E(t,x) := \operatorname{co}\{(f(t,x,u), g(t,x,u)) : u \in P\},$$

- *and the boundary condition*

$$V'(T,x) = \sigma(x), \quad x \in \mathbb{R}^n, \tag{5.8}$$

  holds.

Proof. Necessity of conditions (5.7)–(5.8). Fix a point $(t_0,x_0) \in \Pi_T$. According to the statement of the problem **OCP**, the admissible generalized controls $\mu(\cdot|du) \in \mathbf{M}_{t_0}$ and the corresponding trajectories $x(\cdot) = x(\cdot;t_0,x_0,\mu(\cdot|du))$ satisfy the relations

$$V(t_0,x_0) \le V(t_0+\delta, x(t_0+\delta)) + \int\limits_{t_0}^{t_0+\delta} \int\limits_{P} g(\tau, x(\tau), u)\mu(\tau|du)d\tau, \quad \delta > 0, \tag{5.9}$$

where

$$x(t_0+\delta) = x_0 + \int\limits_{t_0}^{t_0+\delta} \int\limits_{P} f(\tau, x(\tau), u)\mu(\tau|du)d\tau. \tag{5.10}$$

Using (5.9) and the definitions of $\dfrac{d^{\pm}V(t_0,x_0)}{(1,f)}$, we obtain that inequalities

$$\frac{d^{\pm}V(t_0,x_0)}{(1,\tilde f)} + \tilde g \ge 0 \tag{5.11}$$

are valid at any point (t_0,x_0) and for all vectors

$$(\tilde f, \tilde g) = \left(\int\limits_P f(t_0,x_0,u)\mu(t|du), \int\limits_P g(t_0,x_0,u)\mu(t|du) \right) \in E(t_0,x_0).$$

One can see that the equalities

$$V(t, x^0(t)) + \int_{t_0}^{t} \int_P g(\tau, x^0(\tau), u)\mu^0(\tau|du)d\tau = V(t_0, x_0), \quad t \in [t_0, T], \tag{5.12}$$

hold along any optimal trajectory $x^0(\cdot) = x(\cdot; t_0, x_0, \mu^0) \in \mathrm{Sol}(t_0, x_0)$.

Conditions (5.11) and (5.12) imply that the equality

$$\min_{(\tilde{f},\tilde{g})\in E(t_0,x_0)} \frac{d^{\pm}V(t_0, x_0)}{(1, \tilde{f})} + \tilde{g} = 0 \tag{5.13}$$

holds for any point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Thus, the value function satisfies the necessary condition (5.7). The validity of the boundary condition (5.8) follows from the definition of the value function.

Sufficiency of conditions (5.7)–(5.8). If a locally Lipschitz continuous function $V'(t, x)$ satisfies conditions (5.7)–(5.8), then the conditions

$$\min_{u\in P} \frac{d^{+}V'(t, x)}{(1, f(t, x, u))} + g(t, x, u) \geq 0 = \min_{(\tilde{f},\tilde{g})\in \tilde{E}(t,x)} \frac{d^{\pm}V'(t, x)}{(1, \tilde{f})} + \tilde{g} \tag{5.14}$$

hold for any $(t, x) \in \Pi_T$, in accordance with the definitions of $\tilde{E}(t, x)$ and $\dfrac{d^{\pm}V'(t, x)}{(1, f)}$ (see (5.4)). As follows from Assertion II.2, these conditions are sufficient for the equivalence of the function $V'(t, x)$ and the value function $V(t, x)$ of the problem **OCP**. Theorem II.4 is proved. □

Remark II.2. Taking (4.10) into account, one can obtain that Eqs. (5.7) turn out to be the Bellman equation (5.1) at all points (t, x), where the value function is differentiable. Therefore, these equalities can be considered as a generalization of the Bellman equation. According to Theorem II.4 and Assertion II.3, the minimax (and/or viscosity) solution $\mathrm{cl}\,\Pi_T \ni (t, x) \mapsto V'(t, x) \in \mathbb{R}$ of the Cauchy problem (5.2) for the Bellman equation (5.1) satisfies the generalized Bellman equation (5.7) for all points $(t, x) \in \Pi_T = (0, T) \times \mathbb{R}^n$.

Thus, in the present section, the justification of the dynamical programming method for the problem **OCP** is exposed under conditions of local Lipschitz continuity for initial data (3.1), (3.2) and the value function $V(t, x)$ (see (3.3)). A generalization of the Bellman equation is obtained in terms of the *directional Dini semiderivatives* $\dfrac{d^{\pm}V'(t, x)}{(1, f)}$ in directions $(1, f) \in \mathbb{R}^{n+1}$ and relations (5.7) and (5.8) are valid for *all* points $(t, x) \in \Gamma_T$. Using the validity of relations (5.7) and (5.8) for all points, one can construct optimal syntheses (optimal feedbacks) for the problem **OCP** (see Sec. 8 below and [273]).

Remark II.3. Note that Theorem II.4 is proved without assumptions on the functions $f(t, x, u)$ and $g(t, x, u)$ to have neither continuous partial derivatives in (t, x) of first order (which was assumed in [247]) nor continuous partial derivatives in (t, x) of second order (which was assumed in [116]). The proof also does not use an additional requirement on semiconcavity of the functions $f(t, x, u)$ and $g(t, x, u)$ (which was assumed in [42]). Note that stronger assumptions in Secs. 6 and 7 below will provide the directional differentiability for the value function $V(t, x)$ in *all* directions $(1, f) \in \mathbb{R} \times \mathbb{R}^n$. The results presented in this section are obtained for a more general case of locally Lipschitz continuous initial data for the problem **OCP** and, as a consequence, for a more "nonsmooth" locally Lipschitz continuous value function. In this case, the value function *is not to be* differentiable *in all* directions $(1, f) \in \mathbb{R}^{n+1}$.

6. Pontryagin Maximum Principle and Classical Characteristics of the Bellman Equation

6.1. Case of differentiable input data. In Secs. 6 and 7, the problem **OCP** (3.1)–(3.3) is considered under modified assumptions. Namely, conditions **(A1)** and **(A3)** are replaced by the following, stronger requirements on input data of the problem.

(A1′) Functions $f(t,x,u)$ and $g(t,x,u)$ in (3.1) and (3.2) are continuous in $\mathrm{cl}\,\Pi_T \times P$. The partial derivatives $\dfrac{\partial f}{\partial t}$, $\dfrac{\partial f}{\partial x_i}$, $\dfrac{\partial g}{\partial t}$, $\dfrac{\partial g}{\partial x_i}$, $i \in \overline{1,n}$, are defined and continuous in $\Pi_T \times P$.

(A3′) The terminal function $\sigma(x)$ in functional (3.2) and its partial derivatives $\dfrac{\partial \sigma}{\partial x_i}$, $i \in \overline{1,n}$, are continuous in $\mathbb{R}^n$.

The modifications of requirements for input data of the problem **OCP** imply some changes of smoothness of the value function $V(t,x)$ (3.3). The following statements hold.

Theorem II.5. *Let conditions* **(A1′)**, **(A2)**, *and* **(A3′)** *for the problem* ***OCP*** *hold. Then the value function* $V(t,x)$ (3.3) *has the representation*

$$V(t,x) = \min_{\alpha \in \mathbf{A}} \omega(t,x,\alpha) \tag{6.1}$$

at any point $(t,x) \in \mathrm{cl}\,\Pi_T$, *where the parameter* α *accepts values in a metric compact set* $\mathbf{A}$.

The function $\omega(\cdot) : \mathrm{cl}\,\Pi_T \times A \to \mathbb{R}$ *is continuous and the function*

$$\mathrm{cl}\,\Pi_T \to \mathbb{R} : (t,x) \mapsto \omega(t,x,\alpha), \quad \alpha \in \mathbf{A},$$

has partial derivatives $(t,x,\alpha) \mapsto \dfrac{\partial \omega(t,x,\alpha)}{\partial t}, \dfrac{\partial \omega(t,x,\alpha)}{\partial x_i}$, $i \in \overline{1,n}$, *which are uniformly* (*in* α) *continuous.*

The proof of formula (6.1) in Theorem II.5 coincides with the corresponding proof of formulas (4.2) in Theorem II.1. The theorems differ in conclusions about the properties of function $\omega(t,x;\alpha)$.

It follows from the construction that the functions $(t,x) \mapsto \omega(t,x,\alpha)$ (4.4) are superpositions of the terminal function $\sigma(\cdot)$ of **OCP** and the solutions $y(\cdot;0,x,\alpha;t)$ and $z(\cdot;0,0,\alpha;t,x)$ of the ordinary differential equations (4.5)–(4.6). The properties of the function $\omega(t,x,\alpha)$ declared in Theorem II.5 are consequences of assumptions **(A1′)**, **(A2)**, and **(A3′)** and the existence of continuous derivatives of solutions of Eqs. (4.5), (4.6) in parameters and initial states [210, 300].

According to Theorem II.5, the value function is the lower envelope (6.1) of the family of smooth functions $\omega(\cdot,\alpha)$ over a compact set of parameters α. It is well known that the envelope is directionally differentiable. The formulas of the derivatives (see [51, 64]) and Theorem II.4 imply the validity of the following statements.

Theorem II.6. *Let conditions* **(A1′)**, **(A2)**, *and* **(A3′)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Then the value function* $V(t,x)$ *is locally Lipschitz continuous in the strip* $\mathrm{cl}\,\Pi_T$ *and there exists the directional derivative* $\dfrac{dV(t,x)}{(1,f)} = \dfrac{d^{\pm}V(t,x)}{(1,f)}$ *at any point* $(t,x) \in \Pi_T$ *for any vector* $f \in \mathbb{R}^n$. *The formula*

$$\frac{dV(t,x)}{(1,f)} = \min_{\alpha^0 \in \mathbf{A}^0(t,x)} \left[\frac{\partial \omega(t,x,\alpha^0)}{\partial t} + \langle D_x \omega(t,x,\alpha^0), f \rangle \right] \tag{6.2}$$

holds, where

$$\mathbf{A}^0(t,x) = \{\alpha^0 \in \mathbf{A} : \omega(t,x,\alpha^0) = V(t,x)\}. \tag{6.3}$$

The value function satisfies the following generalized Bellman equation everywhere in Π_T:

$$\min_{(f,g) \in \tilde{E}(t,x)} \left[\frac{dV(t,x)}{(1,f)} + g \right] = 0, \tag{6.4}$$

where

$$\tilde{E}(t,x) = \mathrm{co}\{(f(t,x,u), g(t,x,u)) : u \in P\}.$$

Remark II.4. According to the definitions of generalized differentials, the following relations hold for the problem **OCP** for all $(t,x) \in \Pi_T$:

$$\emptyset \neq \partial_C V(t,x) = \partial^+ V(t,x) = \mathrm{co}\left\{ \left(\frac{\partial \omega(t,x,\alpha^0)}{\partial t}, D_x \omega(t,x,\alpha^0) \right) : \alpha^0 \in \mathbf{A}^0(t,x) \right\}. \tag{6.5}$$

6.2. Preliminaries. This section contains preliminary technical material needed for obtaining necessary optimality conditions in Sec. 6.3.

Consider an element $\alpha \in \mathbf{A}$ (4.3) and moments of time t and t_* satisfying the inequalities $0 \leq t_* \leq t \leq T$. Transform it to the element $\alpha[t;t_*] \in \mathbf{A}$ as follows.

Consider the bijective transformation $[t_*, T] \ni \xi \mapsto \tau \in [0,1]$ defined by the formula

$$\xi = t_* + (T - t_*)\tau. \tag{6.6}$$

Choose an element $\mu \in \mathbf{M}_{t_*}$ (see Sec. 3.3) such that

$$\mu(\xi|du) = \alpha(\tau|du). \tag{6.7}$$

Let $\mu[t/t_*](\cdot|du) : [t,T] \to \mathrm{rpm}(P)$ be the restriction of the function $\mu(\cdot|du) : [t_*,T] \to \mathrm{rpm}(P)$. We set

$$\alpha[t;t_*](\tau|du) = \mu[t/t_*](\xi|du) \quad \text{for} \quad \tau \in [0,1], \xi = t + (T-t)\tau. \tag{6.8}$$

It follows from constructions (6.6)–(6.8) that

$$\alpha[t;t_*](\tau|du) = \alpha(\tau_*|du), \tag{6.9}$$

where

$$\tau_* = [(t - t_*) + (T - t)\tau] \cdot (T - t_*)^{-1}, \quad \tau \in [0,1]. \tag{6.10}$$

The function $\tau \mapsto \alpha[t;t_*](\tau|du)$ is measurable and $\alpha[t;t_*] \in \mathbf{A}$. The definition of the convergence on $\mathbf{A}$ (see [300]) and the constructions $[t_*,T] \ni t \mapsto \alpha[t;t_*] \in \mathbf{A}$ (6.6)–(6.8) imply the following assertion.

Lemma II.5. *The transformation $[t_*,T] \ni t \mapsto \alpha[t;t_*] \in \mathbf{A}$ is continuous for all $\alpha \in \mathbf{A}$ and $t_* \in [0,T)$.*

The set

$$\mathbf{A}^0(t,x) = \{\alpha^0 \in \mathbf{A} : \omega(t,x,\alpha^0) = V(t,x)\}$$

is defined for all points $(t,x) \in \Pi_T$ in Theorem II.6. The definition of the set $\mathbf{A}^0(t,x)$, conditions (6.6) and (6.7), and Lemma II.4 imply the following assertion.

Lemma II.6. *Let $\alpha^0 \in \mathbf{A}^0(t_*,x_*)$ and $t_* \in [0,T)$. Then the inclusion*

$$\alpha^0[t;t_*] \in \mathbf{A}^0(t, x^0(t)) \tag{6.11}$$

holds for all $t \in [t_*,T]$, where

$$x^0(t) = x_* + \int_{t_*}^{t} \int_P f(\xi, x^0(\xi), u)\mu^0(\xi|du)d\xi, \tag{6.12}$$

$$\mu^0(\xi|du) = \alpha^0(\tau|du) \quad \mathit{for} \quad \xi \in [t_*,T], \quad \tau = frac\xi - t_*T - t_*. \tag{6.13}$$

We prove the following result by using Lemmas II.5 and II.6.

Theorem II.7. *Let conditions* **(A1′)**, **(A2)**, *and* **(A3′)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Then the equality*

$$\min_{u \in P} \left[\frac{\partial \omega(t,x,\alpha^0)}{\partial t} + \langle D_x \omega(t,x,\alpha^0), f(t,x,u) \rangle + g(t,x,u) \right] = 0 \tag{6.14}$$

holds for all $(t,x) \in \Pi_T$ and $\alpha^0 \in \mathbf{A}^0(t,x)$.

Proof. Assume the contrary, i.e., that there exist $(t_*, x_*) \in \Pi_T$, $\alpha_*^0 \in \mathbf{A}^0(t_*, x_*)$, and a number $d > 0$ such that the following relations hold:

$$\min_{u\in P} \left[\frac{\partial \omega(t_*, x_*, \alpha_*^0)}{\partial t} + \langle D_x \omega(t_*, x_*, \alpha_*^0), f(t_*, x_*, u)\rangle + g(t_*, x_*, u) \right] = d > 0. \tag{6.15}$$

According to the continuity of the functions $\dfrac{\partial \omega(\cdot)}{\partial t}$, $\dfrac{\partial \omega(\cdot)}{\partial x_i}$, $i \in \overline{1,n}$, $f(\cdot)$, and $g(\cdot)$, one can choose a number $\delta > 0$ and a closed δ-neighborhood $B^\delta(\alpha_*^0)$ of $\alpha_*^0 \in \mathbf{A} \subset L^1([0,1], C(P))^*$ such that the inequality

$$\min_{u\in P} \left[\frac{\partial \omega(t, x, \alpha)}{\partial t} + \langle D_x \omega(t, x, \alpha), f(t, x, u)\rangle + g(t, x, u) \right] \geq \frac{d}{2} > 0 \tag{6.16}$$

holds for any (t, x, α) such that $0 \leq t - t_* \leq \delta$, $\|x - x_*\| \leq \delta$ and $\alpha \in B^\delta(\alpha_*^0) \cap \mathbf{A}$.

Consider the optimal trajectory $x^0(\cdot) = x^0(\cdot; t_*, x_*, \mu_*^0)$ of system (3.6) starting at the initial state $x^0(t_*) = x_*$ and generated by the optimal generalized control $\mu_*^0 \in \mathbf{M}_{t_*}$ corresponding to $\alpha_*^0 \in \mathbf{A}^0(t_*, x_*)$ by the rule

$$\mu_*^0(\xi|du) = \alpha_*^0(\tau|du), \quad \tau \in [0,1], \quad \xi = t_* + (T - t_*)\tau.$$

Let us estimate the variation of the value function $V(t, x)$ along the trajectory $x^0(\cdot)$. By Lemmas II.4 and II.6 and Theorems II.5 and II.6, we have

$$\begin{aligned} -\int_{t'}^{t} \int_P g(\xi, x^0(\xi), u)\mu_*^0(\xi|du)d\xi &= V(t, x^0(t)) - V(t', x^0(t')) \\ &= \omega(t, x^0(t), \alpha_*^0[t; t_*]) - \omega(t', x^0(t'), \alpha_*^0[t'; t_*]) \\ &\geq \omega(t, x^0(t), \alpha_*^0[t; t_*]) - \omega(t', x^0(t'), \alpha_*^0[t; t_*]) \end{aligned} \tag{6.17}$$

for all t and t' such that $t_* \leq t' \leq t \leq T$.

Choose moments t' and t'' such that $t_* \leq t' < t'' \leq \min\{T, t_* + \delta\}$ as follows.

According to Lemma II.6 and the continuity of the mapping $t \to \alpha_*^0[t; t_*]$, we have

$$\alpha_*^0[t; t_*] \in B^\delta(\alpha_*^0) \cap \mathbf{A}, \quad t \in [t_*, t'']. \tag{6.18}$$

Consider a Lebesgue point t' of the absolutely continuous function $x^0(\cdot)$. Therefore, the function $x^0(\cdot)$ has the derivative at the point t', namely,

$$\frac{dx^0(t')}{dt} = \int_P f(t', x^0(t'), u)\mu_*^0(t'|du) = f(t', x^0(t'), u'). \tag{6.19}$$

Assume also that

$$\frac{d}{dt} \int_{t'}^{t} \int_P g(\xi, x^0(\xi), u)\mu_*^0(\xi|du) = g(t', x^0(t'), u'). \tag{6.20}$$

The existence of $u' \in P$ in equalities (6.19) and (6.20) follows from the convexity of the vectograms $E(t', x^0(t'))$ by condition **(A4)**. Using (6.19), for $t \in [t', t'']$, one obtain

$$x^0(t) = x^0(t') + f(t', x^0(t'), u') \cdot (t - t') + o_{t'}(t - t'), \tag{6.21}$$

$$\|o_{t'}(t - t')\|/(t - t') \to 0 \quad \text{as} \quad t \downarrow t'. \tag{6.22}$$

Let

$$x'(t) = x^0(t') + f(t', x^0(t'), u') \cdot (t - t'), \quad t \in [t', t'']. \tag{6.23}$$

Choose t' and t'' such that the relations (6.21)–(6.23) imply

$$\|x'(t) - x_*\| \leq \delta \quad \text{for} \quad t \in [t', t'']. \tag{6.24}$$

Consider the function $\omega(\cdot)$ in a closed δ-neighborhood of the graph of the trajectory $x^0(t) : t_* \le t \le T$. Using the smoothness of the function $\omega(\cdot)$, one can easily obtain that this function satisfies the Lipschtz condition in (t, x) with constant $\kappa > 0$, uniform with respect to $\alpha \in \mathbf{A}$. Taking (6.21)–(6.23) into account, we obtain the estimate

$$|\omega(t, x^0(t), \alpha_*^0[t; t_*]) - \omega(t', x^0(t'), \alpha_*^0[t; t_*])| \le \kappa \|x^0(t) - x'(t)\| = \kappa \|o_{t'}(t - t')\| \tag{6.25}$$

for $t \in [t', t'']$. Let

$$\beta_t(\Delta) = \max_{0 \le i \le n} \Omega_i(\Delta), \quad \Delta > 0,$$

where $\Delta \mapsto \Omega_0(\Delta)$ is the module of continuity of the function

$$\Phi_0(t, \cdot) = \frac{\partial \omega(\cdot, x^0(\cdot), \alpha_*^0[t; t_*](\cdot))}{\partial t} : [t', t''] \to \mathbb{R},$$

and $\Delta \mapsto \Omega_i(\Delta)$, $i \in \overline{1, n}$, are the modules of continuity of the functions

$$\Phi_i(t, \cdot) = \frac{\partial \omega(\cdot, x^0(\cdot), \alpha_*^0[t; t_*](\cdot))}{\partial x_i} : [t', t''] \to \mathbb{R}.$$

It follows from the smoothness of the function $\omega(\cdot)$ (6.1) that the following uniform convergence takes place on $t \in [t', t'']$:

$$\beta_t(\Delta) \to 0 \quad \text{as} \quad \Delta \to 0. \tag{6.26}$$

Now choose a moment $t \in (t', t'']$ in accordance with (6.22), (6.26), and (6.20) to satisfy the relations

$$-\kappa \frac{\|o_{t'}(t - t')\|}{t - t'} - \beta_t(t - t') \ge -\frac{d}{8}, \tag{6.27}$$

$$\int_{t'}^{t} \int_{P} g(\xi, x^0(\xi), u) \mu_*^0(\xi | du) d\xi - g(t', x^0(t'), u') \cdot (t - t') = \hat{o}_{t'}(t - t'),$$

$$\frac{\|\hat{o}_{t'}(t - t')\|}{t - t'} \le \frac{d}{8}. \tag{6.28}$$

Use relations (6.25), (6.27), (6.16), (6.18), and (6.24) to continue the estimate (6.17):

$$\begin{aligned}
-\int_{t'}^{t} \int_{P} g(\xi, x^0(\xi), u) \mu_*^0(\xi | du) d\xi &= -g(t', x^0(t'), u')(t - t') + \hat{o}_{t'}(t - t') \\
&= \omega(t, x^0(t), \alpha_*^0[t; t_*]) - \omega(t, x'(t), \alpha_*^0[t; t_*]) + \omega(t, x'(t), \alpha_*^0[t; t_*]) - \omega(t', x^0(t'), \alpha_*^0[t'; t_*]) \\
&\ge \omega(t, x^0(t), \alpha_*^0[t; t_*]) - \omega(t, x'(t), \alpha_*^0[t; t_*]) + \omega(t, x'(t), \alpha_*^0[t; t_*]) - \omega(t', x'(t'), \alpha_*^0[t; t_*]) \\
&\ge \Big\{ -\kappa \frac{\|o_{t'}(t - t')\|}{t - t'} - \beta_t(t - t') + \frac{\partial \omega(t', x^0(t'), \alpha_*^0[t; t_*])}{\partial t} \\
&\qquad + \langle D_x \omega(t', x^0(t'), \alpha_*^0[t; t_*]), f(t', x^0(t'), u') \rangle \Big\} (t - t').
\end{aligned} \tag{6.29}$$

Finally, from (6.28) and (6.29), we obtain the following inequalities for $t > t'$:

$$\begin{aligned}
0 &\ge \Big\{ \frac{\partial \omega(t', x^0(t'), \alpha_*^0[t; t_*])}{\partial t} + \\
&\quad + \min_{u \in P} \Big[\langle D_x \omega(t', x^0(t'), \alpha_*^0[t; t_*]), f(t', x^0(t'), u) \rangle + g(t', x^0(t'), u) \Big] \Big\} (t - t') - 2\frac{d}{8}(t - t') \\
&\ge \frac{d}{2}(t - t') - \frac{d}{4}(t - t') = \frac{d}{4}(t - t') > 0.
\end{aligned} \tag{6.30}$$

The contradiction in (6.30) proves Theorem II.7. □

6.3. Necessary optimality conditions. This section contains a proof of necessary optimality conditions, the Pontryagin maximum principle (see [213]) obtained on the basis of Theorem II.7.

Recall that the function $\omega(\cdot)$ defined by the formula (6.1) for $V(t,x)$ (see Theorems II.5 and II.1) has the representation

$$\omega(t,x,\alpha)=\sigma(y(1;0,x,\alpha;t))+z(1;0,0,\alpha;t,x)=\sigma(y(1))+z(1), \tag{6.31}$$

where the absolutely continuous functions $y(\cdot)=y(\cdot;0,x,\alpha;t):[0,1]\to\mathbb{R}^n$ and $z(\cdot)=z(\cdot;0,0,\alpha;t,x):[0,1]\to\mathbb{R}$ are solutions of the system

$$\frac{d\xi}{d\tau}=(T-t), \tag{6.32}$$

$$\frac{dy}{d\tau}=(T-t)\int\limits_P f(\xi(t,\tau),y(\tau),u)\alpha(\tau|du), \tag{6.33}$$

$$\frac{dz}{d\tau}=(T-t)\int\limits_P g(\xi(t,\tau),y(\tau),u)\alpha(\tau|du) \tag{6.34}$$

with the initial condition $\xi(0)=t$, $y(0)=x$, $z(0)=0$. Here t plays the role of a parameter and $\xi(t,\tau)$ has the form

$$\xi(t,\tau)=t+(T-t)\tau,\quad \tau\in[0,1]. \tag{6.35}$$

Note that assumptions **(A1′)** and **(A3′)** imply (see [210, 300]) the existence and continuity of partial derivatives of solutions of the ordinary differential equations (6.33) and (6.34) in parameters and initial state. Formula (6.31) implies that there exist the continuous partial derivatives $\dfrac{\partial\omega(t,x,\alpha)}{\partial x_i}$, $i\in\overline{1,n}$, and $\dfrac{\partial\omega(t,x,\alpha)}{\partial t}$.

Lemma II.7. *Let conditions* **(A1′)**, **(A2)**, **(A3′)**, *and* **(A4)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Let a control $\mu^0(\cdot)\in\mathbf{M}_{t_0}$ and the corresponding trajectory $x^0(\cdot)=x^0(\cdot;t_0,x_0,\mu^0(\cdot))$ of system* (3.1) *be optimal* (3.3) *at the point $(t_0,x_0)\in\Pi_T$. Let $\alpha^0\in\mathbf{A}^0(t_0,x_0)$ be the standardized control of the form*

$$\alpha^0=\alpha^0(\tau|du)=\mu^0(\xi|du),\quad \xi=\xi(t_0,\tau)=t_0+(T-t_0)\cdot\tau,\quad \tau\in[0,1], \tag{6.36}$$

and the standardized generalized controls $\alpha^0\in\mathbf{A}^0(t_0,x_0)$ constructed by $\alpha^0\in\mathbf{A}^0(t,x^0(t))$ by the rule (6.6)–(6.8) *for all $t\in[t_0,T]$. Consider the mappings*

$$\begin{aligned}&[t_0,T)\to\mathbb{R}^n:\quad t\mapsto p^0(t)=D_x\omega(t,x^0(t),\alpha^0[t;t_0]),\\ &[t_0,T)\to\mathbb{R}:\quad t\mapsto\lambda^0(t)=\frac{\partial\omega(t,x^0(t),\alpha^0[t;t_0])}{\partial t}:\end{aligned} \tag{6.37}$$

The following relations hold for any $t\in[t_0,T]$:

$$p^0(t)=D_x\sigma(x^0(T))+\int\limits_t^T\left(\int\limits_P\frac{\partial f(\xi,x^0(\xi),u)}{\partial x}\mu^0(\xi|du)\right)^{\top}p^0(\xi)d\xi+\int\limits_t^T\int\limits_P D_xg(\xi,x^0(\xi),u)\mu^0(\xi|du)d\xi,$$

$$\begin{aligned}\lambda^0(t)=&-\min_{u\in P}[\langle p^0(T),f(T,x^0(T),u)\rangle+g(T,x^0(T),u)]\\ &+\int\limits_t^T\left(\int\limits_P\frac{\partial f(\xi,x^0(\xi),u)}{\partial\xi}\mu^0(\xi|du)\right)^{\top}p^0(\xi)d\xi+\int\limits_t^T\int\limits_P\frac{\partial g(\xi,x^0(\xi),u)}{\partial\xi}\mu^0(\xi|du)d\xi.\end{aligned}$$

Proof. First, note that the symbols

$$x^0(\cdot) = (x_1^0(\cdot), \dots, x_n^0(\cdot)), \quad D_x\omega(t,x,\alpha) = \left(\frac{\partial\omega(t,x,\alpha)}{\partial x_1}, \dots, \frac{\partial\omega(t,x,\alpha)}{\partial x_n}\right),$$

$$D_x\sigma(x) = \left(\frac{\partial\sigma(x)}{\partial x_1}, \dots, \frac{\partial\sigma(x)}{\partial x_n}\right), \quad \frac{\partial f(\xi,x,u)}{\partial\xi} = \left(\frac{\partial f_1(\xi,\cdot)}{\partial\xi}, \dots, \frac{\partial f_n(\xi,\cdot)}{\partial\xi}\right)$$

denote finite-dimensional column-vectors. The symbol $\frac{\partial f}{\partial x}$ means the matrix of partial derivatives of a vector $f(\xi,x,u)$ in x_i, $i \in \overline{1,n}$, i.e.,

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

The symbol $^\top$ means the transposition.

Passing in formulas (6.31)–(6.36) from the variable $\tau \in [0,1]$ to the variable $\xi \in [t,T]$ by using the linear transformation $\xi = \xi(t,\tau)$ (6.35), we obtain

$$\alpha \in \mathbf{A}, \alpha(\tau|du) = \mu(\xi|du) \quad \Rightarrow \quad \mu = \mu(\cdot|du) \in \mathbf{M}_t,$$
$$\xi(\tau,t) = \tilde{\xi}(\xi;t,t) = \xi,$$
$$y(\tau;0,x,\alpha;t) = \tilde{x}(\tilde{\xi}(\xi;t,t);t,x,\mu),$$
$$z(\tau;0,0,\alpha;t,x) = \tilde{z}(\tilde{\xi}(\xi;t,t);t,0,\mu;t,x),$$

where $\tilde{\xi}(\cdot) = \tilde{\xi}(\cdot;t,t)$, $\tilde{x}(\cdot) = \tilde{x}(\cdot;t,x,\mu)$, $\tilde{z}(\cdot) = \tilde{z}(\cdot;t,0,\mu;t,x)$ is a solution of the system

$$\begin{aligned} &\frac{\tilde{\xi}(\xi)}{d\xi} = 1, && \xi \in (t,T), \quad \tilde{\xi}(t) = t, \\ &\frac{d\tilde{x}(\xi)}{d\xi} = \int_P f(\xi,\tilde{x}(\xi),u)\mu(\xi|du), && \xi \in (t,T), \quad \tilde{x}(t) = x, \\ &\frac{d\tilde{z}(\xi)}{d\xi} = \int_P g(\xi,\tilde{x}(\xi),u)\mu(\xi|du), && \xi \in (t,T), \quad \tilde{z}(t) = 0. \end{aligned} \tag{6.38}$$

Rewrite formula (6.31) for $\omega(t,x,\alpha)$ in this notation:

$$\omega(t,x,\alpha) = \sigma(\tilde{x}(T)) + \tilde{z}(T) = \sigma(\tilde{x}(T)) + \int_t^T \int_P g(\xi,\tilde{x}(\xi),u)\mu(\xi|du)d\xi. \tag{6.39}$$

It was mentioned above that condition **(A1′)** for the considered problem **OCP** implies that solutions $\tilde{x}(\cdot;t,x,\mu)$ of system (6.38) have continuous partial derivatives $\frac{\partial\tilde{x}(\cdot;t,x,\mu)}{\partial x}$ and $\frac{\partial\tilde{x}(\cdot;t,x,\mu)}{\partial t}$ with respect to the initial state (t,x). These derivatives satisfy the *system in variations*

$$\frac{d}{d\xi}\frac{\partial\tilde{x}(\xi;t,x,\mu)}{\partial x} = \left(\int_P \frac{\partial f(\xi,\tilde{x}(\xi),u)}{\partial x}\mu(\xi|du)\right)\frac{\partial\tilde{x}(\xi;t,x,\mu)}{\partial x}, \tag{6.40}$$

$$\frac{d}{d\xi}\frac{\partial\tilde{x}(\xi;t,x,\mu)}{\partial t} = \left(\int_P \frac{\partial f(\xi,\tilde{x}(\xi),u)}{\partial x}\mu(\xi|du)\right)\frac{\partial\tilde{x}(\xi;t,x,\mu)}{\partial t} + \frac{\partial f(\xi,\tilde{x}(\xi),u)}{\partial\xi} \tag{6.41}$$

on the interval (t, T) (see, e.g., [210, 233]) and the boundary conditions

$$\frac{\partial \tilde{x}(t; t, x, \mu)}{\partial x} = \mathbf{E}_n, \quad \frac{\partial \tilde{x}(t; t, x, \mu)}{\partial t} = \theta_n. \tag{6.42}$$

The symbol $\dfrac{\partial \tilde{x}(t; t, x, \mu)}{\partial x}$ denotes the matrix

$$\frac{\partial \tilde{x}}{\partial x} = \begin{pmatrix} \dfrac{\partial \tilde{x}_1}{\partial x_1} & \dots & \dfrac{\partial \tilde{x}_1}{\partial x_n} \\ \dots & \dots & \dots \\ \dfrac{\partial \tilde{x}_n}{\partial x_1} & \dots & \dfrac{\partial \tilde{x}_n}{\partial x_n} \end{pmatrix},$$

$\mathbf{E}_n$ is the identity $(n \times n)$-matrix, and θ_n is the zero-vector in the space $\mathbb{R}^n$.

It is easy to see that

$$\frac{\partial \tilde{\xi}(\xi; t, t)}{\partial t} = 1, \quad \xi \in [t, T]. \tag{6.43}$$

We also obtain the following expressions by using the definition of $\tilde{z}(\xi; t, 0, \mu; t, x)$ and differentiating integrals in parameters:

$$D_x \tilde{z}(T; t, 0, \mu; t, x)^\top = \int_t^T \left(\int_P D_x g(\xi, \tilde{x}(\xi), u) \mu(\xi | du) \right)^\top \frac{\partial \tilde{x}(\xi; t, x, \mu)}{\partial x} d\xi, \tag{6.44}$$

$$\frac{\partial \tilde{z}(T; t, 0, \mu; t', x)}{\partial t'} = \int_t^T \int_P \frac{\partial g(\xi, \tilde{x}(\xi), u)}{\partial \xi} \mu(\xi | du) d\xi + \\ + \int_t^T \left(\int_P D_x g(\xi, \tilde{x}(\xi), u) \mu(\xi | du) \right)^\top \frac{\partial \tilde{x}(\xi; t', x, \mu)}{\partial t'} d\xi. \tag{6.45}$$

By the formulas for the derivatives of solutions of system (6.38) in parameters and initial state t, x and formula (6.39), we obtain the following expressions for the corresponding derivatives of the function $\omega(t, x, \alpha)$:

$$D_x \omega(t, x, \alpha)^\top = \left\{ \frac{\partial \omega(t, x, \alpha)}{\partial x_1}, \dots, \frac{\partial \omega(t, x, \alpha)}{\partial x_n} \right\} \\ = D_x \sigma(\tilde{x}(T; t, x, \mu))^\top \frac{\partial \tilde{x}(T; t, x, \mu)}{\partial x} + D_x \tilde{z}(T; t, 0, \mu; t, x)^\top, \tag{6.46}$$

$$\frac{\partial \omega(t, x, \alpha)}{\partial t} = D_x \sigma(\tilde{x}(T; t, x, \mu))^\top \frac{\partial \tilde{x}(T; t, x, \mu))}{\partial t} + D_x \sigma(\tilde{x}(T; t, x, \mu))^\top \frac{d\tilde{x}(T; 0, x, \mu))}{d\xi} \cdot \frac{\partial \tilde{\xi}(T; t, t)}{\partial t} \\ + \frac{\partial \tilde{z}(T; t, 0, \mu; t, x)}{\partial t} + \frac{d\tilde{z}(T; t, 0, \mu; t, x)}{d\xi} \cdot \frac{\partial \tilde{\xi}(T; t, t)}{\partial t}. \tag{6.47}$$

According to (6.40) and (6.42), the matrix $\dfrac{\partial \tilde{x}(\xi; t, x, \mu)}{\partial x}$ coincides with the fundamental matrix $X(\xi, t)$ of solutions of the system in variations. As is known, the matrix $(X(\xi, t)^\top)^{-1} = \Phi(\xi, t)$ is the fundamental matrix of solutions of the system, which is conjugate to the system in variations, i.e., it satisfies the system

$$\frac{d\Phi(\xi,t)}{d\xi} = -\left(\int\limits_P \frac{\partial f(\xi,\tilde{x}(\xi),u)}{\partial x}\mu(\xi|du)\right)^{\top} \Phi(\xi,t) \tag{6.48}$$

for all $\xi \in (t,T)$ and the boundary condition

$$\Phi(t,t) = \mathbf{E}_n. \tag{6.49}$$

It is also known that the matrix $\Phi(\xi,t)$ possesses the following semigroup property:

$$\Phi(T,t) = \Phi(T,\xi)\Phi(\xi,t), \quad \xi \in [t,T]. \tag{6.50}$$

Transform relations (6.46) using these remarks and notation:

$$\begin{aligned}\Phi(T,t)D_x\omega(t,x,\alpha) &= \left(X(T,t)^{\top}\right)^{-1} D_x\omega(t,x,\alpha)\\ &= \left(X(T,t)^{\top}\right)^{-1} X(T,t)^{\top} D_x\sigma(\tilde{x}(T;t,x,\mu))\\ &+ \int\limits_t^T \left(X(T,t)^{\top}\right)^{-1} X(\xi,t)^{\top}\left(\int\limits_P D_x g(\xi,\tilde{x}(\xi),u)\mu(\xi|du)\right)d\xi\\ &= D_x\sigma(\tilde{x}(T;t,x,\mu)) + \int\limits_t^T \Phi(T,\xi)D_x g(\xi,\tilde{x}(\xi),u)\mu(\xi|du)d\xi. \end{aligned} \tag{6.51}$$

Applying the Cauchy formula for solutions of systems of linear equations, we obtain from (6.51) that there exists an absolutely continuous function $p(\cdot) : [t,T] \mapsto \mathbb{R}^n$ satisfying the following conditions:

- for $\xi = T$, we have

$$p(T) = D_x\sigma(\tilde{x}(T;t,x,\mu)); \tag{6.52}$$

- for almost all $\xi \in (t,T)$, we have

$$\frac{dp(\xi)}{d\xi} = -\left(\int\limits_P \frac{\partial f(\xi,\tilde{x}(\xi),u)}{\partial x}\mu(\xi|du)\right)^{\top} p(\xi) - \int\limits_P D_x g(\xi,\tilde{x}(\xi),u)\mu(\xi|du); \tag{6.53}$$

- for $\xi = t$, we have

$$p(t) = D_x\omega(t,x,\alpha). \tag{6.54}$$

Now transform expression (6.47) using properties of $p(\xi)$, (6.43), (6.45), and the notation

$$\lambda(T) = p(T)^{\top}\frac{d\tilde{x}(T;t',x,\mu))}{dt'} + \frac{d\tilde{z}(T;t,0,\mu;t',x)}{dt'}. \tag{6.55}$$

We have

$$\begin{aligned}\frac{\partial\omega(t,x,\alpha)}{\partial t}\\ &= p(T)^{\top}\frac{d\tilde{x}(T;0,x,\mu)}{d\xi} + \frac{d\tilde{z}(T;t,0,\mu;t,x)}{d\xi} + p(T)^{\top}\frac{\partial\tilde{x}(T;t,x,\mu)}{\partial t} + \frac{\partial\tilde{z}(T;t,0,\mu;t,x)}{\partial t}\\ &= \lambda(T) + p(T)^{\top}\frac{\partial\tilde{x}(T;t,x,\mu)}{\partial t} - p(t)^{\top}\frac{\partial\tilde{x}(t;t,x,\mu)}{\partial t}\\ &+ \int\limits_t^T\left(\int\limits_P D_x g(\xi,\tilde{x}(\xi),u)\mu(\xi|du)\right)^{\top}\frac{\partial\tilde{x}(\xi;t,x,\mu)}{\partial t}d\xi + \int\limits_t^T\int\limits_P \frac{\partial g(\xi,\tilde{x}(\xi),u)}{\partial\xi}\mu(\xi|du)d\xi\end{aligned}$$

$$= \lambda(T) + \int\limits_t^T \frac{d}{d\xi} \left\langle p(\xi)), \frac{\partial \tilde{x}(\xi; t, x, \mu)}{\partial t} \right\rangle d\xi$$

$$+ \int\limits_t^T \left(\int\limits_P D_x g(\xi, \tilde{x}(\xi), u) \mu(\xi|du) \right)^\top \frac{\partial \tilde{x}(\xi; t, x, \mu)}{\partial t} d\xi + \int\limits_t^T \int\limits_P \frac{\partial g(\xi, \tilde{x}(\xi), u)}{\partial \xi} \mu(\xi|du) d\xi$$

$$= \lambda(T) + \int\limits_t^T \left(\int\limits_P \frac{\partial f(\xi, \tilde{x}(\xi), u)}{\partial \xi} \mu(\xi|du) \right)^\top p(\xi) d\xi + \int\limits_t^T \int\limits_P \frac{\partial g(\xi, \tilde{x}(\xi), u)}{\partial \xi} \mu(\xi|du) d\xi.$$

Thus, we have defined an absolutely continuous function $\lambda(\cdot) : [t, T] \to \mathbb{R}$ satisfying the following conditions:

- for almost all $\xi \in (t, T)$, we have

$$\frac{d\lambda(\xi)}{d\xi} = -\left(\int\limits_P \frac{\partial f(\xi, x(\xi), u)}{\partial t} \mu(\xi|du) \right)^\top p(\xi) - \int\limits_P \frac{\partial g(\xi, x(\xi), u)}{\partial t} \mu(\xi|du); \tag{6.56}$$

- for $\xi = t$, we have

$$\lambda(t) = \frac{\partial \omega(t, x, \alpha)}{\partial t}. \tag{6.57}$$

Fix an initial position $(t_0, x_0) \in \Pi_T$, an element $\alpha^0 \in \mathbf{A}^0(t_0, x_0)$, and a solution $x^0(\cdot) = x^0(\cdot; t_0, x_0, \mu^0)$, where $\mu^0 \in \mathbf{M}_{t_0}$, $\mu^0(\xi|du) = \mu^0(\xi(t_0, \tau)|du) = \alpha^0(\tau|du)$, $\tau \in [0, 1]$, and $\xi(t_0, \tau) = t_0 + (T - t_0)\tau$.

Lemma II.6 implies that for all $t \in (t_0, T)$ and $\xi \in [t, T]$, we have

$$x^0(\xi; t_0, x_0, \mu^0) = x^0(\xi; t, x^0(t), \mu^0[t, t_0]), \tag{6.58}$$

where $\mu^0[t, t_0] : [t, T] \to \text{rpm}(P)$ is the restriction of $\mu^0 : [t_0, T] \to \text{rpm}(P)$, namely,

$$\mu^0[t, t_0](\xi|du) = \mu^0(\xi|du) \quad \text{for} \quad \xi \in [t, T].$$

Fix a moment $t \in [t_0, T]$. Consider a point on the optimal trajectory $x = x^0(t)$ and $\alpha^0[t, t_0]$, which is an element of $\mathbf{A}^0(t, x^0(t))$, corresponding to the restriction $\mu^0[t, t_0]$ of the considered optimal control μ^0:

$$\mu^0[t, t_0](\xi(t, \tau)|du) = \alpha^0[t, t_0](\mu|du), \quad \tau \in [0, 1].$$

Consider a solution $p^0(\xi), \lambda^0(\xi) : [t_0, T] \to \mathbb{R}^n \times \mathbb{R}$ of the system

$$\frac{dp^0(\xi)}{d\xi} = -\left(\int\limits_P \frac{\partial f(\xi, x^0(\xi), u)}{\partial x} \mu^0(\xi|du) \right)^\top p^0(\xi) - \int\limits_P D_x g(\xi, x^0(\xi), u) \mu^0(\xi|du); \tag{6.59}$$

$$\frac{d\lambda^0(\xi)}{d\xi} = -\int\limits_P \left(\frac{\partial f(\xi, x^0(\xi), u)}{\partial t} \right)^\top p^0(\xi) \mu^0(\xi|du) - \int\limits_P \frac{\partial g(\xi, x^0(\xi), u)}{\partial t} \mu^0(\xi|du) \tag{6.60}$$

Assume that the following initial conditions hold:

$$p^0(t_0) = D_x \omega(t_0, x_0, \alpha^0), \tag{6.61}$$

$$\lambda^0(t_0) = \frac{\partial \omega(t_0, x_0, \alpha^0)}{\partial t}. \tag{6.62}$$

It is easy to see that the solutions coincide with the solutions $p(\xi)$ of (6.53), (6.54) and $\lambda(\xi)$ of (6.56), (6.57) on the interval $[t, T)$ for all $t \in [t_0, T)$.

It follows from Theorem II.7 and Lemma II.6 that the following relations hold for any $t \in [t_0, T)$:

$$p^0(t) = p(t) = D_x\omega(t, x^0(t), \alpha^0[t, t_0]), \tag{6.63}$$

$$\lambda^0(t) = \lambda(t) = \frac{\partial\omega(t, x^0(t), \alpha^0[t, t_0])}{\partial t} = -\min_{u\in P}\Big[\big\langle p^0(t), f(t, x^0(t), u)\big\rangle + g(t, x^0(t), u)\Big]. \tag{6.64}$$

Consider continuous extensions of the functions $p^0(\cdot)$, $\lambda^0(\cdot)$ on the interval $[t_0, T]$. The extended functions satisfy the boundary conditions

$$\begin{gathered} p^0(T) = D_x\sigma(x^0(T)), \\ \lambda^0(T) = -\min_{u\in P}\Big[\big\langle p^0(T), f(T, x^0(T), u)\big\rangle + g(T, x^0(T), u)\Big]. \end{gathered} \tag{6.65}$$

The boundary conditions are called the *transversality conditions* for the problem **OCP**. Lemma II.7 is proved. □

Fix $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Let $u^0(\cdot) \in \mathbf{U}_{t_0}$ and $x^0(\cdot) = x^0(\cdot; t_0, x_0, u^0(\cdot))$ be an optimal control and the corresponding optimal trajectory of system (3.1), i.e.,

$$I_{t_0,x_0}(x^0(\cdot), u^0(\cdot)) = V(t_0, x_0). \tag{6.66}$$

The convexity of the vectograms $E(t, x)$ (see assumption **(A4)**) implies, by the Filippov lemma [75, 300], that the equation

$$f(\xi, x^0(\xi), u^0(\xi)) = \int\limits_P f(\xi, x^0(\xi), u)\mu^0(\xi|du) \tag{6.67}$$

holds for almost all $\xi \in [t_0, T]$. One can consider values $\mu^0(\cdot|du) \in \mathbf{M}_{t_0}$ as regular probability measures on P concentrated at points $u^0(\xi)$. Thus, the optimal trajectory $x^0(\cdot)$ of system (3.1) coincides with the trajectory of system (3.6):

$$x^0(\cdot) = x^0(\cdot; t_0, x_0, u^0(\cdot)) = x^0(\cdot; t_0, x_0, \mu^0(\cdot)) \tag{6.68}$$

and

$$I_{t_0,x_0}\big(x^0(\cdot; t_0, x_0, \mu^0(\cdot|du)), \mu^0(\cdot|du)\big) = V(t_0, x_0).$$

Consider

$$\alpha^0 = \alpha^0(\tau|du) = \mu^0(\xi|du) \tag{6.69}$$

for

$$\xi = \xi(t_0, \tau) = t_0 + (T - t_0)\cdot\tau, \quad \tau \in [0, 1], \quad \alpha^0 \in \mathbf{A}^0(t_0, x_0) \subset \mathbf{A}.$$

The element $\alpha^0 \in \mathbf{A}$ corresponds to the optimal control $u^0(\cdot) \in \mathbf{U}_{t_0}$.

According to Lemma II.4, we have

$$\frac{dV(t, x^0(t))}{\Big(1, \int\limits_P f(t, x^0(t), u)\mu^0(t|du)\Big)} = -\int\limits_P g(t, x^0(t), u))\mu^0(t|du). \tag{6.70}$$

It is known (see, e.g., [51, 64]) that the directional derivative $\dfrac{dV(t,x)}{(1,f)}$ for the lower envelope of a family of smooth functions (6.1) has the form (6.2):

$$\frac{dV(t, x)}{(1, f)} = \min_{\alpha^0\in\mathbf{A}^0(t,x)}\left[\frac{\partial\omega(t, x, \alpha^0)}{\partial t} + \langle D_x\omega(t, x, \alpha^0), f\rangle\right], \tag{6.71}$$

where

$$\mathbf{A}^0(t, x) = \{\alpha^0 \in \mathbf{A} : \omega(t, x, \alpha^0) = V(t, x)\}. \tag{6.72}$$

It follows from Theorem II.7 and Lemma II.7 (see (6.63)–(6.64)) that the optimal control $u^0(\cdot)$ and the corresponding trajectory $x^0(\cdot)$ satisfy the Pontryagin maximum principle [213]. It can be presented to the considered problem **OCP** as the following assertion.

Theorem II.8. *Let conditions* **(A1′)**, **(A2)**, **(A3′)**, *and* **(A4)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Let a control* $u^0(\cdot) \in \mathbf{U}_{t_0}$ *and the corresponding trajectory* $x^0(\cdot) = x^0(\cdot; t_0, x_0, u^0(\cdot))$ *of system* (3.1) *be optimal* (3.3) *for* $(t_0, x_0) \in \Pi_T$. *Let an absolutely continuous vector function* $p^0(\cdot), \lambda^0(\cdot) : [t_0, T] \to \mathbb{R}^n \times \mathbb{R}$ *satisfy the following full conjugate system on* $[t_0, T]$:

$$\frac{dp^0(\xi)}{d\xi} = -\left(\frac{\partial f(\xi, x^0(\xi), u^0(\xi))}{\partial x}\right)^{\top} p^0(\xi) - D_x g(\xi, x^0(\xi), u^0(\xi)), \tag{6.73}$$

$$\frac{d\lambda^0(\xi)}{d\xi} = -\left(\frac{\partial f(\xi, x^0(\xi), u^0(\xi))}{\partial t}\right)^{\top} p^0(\xi) - \frac{\partial g(\xi, x^0(\xi), u^0(\xi))}{\partial t}. \tag{6.74}$$

Let the following boundary conditions hold:

$$\begin{gathered} p^0(T) = D_x\sigma(x^0(T)) = p^0, \\ \lambda^0(T) = -\min_{u\in P}\Big[\big\langle p^0, f(T, x^0(T), u)\big\rangle + g(T, x^0(T), u)\Big] = \lambda^0. \end{gathered} \tag{6.75}$$

Then the equality

$$\begin{aligned} &\lambda^0(\xi)\cdot 1 + \min_{u\in P}\Big[\big\langle p^0(\xi), f(\xi, x^0(\xi), u)\big\rangle + g(\xi, x^0(\xi), u)\Big] \\ &\qquad = \lambda^0(\xi)\cdot 1 + \Big[\big\langle p^0(\xi), f(\xi, x^0(\xi), u^0(\xi))\big\rangle + g(\xi, x^0(\xi), u^0(\xi))\Big] = 0. \end{aligned} \tag{6.76}$$

is valid for almost all $\xi \in [t_0, T]$.

Integrate Eq. (6.74) and consider the boundary condition (6.75) and formulas (6.63)–(6.66), to supply the standard conditions in the maximum principle and obtain the equalities

$$\begin{aligned} H(t, x^0(t), p^0(t)) &= -\lambda^0(t) = -\frac{\partial\omega(t, x^0(t), \alpha^0[t; t_0])}{\partial t} \\ &= -\int_t^T \left\langle p^0(\xi), \int_P \frac{\partial f(\xi, x^0(\xi), u)}{\partial \xi}\mu^0(\xi|du)\right\rangle d\xi - \int_t^T\int_P \frac{\partial g(\xi, x^0(\xi), u)}{\partial \xi}\mu^0(\xi|du)d\xi \\ &\qquad + \min_{u\in P}[\langle p^0, f(T, x^0(T), u)\rangle + g(T, x^0(T), u))]. \end{aligned}$$

These equalities hold for any $t \in [t_0, T]$.

Definition II.1. An *extremal* for a given point $(t_0, x_0) \in \Pi_T$ to the problem **OCP** (3.1)–(3.3) is a trajectory $x^e(\cdot) = x(\cdot; t_0, x_0, u^e(\cdot)) : [t_0, T] \to \mathbb{R}^n$ of system (3.1) starting at the initial point $x^e(t_0) = x_0$ and generated by the control $u^e(\cdot) \in \mathbf{U}_{t_0}$:

$$\begin{aligned} &\frac{dx^e(t)}{dt} = f(t, x^e(t), u^e(t)), \quad t \in [t_0, T], \\ &x^e(t_0) = x_0, \end{aligned} \tag{6.77}$$

and satisfying the Pontryagin maximum principle condition

$$\min_{u\in P}\Big[\big\langle p^e(t), f(t, x^e(t), u)\big\rangle + g(t, x^e(t), u)\Big] = \Big[\big\langle p^e(t), f(t, x^e(t), u^e(t))\big\rangle + g(t, x^e(t), u^e(t))\Big] \tag{6.78}$$

for almost all $t \in [t_0, T]$. The vector-valued function $p^e(\cdot):\ [t_0, T] \to \mathbb{R}^n$ in (6.78) is called the *coextremal* or the *adjoint variable*. It satisfies the conjugate system (6.73):

$$\begin{aligned} \frac{dp^e(t)}{dt} &= -\left(\frac{\partial f(t, x^e(t), u^e(t))}{\partial x}\right)^\top p^e(t) - D_x g(t, x^e(t), u^e(t)),\\ p^e(T) &= D_y \sigma(x^e(T)). \end{aligned} \tag{6.79}$$

6.4. Connection between the Pontryagin maximum principle and the Cauchy method of characteristics for the Bellman equation. Recall that the Hamiltonian in the Bellman equation (5.1) for the considered problem **OCP** (3.1)–(3.3) has the form

$$H(t, x, p) = \min_{u \in P} [\langle p, f(t, x, u)\rangle + g(t, x, u)]. \tag{6.80}$$

In this section, we consider the problem **OCP** under assumptions **(A1′)**, **(A2)**, and **(A3′)** and the following two assumptions strengthening **(A2)** and **(A4)**.

(A2′) Condition **(A2)** is supplied with the following extendability conditions:

$$\begin{aligned} \left\|\frac{\partial f_i(t, x, u)}{\partial x_j}\right\| &\le K_1(1 + \|x\|), & \left\|\frac{\partial f_i(t, x, u)}{\partial t}\right\| &\le K_1(1 + \|x\|),\\ \left\|\frac{\partial g(t, x, u)}{\partial x_j}\right\| &\le K_1(1 + \|x\|), & \left\|\frac{\partial g(t, x, u)}{\partial t}\right\| &\le K_1(1 + \|x\|) \end{aligned}$$

for all $i \in \overline{1, n}$, $j \in \overline{1, n}$, $(t, x, u) \in \Pi_T \times P$, where $K_1 > 0$ is a constant.

(A4′) The vectograms

$$E(t, x) = \{(f(t, x, P),\ g(t, x, P))\} \subset \mathbb{R}^n \times \mathbb{R}$$

are *strictly* convex sets for all $(t, x) \in \Pi_T$.

Condition **(A4′)** implies that the minimum in formula (6.80) for the Hamiltonian $H(t, x, p)$ is achieved at a unique element $u^0 = u^0(t, x, p)$:

$$H(t, x, p) = [\langle p, f(t, x, u^0)\rangle + g(t, x, u^0)]. \tag{6.81}$$

The mapping $(t, x, p) \to u^0(t, x, p)$ is continuous, and the Hamiltonian has continuous partial derivatives with respect to the impulse variables p_i, $i \in \overline{1, n}$. The vector of the partial derivatives has the form

$$D_p H(t, x, p) = f(t, x, u^0). \tag{6.82}$$

In addition, using (6.81), one can easily obtain the following expressions for the partial derivatives of the Hamiltonian with respect to x and t:

$$D_x H(t, x, p) = \left(\frac{\partial f(t, x, u^0)}{\partial x}\right)^\top \cdot p + D_x g(t, x, u^0), \tag{6.83}$$

$$\frac{\partial H(t, x, p)}{\partial t} = \left\langle \frac{\partial f(t, x, u^0)}{\partial t}^\top, p \right\rangle + \frac{\partial g(t, x, u^0)}{\partial t}. \tag{6.84}$$

Assumptions **(A1′)** and formulas (6.82)–(6.84) imply that the mapping

$$\Pi_T \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \quad (t, x) \mapsto \left(D_p H(t, x, u^0), D_x H(t, x, u^0), \frac{\partial H(t, x, u^0)}{\partial t}\right)$$

is continuous and bounded on any compact set $Q \in \Pi_T$.

The last property and condition **(A2′)** for initial data for the problem **OCP** are sufficient for the existence and uniqueness of classical characteristics [62] for the Bellman equation (5.1) in the boundary Cauchy problem (5.2). Recall that characteristics are the family of absolutely continuous functions

$$(\hat{x}(\cdot, y),\ \hat{p}(\cdot, y),\ \hat{z}(\cdot, y)) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

depending on the parameter $y \in \mathbb{R}^n$ and satisfying the system of ordinary differential equations on $(0, T)$

$$\frac{d\hat{x}}{dt} = D_p H(t, \hat{x}, \hat{p}),$$
$$\frac{d\hat{p}}{dt} = -D_x H(t, \hat{x}, \hat{p}), \tag{6.85}$$
$$\frac{d\hat{z}}{dt} = \langle D_p H(t, \hat{x}, \hat{p}), \hat{p} \rangle - H(t, \hat{x}, \hat{p})$$

and the following boundary conditions at $t = T$:

$$\hat{x}(T, y) = y, \quad \hat{p}(T, y) = D_y \sigma(y), \quad \hat{z}(T, y) = \sigma(y). \tag{6.86}$$

Formulas (6.82)–(6.85) for characteristics and Definition II.1 of extremals and coextremals (6.77)–(6.79) of the problem **OCP** imply the following theorem.

Theorem II.9. *Let conditions* **(A1′)**–**(A4′)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Then for any point* $(t_0, x_0) \in \Pi_T$*, the set of all extremals* $x^e(\cdot) = x(\cdot; t_0, x_0, u^e(\cdot))$ *coincides with the set* $X(t_0, x_0)$ *of all components* $\hat{x}(\cdot, y_0)$ *of classical characteristics for the Bellman equation* (5.1)*, which intersect at the point* (t_0, x_0)*, namely,*

$$X(t_0, x_0) = \{\hat{x}(\cdot, y_0) : y_0 \in Y(t_0, x_0) \subset \mathbb{R}^n, \ Y(t_0 x_0) \neq \emptyset\},$$
$$Y(t_0, x_0) = \{\forall y_0 : \hat{x}(t_0, y_0) = x_0\}.$$

The set of all corresponding coextremals $p^e(\cdot)$ *coincides with the set of all components* $\hat{p}(\cdot, y_0)$*,* $y_0 \in Y(t_0, x_0)$ *of classical characteristics.*

Note that by Theorem II.9 and definitions of characteristics, one can consider the function $\hat{x}(\cdot, y_0)$ as the trajectory of system (3.1) generated by the control $\hat{u}(\cdot, y_0)$:

$$\hat{u}(t, y_0) = u^0(t, \hat{x}(t; y_0), \hat{p}(t; y_0)), \quad t \in [t_0, T], \quad y_0 \in Y(t_0, x_0),$$

where $u^0(t, x, p)$ is defined by (6.81). It is easy to see that $\hat{u}(\cdot, y_0) \in \mathbf{U}_{t_0}$.

By Theorem II.9 and definitions of characteristics, extremals, and coextremals, we obtain the relations

$$\begin{aligned} \frac{d\hat{z}(t, y_0)}{dt} &= \langle f(t, \hat{x}(t, y_0), \hat{u}(t, y_0)), \hat{p}(t, y_0) \rangle + H(t, \hat{x}(t, y_0), \hat{p}(t, y_0)) \\ &= -g(t, \hat{x}(t, y_0), \hat{u}(t, y_0)), \qquad t \in (t_0, T), \\ y_0 &= \hat{x}(T, y_0) = x^e(T), \quad y_0 \in Y(t_0, x_0), \\ \hat{z}(T, y_0) = \sigma(y_0) &= \sigma(x^e(T)), \quad \hat{z}(t_0, y_0) = I_{t_0, x_0}(\hat{x}(\cdot, y_0), \hat{u}(\cdot, y_0)) \end{aligned} \tag{6.87}$$

The conditions of Theorem II.8 (the Pontryagin maximum principle) are necessary optimality conditions. Obviously, the value function $V(t_0, x_0)$ calculated as the minimum over the set of all admissible program controls $u(\cdot) \in \mathbf{U}_{t_0}$ and the corresponding trajectories $x(\cdot; t_0, x_0, u(\cdot))$ of system (3.1) can also be successfully calculated as the minimum over the set of extremal controls $u^e(\cdot) \in \mathbf{U}_{t_0}$ and the corresponding trajectories $x^e(\cdot)$ satisfying the conditions of the Pontryagin maximum principle.

Relations (6.87) imply that the value function $V(t_0, x_0)$ (3.3) can also be successfully calculated as the minimum of the cost functional $I_{t_0, x_0}(x(\cdot), u(\cdot))$ (3.2) over all components $\hat{x}(\cdot, y_0)$ of characteristics, which intersect at the point (t_0, x_0), (namely, over all $y_0 \in Y(t_0, x_0)$, see (6.85)–(6.86)) and over the corresponding program controls $\hat{u}(\cdot, y_0)$:

$$V(t_0, x_0) = \min_{y_0 \in Y(t_0, x_0)} I_{t_0, x_0}(\hat{x}(\cdot, y_0), \hat{u}(\cdot, y_0)). \tag{6.88}$$

Note that assumption **(A4′)** implies the existence of an optimal control $u^0(\cdot) \in \mathbf{U}_{t_0}$. According to Theorems II.8 and II.9, the control belongs to the set of all $\hat{u}(\cdot, y_0)$. Thus, the operation of minimum in

(6.88) is correct. Moreover, definitions (6.85)–(6.86) and assumptions **(A1′)**, **(A2′)**, **(A3′)**, and **(A4′)** imply that the sets $Y(t_0, x_0)$ are compact for all $(t_0, x_0) \in \hat{\Pi}_T$.

As above, let us transform the interval $[t_0, T]$ to the standard interval $[0, 1]$ by the linear transformation $\xi(t_0, \tau)$ (4.7):

$$\xi(t_0, \tau) = t_0 + (T - t_0)\tau, \quad \tau \in [0, 1]. \tag{6.89}$$

Thus, controls $\hat{u}(\cdot, y_0)$ are transformed to the standardized generalized controls $\hat{\alpha}(t_0, y_0) \in \mathbf{A}$,

$$\hat{\alpha}(t_0, y_0) : \hat{\alpha}(t_0, y_0)(\tau | du) = \hat{u}(\xi(t_0, \tau), y_0), \quad \tau \in [0, 1]. \tag{6.90}$$

Let

$$\hat{\mathbf{A}}(t_0, x_0) = \{\hat{\alpha}(t_0, y_0) : y_0 \in Y(t_0, x_0)\}. \tag{6.91}$$

According to formulas (4.4)–(4.6), we have

$$I_{t_0,x_0}(\hat{x}(\cdot, y_0), \hat{u}(\cdot, y_0)) = \omega(t_0, x_0, \hat{\alpha}(t_0, y_0)).$$

Thus, using (6.88), one can obtain the following specification of formula (6.1) (see Theorem II.5) defining the value function.

Lemma II.8. *Let conditions* **(A1′)**–**(A4′)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. Then the value function* $V(t, x)$ (3.3) *has the representation*

$$V(t, x) = \min_{\hat{\alpha} \in \hat{\mathbf{A}}(t,x)} \omega(t, x, \hat{\alpha}), \quad (t, x) \in \Pi_T, \tag{6.92}$$

for any $(t, x) \in \operatorname{cl} \Pi_T$, *where* $\mathbf{A} \supseteq \hat{\mathbf{A}}(t, x) \supseteq \mathbf{A}^0(t, x)$ (*see* (6.91) *and* (6.3)).

The function $\omega(\cdot) : \operatorname{cl} \Pi_T \times \mathbf{A} \to \mathbb{R}$ *is continuous. The function*

$$\operatorname{cl} \Pi_T \to \mathbb{R} : (t, x) \mapsto \omega(t, x, \hat{\alpha}), \quad \hat{\alpha} \in \hat{\mathbf{A}}(t, x),$$

has the partial derivatives $\dfrac{\partial \omega(t, x, \hat{\alpha})}{\partial t}$, $\dfrac{\partial \omega(t, x, \hat{\alpha})}{\partial x_i}$, $i \in \overline{1, n}$, *which are uniformly continuous relative to* $\hat{\alpha}$.

7. Necessary and Sufficient Optimality Conditions

7.1. The Pontryagin maximum principle and the superdifferential of the value function. Theorem II.6, Remark II.4, and formulas (6.63) and (6.64) imply that the superdifferential of the value function $\partial^+ V(t, x)$ is nonempty and is defined at any point $(t, x) \in \Pi_T$ as follows:

$$\begin{gathered}\partial^+ V(t, x) = \operatorname{co} \Big\{ (\lambda^0, p^0) \in \mathbb{R} \times \mathbb{R}^n : \\ \lambda^0 = \frac{\partial \omega(t, x, \alpha^0)}{\partial t},\ p^0 = D_x \omega(t, x, \alpha^0),\ \alpha^0 \in \mathbf{A}^0(t, x) \Big\}.\end{gathered} \tag{7.1}$$

According to Theorem II.9, the following equalities hold:

$$p^0 = \hat{p}(t, y_0), \quad \lambda^0 = \hat{\lambda}(t, y_0) = -H(t, \hat{x}(t, y_0), \hat{p}(t, y_0)), \quad y_0 \in Y(t, x). \tag{7.2}$$

Let us complete necessary optimality conditions (see Theorem II.8) up to sufficient conditions using representations (7.1) and (7.2) for the superdifferential of the value function $\partial^+ V(t, x)$. We prove the following assertions.

Theorem II.10. *Let conditions* **(A1′)**–**(A4′)** *for the problem* ***OCP*** (3.1)–(3.3) *hold. A control* $u^e(\cdot) \in \mathbf{U}_{t_0}$ *and the corresponding trajectory* $x^e(\cdot) = x(\cdot; t_0, x_0, u^e(\cdot))$ *of system* (3.1) *are optimal* (3.3) *at a point* $(t_0, x_0) \in \Pi_T$ *if and only if the absolutely continuous vector-valued function* $p^e(\cdot), \lambda^e(\cdot) : [t_0, T] \to \mathbb{R}^n \times \mathbb{R}$

satisfying the full conjugate system (6.73)–(6.75) *satisfy also the Pontryagin maximum principle condition for almost all* $t \in [t_0, T]$:

$$\lambda^e(t) \cdot 1 + \min_{u \in P} \Big[\big\langle p^e(t), f(t, x^e(t), u) \big\rangle + g(t, x^e(t), u) \Big]$$
$$= \lambda^e(t) \cdot 1 + \Big[\big\langle p^e(t), f(t, x^e(t), u^e(t)) \big\rangle + g(t, x^e(t), u^e(t)) \Big] = 0, \quad (7.3)$$

and the inclusion

$$(\lambda^e(t), p^e(t)) \in \partial^+ V(t, x^e(t)) \quad (7.4)$$

holds for all $t \in [t_0, T]$.

Proof. Necessity. According to Theorems II.8 and II.9 and formulas (7.1) and (7.2), conditions (7.3) and (7.4) are necessary optimality conditions.

Sufficiency. Now assume that conditions (7.3) and (7.4) hold for a pair $(x^e(\cdot), u^e(\cdot))$, a control $u^e(\cdot) \in \mathbf{U}_{t_0}$, and the corresponding trajectory $x^e(\cdot) = x(\cdot; t_0, x_0, u^e(\cdot))$ of system (3.1). Consider the realization of the value function $V(t, x)$ along the trajectory $x^e(t)$ (see Theorem II.6). One can easily obtain that the following inequalities hold for almost all $t \in [t_0, T]$:

$$0 = \min_{u \in P} \left[\frac{dV(t, x^e(t))}{(1, f(t, x^e(t), u))} + g(t, x^e(t), u) \right] \le \left[\frac{dV(t, x^e(t))}{(1, f(t, x^e(t), u^e(t)))} + g(t, x^e(t), u^e(t)) \right]. \quad (7.5)$$

Theorems II.6, II.8, and II.9 and formulas (7.1) and (7.2) imply

$$\frac{dV(t, x)}{(1, f)} = \min_{\alpha^0 \in \mathbf{A}^0(t,x)} \left[\frac{\partial \omega(t, x, \alpha^0)}{\partial t} + \langle D_x \omega(t, x, \alpha^0), f \rangle \right]$$
$$= \min_{y_0 \in Y(t,x)} \big[\hat{\lambda}(t, y_0) + \langle \hat{p}(t, y_0), f \rangle \big] = \min_{(\lambda^0, p^0) \in \partial^+ V(t,x)} \big[\lambda^0 + \langle p^0, f \rangle \big], \quad (7.6)$$

where

$$\mathbf{A}^0(t, x) = \{\alpha^0 \in \mathbf{A} : \omega(t, x, \alpha^0) = V(t, x)\},$$
$$Y(t, x) = \{y^0 \in \mathbb{R}^n : \hat{x}(t, y_0) = x,\ \hat{x}(T, y_0) = y_0\}.$$

The extremal $x^e(\cdot)$ generated by the control $u^e(\cdot)$ and the corresponding coextremal $p^e(\cdot)$ supplied by $\lambda^e(\cdot)$ via (6.75)–(6.77), satisfy the relations

$$\frac{\partial \omega(t, x^e(t), \alpha^e[t, t_0]}{\partial t} = \hat{\lambda}(t, y_0) = \lambda^e(t), \quad (7.7)$$
$$D_x \omega(t, x^e(t), \alpha^e[t, t_0]) = \hat{p}(t, y_0) = p^e(t). \quad (7.8)$$

The element $\alpha^e[t, t_0] \in \mathbf{A}$ is constructed as the restriction of the control $u^e(\cdot) : [t_0, T] \to P$ to a smaller interval $[t, T]$, $t_0 \le t \le T$, which is transformed by the linear transformation $\xi(t, \tau) = t + (T - t)\tau : [t, T] \to [0, 1]$. According to conditions (7.3) and (7.4), we have

$$\lambda^e(t) \cdot 1 + \Big[\big\langle p^e(t), f(t, x^e(t), u^e(t)) \big\rangle + g(t, x^e(t), u^e(t)) \Big] = 0, \quad (7.9)$$
$$(\lambda^e(t), p^e(t)) \in \partial^+ V(t, x^e(t)). \quad (7.10)$$

Using (7.6), (7.9), and (7.10), let us continue estimate (7.5):

$$0 \le \frac{dV(t, x^e(t))}{(1, f(t, x^e(t), u^e(t)))} + g(t, x^e(t), u^e(t))$$
$$= \min_{(\lambda^0, p^0) \in \partial^+ V(t, x^e(t))} \lambda^0 + \big\langle p^0, f(t, x^e(t), u^e(t)) \big\rangle + g(t, x^e(t), u^e(t))$$
$$\le \lambda^e(t) + \big\langle p^e(t), f(t, x^e(t), u^e(t)) \big\rangle + g(t, x^e(t), u^e(t)) = 0.$$

This implies that for almost all $t \in [t_0, T]$, we have

$$\frac{dV(t, x^e(t))}{(1, f(t, x^e(t), u^e(t)))} + g(t, x^e(t), u^e(t)) = 0, \tag{7.11}$$

$$\alpha^e[t, t_0] \in \mathbf{A}^0(t, x^e(t)). \tag{7.12}$$

By Lemma II.4 (see Sec. 4.2, (4.16)), these relations are sufficient optimality conditions for the considered control $u^e(\cdot) \in \mathbf{U}_{t_0}$ and the coextremal $x^e(\cdot)$ generated by the control. Theorem II.10 is proved. □

Remark II.5. Note that Theorem II.10 remains valid if inclusion (7.4) is replaced by the inclusion

$$(\lambda^e(t), p^e(t)) \in \Big\{ \big(-H(t, \hat{x}(t, y_0), \hat{p}(t, y_0)), \hat{p}(t, y_0) \big) : y_0 \in Y(t, x^e(t)) \Big\}. \tag{7.13}$$

This follows from formulas (7.3) and (7.6) and the linearity of the inner product.

Remark II.6. Note that condition (7.4) is a generalization of the following well-known *necessary* optimality condition:

$$\forall t \in [t_0, T]: \quad p^e(t) \in \partial_x V(t, x^e(t)) = \operatorname{co} \Big\{ p^e = \lim_{\substack{t_k \to t \\ x_k \to x^e(t)}} D_x V(t_k, x_k) : (t_k, x_k) \in \Pi_T \Big\}, \tag{7.14}$$

where (t_k, x_k) are points of differentiability of the locally Lipschitz function $V(\cdot)$. Inclusion (7.14) was obtained in [56].

Remark II.7. Note that relations (7.4) and (7.14) are *first-order* conditions. These conditions are effective for obtaining necessary and sufficient optimality conditions, simultaneously. This can be understood as a consequence of full information about the control system, which is contained in the definition of the value function.

Similar results can be found in [25, 52, 56, 80, 306].

7.2. Necessary and sufficient optimality conditions in the case of nonconvex vectograms. In this section, the problem **OCP** is considered under assumptions **(A1′)**–**(A3′)**. Condition **(A4)** on full vectograms of admissible speeds $E(t, x)$ (3.4) of system (3.1)–(3.2) is omitted. As was mentioned above (see Sec. 3.3), to guarantee the attainability of the optimal result $V(t_0, x_0)$ (3.3), it is necessary to extend classes of admissible program controls $\mathbf{U}_{t_0}$ up to classes of generalized program controls $\mathbf{M}_{t_0}$. The last classes are formed by measurable functions on $[t_0, T]$ with values in the set $\operatorname{rpm}(P)$ of regular probability measures defined on the compact set P, which is the given constraint for values of control parameters to system (3.1).

Recall that under generalized program controls $\mu(\cdot|du) \in \mathbf{M}_{t_0}$, system (3.1) has the dynamics described by (3.6), i.e.,

$$\dot{x}(t) = \int\limits_P f(t, x(t), u)\mu(t|du), \quad x(t_0) = x_0. \tag{7.15}$$

The corresponding value of the cost functional $I_{t_0,x_0}(x(\cdot), \mu(\cdot|du))$ (3.7) is calculated as follows:

$$I_{t_0,x_0}(x(\cdot), \mu(\cdot|du)) = \sigma(x(T; t_0, x_0, \mu(\cdot|du))) + \int\limits_{t_0}^{T} \int\limits_P g(t, x(t), u)\mu(t|du)dt, \tag{7.16}$$

where $x(\cdot) = x(\cdot, t_0, x_0, \mu(\cdot|du))$ is the trajectory of system (7.15) generated by the control $\mu := \mu(\cdot|du) \in \mathbf{M}_{t_0}$.

Assertion II.1 implies that the optimal result

$$\tilde{V}(t_0, x_0) = \min_{\mu(\cdot|du) \in \mathbf{M}_{t_0}} I_{t_0,x_0}\big(x(\cdot; t_0, x_0, \mu(\cdot|du)), \mu(\cdot|du)\big) \tag{7.17}$$

of problem (7.15), (7.16) is achieved; it coincides with the optimal result $V(t_0, x_0)$ (3.3) of problem (3.1), (3.2).

The Hamiltonian $\tilde{H}(t, x, p)$ of problem (7.15)–(7.17) has the form

$$\tilde{H}(t,x,p) = \min_{(f,g)\in\tilde{E}(t,x)} \big[\langle p, f\rangle + g\big], \tag{7.18}$$

where

$$\begin{aligned}\tilde{E}(t,x) &= \operatorname{co} E(t,x) = \operatorname{co}\big\{(f(t,x,u), g(t,x,u)) : u \in P\big\} \\ &= \left\{ \left(\int\limits_P f(t,x,u)\mu(du),\ \int\limits_P g(t,x,u)\mu(du) \right) : \mu(\cdot) \in \operatorname{rpm}(P) \right\}.\end{aligned} \tag{7.19}$$

Formulas (7.18), (7.19), and (6.80) and the linearity of the inner product imply the relation $\tilde{H}(t,x,p) = H(t,x,p)$. Note that the absence of condition **(A4)** implies that the Hamiltonian $\tilde{H}(t,x,p)$ is differentiable with rspect to the variable p.

We use the fact (see [74, 87, 300]) that any element $(\tilde{f}, \tilde{g}) \in \tilde{E}(t,x)$ is presented in the form

$$\tilde{f} = \int\limits_P f(t,x,u)\mu(du), \quad \tilde{g} = \int\limits_P g(t,x,u)\mu(du) \tag{7.20}$$

with a measure $\mu \in \operatorname{rpm}(P)$.

According to condition **(A1′)** and well-known results of analysis (see [300]), we obtain the relation

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial t} &= \frac{\partial}{\partial t}\int\limits_P f(t,x,u)\mu(du) = \int\limits_P \frac{\partial \tilde{f}(t,x,u)}{\partial t}\mu(du), \\ \frac{\partial \tilde{f}}{\partial x} &= \frac{\partial}{\partial x}\int\limits_P f(t,x,u)\mu(du) = \int\limits_P \frac{\partial \tilde{f}(t,x,u)}{\partial x}\mu(du), \\ \frac{\partial \tilde{g}}{\partial t} &= \frac{\partial}{\partial t}\int\limits_P g(t,x,u)\mu(du) = \int\limits_P \frac{\partial \tilde{g}(t,x,u)}{\partial t}\mu(du), \\ D_x\tilde{g} &= D_x\int\limits_P g(t,x,u)\mu(du) = \int\limits_P D_x g(t,x,u)\mu(du)\end{aligned} \tag{7.21}$$

for any $(t,x) \in \Pi_T$, $\mu \in \operatorname{rpm}(P)$, and $(\tilde{f}, \tilde{g}) \in \tilde{E}(t,x)$.

Using the definition of $\tilde{H}(t,x,p)$ (7.18), we define the set

$$\begin{aligned}\tilde{F}(t,x,p) &= \Big\{(\tilde{f}^0, \tilde{g}^0) \in \tilde{E}(t,x) : \tilde{H}(t,x,p) = \big\langle \tilde{f}^0, p\big\rangle + \tilde{g}^0\Big\} \\ &= \operatorname{co}\Big\{(f^0, g^0) : (f^0, g^0) = \big(f(t,x,u^0), g(t,x,u^0)\big),\ u^0 \in P, \\ &\qquad \big\langle f(t,x,u^0), p\big\rangle + g(t,x,u^0) = H(t,x,p)\Big\}\end{aligned} \tag{7.22}$$

for any $(t,x) \in \operatorname{cl}\Pi_T$. Clearly, this set is convex and compact. It follows from (7.21) and (7.22) that any element $(\tilde{f}^0, \tilde{g}^0) = (\tilde{f}(t,x,p), \tilde{g}(t,x,p))$ of the set $\tilde{F}(t,x,p)$ has the partial derivatives

$$\frac{\partial \tilde{f}_i(t,x,p)}{\partial t}, \quad \frac{\tilde{g}(t,x,p)}{\partial t}, \quad \frac{\partial \tilde{f}_i(t,x,p)}{\partial x_j}, \quad D_x\tilde{g}(t,x,p), \quad i,j \in \overline{1,n}.$$

Thus (see [52, 65, 186, 215, 217]), the Hamiltonian $\tilde{H}(t,x,p)$ is a nonsmooth function having the partial superdifferentials $\partial_t\tilde{H}(t,x,p)$, $\partial_x\tilde{H}(t,x,p)$, and $\partial_p\tilde{H}(t,x,p)$ and the following formulas hold:

$$\begin{aligned}\partial_p\tilde{H}(t,x,p) &= \left\{\tilde{f}^0 \in \mathbb{R}^n : \exists \tilde{g}^0 \in \mathbb{R},\ (\tilde{f}^0,\tilde{g}^0) \in \tilde{F}(t,x,p)\right\},\\ \partial_t\tilde{H}(t,x,p) &= \left\{\left(\frac{\partial \tilde{f}^0}{\partial t}\right)^{\top} p + \frac{\partial \tilde{g}^0}{\partial t} : (\tilde{f}^0,\tilde{g}^0) \in \tilde{F}(t,x,p)\right\},\\ \partial_x\tilde{H}(t,x,p) &= \left\{\left(\frac{\partial \tilde{f}^0}{\partial x}\right)^{\top} p + D_x\tilde{g}^0 : (\tilde{f}^0,\tilde{g}^0) \in \tilde{F}(t,x,p)\right\}.\end{aligned} \tag{7.23}$$

Remark II.8. It follows from definition (7.23) of partial superdifferentials that the multi-valued mappings

$$(t,x) \to \partial_p\tilde{H}(t,x,p), \quad (t,x) \to \partial_x\tilde{H}(t,x,p), \quad (t,x) \to \partial_t\tilde{H}(t,x,p)$$

have nonempty, convex, compact values. The mappings are lower semicontinuous.

Now we define quasi-characteristics for problem (7.15)–(7.17) as an n-parametric family of functions

$$(\tilde{x}(\cdot,y),\tilde{p}(\cdot,y),\tilde{z}(\cdot,y)) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \quad y \in \mathbb{R}^n,$$

which depend on the parameter $y \in \mathbb{R}^n$ and satisfy the differential inclusions

$$\begin{aligned}\frac{d\tilde{x}(t)}{dt} &\in \partial_p\tilde{H}(t,\tilde{x}(t),\tilde{p}(t)),\\ \frac{d\tilde{p}(t)}{dt} &\in -\partial_x\tilde{H}(t,\tilde{x}(t),\tilde{p}(t)),\\ \frac{d\tilde{z}(t)}{dt} &= \left\langle \frac{d\tilde{x}(t)}{dt}, \tilde{p}(t)\right\rangle - H(t,\tilde{x}(t),\tilde{p}(t))\end{aligned} \tag{7.24}$$

for almost all $t \in [0,T]$ and the boundary condition at $t = T$:

$$\tilde{x}(T,y) = y, \quad \tilde{p}(T,y) = D_y\sigma(y), \quad \tilde{z}(T,y) = \sigma(y). \tag{7.25}$$

Note that condition **(A1′)** and Remark II.8 guarantee the existence of quasi-characteristics (7.24), (7.25). Condition **(A2′)** also provides extendability on the interval $[0,T]$. Note that, as a rule, the boundary condition (7.25) is satisfied by a *set* of quasi-characteristics.

Let us introduce the notions of *extemals* and *coextremals* for problem (7.15)–(7.17).

Definition II.2. An *extremal* of problem (7.15)–(7.17) at a point $(t_0,x_0) \in \Pi_T$ is a trajectory $x^e(\cdot) = x(\cdot;t_0,x_0,\mu^e(\cdot|du)) : [t_0,T] \to \mathbb{R}^n$ of system (7.15) generated by the control $\mu^e(\cdot|du) \in \mathbf{M}_{t_0}$, i.e.,

$$\frac{dx^e(t)}{dt} = \int_P f(t,x^e(t),u)\mu^e(t|du), \quad t \in [t_0,T], \quad x^e(t_0) = x_0, \tag{7.26}$$

which satisfies the Pontryagin maximum principle for almost all $t \in [t_0,T]$:

$$\begin{aligned}\min_{\mu \in \mathrm{rpm}(P)} &\left[\left\langle p^e(t), \int_P f(t,x^e(t),u)\mu(du)\right\rangle + \int_P g(t,x^e(t),u)\mu(du)\right]\\ &= \left[\left\langle p^e(t), \int_P f(t,x^e(t),u)\mu^e(t|du)\right\rangle + \int_P g(t,x^e(t),u)\mu^e(t|du)\right],\end{aligned} \tag{7.27}$$

The variable $p^e(\cdot) : [t_0, T] \to \mathbb{R}^n$ is called the *coextremal* or the *adjoint variable*; it is a solution of the conjugate system

$$\begin{gathered}\frac{dp^e(t)}{dt} = -\int\limits_P \left(\frac{\partial f(t, x^e(t), u)}{\partial x}\right)^\top \mu^e(t|du) \cdot p^e(t) - \int\limits_P D_x g(t, x^e(t), u)\mu^e(t|du),\\ p^e(T) = D_y\sigma(x^e(T)).\end{gathered} \tag{7.28}$$

Complete the coextremal $p^e(\cdot)$ with the variable $\lambda^e(\cdot)$, which is an absolutely continuous function satisfying the differential equation (7.29) for almost all $t \in [t_0, T]$:

$$\frac{d\lambda^e(t)}{dt} = -\int\limits_P \left\langle \left(\frac{\partial f(t, x^e(t), u)}{\partial t}\right)^\top \mu^e(t|du),\ p^e(t)\right\rangle - \int\limits_P \frac{\partial g(t, x^e(t), u)}{\partial t}\mu^e(t|du), \tag{7.29}$$

$$\lambda^e(T) = -\tilde{H}(T, x^e(T), p^e(T)). \tag{7.30}$$

These equation together with (7.28) are the full conjugate system for problem (7.15)–(7.17). According to (7.18) and (7.26)–(7.30), for any $t \in [t_0, T]$, the following relation holds:

$$\lambda^e(t) = -\tilde{H}(t, x^e(t), p^e(t)). \tag{7.31}$$

As well as in the convex case, where condition **(A4′)** holds, formulas (7.22)–(7.28) and results of the convex analysis [52] imply that

$$\begin{gathered}\frac{dx^e(t)}{dt} \in \partial_p\tilde{H}(t, x^e(t), p^e(t))), \quad x^e(t_0) = x_0,\\ \frac{dp^e(t)}{dt} \in -\partial_x\tilde{H}(t, x^e(t), p^e(t)), \quad p^e(T) = \frac{\partial\sigma(x^e(T))}{\partial x}.\end{gathered} \tag{7.32}$$

Since the triple $(x^e(t), p^e(t), \lambda^e(t) = -H(t, x^e(t), p^e(t)))$ satisfies Eq. (6.76) (the Pontryagin maximum principle), we have

$$\begin{gathered}x^e(t) = \tilde{x}(t, y_0), \quad p^e(t) = \tilde{p}(t, y_0), \quad t \in [t_0, T],\\ y_0 = x^e(T) \in Y(t_0, x_0) \neq \emptyset,\\ \tilde{z}(t_0, y_0) = I_{t_0,x_0}(x^e(\cdot), \mu^e(\cdot|du)).\end{gathered} \tag{7.33}$$

On the contrary, by definition, any quasi-characteristic $\tilde{x}(\cdot, y_0)$, $\tilde{p}(\cdot, y_0)$, $\tilde{z}(\cdot, y_0)$: $y_0 \in Y(t_0, x_0) \neq \emptyset$ is connected to some extremal $x^e(\cdot)$ and the corresponding coextremal $p^e(\cdot)$ by relations (7.33). This means that the following assertion, which is equivalent to Theorem II.9, holds.

Theorem II.11. *Let conditions* **(A1′)**–**(A3′)** *for the problem* (7.15)–(7.17) *hold. Then for any point* $(t_0, x_0) \in \Pi_T$*, the set of all extremals* $x^e(\cdot) = x(\cdot; t_0, x_0, \mu^e(\cdot|du))$ *coincides with the set* $\tilde{X}(t_0, x_0)$ *of all components* $\tilde{x}(\cdot, y_0)$ *of quasi-characteristics* (7.24)–(7.25) *of the Bellman equation* (5.1)*, which intersect at the point* (t_0, x_0)*, namely,*

$$\begin{gathered}\tilde{X}(t_0, x_0) = \{\tilde{x}(\cdot, y_0) : y_0 \in Y(t_0, x_0) \subset \mathbb{R}^n, Y(t_0, x_0) \neq \emptyset\},\\ Y(t_0, x_0) = \{\forall y_0; \tilde{x}(t_0, y_0) = x_0\}.\end{gathered}$$

The set of all corresponding coextremals $p^e(\cdot)$ *coincides with the set of all components* $\tilde{p}(\cdot, y_0)$*,* $y_0 \in Y(t_0, x_0)$ *of quasi-characteristics.*

Note that Theorem II.11 expresses the equivalence of necessary optimality conditions obtained within the frameworks proposed by Pontryagin [213] and Clarke [52].

Using Theorem II.11 and Eq. (7.31), one can repeat arguments similar to that applied in the proof of Theorem II.10 and obtain the following assertion (the unified form of the optimality criterion).

Theorem II.12. *Let conditions* **(A1′)**–**(A3′)** *for problem* (7.15)–(7.17) *hold. A control* $\mu^e(\cdot|du) \in \mathbf{M}_{t_0}$ *and the corresponding trajectory* $x^e(\cdot) = x(\cdot; t_0, x_0, \mu^e(\cdot|du))$ *of system* (7.15) *are optimal* (7.17) *for a point* $(t_0, x_0) \in \Pi_T$ *if and only if the following conditions hold*:

- *the trajectory* $x^e(\cdot)$ *coincides with the phase component* $\tilde{x}(\cdot, y_0)$ *of quasi-characteristics* (7.24)–(7.25) *corresponding to the parameter* $y_0 \in Y(t_0, x_0)$;
- *the quasi-characteristic satisfies the inclusion*

$$(-\tilde{H}(t, \tilde{x}(t, y_0), \tilde{p}(t, y_0), \tilde{p}(t, y_0)) \in \partial\tilde{V}(t, \tilde{x}(t, y_0)) \tag{7.34}$$

for all $t \in [t_0, T]$.

Remark II.9. It was mentioned above that the Hamiltonians $\tilde{H}(t, x, p)$ and $H(t, x, p)$ and also the optimal results $\tilde{V}(t, x)$ and $V(t, x)$ for problems (7.15)–(7.17) and (3.1)–(3.3) coincide. Let a generalized optimal control $\mu^0(\cdot|du) \in M_{t_0}$ generate an optimal trajectory $x^0(\cdot)$, which cannot be generated by program controls of class $\mathbf{U}_{t_0}$. This is possible if the set $E(t, x)$ is nonconvex. The situation can be understood as a sliding motion of system (3.1) along the phase component $\tilde{x}(\cdot, y_0)$ of the quasi-characteristic (7.24)–(7.25) satisfying condition (7.34).

7.3. The representative formula for the minimax solution of the Hamilton–Jacobi–Bellman equation in terms of classical characteristics. In this section, results of Secs. 6.4, 7.1, and 7.2 are used for the construction of the generalized solution of the Cauchy problem for the equation

$$\frac{\partial u(t, x)}{\partial t} + H(t, x, D_x u(t, x)) = 0, \quad 0 < t < T, \quad x \in \mathbb{R}^n, \tag{7.35}$$

with the boundary condition at the terminal moment $t = T$:

$$u(T, x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{7.36}$$

Recall that

$$D_x u(t, x) = \left(\frac{\partial u(t, x)}{\partial x_1}, \dots, \frac{\partial u(t, x)}{\partial x_n} \right) = p \in \mathbb{R}^n.$$

Consider this problem for sufficiently smooth input data $H(t, x, p)$ and $\sigma(x)$. The specific assumption in the considered problem is the *concavity* of the Hamiltonian $H(t, x, p)$ with respect to the variable p. Note that this assumption takes place for the Bellman equation [29].

If a classical solution $u(t, x)$ of problem (7.35), (7.36) exists globally on $\operatorname{cl}\Pi_T = [0, T] \times \mathbb{R}^n$ or even locally, in a small neighborhood of a hyperplane $t = T$, it can be constructed (see [62, 207, 233]) by the Cauchy method of characteristics. Namely, one can use the family of functions

$$(\hat{x}(\cdot, y), \hat{p}(\cdot, y), \hat{z}(\cdot, y)) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \tag{7.37}$$

depending on the parameter $y \in \mathbb{R}^n$ and satisfying the system of equations

$$\begin{aligned} \frac{d\hat{x}}{dt} &= D_{\hat{p}} H(t, \hat{x}, \hat{p}), \\ \frac{d\hat{p}}{dt} &= -D_{\hat{x}} H(t, \hat{x}, \hat{p}), \\ \frac{d\hat{z}}{dt} &= \langle \hat{p}, D_{\hat{p}} H(t, \hat{x}, \hat{p}) \rangle - H(t, \hat{x}, \hat{p}), \end{aligned} \tag{7.38}$$

where $0 < t < T$. The classical characteristics satisfy also the boundary condition at $t = T$:

$$\hat{x}(T, y) = y, \quad \hat{p}(T, y) = D_y \sigma(y), \quad \hat{z}(T, y) = \sigma(y). \tag{7.39}$$

Here $H(t, x, p)$ and $\sigma(x)$ are input data of problem (7.35), (7.36).

For $(t, x) \in \operatorname{cl}\Pi_T$, define the set

$$Y(t, x) = \{y \in \mathbb{R}^n : \hat{x}(t, y) = x\}. \tag{7.40}$$

In the case where problem (7.35), (7.36) has a classical solution $u(t,x)$ of class C^1, this set is a singleton, i.e., $Y(t,x) = \{y(t,x)\}$. According to the Cauchy method, we have

$$u(t,x) = \hat{z}(t, y(t,x)), \quad D_t u(t,x) = -H\big(t, \hat{x}(t, y(t,x)), \hat{p}(t, y(t,x))\big), \quad D_x u(t,x) = \hat{p}(t, y(t,x)).$$

It is known that, as a rule, a classical solution of problem (7.35), (7.36) does not exist (see, e.g., [62, 207]). However, there is a *globally* defined, nonsmooth function coinciding with the value function of an appropriate local optimal control problem. At points of differentiability, this function satisfies Eq. (7.35) and the boundary condition (7.36) at $t = T$. It is proved in the theory of minimax solutions of first-order partial differential equations [236, 238] that this function coincides with the minimax solution of problem (7.35), (7.36).

In this section, the following construction of a minimax solution of problem (7.35), (7.36) by using classical characteristics (7.37)–(7.40) will be justified:

$$u(t,x) = \min\big\{\hat{z}(t,y) : y \in Y(t,x)\big\} \quad \forall (t,x) \in \operatorname{cl}\Pi_T. \tag{7.41}$$

It is important to note the following.

- If the assumptions for problem (7.35), (7.36) described in [122, 139, 156, 190] hold, then the formula (7.41) is valid for generalized solutions defined in these works.
- Obviously, this formula is compatible to the above-mentioned formula for classical solutions $u(\cdot) \in C^1$ obtained by the Cauchy method.
- Finally, the equivalence of minimax and viscosity solutions of the problem (7.35), (7.36) implies that formula (7.41) holds also for viscosity solutions.

Thus, problem (7.35), (7.36) is considered under the following assumptions.

(*A1*) Functions $H(\cdot)$, $D_tH(\cdot)$, $D_xH(\cdot)$, $D_pH(\cdot)$, $D_{tp}H(\cdot)$, $D_{pt}H(\cdot)$, $D_{xp}H(\cdot)$, and $D_{px}H(\cdot)$ are defined and continuous on $\operatorname{cl}\Pi_T \times \mathbb{R}^n$. The functions $\sigma(\cdot)$ and $D_x\sigma(\cdot)$ are defined and continuous on $\mathbb{R}^n$.

(*A2*) Functions $D_{tp}H(t,x,p)$ and $D_{xp}H(t,x,p)$ are bounded on each compact set $D \subset \operatorname{cl}\Pi_T$ uniformly in p.

(*A3*) Function $p \to H(t,x,p)$ is concave for any $(t,x) \in \operatorname{cl}\Pi_T$.

(*A4*) There exists a constant $C > 0$ such that

$$\|D_pH(t,x,p)\| \le C(1 + \|x\|) \quad \forall (t,x,p) \in \operatorname{cl}\Pi_T \times \mathbb{R}^n,$$
$$|H^*(t,x,f)| \le C(1 + \|x\|) \quad \forall (t,x) \in \operatorname{cl}\Pi_T, \quad \forall f \in \operatorname{dom} H^*(t,x,\cdot),$$

where $H^*(t,x,\cdot)$ is the *conjugate function to* $H(t,x,\cdot)$ and $\operatorname{dom} H^*(t,x,\cdot)$ is the *effective domain* of the function $H^*(t,x,\cdot)$ (see, e.g., [217, 218]).

(*A5*) For any compact set $D \subset \operatorname{cl}\Pi_T$, there is a constant $C(D) > 0$ such that

$$|H_t(t,x,p)| \le C(D)(1 + \|p\|), \quad \|H_x(t,x,p)\| \le C(D)(1 + \|p\|)$$

for any $p \in \mathbb{R}^n$ and $(t,x) \in D$.

Assumptions (*A1*)–(*A5*) and results of the convex analysis [217] imply the following properties of problem (7.35), (7.36).

Lemma II.9. *Let conditions (A1)–(A5) for problem* (7.35), (7.36) *hold. Then the effective domain* $\operatorname{dom} H^*(t,x,\cdot)$ *of the conjugate function to the Hamiltonian is a convex compact set for any* $(t,x) \in \operatorname{cl}\Pi_T$ *and it has the form*

$$\operatorname{dom} H^*(t,x,\cdot) = \operatorname{co}\{D_sH(t,x,s) : s \in \mathbb{R}^n\}. \tag{7.42}$$

For any $(t,x) \in \operatorname{cl}\Pi_T$ *and* $p \in \mathbb{R}^n$, *the Hamiltonian* $H(t,x,p)$ (7.35) *can be presented as follows*:

$$\begin{aligned} H(t,x,p) &= \min\Big\{\big\langle f,p\big\rangle - H^*(t,x,f) : f \in \operatorname{dom} H^*(t,x,\cdot)\Big\} \\ &= \min_{s\in\mathbb{R}^n}\Big[\big\langle p, D_sH(t,x,s)\big\rangle - H^*\big(t,x,D_sH(t,x,s)\big)\Big] \\ &= \big\langle p, D_pH(t,x,p)\big\rangle - H^*\big(t,x,D_pH(t,x,p)\big). \end{aligned} \tag{7.43}$$

Lemma II.10. *For any compact set* $D_0 \subset \operatorname{cl}\Pi_T$, *there exists a constant* $K(D_0) > 0$ *such that all characteristics* (7.37)–(7.39) *satisfying the relations*

$$y \in Y(D_0) = \bigcup_{(t_0,x_0)\in D_0} Y(t_0,x_0) \tag{7.44}$$

at $\tau = T$ *can be estimated for* $\tau \in [t_0^*(D_0), T]$ *as follows*:

$$\|\hat{x}(\tau,y)\| \le K(D_0), \quad \|\hat{p}(\tau,y)\| \le K(D_0), \tag{7.45}$$

where $t_0^*(D_0) = \min\{t \in [0,T] : \exists (t,x) \in D_0\}$.

Proof. Consider the set $\hat{X}(D_0)$ of all components $\hat{x}(\cdot)$ of characteristics (7.37)–(7.39) satisfying condition (7.44). According to condition (*A4*), we have

$$\left\|\frac{d\hat{x}(\tau,y)}{d\tau}\right\| \le C(1 + \|\hat{x}(\tau,y)\|)$$

for any $\hat{x}(\cdot) \in \hat{X}(D_0)$ and $t_0^* \le t_0 \le \tau \le T$.

Using this inequality and the Gronwall lemma [300], we obtain the estimate

$$\|\hat{x}(\tau,y)\| \le \big(e^{C(t-t_0)} - 1\big) + \|x_0\| \cdot e^{C(t-t_0)} \le K_1(D_0), \tag{7.46}$$

which holds for any $\hat{x}(\cdot) \in \hat{X}(D_0)$ and $t_0^* \le t_0 \le \tau \le T$, where

$$K_1(D_0) = \max_{(t_0,x_0)\in D_0} (1 + \|x_0\|) \cdot e^{CT} < \infty. \tag{7.47}$$

In particular, estimates (7.46) and (7.47) imply the following estimate for the corresponding boundary values of the components $\{\hat{p}(T,y) : y \in Y(D_0)\}$:

$$\max_{y\in Y(D_0)} \|p(T,y)\| \le \max_{\|y\|\le K_1(D_0)} \|D_y\sigma(y)\| = K_2(D_0). \tag{7.48}$$

Consider the set

$$D_0' := [t_0^*, T] \times \{x : \|x\| \le 3K_1(D_0)\} \subset \operatorname{cl}\Pi_T. \tag{7.49}$$

According to condition (*A5*), there is a constant $C_0' = C(D_0') > 0$ such that

$$\|H_x(t,x,p)\| \le C_0'(1 + \|p\|)$$

for all $p \in \mathbb{R}^n$ and $(t,x) \in D_0'$.

According to (7.46) and (7.47) and the last inequality, we estimate the components $\hat{p}(\tau,y)$ of all characteristics (7.37)–(7.39) satisfying condition (7.44) as follows:

$$\left\|\frac{d\hat{p}(\tau,y)}{d\tau}\right\| \le C_0'(1 + \|\hat{p}(\tau,y)\|) \tag{7.50}$$

for any $y \in Y(D_0)$ and $t_0^* \le t_0 \le \tau \le T$.

By inequalities (7.48) and (7.50) and the Gronwall lemma, we obtain the estimate

$$\|\hat{p}(\tau,y)\| \le \big(e^{C_0'(\tau-t_0)} - 1\big) + \|\hat{p}(T,y)\| \cdot e^{C_0'(t-t_0)} \le K_3(D_0) \tag{7.51}$$

for any $y \in Y(D_0)$ and $t_0^* \le t_0 \le \tau \le T$, where

$$K_3(D_0) = (1 + K_2(D_0)) \cdot e^{C_0' T} < \infty. \tag{7.52}$$

Finally, define a constant $K(D_0)$ declared in the statement of Lemma II.10 as follows:

$$K(D_0) = \max\{K_1(D_0), K_3(D_0)\}. \tag{7.53}$$

Lemma II.10 is proved. □

Now we consider the auxiliary control system

$$\frac{dx(\tau)}{d\tau} = D_s H(\tau, x(\tau), s(\tau)), \quad \tau \in [t_0, T], \quad x(t_0) = x_0, \tag{7.54}$$

where $s(\cdot) : [t_0, T] \to \mathbb{R}^n$ is a measurable control, $t_0 \in [0, T]$. Denote the set of all such controls by $\mathbf{S}_{t_0}$. Let $G_T(D_0)$ be the attainability set at the moment T for system (7.54) starting from any initial state $(t_0, x_0) \in D_0 \subset \operatorname{cl} \Pi_T$ under any measurable control $s(\cdot)$, namely,

$$G_T(D_0) = \bigcup_{(t_0,x_0)\in D_0} \; \bigcup_{s(\cdot)\in \mathbf{S}_{t_0}} \big\{x(T) = x(T; t_0, x_0, s(\cdot)) \in \mathbb{R}^n\big\},$$

where $x(\cdot; t_0, x_0, s(\cdot)) : [t_0, T] \to \mathbb{R}^n$ is a trajectory of system (7.54) starting at (t_0, x_0) under a control $s(\cdot) \in \mathbf{S}_{t_0}$.

Using the input data H and σ of problem (7.35), (7.36) and the given set of initial states $D_0 \subset [0, T] \times \mathbb{R}^n$, we choose an auxiliary optimal control problem $\mathbf{OCP'}$ for system (7.54) for which the optimal result defined at $(t, x) \in D_0$ coincides with $u(t, x)$ (7.41).

Consider the cost functional of the form

$$J_{(t_0,x_0)}(x(\cdot), s(\cdot)) = \sigma(x(T)) - \int_{t_0}^{T} H^*(\tau, x(\tau), D_p H(\tau, x(\tau), s(\tau)))d\tau, \tag{7.55}$$

where $x(\cdot) = x(\cdot, t_0, x_0, s(\cdot))$ is a trajectory of system (7.54) generated by a control $s(\cdot) \in \mathbf{S}_{t_0}$ and the integrand $H^*(\tau, x, f)$ is the conjugate function to the Hamiltonian H.

Also, consider the set $U_{t_0}(P^*) \subset \mathbf{S}_{t_0}$ of all measurable controls $s(\cdot)$ with values in the compact set $P^* \subset \mathbb{R}^n$.

The problem $\mathbf{OCP'}$ is as follows: minimize the cost functional (7.55) on the set of all controls $s(\cdot) \in \mathbf{U}_{t_0}(P^*)$ and corresponding trajectories $x(\cdot)$ of system (7.54). According to Assertion II.1, there exists a solution to the problem $\mathbf{OCP'}$ (7.54), (7.55) for any P^*, anyway, in the class of all generalized program controls.

Denote by $V'^0(t_0, x_0)$ the optimal result

$$V'^0(t_0, x_0) = \inf \big\{J_{(t_0,x_0)}(x(\cdot), s(\cdot)) : s(\cdot) \in U_{t_0}(P^*)\big\}. \tag{7.56}$$

For any initial set $D_0 \subset \operatorname{cl} \Pi_T$, we define, by Lemma II.10, a constant $K(D_0) > 0$ (7.53) and the sets

$$D_1 := [0, T] \times \{x : \|x\| < 2K_1(D_0)\}, \tag{7.57}$$

$$P^* = P_1 = P^*(D_0) = \{p \in \mathbb{R}^n : \|p\| \le K(D_0)\}. \tag{7.58}$$

Lemmas II.1 and II.2 and conditions (*A1*)–(*A5*) imply that the following assertion holds.

Lemma II.11. *If conditions* (*A1*)–(*A5*) *for input data of the Cauchy problem* (7.35), (7.36) *hold, then conditions* **(A1′)**–**(A3′)** (*see Secs.* 6.1 *and* 6.4) *for the auxiliary optimal control prollems* ***OCP′*** (7.54), (7.55), (7.58) *considered on the sets* D_1 (7.57) *hold.*

Using the material of Sec. 7.2, properties of partial superdifferentials $\partial_t \tilde{H}(t, x, p)$, $\partial_x \tilde{H}(t, x, p)$, and $\partial_p \tilde{H}(t, x, p)$ (see (7.23)), and Lemma II.11, we obtain the following lemma.

Lemma II.12. *For the auxiliary optimal control problem* ***OCP′*** (7.54), (7.55), (7.58) *considered on the set* D_1 (7.57), *the following relations hold*:

$$\begin{aligned} H(t,x,p) = \tilde{H}(t,x,p) &= \min_{s\in P^*}\Big[\big\langle p, D_sH(t,x,s)\big\rangle - H^*\big(t,x,D_sH(t,x,s)\big)\Big] \\ &= \big\langle p, D_pH(t,x,p)\big\rangle - H^*\big(t,x,D_pH(t,x,p)\big); \end{aligned} \tag{7.59}$$

$$\partial_t\tilde{H}(t,x,p) = \left\{\frac{\partial H(t,x,p)}{\partial t}\right\}, \quad \partial_x\tilde{H}(t,x,p) = \{D_xH(t,x,p)\}, \quad \partial_p\tilde{H}(t,x,p) = \{D_pH(t,x,p)\}. \tag{7.60}$$

Theorem II.12 and Lemma II.12 imply the following lemma.

Lemma II.13. *For any* $(t_0,x_0)\in D_0$, *any optimal trajectory* $x^0(\cdot;t_0,x_0,\mu^0(\cdot|ds))$, $\mu^0(\cdot|ds)\in \mathbf{M}_{t_0}$, *of the problem* ***OCP′*** (7.54)–(7.56), *coincides with the phase component* $\hat{x}(\cdot,y_0)$ *of the classical characteristic* (7.38)–(7.39) *under the condition* $y_0\in Y(t_0,x_0)$ (7.40).

The main result of this section is the following theorem.

Theorem II.13. *A minimax solution* $u(t,x)$ *of problem* (7.35), (7.36) *is defined by formulas* (7.41), (7.37)–(7.40) *for any* $(t,x)\in \operatorname{cl}\Pi_T$, *where* $Y(t,x)\neq\emptyset$ *everywhere on* $\operatorname{cl}\Pi_T$. *For any* $(t,x)\in\Pi_T$, *the superdifferential* $\partial^+u(t,x)\subset\mathbb{R}^{n+1}$ *of the function* $u(\cdot)$ *has the form*

$$\partial^+u(t,x) = \operatorname{co}\Big\{\big(-H(t,x,p(t,y^0)),p(t,y^0)\big) : y^0\in Y^0(t,x)\Big\}, \tag{7.61}$$

where

$$Y^0(t,x) = \big\{y^0\in Y(t,x) : z(t,y^0) = u(t,x)\big\}\neq\emptyset. \tag{7.62}$$

Proof. Fix any point $(t_0,x_0)\in\operatorname{cl}\Pi_T$ and any number $\varepsilon\in(0,T-t_0)$. Let $D_0 = D_\varepsilon(t_0,x_0)\cap\operatorname{cl}\Pi_T$, where $D_\varepsilon(t_0,x_0)$ is a closed ε-neighborhood of the point (t_0,x_0) in $\mathbb{R}^{n+1}$, and a number $K(D_0)>0$ be chosen in accordance with Lemma II.10.

Define the set $P^* = P_1$ for **OCP′** similarly to (7.58):

$$P^* = P_1 = P^*(D_0) = \big\{p\in\mathbb{R}^n : \|p\|\le K(D_0)\big\}.$$

Integrate the equation for $\hat{z}(\cdot,y)$ in (7.38), (7.39), where $y\in Y(t,x)$, $(t,x)\in D_0$, taking into account the formula (7.59) for $H(t,x,p)$ in the domain D_1 (7.57):

$$H(t,x,p) = \big\langle p, D_pH(t,x,p)\big\rangle - H^*\big(t,x,D_pH(t,x,p)\big).$$

For $t<T$, we have

$$\hat{z}(t,y) = \sigma(y) - \int_t^T H^*\Big(\tau,\hat{x}(\tau,y),D_pH\big(\tau,\hat{x}(\tau,y),\hat{p}(\tau,y)\big)\Big)d\tau.$$

It follows from Theorem II.11 and Lemma II.13 that for any point $(t,x)\in D_0$ and any pair consisting of an extremal and the corresponding coextremal of the problem **OCP′**, namely, $x^e(\cdot) = x^e(\cdot;t,x,\mu^e(\cdot|ds))$ and $p^e(\cdot) = p^e(\cdot;T,\partial\sigma(x^e(T))/\partial x,\mu^e(\cdot|ds)) : [t,T]\to\mathbb{R}^n\times\mathbb{R}^n$, the following relations hold:

$$\begin{gathered} Y(t,x)\neq\emptyset, \\ y\in Y(t,x) \quad\Leftrightarrow\quad y = x^e(T,t,x,\mu^e(\cdot|ds)) = x^e(T), \\ x^e(\tau,t,x,\mu^e(\cdot|ds)) = \hat{x}(\tau,x^e(T)), \quad p^e(\tau) = \hat{p}(\tau,x^e(T)) = \mu^e(\tau|ds) \quad \text{for} \quad \tau\in[t,T], \\ \hat{z}(t,x^e(T)) = J_{(t,x)}\big(\hat{p}(\cdot,x^e(T)),\hat{x}(\cdot,x^e(T))\big), \\ V'^0(t,x) = \min_{y\in Y(t,x)} J_{(t,x)}\big(\hat{p}(\cdot,y),\hat{x}(\cdot,y)\big) = \min\big\{\hat{z}(t,y) : y\in Y(t,x)\big\} = u(t,x). \end{gathered}$$

Thus, in the domain D_0, the function $u(t,x)$ of the form (7.41) coincides with the value function $V'^0(t,x)$ of the auxiliary problem $\mathbf{OCP'}$ (7.54), (7.55), (7.58). According to (7.59), it is possible to treat Eq. (7.35) as the Bellman equation of the problem.

Apply the results of the theory of minimax solutions of the Hamilton–Jacobi equation [235, 238] to obtain that the function $(t,x) \mapsto u(t,x)$ coincides with a unique minimax solution of the Cauchy problem (7.35), (7.36) in the domain D_0. Thus, for any $(t,x) \in D_0$, formula (7.61) holds, namely,

$$\partial V'^0(t,x) = \operatorname{co}\Big\{\big(-H(t,x,p),p\big) : p \in \hat{P}^0(t,x)\Big\}$$
$$= \operatorname{co}\Big\{\big(-H(t,\hat{x}(t,y),\hat{p}(t,y))\big) : y \in Y^0(t,x)\Big\} = \partial u(t,x) \neq \emptyset,$$

where

$$\hat{P}^0(t,x) = \Big\{p = p^e(t) : J_{(t,x)}\big(p^e(\cdot),x^e(\cdot)\big) = V'^0(t,x)\Big\}$$

and the set $Y^0(t,x)$ is determined in (7.62).

Since (t_0,x_0) and $\varepsilon \in (0,T-t_0)$ are arbitrary, the function $(t,x) \mapsto u(t,x)$ (see (7.41) and (7.37)–(7.40)) coinciding with the value function of the auxiliary problem $\mathbf{OCP'}$ (7.54), (7.55), (7.58) is a global minimax solution of the Cauchy problem (7.35), (7.36). In other words, it is the solution "on the whole."

8. Method of Dynamical Programming and Optimal Synthesis in Feedback Optimal Control Problems

8.1. Statement of feedback optimal control problems. A basic approach to studying and solving the problem **OCP** is a feedback statement, in which one constructs an optimal control as an optimal feedback, in other words, as an optimal synthesis or an optimal positional strategy. Feedbacks are functions $[0,T] \times \mathbb{R}^n \ni (t,x) \mapsto U(t,x) \in P$, possibly discontinuous [29, 135, 213].

To define motions of system (3.1) under feedbacks, the following concepts are known:

- formalizations using solutions of differential inclusions constructed by discontinuous right-hand sides of the dynamical equations (see [14, 75]);
- formalizations using limits of step-by-step motions (or Euler polygons with discrete feedbacks; see [82, 128, 133]);
- formalizations using multi-valued strategies constructed by feedbacks (see [35]), and others.

In the framework of these formalizations, a feedback generates a whole set of motions. Therefore, a necessity of defining the notion of guaranteed result arises. Recall that the guaranteed result is the *worst* value of the cost functional on the set of all motions generated by the feedback.

Remark II.10. Note that the set of motions obtained as limits of Euler polygons turns out to be the *least*, and the corresponding guaranteed result is the *best* of all the above-mentioned results. This circumstance is a reason for applying this approach in the research presented in the present monograph.

Recall the formalization of the feedback problem **OCP** introduced and developed in the theory of positional differential games (see, e.g., [128, 133]).

Consider a feedback

$$\operatorname{cl}\Pi_T = [0,T] \times \mathbb{R}^n \ni (t,x) \mapsto U(t,x) \subset P.$$

Let $\Delta := \{t_0 = \tau_0 < \dots < \tau_i < \dots < \tau_{N+1} = T\}$ be a partition of the interval $[t_0,T]$ with diameter $\operatorname{diam}\Delta := \max\{(\tau_{i+1}-\tau_i) : 0 \le i \le N\}$.

Definition II.3. The *Euler solution*, or the *Euler polygon*, or the *step-by-step motion* $x_\Delta(\cdot) = x_\Delta(\cdot;t_0,x_0,U) : [t_0,T] \to \mathbb{R}^n$ generated by the feedback $U(t,x)$ and corresponding to the partition Δ is the trajectory of system (3.1) described on any subinterval $[\tau_i,\tau_{i+1})$, $i = 0,1,\dots,N-1$, as follows:

$$\dot{x}_\Delta(t) = f(t,x_\Delta(t),u_i), \quad t \in [\tau_i,\tau_{i+1}), \tag{8.1}$$

where the control $u_i = U(\tau_i, x_\Delta(\tau_i))$ is constant. The initial state is $x_\Delta(t_0) = x_0$. The function $U_\Delta[\cdot] : [t_0, T] \to P$ such that $U_\Delta[t] = u_i$ for $t \in [\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots, N-1$, is called a *discrete realization* of the feedback $U(t,x)$. The corresponding value of the cost functional $I_{t_0,x_0}(x_\Delta(\cdot), U_\Delta[\cdot])$ is calculated as follows:

$$I_{t_0,x_0}(x_\Delta(\cdot), U_\Delta[\cdot]) = \sigma(x_\Delta(T)) + \int\limits_{t_0}^{T} g(t, x_\Delta(t), U_\Delta[t])dt. \tag{8.2}$$

Definition II.4. The uniform limit of Euler solutions $x_\Delta(\cdot)$ as $\operatorname{diam}\Delta = \max\limits_{i\in\overline{1,N}}(\tau_i - \tau_{i-1})$ tends to 0 is the *motion* $x(\cdot) = x(\cdot; t_0, x_0, U) : [t_0, T] \to \mathbb{R}^n$ of system (3.1) starting at an initial point (t_0, x_0) under the feedback $U(t,x)$.

We denote by $\mathrm{Sol}(t_0, x_0, U)$ the set of all motions $x(t)$ of system (3.1) starting at the initial point (t_0, x_0) under the feedback $U(t,x)$.

Remark II.11. The definition of motions $x(\cdot) \in \mathrm{Sol}(t_0, x_0, U)$, the compactness of the set $\mathbf{M}$, and conditions **(A1)**–**(A3)** imply that any motion $x(\cdot) \in \mathrm{Sol}(t_0, x_0, U)$ coincides with a trajectory of system (3.1) generated by a generalized control $\mu(\cdot|du) : [t_0, T] \to \mathrm{rpm}(P)$.

Definition II.5. The *guaranteed result* $\Gamma(t_0, x_0, U)$ for a feedback $U(t,x)$ of the problem **OCP** at an initial point $(t_0, x_0) \in \Pi_T$ is defined as follows:

$$\begin{aligned} \Gamma(t_0, x_0, U) &= \limsup_{\operatorname{diam}\Delta\to 0} I_{t_0,x_0}(x_\Delta(\cdot), U_\Delta[\cdot]) \\ &= \sup_{x(\cdot)\in \mathrm{Sol}(t_0,x_0,U)} I_{t_0,x_0}\big(x(\cdot; t_0, x_0, \mu(\cdot|du)), \mu(\cdot|du)\big), \end{aligned} \tag{8.3}$$

The limit is calculated over all converging sequences of Euler solutions $x_\Delta(\cdot) = x_\Delta(\cdot; t_0, x_0, U)$ paired with corresponding discrete realizations $U_\Delta[\cdot]$. The supremum in (8.3) is calculated over all limits $x(\cdot)$ of sequences paired with generalized controls $\mu(\cdot)$ generating the limit motions.

Let $\varepsilon > 0$ be a small parameter.

Definition II.6. A feedback $U^\varepsilon : [0, T] \times \mathbb{R}^n \to P$ is said to be ε*-optimal* for an initial point $(t_0, x_0) \in \operatorname{cl}\Pi_T$ if it satisfies the following relation:

$$\Gamma(t_0, x_0, U^\varepsilon) = \limsup_{\operatorname{diam}\Delta\to 0} I_{t_0,x_0}\big(x^\varepsilon_\Delta(\cdot), U^\varepsilon_\Delta[\cdot]\big) \le V(t_0, x_0) + \varepsilon, \tag{8.4}$$

where $x^\varepsilon_\Delta(\cdot) = x_\Delta(\cdot; t_0, x_0, U^\varepsilon)$ and $U^\varepsilon_\Delta[\cdot]$ is the corresponding discrete realization of the feedback.

Definition II.7. A feedback $U^0 : [0, T] \times \mathbb{R}^n \to P$ is said to be *optimal* for an initial point $(t_0, x_0) \in \operatorname{cl}\Pi_T$ if it satisfies the following relation:

$$\Gamma(t_0, x_0, U^0) = \limsup_{\operatorname{diam}\Delta\to 0} I_{t_0,x_0}\big(x^0_\Delta(\cdot), U^0_\Delta[\cdot]\big) = V(t_0, x_0), \tag{8.5}$$

where $x^0_\Delta(\cdot) = x_\Delta(\cdot; t_0, x_0, U^0)$ and $U^0_\Delta[\cdot]$ is the corresponding discrete realization of the feedback.

Definition II.8. A feedback $U^0 : [0, T] \times \mathbb{R}^n \to P$ is said to be *universal optimal* in a domain $D \subset \operatorname{cl}\Pi_T$ if it satisfies the relation

$$\Gamma(t_0, x_0, U^0) = \limsup_{\operatorname{diam}\Delta\to 0} I_{t_0,x_0}\big(x^0_\Delta(\cdot), U^0_\Delta[\cdot]\big) = V(t_0, x_0)$$

for all initial points $(t_0, x_0) \in D$, where $x^0_\Delta(\cdot) = x_\Delta(\cdot; t_0, x_0, U^0)$ and $U^0_\Delta[\cdot]$ is the corresponding discrete realization of the feedback.

Recall the following result of the theory of feedback control [135].

Assertion II.4. *Let conditions* **(A1)**–**(A3)** *for the problem* ***OCP*** *hold. Then the optimal program result and the optimal guaranteed result coincide for all* $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, *i.e.*,

$$\begin{aligned} V(t_0, x_0) &= \inf_{u(\cdot)\in \mathbf{U}_{t_0}} I_{t_0,x_0}\big(x(\cdot; t_0, x_0, u(\cdot)), u(\cdot)\big) \\ &= \inf_U \Gamma(t_0, x_0, U) = \inf_U \limsup_{\operatorname{diam}\Delta\to 0} I_{t_0,x_0}\big(x_\Delta(\cdot; t_0, x_0, U), U_\Delta[\cdot]\big), \end{aligned} \tag{8.6}$$

where the last infimum is calculated over the set of all feedbacks $U = U(t, x)$.

8.2. Classical dynamical programming method and continuous optimal synthesis. As is well known (see, e.g., [21, 29, 35, 42, 55, 80, 238, 247], etc.), the value function $V(t, x)$ (see (3.3)) plays a key role in the study and solution of the feedback **OCP**. One of the classical methods of solution of the problem **OCP** is the Bellman dynamical programming method [29]. This method is based on the following properties of the value function (see, e.g., [80]).

Assertion II.5. *Let a function* $V^*(\cdot) : \operatorname{cl}\Pi_T \mapsto \mathbb{R}$ *be a classical solution of the following boundary-value Cauchy problem*:

$$V^*(T, x) = \sigma(x), \quad x \in \mathbb{R}^n, \tag{8.7}$$

for the Bellman equation

$$\frac{\partial V^*(t,x)}{\partial t} + \min_{u\in P}\left[\left\langle \frac{\partial V^*(t,x)}{\partial x}, f(t,x,u)\right\rangle + g(t,x,u)\right] = 0 \tag{8.8}$$

in the strip $(t, x) \in \Pi_T$. *Let a feedback* $(t, x) \mapsto U^0(t, x) : \Pi_T \to P$ *have the form*

$$U^0(t,x) = \operatorname{Arg}\min_{u\in P}\left\{\left\langle \frac{\partial V^*(t,x)}{\partial x}, f(t,x,u)\right\rangle + g(t,x,u)\right\}. \tag{8.9}$$

Let it be single-valued and continuous. Then trajectories $x^0(\cdot) = x^0(\cdot, t_0, x_0, U^0)$ *of the closed-loop system*

$$\begin{aligned} \dot{x}^0(t) &= f\big(t, x^0(t), U^0(t, x^0(t))\big), \quad t \in [t_0, T], \\ x^0(t_0) &= x_0 \end{aligned} \tag{8.10}$$

and the corresponding realizations $u^0(t) = U^0(t, x^0(t))$ *of the feedback* $U^0(t, x)$ (8.9) *satisfy the relation*

$$I_{t_0,x_0}\big(x^0(\cdot, t_0, x_0, U^0), u^0(\cdot)\big) = V^*(t_0, x_0) = V(t_0, x_0). \tag{8.11}$$

for all $(t_0, x_0) \in \operatorname{cl}\Pi_T$.

Therefore, in the classical dynamical programming method, the differentiability of the value function and the continuity of the optimal feedback $U^0(t, x)$ (8.9) are assumed.

It is well known [213] that applications of the classical dynamical programming method are restricted to the construction of optimal synthesis for the problem **OCP**. As a rule, the classical solution of the Cauchy problem (8.7), (8.8), which is equal to the value function, does not exist in the whole domain Π_T [140]. The relations of type (8.9) for $U^0(t, x)$ are not defined at singular points of the value function, where it is not differentiable. By the way, the mapping (8.9) often turns out to be multi-valued and lower semicontinuous at regular points, where the classical solution is defined [35, 42].

As was mentioned above in Sec. 5.2, the value function $V(t, x)$ (3.3) coincides with the minimax [238] and/or the viscosity [59] solution $V'(t, x)$ of the Cauchy problem. Therefore, according to Theorem II.4 (see Sec. 5.2), inequalities (5.7) hold at singular points of the function $V'(t, x)$. One can use these inequalities to define a single-valued optimal feedback $U^0(t, x)$ everywhere in the domain Π_T.

8.3. Necessary and sufficient optimality conditions for feedbacks. Note that $U^0(t,x)$ (8.9) is a *universal* feedback or *optimal synthesis* for the problem **OCP** for any initial point in the strip $D \subset [0,T] \times \mathbb{R}^n$.

If a system is under the influence of a control and a disturbance, the situation can be considered as a problem of the theory of differential games. It is proved in this theory that, as a rule, optimal syntheses are impossible to construct, and the corresponding examples exist [135, 252]. However, for any given accuracy $\varepsilon > 0$, universal ε-optimal feedbacks always exist and guarantee this accuracy at any initial point in the given domain uniformly. One can find discussions and examples of constructions of universal ε-optimal feedbacks in [123, 126, 135].

If universal optimal feedbacks exist in a differential-game problem, then this structure and values are close to arguments in (8.9) almost everywhere in the set of regular points of the value function (see Assertion II.6 below and [135, 252]).

Recall the necessary conditions for optimal synthesis, which, in general, is discontinuous. These conditions are obtained in the theory of differential games [135, 252] and are modified below for the optimal control problem **OCP**. As was mentioned in Sec. 5.1, the problem **OCP** can be interpreted as a differential game with a fictitious second player.

Assertion II.6. *Let Ω and Φ be open domains in $[0,T] \times \mathbb{R}^n$. Let a feedback $U^0(t,x)$ be universal optimal in the domain Ω for the problem **OCP**. Assume also that the value function $V(t,x)$ is continuously differentiable in the domain Φ and the inclusion $\Omega \supset \Phi$ holds.*

Then for any small $\varepsilon > 0$, there is a set Φ_ε everywhere dense in Φ such that at any point $(t,x) \in \Phi_\varepsilon$, we have

$$\left\langle \frac{\partial V(t,x)}{\partial x}, f\big(t,x,U^0(t,x)\big) \right\rangle + g\big(t,x,U^0(t,x)\big) \le \min_{u \in P} \left[\left\langle \frac{\partial V(t,x)}{\partial x}, f(t,x,u) \right\rangle + g(t,x,u) \right] + \varepsilon. \quad (8.12)$$

This result and also relations (8.9) serve as a guideline of optimal synthesis $U^0(t,x)$ in the case where the value function $V(t,x)$ is locally Lipschitz continuous.

Introduce the following objects.

Let $D \in \operatorname{cl} \Pi_T$ be a given compact set. Consider points (t_*, x_*), where $t_* \in [t_0, T]$ and $x_* = x(t_*)$ are points on trajectories $x(\cdot) \in \mathrm{Sol}(t_0, x_0)$, $(t_0, x_0) \in D$ (see Sec. 3.3).

Remark II.12. For any compact set $D \in \operatorname{cl} \Pi_T$, Definitions II.3–II.4 and consitions **(A1)** and **(A2)** imply that all points

$$\begin{gathered} (t_*, x_*, z_*): \quad t_* \in [t_0, T], \quad x_* = x_*(t_*), \quad z_* = z_*(t_*), x_*(t_0) = x_0, \quad z_*(t_0) = 0, \\ z_*(t_*) = \int_{t_0}^{t_*} \int_P g(t, x_*(t), u) \mu_*(t|du)dt, \quad x(\cdot) \in \mathrm{Sol}(t_0, x_0), \end{gathered} \quad (8.13)$$

are contained in a compact set $Q^*_{n+2}(D) \ni (t_*, x_*, z_*)$, $Q^*_{n+2}(D) \subset \mathbb{R}^{n+2}$ (see Lemma II.10). Denote the projection of the compact set $Q^*_{n+2}(D)$ to the space $\mathbb{R}^{n+1}$ by $Q^*(D)$.

For any given number $\delta > 0$, introduce the average speed

$$\frac{x_*(t_* + \delta) - x(t_*)}{\delta}, \qquad \frac{1}{\delta} \int_{t_*}^{t_*+\delta} \int_P g(t, x_*(t), u) \mu_*(t|du)dt$$

and their limits along trajectories $x_*(\cdot) \in \mathrm{Sol}(t_*, x_*)$:

$$\tilde{f}(t_*, x_* | x_*(\cdot)) = \lim_{\delta \to 0} \frac{x_*(t_* + \delta) - x(t_*)}{\delta},$$

$$\tilde{g}(t_*, x_*|x_*(\cdot)) = \lim_{\delta\to 0} \frac{1}{\delta} \int_{t_*}^{t_*+\delta} \int_P g(t, x_*(t), u)\mu_*(t|du)dt.$$

On the compact set $Q^*(D) \ni (t_*, x_*)$, we define the following quantities (*moduli of inertiality*):

$$\beta_f(\delta, t_*, x_*) = \sup_{x_*(\cdot)\in \mathrm{Sol}(t_*,x_*)} \left\| \tilde{f}\big(t_*, x_*|x_*(\cdot)\big) - \frac{x_*(t_*+\delta) - x(t_*)}{\delta} \right\|,$$

$$\beta_g(\delta, t_*, x_*) = \sup_{x_*(\cdot)\in \mathrm{Sol}(t_*,x_*)} \left\| \tilde{g}\big(t_*, x_*|x_*(\cdot)\big) - \frac{1}{\delta} \int_{t_*}^{t_*+\delta} \int_P g\big(t, x_*(t), u\big)\mu_*(t|du)dt \right\|,$$

and

$$\beta_f(\delta, D) = \sup_{(t_*,x_*)\in Q^*(D)} \beta_f(\delta, t_*, x_*), \qquad \beta_g(\delta, D) = \sup_{(t_*,x_*)\in Q^*(D)} \beta_g(\delta, t_*, x_*). \tag{8.14}$$

By the definitions, the quantities $\beta_f(\delta, D) > 0$ and $\beta_g(\delta, D) > 0$ (see (8.14)) satisfy the conditions

$$\beta_f(\delta, D) \downarrow 0, \quad \beta_g(\delta, D) \downarrow 0 \tag{8.15}$$

as $\delta \to 0$.

Using the representative formula (4.2) for the value function,

$$V(t, x) = \min_{\alpha\in \mathbf{A}} \omega(t, x, \alpha)$$

(see Theorem II.1) and using condition (8.15), We prove the following auxiliary fact.

Theorem II.14. *Let conditions* **(A1)**–**(A4)** *for the problem* ***OCP*** *hold. Then at any point* $(t_0, x_0) \in \Pi_T$ *and for any parameter* $\alpha^0 \in A^0(t_0, x_0)$*, where*

$$A^0(t, x) := \big\{\alpha^0 \in A : V(t, x) = \omega(t, x, \alpha^0)\big\}, \tag{8.16}$$

there exists a value of control $u^0 = u(t_0, x_0, \alpha^0) \in P$ *such that*

$$\frac{d\omega(t_0, x_0, \alpha^0)}{(1, f(t_0, x_0, u^0))} + g(t_0, x_0, u^0) = \frac{dV(t_0, x_0)}{(1, f(t_0, x_0, u^0))} + g(t_0, x_0, u^0) = 0. \tag{8.17}$$

Proof. For any initial point $(t_0, x_0) \in \mathrm{cl}\,\Pi_T$, Assertion II.1 for the problem **OCP** implies (see (3.9)) that there is a generalized control $\mu^0(\cdot|du) \in \mathbf{M}_{t_0}$ and the corresponding trajectory $x^0(\cdot) = x^0(\cdot;\, t_0, x_0, \mu^0(\cdot))$ of system (3.6) starting at the point (t_0, x_0) such that

$$I_{t_0,x_0}\Big(x^0\big(\cdot; t_0, x_0, \mu^0(\cdot|du), \mu^0(\cdot|du)\big)\Big) = V(t_0, x_0), \tag{8.18}$$

where the cost functional (3.7) is calculated for this pair. This means that the pair is optimal for the initial point (t_0, x_0).

Use constructions (4.4) and (4.5) from Theorem II.1, the linear transformation (4.7): $[t_0, T] \to [0, 1]$, and the bijective correspondence between generalized controls $\mu(\cdot|du) \in \mathbf{M}_{t_0}$ and elements $\alpha \in A$. Thus, any optimal control $\mu^0(\cdot|du)$ is transformed to an element $\alpha^0 \in A^0(t, x)$ satisfying the relation

$$V(t_0, x_0) = \omega(t_0, x_0, \alpha^0). \tag{8.19}$$

According to the optimality principle, for any moment $t \in [t_0, T]$, the restriction $(x^0_t(\cdot), \mu^0_t(\cdot|du))$: $[t, T] \to \mathbb{R}^n \times \mathrm{rpm}(P)$ of the optimal pair $(x^0(\cdot), \mu^0(\cdot|du)) : [T_0, T] \to \mathbb{R}^n \times \mathrm{rpm}(P)$ satisfies the relation

$$I_{t,x^0(t)}\Big(x^0_t\big(\cdot; t, x^0(t), \mu^0_t(\cdot|du)\big)\Big) = V(t, x^0(t)). \tag{8.20}$$

Let α_t^0 be the element of the compact set A, which is obtained from $\mu_t^0(\cdot|du)$ under the linear transformation (4.7): $[t,T] \to [0,1]$. Similarly to (8.19), we have

$$V(t, x^0(t)) = \omega(t, x^0(t), \alpha_t^0), \quad t \in [t_0, T]. \tag{8.21}$$

For any moment $t \in [t_0, T]$ and $\delta \in (0, T-t)$, Eq. (8.20) implies

$$V(t+\delta, x^0(t+\delta)) = \omega(t+\delta, x^0(t+\delta), \alpha_{t+\delta}^0) = \omega(t, x^0(t), \alpha_t^0) = V(t, x^0(t)), \tag{8.22}$$

where the optimal trajectory $x^0(\cdot)$ has the representation

$$x^0(t+\delta) = x^0(t) + \int\limits_t^{t+\delta} \int\limits_P f(\tau, x^0(\tau), u)\mu^0(\tau|du)d\tau. \tag{8.23}$$

Consider the sets

$$E(t,x) = \{(f(t,x,u), g(t,x,u)) : u \in P\},$$

which are convex and compact according to assumptions **(A1)** and **(A4)**. These properties imply the following relations.

- For any moment $t \in [t_0, T]$ and for almost all $\tau \in (t, t+\delta)$, the set $E(t, x^0(t))$ contains points $(f^0[t,\tau], g^0[t,\tau])$ of the form

$$f^0[t,\tau] = \int\limits_P f(t, x^0(t), u)\mu^0(\tau|du), \quad g^0[t,\tau] = \int\limits_P g(t, x^0(t), u)\mu^0(\tau|du). \tag{8.24}$$

  The *average* vectors $(f_\delta^0(t), g_\delta^0(t)) \in E(t, x^0(t))$, where

$$f_\delta^0(t) = \frac{1}{\delta}\int\limits_t^{t+\delta} f^0[t,\tau]d\tau, \quad g_\delta^0(t) = \frac{1}{\delta}\int\limits_t^{t+\delta} g^0[t,\tau]d\tau. \tag{8.25}$$

  satisfy the inclusion $(f_\delta^0(t), g_\delta^0(t)) \in E(t, x^0(t))$.
- Any *limit* average vector $(f^0(t), g^0(t))$ of the form

$$f^0(t) = \lim_{\delta_k \to 0}\{f_{\delta_k}^0(t)\}, \quad g^0(t) = \lim_{\delta_k \to 0}\{g_{\delta_k}^0(t)\} \tag{8.26}$$

  satisfies the inclusion $(f^0(t), g^0(t)) \in E(t, x^0(t))$. It follows from condition **(A1)** and (8.15) that the following estimates hold:

$$\begin{aligned} \|f_{\delta_k}^0(t) - f^0(t)\| &\le \beta_f(\delta_k, D) + L_1(1+K^0)\delta_k, \\ \|g_{\delta_k}^0(t) - g^0(t)\| &\le \beta_g(\delta_k, D) + L_1(1+K^0)\delta_k, \end{aligned} \tag{8.27}$$

  where $(t_0, x_0) \in D \subset \operatorname{cl}\Pi_T$, D is a compact set, and $(t, x^0(t)) \in Q^*(D)$, and

$$K^0 := \max_{(t,x,u)\in Q^*(D)\times P} \big\{\|f(t,x,u)\|,\ |g(t,x,u)|\big\}.$$

As follows from conditions **(A1)** and **(A2)** and (8.15), estimates (8.27) are uniform with respect to points $(t,x) = (t, x_*^0(t)) \in Q^*(D)$ and trajectories $x_*^0(\cdot) \in \operatorname{Sol}(t_*, x^0(t_*))$, $t_0 \le t_* \le t \le T$.

Properties (8.24)–(8.26) imply that

$$\begin{aligned} x_*^0(t+\delta_k) - x^0(t) &= f^0(t)\cdot\delta_k + h_f(\delta_k), \\ z_*^0(t+\delta_k) - z^0(t) &= g^0(t)\cdot\delta_k + h_g(\delta_k). \end{aligned} \tag{8.28}$$

As follows from assumption **(A4)** on the convexity of $E(t,x)$, there is a control parameter $u_*^0 = u_*^0(t, x, \alpha_t^0) \in P$ such that

$$f^0(t) = f(t, x^0(t), u_*^0), \quad g^0(t) = g(t, x^0(t), u_*^0). \tag{8.29}$$

According to (8.14), (8.24)–(8.26), (8.28), and (8.29), the vector $h(\delta_k) = (h_f(\delta_k), h_g(\delta_k))$ has the form

$$h_f(\delta_k) = [x_*^0(t+\delta_k) - x^0(t)] - f(t, x^0(t), u_*^0)\delta_k,$$
$$h_g(\delta_k) = [z_*^0(t+\delta_k) - z^0(t)] - g(t, x^0(t), u_*^0)\delta_k.$$

Using (8.15), we estimate $\|h(\delta)\|$:

$$\|h(\delta)\| \le \beta_f(\delta, D)\cdot\delta + \beta_g(\delta, D)\cdot\delta. \tag{8.30}$$

In particular, for any point $(t_0, x_0) \in \Pi_T$, it follows from relations (8.22), (8.23), and (8.28)–(8.30) that the locally Lipschitz continuous value function $V(t,x)$ satisfies the required relations (8.17), namely,

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\frac{1}{\delta}\left[V(t_0+\delta, x^0(t_0+\delta)) - V(t_0,x_0) + \int_{t_0}^{t_0+\delta}\int_P g\big(\tau, x^0(\tau), u\big)\mu^0(\tau|du)d\tau\right] \\ &= \lim_{\delta_k\to 0}\frac{V(t_0+\delta_k, x_0 + f(t_0,x_0,u^0)\delta_k) - V(t_0,x_0)}{\delta_k} + g(t_0,x_0,u^0) \\ &= \frac{d\omega(t_0,x_0,\alpha^0)}{(1, f(t_0,x_0,u^0))} + g(t_0,x_0,u^0) = \frac{dV(t_0,x_0)}{(1, f(t_0,x_0,u^0))} + g(t_0,x_0,u^0) = 0. \end{aligned} \tag{8.31}$$

Theorem II.14 is proved. □

Note the following. According to the optimality principle [29, 255], for any point $(t_0, x_0) \in (0,T)\times\mathbb{R}^n$ and any pair consisting of a generalized control $\mu(\cdot|du) \in \mathbf{M}$ and the corresponding trajectory $x(\cdot) = x(\cdot; t_0, x_0,\ \mu(\cdot|du))$, the following inequality holds:

$$V(t_0+\delta, x(t_0+\delta)) \ge V(t_0,x_0) - \int_{t_0}^{t_0+\delta}\int_P g\big(\tau, x(\tau), u\big)\mu(\tau|du)d\tau, \tag{8.32}$$

where

$$x(t_0+\delta) = x_0 + \int_{t_0}^{t_0+\delta}\int_P f\big(\tau, x(\tau), u\big)\mu(\tau|du)d\tau. \tag{8.33}$$

Relations (8.17) and (8.31) in Theorem II.14 and inequalities (8.32) and (8.33) imply the validity of the following assertion.

Corollary II.1. *Let assumptions* **(A1)**–**(A4)** *for the problem* ***OCP*** *hold. Then at any point* $(t,x) \in \operatorname{cl}\Pi_T$*, we have*

$$\min_{u\in P}\min_{\alpha^0\in A^0(t,x)}\frac{d^{\pm}\omega(t,x,\alpha^0)}{(1, f(t,x,u))} + g(t,x,u) = \frac{d\omega(t,x,\alpha_*^0)}{(1, f(t,x,u_*^0))} + g(t,x,u_*^0) = \frac{dV(t,x)}{(1, f(t,x,u_*^0))} + g(t,x,u_*^0), \tag{8.34}$$

where $u_*^0 = u_*^0(t,x,\alpha_*^0)$*.*

The following theorem is the main result of this section.

Theorem II.15. *Let conditions* **(A1)**–**(A4)** *for the problem* ***OCP*** *hold. Then for all* $(t,x) \in \Pi_T = (0,T)\times\mathbb{R}^n$*, universal optimal feedbacks* $U^0(t,x)$ *can be defined by the relation*

$$U^0(t,x) \in \operatorname{Arg}\min_{u\in P}\left\{\frac{d^{\pm}V(t,x)}{(1, f(t,x,u))} + g(t,x,u)\right\}, \tag{8.35}$$

where $V(t,x)$ *(see(3.3)) is the value function of the problem* ***OCP****.*

Proof. According to Theorem II.4 and the definition (8.35) of a feedback $U^0(t,x)$, the equality

$$\frac{d^{\pm}V(t,x)}{\big(1, f(t,x,U^0(t,x))\big)} + g\big(t,x,U^0(t,x)\big) = 0 \tag{8.36}$$

holds for all $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$.

Moreover, it follows from the proof of Theorem II.3 and the definitions of directional Dini semiderivatives $\dfrac{d^{\pm}V(t,x)}{(1,f)}$ that the locally Lipshitz continuous value function is *directional differentiable in the directions* $(1, f(t,x,U^0(t,x))$ *of shift along optimal trajectories.* Hence, for all $(t,x) \in \Pi_T$, we have

$$\frac{dV(t,x)}{\big(1, f(t,x,U^0(t,x))\big)} + g(t,x,U^0(t,x))) = 0. \tag{8.37}$$

Let $D \subset \operatorname{cl}\Pi_T = [0,T] \times \mathbb{R}^n$ be a compact set in $\mathbb{R}^{n+1}$. Fix a point $(t_0,x_0) \in D$ and consider a motion $x(\cdot) \in \operatorname{Sol}(t_0,x_0,U^0)$ starting at an initial point $x_\Delta(t_0) = x_0$ under a feedback $U^0(t,x)$ (8.35). Let us consider Euler polygons $x_\Delta(\cdot)$ approximating this motion $x(\cdot)$. Recall that the Euler polygons are generated under piecewise constant controls $U^0_\Delta(\cdot) : [t_0,T] \to P$, which are defined at nodes τ_j, $j = 0,1,\ldots,N(\Delta)$, of a partition Δ of the interval $[t_0,T]$. The controls have the form

$$U^0_\Delta(t) = U^0(\tau_j, x_\Delta(\tau_j)) \quad \text{for} \quad t \in [\tau_j, \tau_{j+1}). \tag{8.38}$$

Therefore, the controls are realizations of the feedback $U^0(\cdot)$.

Remark II.13. Definitions II.3 and II.4 and conditions **(A1)** and **(A2)** imply that a compact set $Q^*(D)$ contains all $(t, x_\Delta(t))$, where $t \in [t_0,T]$, and $x_\Delta(\cdot)$ are the Euler polygons starting at the points $x_\Delta(t_0) = x_0$ for any initial point $(t_0,x_0) \in D$, any partition Δ of the interval $[t_0,T]$ with $\operatorname{diam}\Delta \le 1$, and any feedback with values in P. The compact set $Q^*(D)$ is the projection of the compact set $Q^*_{n+2}(D) \subset \mathbb{R}^{n+2}$ (see Remark II.12) to space $\mathbb{R}^{n+1}$, where $z(t_0) = 0$.

Let us introduce the following notation:

$$\begin{gathered} Q^* := Q^*(D), \quad K^* = \max_{(t,x,u)\in Q^*\times P} \big\{\|f(t,x,u)\|,\ |g(t,x,u)|\big\}, \\ \big|V(t',x') - V(t'',x'')\big| \le L^*\big\|(t',x') - (t'',x'')\big\| \quad \forall (t',x') \in Q^*, \quad (t'',x'') \in Q^*, \end{gathered} \tag{8.39}$$

where $L^* = L^*(Q^*) > 0$ is the Lipschitz constant for the value function $V(t,x)$ on the compact set Q^*.

Consider the difference

$$0 \le I_{t_0,x_0}(x(\cdot)) - V(t_0,x_0) = \lim_{\operatorname{diam}\Delta\to 0} \left[\sigma\big(x_\Delta(T)\big) + \int_{t_0}^{T} g\big(t, x_\Delta(t), U^0_\Delta(t)\big)dt\right] - V(t_0,x_0) \tag{8.40}$$

and estimate the current differences minimizing in (8.40):

$$\begin{aligned} &\left[\sigma\big(x_\Delta(T)\big) + \int_{t_0}^{T} g\big(t, x_\Delta(t), U^0_\Delta(t)\big)dt\right] - V(t_0,x_0) \\ &\qquad = \sum_{j=0}^{N-1}\left[V\big(\tau_{j+1}, x_\Delta(\tau_{j+1})\big) - V\big(\tau_j, x_\Delta(\tau_j)\big) + \int_{\tau_j}^{\tau_{j+1}} g\Big(t, x_\Delta(t), U^0\big(\tau_j, x_\Delta(\tau_j)\big)\Big)dt\right], \end{aligned} \tag{8.41}$$

where $\tau_N = T$ and $\sigma(x(T)) = V(T, x(T)))$.

Let us introduce the notation

$$x_j = x_\Delta(\tau_j), \quad U_j^0 = U^0(\tau_j, x_j),$$
$$f_j^0 = f\big(\tau_j, x_j, U^0(\tau_j, x_j)\big), \quad g_j^0 = g\big(\tau_j, x_j, U^0(\tau_j, x_j)\big). \tag{8.42}$$

First, let us estimate the term of the sum in Eq. (8.41) and rewrite it in the new notation:

$$0 \le V\left(\tau_{j+1}, x_\Delta(\tau_{j+1})\right) + \int\limits_{\tau_j}^{\tau_{j+1}} g\Big(t, x_\Delta(t), U^0\big(\tau_j, x_\Delta(\tau_j)\big)\Big)dt - V(\tau_j, x_j)$$
$$= \Big[V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big) - V(\tau_j, x_j)\Big] + g_j^0 \cdot (\tau_{j+1} - \tau_j)$$
$$+ \Big[V\big(\tau_{j+1}, x_\Delta(\tau_{j+1})\big) - V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big)\Big]$$
$$+ \left[\int\limits_{\tau_j}^{\tau_{j+1}} g\Big(t, x_\Delta(t), U^0\big(\tau_j, x_\Delta(\tau_j)\big)\Big)dt - g_j^0 \cdot (\tau_{j+1} - \tau_j)\right] = I_j + I_j' + I_j''. \tag{8.43}$$

Using conditions **(A1)** and **(A2)** and notation (8.39), we obtain the following estimates:

$$x_\Delta(\tau_{j+1}) = x_j + \int\limits_{\tau_j}^{\tau_{j+1}} f\big(t, x_\Delta(t), U^0(\tau_j, x_\Delta(\tau_j))\big)dt \le x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j) + \Delta x_j,$$
$$\|\Delta x_j\| \le \frac{L_1(1+K^*)}{2}(\tau_{j+1} - \tau_j)^2, \tag{8.44}$$

$$I_j' = \Big[V\big(\tau_{j+1}, x_\Delta(\tau_{j+1})\big) - V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big)\Big]$$
$$\le L^* \big\|x_\Delta(\tau_{j+1}) - x_j - f_j^0 \cdot (\tau_{j+1} - \tau_j)\big\| \le L^* L_1 \frac{1+K^*}{2}(\tau_{j+1} - \tau_j)^2, \tag{8.45}$$

$$I_j'' = \left[\int\limits_{\tau_j}^{\tau_{j+1}} g\Big(t, x_\Delta(t), U^0\big(\tau_j, x_\Delta(\tau_j)\big)\Big)dt - g_j^0 \cdot (\tau_{j+1} - \tau_j)\right] \le L_1 \frac{1+K^*}{2}(\tau_{j+1} - \tau_j)^2. \tag{8.46}$$

Estimate terms of I_j, which are defined on the right-hand side of relations (8.43). Using pieces of optimal trajectories

$$x_j^0(\cdot) : [\tau_j, \tau_{j+1}] \to \mathbb{R}^n, \quad x_j^0(\cdot) = x(\cdot; \tau_j, x_j, \mu_j^0(\cdot|du)),$$

starting at initial points (τ_j, x_j) under optimal generalized controls $\mu_j^0(\cdot|du) : [\tau_j, T] \to \mathrm{rpm}(P)$, we obtain

$$I_j = \Big[V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big) - V(\tau_j, x_j)\Big] + g_j^0 \cdot (\tau_{j+1} - \tau_j)$$
$$\le \Big[V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big) - V\big(\tau_{j+1}, x_j^0(\tau_{j+1})\big)\Big]$$
$$+ \Big[V\big(\tau_{j+1}, x_j^0(\tau_{j+1})\big) - V(\tau_j, x_j)\Big] + g_j^0 \cdot (\tau_{j+1} - \tau_j)$$
$$= \Big[V\big(\tau_{j+1}, x_j + f_j^0 \cdot (\tau_{j+1} - \tau_j)\big) - V\big(\tau_{j+1}, x_j^0(\tau_{j+1})\big)\Big]$$
$$- \int\limits_{\tau_j}^{\tau_{j+1}} \int\limits_P g\big(\tau, x_j^0(\tau), u\big)\mu_j^0(\tau|du)d\tau + g\big(\tau_j, x_j, U^0(\tau_j, x_j)\big) \cdot (\tau_{j+1} - \tau_j). \tag{8.47}$$

Using the Lipschitz continuity of the value function $V(t,x)$ and the functions $f(\cdot,u)$ and $g(\cdot,u)$ and the uniform boundedness of $f(\cdot)$ and $g(\cdot)$ on Q^* (8.39), we estimate the terms on the right-hand side of inequalities (8.47):

$$\Big[V\big(\tau_{j+1}, x_j + f_j^0\cdot(\tau_{j+1}-\tau_j)\big) - V\big(\tau_{j+1}, x_j^0(\tau_{j+1})\big)\Big]$$
$$\le L^*\left\|\int\limits_{\tau_j}^{\tau_{j+1}} \|f(\tau_j, x_j, U_j^0) - \int\limits_P f\big(\tau, x_j^0(\tau), u\big)\mu_j^0(\tau|du)d\tau\right\|$$
$$\le L^*\left\|f_j^0 - \frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P f(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau\right\|\cdot(\tau_{j+1}-\tau_j)$$
$$+ L^*\int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P \Big\|f(\tau_j, x_j, u) - f\big(\tau_j, x_j^0(\tau), u\big)\Big\|\mu_j^0(\tau|du)d\tau$$
$$\le L^*\left\|f_j^0 - \frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P f(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau\right\|\cdot(\tau_{j+1}-\tau_j)$$
$$+ L^*L_1(1+K^*)\cdot(\tau_{j+1}-\tau_j)^2. \quad (8.48)$$

Similarly, let us estimate the second part of the terms on the right-hand side of inequalities (8.47):

$$\left|g_j^0\cdot(\tau_{j+1}-\tau_j) - \int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P g\big(\tau, x_j^0(\tau), u\big)\mu_j^0(\tau|du)d\tau\right|$$
$$\le \left|g_j^0 - \frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P g(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau\right|\cdot(\tau_{j+1}-\tau_j)$$
$$+ \int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P \big|g(\tau_j, x_j, u) - g(\tau_j, x_j^0(\tau), u)\big|\mu_j^0(\tau|du)d\tau$$
$$\le \left|g_j^0 - \frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_{j+1}}\int\limits_P g(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau\right|\cdot(\tau_{j+1}-\tau_j) + L_1(1+K^*)\cdot(\tau_{j+1}-\tau_j)^2. \quad (8.49)$$

According to the constructions presented in the proof of Theorem II.14, we choose $x_j^0(\cdot)$ and $\mu_j^0(\cdot|du)$ satisfying the relations

$$f_j^0 = f_j^0(\tau_j) = \lim_{\delta_{k_j}\to 0}\frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_j+\delta_{k_j}}\int\limits_P f(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau,$$
$$g_j^0 = g_j^0(\tau_j) = \lim_{\delta_{k_j}\to 0}\frac{1}{\delta_{k_j}}\int\limits_{\tau_j}^{\tau_j+\delta_{k_j}}\int\limits_P g(\tau_j, x_j, u)\mu^0(\tau|du)d\tau.$$

Using relations (8.27) for all points $(\tau_j, x_j) \in Q^*$, we continue estimates (8.48) and (8.49):

$$\left\| f_j^0 - \frac{1}{\delta_{k_j}} \int_{\tau_j}^{\tau_{j+1}} \int_P f(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau \right\| \cdot (\tau_{j+1} - \tau_j)$$
$$\leq \Big[\beta_f\big((\tau_{j+1} - \tau_j), D\big) + L_1(1 + K^*)(\tau_{j+1} - \tau_j)\Big] \cdot (\tau_{j+1} - \tau_j), \quad (8.50)$$

$$\left| g_j^0 - \frac{1}{\delta_{k_j}} \int_{\tau_j}^{\tau_{j+1}} \int_P g(\tau_j, x_j, u)\mu_j^0(\tau|du)d\tau \right| \cdot (\tau_{j+1} - \tau_j)$$
$$\leq \Big[\beta_g\big((\tau_{j+1} - \tau_j), D\big) + L_1(1 + K^0)(\tau_{j+1} - \tau_j)\Big] \cdot (\tau_{j+1} - \tau_j), \quad (8.51)$$

where

$$\beta_f\big((\tau_{j+1} - \tau_j), D\big) \downarrow 0, \quad \beta_g\big((\tau_{j+1} - \tau_j), D\big) \downarrow 0 \quad \text{as} \quad \operatorname{diam}\Delta \to 0.$$

in accordance with (8.15).

Summing the inequalities (8.48)–(8.51), we obtain the final estimate for I_j (8.47):

$$I_j \leq \psi(\tau_{j+1} - \tau_j) \cdot (\tau_{j+1} - \tau_j), \quad (8.52)$$

where $\psi = \psi(\tau_{j+1} - \tau_j)$ is defined by the relation

$$\psi := \left[\frac{5}{2} L_1 \cdot (1 + L^*)(1 + K^*) \cdot (\tau_{j+1} - \tau_j) + \beta_f\big((\tau_{j+1} - \tau_j), D\big) + \beta_g\big((\tau_{j+1} - \tau_j), D\big)\right] \quad (8.53)$$

and $\psi(\tau_{j+1} - \tau_j) \leq \psi(\operatorname{diam}(\Delta))$, $j = 0, 1, \ldots, N-1(\Delta)$.

Using (8.43), (8.45), (8.46), and (8.53), we obtain the final estimate:

$$0 \leq I_{t_0,x_0}(x_\Delta(\cdot)) - V(t_0, x_0) = \sum_{j=0}^{N-1} \Big[V\big(\tau_{j+1}, x_\Delta(\tau_{j+1})\big) - V\big(\tau_j, x_\Delta(\tau_j)\big)$$
$$+ \int_{\tau_j}^{\tau_{j+1}} g\Big(t, x_\Delta(t), U^0\big(\tau_j, x_\Delta(\tau_j)\big)\Big) dt \Big] \leq (T - t_0) \cdot \psi(\operatorname{diam}(\Delta)). \quad (8.54)$$

Obviously, $\psi(\operatorname{diam}(\Delta)) \to 0$ as $\operatorname{diam}(\Delta) \to 0$. Hence

$$0 \leq I_{t_0,x_0}(x_\Delta(\cdot)) - V(t_0, x_0) \to 0 \quad \text{as} \quad \operatorname{diam}(\Delta) \to 0. \quad (8.55)$$

Estimate (8.55) and Definitions II.3 and II.4 imply Theorem II.15. □

Chapter III

GENERALIZATION OF THE METHOD OF CHARACTERISTICS IN THE THEORY OF MINIMAX SOLUTIONS OF SINGULARLY PERTURBED HAMILTON–JACOBI EQUATIONS

As a rule, nonlinear Hamilton-Jacobi equations have no global classical solutions and solutions presented by formulas using *classical* characteristics. Chapters III and IV are devoted to the further development of the method of *generalized* characteristics for the study and construction of global nonsmooth solutions of

Hamilton–Jacobi equations. A new class of singularly perturbed Hamilton–Jacobi equations is considered; these equations are first-order partial differential equations having a small parameter in the *denominators* of some terms containing impulse variables.

In Chap. III, we investigate the problem on the possibility of *singular approximations* for minimax solutions of nonperturbed Hamilton–Jacobi equations, i.e., approximations by minimax solutions of singularly perturbed Hamilton–Jacobi equations considered in the *extended* phase space. Singularly perturbed Hamiltonians have a small parameter in the denominators of coefficients of the corresponding *additional* impulse variables.

Sufficient conditions for the convergence of mimimax solutions of singularly perturbed Hamilton–Jacobi equations are obtained. We prove that the limit of the minimax solutions is the minimax solution of the limit unperturbed Hamilton–Jacobi equation called *asymptotics*. The key condition of the convergence is the existence of attractors in the subspace of singularly perturbed (fast) phase variables of generalized characteristics. The singular approximation can be considered as a development of the reduction techniques of Tikhonov [281] suggested for singularly perturbed dynamics problems.

9. Singularly Perturbed Hamilton–Jacobi Equations

9.1. Statement of the Cauchy problem $\mathbf{P}^\varepsilon$ for singularly perturbed Hamilton–Jacobi equations. We consider two Cauchy problems for Hamilton–Jacobi equations: the unperturbed problem $\mathbf{P}$ and the sinqularly perturbed problem $\mathbf{P}^\varepsilon$.

The unperturbed problem $\mathbf{P}$ in the basic phase space $\mathbb{R}^{n+1}$ with the variables (t, x) has the form

$$\frac{\partial u(t,x)}{\partial t} + H(t, x, D_x u(t,x)) = 0, \quad t \in (0,T), \quad x \in \mathbb{R}^n, \tag{9.1}$$

$$u(T,x) = \sigma(x), \quad x \in \mathbb{R}^n, \tag{9.2}$$

where $D_x u = \left(\dfrac{\partial u}{\partial x_1}, \dots, \dfrac{\partial u}{\partial x_n}\right)$.

The singularly perturbed problem $\mathbf{P}^\varepsilon$ in the augmented phase space $\mathbb{R}^{n+k+1}$ with the variables (t, x, y) has the form

$$\frac{\partial u^\varepsilon(t,x,y)}{\partial t} + H^\varepsilon\left(t, x, y, D_x u^\varepsilon(t,x,y), \frac{1}{\varepsilon} D_y u^\varepsilon(t,x,y)\right) = 0, \quad (t,x,y) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^k, \tag{9.3}$$

$$u^\varepsilon(T,x,y) = \sigma^\varepsilon(x), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^k, \tag{9.4}$$

where $\varepsilon > 0$ is a small parameter. The impulse variables are denoted by the symbols

$$p = D_x u^\varepsilon = \left(\frac{\partial u^\varepsilon}{\partial x_1}, \dots, \frac{\partial u^\varepsilon}{\partial x_n}\right) \in \mathbb{R}^n, \quad q = D_y u^\varepsilon = \left(\frac{\partial u^\varepsilon}{\partial y_1}, \dots, \frac{\partial u^\varepsilon}{\partial y_k}\right) \in \mathbb{R}^k,$$

respectively. Note that a part of the impulse variables, namely, components of the vector q, have coefficients $1/\varepsilon$ in the Hamiltonian $H^\varepsilon(t, x, y, p, \frac{1}{\varepsilon} q)$. The additional phase variables $y = (y_1, \dots, y_k)$ are called *fast variables* and the variables $x = (x_1, \dots, x_n)$ in the basic phase space are called *slow variables.* This terminology will be clarified below.

The results presented in this chapter were obtained in the framework of the theory of minimax solutions of Hamilton–Jacobi equations [235, 236, 238]. It was mentioned in Chap. I that the concepts of *minimax* and *viscosity* solutions are equivalent. It is proved in the theory of viscosity solutions that these solutions can be approximated by using the method of vanishing viscosity. This means that the viscosity solution of Hamilton–Jacobi equations coincides with the limit of smooth solutions of quasi-linear parabolic partial differential equations with a small parameter at the Laplace operator, as the parameter tends to zero. According to the terminology of the theory of perturbations, one can consider the method of vanishing viscosity as the *regular* approximation for viscosity and/or minimax solutions of Hamilton–Jacobi equations (9.1). A small parameter appears in the *numerators* of terms containing senior derivatives in the regularly perturbed Hamilton–Jacobi equations. The regular approximation is considered in the phase space $\mathbb{R}^{n+1}$ of variables (t, x).

Chapter III is devoted to another type of approximations, namely, to the construction of a generalized solution of Hamilton–Jacobi equation (9.1) by using *singular* approximations (9.3) considered in the extended phase space $\mathbb{R}^{n+k+1}$ with the variables (t, x, y).

9.2. Minimax solution of the problem $\mathbf{P}^\varepsilon$. Definition III.2 of the minimax solution of singularly perturbed Hamilton–Jacobi equation suggested below differs from the canonical definition (see Sec. 2.3, Definition I.10, and [235, 236, 238]). It contains a modification taking into account the existence of singular (fast) components of generalized characteristics. Consider the definition.

Fix $\varepsilon > 0$. Let S^ε be a nonempty set and M^ε be a multi-valued mapping $[0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times S^\varepsilon \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$:

$$(t, x, y, s') \mapsto M^\varepsilon(t, x, y, s'). \tag{9.5}$$

The pair $(S^\varepsilon, M^\varepsilon)$ is called the *characteristic complex* (or, briefly, the complex) for the singularly perturbed problem $\mathbf{P}^\varepsilon$ (9.3)–(9.4) if the following conditions hold:

(1°) For any $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k$ and $s' \in S^\varepsilon$, the sets

$$M^\varepsilon(t, x, y, s') = \{(f, h, g)\} \in \operatorname{comp}(\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R})$$

are nonempty, convex, and closed; elements $(f, h, g) \in M^\varepsilon(t, x, y, s')$ satisfy the inequality

$$\|f\| + \|h\| + |g| \le \mu^\varepsilon(1 + \|x\| + \|y\|)$$

for all $s' \in S^\varepsilon$, where $\mu^\varepsilon > 0$ are constants; the multi-valued mappings $(t, x, y) \mapsto M^\varepsilon(t, x, y, s')$ are upper semicontinuous for all $s' \in S^\varepsilon$.

(2°a) For any $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k$ and $(p, q) \in \mathbb{R}^n \times \mathbb{R}^k$, we have

$$\max_{s' \in S^\varepsilon} \min \left\{ \langle f, p \rangle + \frac{1}{\varepsilon}\langle h, q \rangle - g : (f, h, g) \in M^\varepsilon(t, x, y, s') \right\} = H^\varepsilon\left(t, x, y, p, \frac{1}{\varepsilon} q\right). \tag{9.6}$$

(2°b) For any $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k$ and $(p, q) \in \mathbb{R}^n \times \mathbb{R}^k$, we have

$$\min_{s' \in S^\varepsilon} \max \left\{ \langle f, p \rangle + \frac{1}{\varepsilon}\langle h, q \rangle - g : (f, h, g) \in M^\varepsilon(t, x, y, s') \right\} = H^\varepsilon\left(t, x, y, p, \frac{1}{\varepsilon} q\right). \tag{9.7}$$

The set of all characteristic complexes $(S^\varepsilon, M^\varepsilon)$ is denoted by $\mathcal{C}(H^\varepsilon)$.

Pairs $(S^\varepsilon_+, M^\varepsilon_+)$ and $(S^\varepsilon_-, M^\varepsilon_-)$ are called *upper* (respectively, *lower*) characteristic complexes if conditions **(1°)** and **(2°a)** (respectively, conditions **(1°)** and **(2°b)**) hold. The sets of all upper and lower characteristic complexes $(S^\varepsilon_+, M^\varepsilon_+)$ and $(S^\varepsilon_-, M^\varepsilon_-)$ are denoted $\mathcal{C}^\uparrow(H^\varepsilon)$ and $\mathcal{C}^\downarrow(H^\varepsilon)$, respectively.

For any $(S^\varepsilon, M^\varepsilon) \in \mathcal{C}(H^\varepsilon)$ and $s' \in S^\varepsilon$, we denote by $\operatorname{Sol}(t_0, x_0, y_0, z_0, s')$ the set of all absolutely continuous functions $(x(\cdot), y(\cdot), z(\cdot)) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ satisfying the initial condition $(x(t_0), y(t_0), z(t_0)) = (x_0, y_0, z_0)$ and the *characteristic differential inclusion*

$$(\dot{x}(t), \varepsilon\dot{y}(t), \dot{z}(t)) \in M^\varepsilon(t, x(t), y(t), s'). \tag{9.8}$$

Solutions of the characteristic differential inclusions are called *generalized characteristics.* Note that speeds $\dot{y}(t)$ of singular components $y(t)$ of the characteristics have a small coefficient ε on the left-hand side of differential inclusion (9.8).

Definition III.1. A lower semicontinuous function $[0, T] \times \mathbb{R}^n \times \mathbb{R}^k \ni (t, x, y) \mapsto v^\varepsilon(t, x, y) \in \mathbb{R}$ is called an *upper minimax solution* of Hamilton–Jacobi equation (9.3) if the following condition holds: for any

$(t_0, x_0, y_0, z_0) \in \operatorname{epi} v^\varepsilon$ and $s' \in S_+^\varepsilon$, there is a trajectory $(x(\cdot), y(\cdot), z(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s')$ satisfying the inclusion

$$(\tau, x(\tau), y(\tau), z(\tau)) \in \operatorname{epi} v^\varepsilon, \quad \tau \in [t_0, T].$$

An upper semicontinuous function $[0,T] \times \mathbb{R}^n \times \mathbb{R}^k \ni (t, x, y) \mapsto w^\varepsilon(t, x, y) \in \mathbb{R}$, is called a *lower minimax solution* of Hamilton–Jacobi equation (9.3) if the following condition holds: for any $(t_0, x_0, y_0, z_0) \in \operatorname{hypo} w^\varepsilon$ and $s' \in S_-^\varepsilon$, there is a trajectory $(x(\cdot), y(\cdot), z(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s')$ satisfying the inclusion

$$(\tau, x(\tau), y(\tau), z(\tau)) \in \operatorname{hypo} w^\varepsilon, \quad \tau \in [t_0, T].$$

Here, we assume that $(S_+^\varepsilon, M_+^\varepsilon) \in \mathcal{C}^\uparrow(H^\varepsilon)$ and $(S_-^\varepsilon, M_-^\varepsilon) \in \mathcal{C}^\downarrow(H^\varepsilon)$. The symbols $\operatorname{epi} v^\varepsilon$ and $\operatorname{hypo} w^\varepsilon$ denote the epigraph of the function v^ε and the hypograph of the function w^ε, respectively.

Remark III.1. We emphasize that upper and lower minimax solutions are independent of the choice of characteristic complexes $(S_+^\varepsilon, M_+^\varepsilon) \in \mathcal{C}^\uparrow(H^\varepsilon)$ or $(S_-^\varepsilon, M_-^\varepsilon) \in \mathcal{C}^\downarrow(H^\varepsilon)$ in Definition III.1 (see [236, 238]).

Discussions on various characteristic complexes are given in Sec. 10.2 in this chapter and in the next chapter devoted to the Isaacs equations arising in the theory of antagonistic differential games. For example, the well-known differential inclusions defining properties of the u-stability and v-stability of the value function in the theory of differential games [133, 135] can be considered as the characteristic inclusions for the Cauchy problem $\mathbf{P}$ (9.1), (9.2) or $\mathbf{P}^\varepsilon$ (9.3), (9.4) for the corresponding Isaacs equations.

Definition III.2. A continuous function $[0,T] \times \mathbb{R}^n \times \mathbb{R}^k \ni (t, x, y) \mapsto u^\varepsilon(t, x, y) \in \mathbb{R}$ is called a *minimax solution* of Hamilton–Jacobi equation (9.3) if it is an upper and lower minimax solution simultaneously.

Remark III.2. Note that characteristic differential inclusions (2.2) consistent with the Hamiltonian H^ε of the singularly perturbed problem $\mathbf{P}^\varepsilon$ by relations **(2°a)** and **(2°b)** contain a small parameter ε at the speed of the variable $y(t)$. Therefore, the speed has order $1/\varepsilon$ and it increases as $\varepsilon \to 0$. Hence, the singularly perturbed components $y(t)$ of the generalized characteristics $(x(t), y(t), z(t))$ are called *fast variables*. The phase variable $x(t)$ in the basic space $\mathbb{R}^n$ and the component $z(t)$ are called *slow variables*.

10. Conditions for Singular Approximation

10.1. Sufficient conditions for the convergence of generalized solutions. Let us consider the Cauchy problems $\mathbf{P}$ and $\mathbf{P}^\varepsilon$ under the following assumptions.

(A$^\varepsilon$1) The functions $\sigma^\varepsilon(\cdot) : \mathbb{R}^n \to \mathbb{R}$ are continuous and uniformly bounded; moreover, $\sigma^\varepsilon(x) \to \sigma(x)$ as $\varepsilon \to 0$.

(A$^\varepsilon$2) For any $(t, x, y, p, q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k$ and $\varepsilon \in (0, 1]$, the Hamiltonian $H^\varepsilon(t, x, y, p, \frac{1}{\varepsilon} q)$ is continuous and satisfies the estimate

$$\sup_{(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^k} \frac{|H^\varepsilon(t, x, y, 0, 0)|}{|(1 + \|x\| + \|y\|)|} < \infty. \tag{10.1}$$

(A$^\varepsilon$3) The Hamiltonian H^ε satisfies the following Lipschitz conditions with respect to the impulse variables (p, q):

$$\left| H^\varepsilon\left(t, x, y, p', \frac{1}{\varepsilon} q'\right) - H^\varepsilon\left(t, x, y, p'', \frac{1}{\varepsilon} q''\right) \right| \le \lambda^\varepsilon(x, y) \left(\|p' - p''\| + \frac{1}{\varepsilon} \|q' - q''\| \right) \tag{10.2}$$

for any $(t, x, y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, $(p', q'), (p'', q'') \in \mathbb{R}^n \times \mathbb{R}^k$, and $\lambda^\varepsilon(x, y) = \mu^\varepsilon(1 + \|x\| + \|y\|)$, where $\mu^\varepsilon > 0$ is a constant (see Sec. 9.2, condition **(1°)**).

(A$^\varepsilon$4) The Hamiltonian H^ε satisfies the following local Lipschitz conditions with respect to the phase variables (x,y):

$$\sup_{(t',t'',x',x'',y',y'')} \frac{\left| H^\varepsilon\left(t',x',y',p,\dfrac{1}{\varepsilon}q\right) - H^\varepsilon\left(t'',x'',y'',p,\dfrac{1}{\varepsilon}q\right)\right|}{\left(\|x'-x''\|+\|y'-y''\|\right)\left(1+\|p\|+\dfrac{1}{\varepsilon}(1+\|q\|)\right)} < L^\varepsilon \tag{10.3}$$

for any (t',x',y'), $(t'',x'',y'') \in B \subset [0,T]\times\mathbb{R}^n\times\mathbb{R}^k$ and any compact set B, where $L^\varepsilon = L^\varepsilon(B) \in (0,\infty)$ is a constant.

It is known [236, 238] that conditions **(A$^\varepsilon$1)**–**(A$^\varepsilon$4)** guarantee the existence, uniqueness, and equivalence of minimax and viscosity solutions $u^\varepsilon(t,x,y)$ of the problems $\mathbf{P}^\varepsilon$ for any $\varepsilon > 0$. Similar conditions are assumed for the unperturbed problem $\mathbf{P} = \mathbf{P}^0$ below.

To provide the convergence of minimax solutions $u^\varepsilon(t,x,y)$ as $\varepsilon \to 0$, we impose the following conditions on the initial data of the problem.

(A$^\varepsilon$5) For any $\varepsilon > 0$, there exist upper and lower characteristic complexes (S_+, M_+^ε) and (S_-, M_-^ε) such that

- the sets of parameters S_+ and S_- are independent of ε,
- the multi-valued mappings $(t,x,y) \mapsto M_+^\varepsilon(t,x,y,s_+)$ and $(t,x,y) \mapsto M_-^\varepsilon(t,x,y,s_-)$ are Lipshitz continuous in the Hausdorff metric, namely,

$$\operatorname{dist}\left(M_\pm^\varepsilon(t',x',y',s_\pm), M_\pm^\varepsilon(t'',x'',y'',s_\pm)\right) \le r^\varepsilon\left(|t'-t''|+\|x'-x''\|+\|y'-y''\|\right) \tag{10.4}$$

  for all $(t',x',y'),(t'',x'',y'') \in B \subset [0,T]\times\mathbb{R}^n\times\mathbb{R}^k$ and any compact set B, where $r^\varepsilon = r^\varepsilon(B) \in (0,\infty)$ is a constant and $\operatorname{dist}(M^1,M^2)$ is the Hausdorff distance between two sets M^1 and M^2;
- the mappings $\varepsilon \mapsto M_+^\varepsilon$ and $\varepsilon \mapsto M_-^\varepsilon$ are continuously extended onto the interval $\varepsilon \in [0,1]$.

Let us consider the following construction:

$$s_+ \in S_+, \quad Y_+^\varepsilon = Y_+^\varepsilon(t,x,s_+) \subset \{y^0 \in \mathbb{R}^k : \operatorname{proj}_y M_+^\varepsilon(t,x,y^0,s_+) \ni 0\}, \tag{10.5}$$

$$s_- \in S_-, \quad Y_-^\varepsilon = Y_-^\varepsilon(t,x,s_-) \subset \{y_0 \in \mathbb{R}^k : \operatorname{proj}_y M_-^\varepsilon(t,x,y_0,s_-) \ni 0\}, \tag{10.6}$$

where $\operatorname{proj}_y M_\pm^\varepsilon$ is the projection of the set $M_\pm^\varepsilon \subset \mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}$ onto the subspace $\mathbb{R}^k$ of the fast variables y.

Let us assume that the following conditions hold.

(A$^\varepsilon$6) For any $(t,x) \in [0,T]\times\mathbb{R}^n$ and $s_\pm \in S_\pm$, the sets $Y_\pm^\varepsilon$ are nonempty, closed, and *bounded*, i.e.,

$$\forall y \in Y_\pm^\varepsilon(t,x,s_\pm) : \|y\| \le \chi^\varepsilon(1+\|x\|), \tag{10.7}$$

where χ^ε are constants, $\chi^\varepsilon \in (0,\mu^\varepsilon]$. The sets $Y_\pm^\varepsilon$ are continuous with respect to the parameter $\varepsilon \in [0,1]$.

(A$^\varepsilon$7) For any $(t',x'),(t'',x'') \in [0,T]\times\mathbb{R}^n$ and $s_\pm \in S_\pm$, the following Lipschitz condition holds:

$$\operatorname{dist}(Y_\pm^\varepsilon(t',x',s_\pm), Y_\pm^\varepsilon(t'',x'',s_\pm)) \le K^\varepsilon(|t'-t''|+\|x'-x''\|), \tag{10.8}$$

where K^ε is a constant, $K^\varepsilon \in (0,L^\varepsilon]$.

(A$^\varepsilon$8) There are subsets of parameters $\{s_\pm\} = S'_\pm \subset S_\pm$ satisfying the following conditions. For any compact sets $D \subset [0,T]\times\mathbb{R}^n$, $D^0 \subset \mathbb{R}^k$, and $D_0 \subset \mathbb{R}^k$ satisfying the conditions

$$\begin{aligned} D^0 \supset D_+^\varepsilon &= \bigcup_{(t_0,x_0)\in D^1, s_+\in S'_+} Y_+^\varepsilon(t_0,x_0,s_+) + B_k^\varepsilon, \\ D_0 \supset D_-^\varepsilon &= \bigcup_{(t_0,x_0)\in D^1, s_-\in S'_-} Y_-^\varepsilon(t_0,x_0,s_-) + B_k^\varepsilon, \end{aligned} \qquad \forall \varepsilon \in [0,1],$$

there exist numbers $C^\varepsilon = C^\varepsilon(D,D^0,D_0) > 0$ and $\delta(\varepsilon) > 0$ such that

– $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$;
– for any $(t_0, x_0, y_0) \in D^1 \times (D_0 \cup D^0)$ and any $s_\pm \in S'_\pm$, all generalized characteristics $(x^\varepsilon_\pm(\cdot), y^\varepsilon_\pm(\cdot), z^\varepsilon_\pm(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s_\pm)$ satisfying the inequalities

$$z^\varepsilon_+(t) \geq u^\varepsilon(t, x^\varepsilon_+(t), y^\varepsilon_+(t)), \qquad \forall t \geq t_0,$$
$$z^\varepsilon_-(t) \leq u^\varepsilon(t, x^\varepsilon_-(t), y^\varepsilon_-(t)), \qquad \forall t \geq t_0,$$

possess the following properties:

$$\mathrm{dist}\left(y^\varepsilon_\pm(t), Y^\varepsilon_\pm(t, x^\varepsilon_\pm(t), s_\pm) + B^\varepsilon_k\right) \leq C^\varepsilon \cdot \mathrm{diam}\, D_0 \cup D^0 = d^\varepsilon_0, \quad \forall t \in [t_0, T], \tag{10.9}$$
$$y^\varepsilon_\pm(t) \in Y^\varepsilon_\pm(t, x^\varepsilon_\pm(t), s_\pm) + B^\varepsilon_k, \quad \forall t \in [t_0 + \delta(\varepsilon), T], \tag{10.10}$$

where

$$B^\varepsilon_k = \{y \in \mathbb{R}^k : \|y\| \leq \varepsilon\}, \quad D^1 = D + B^1_{n+1}, \quad B^1_{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \|(t, x)\| \leq 1\}.$$

(A$^\varepsilon$9) All constants in **(A$^\varepsilon$1)**–**(A$^\varepsilon$8)** and the regular part of the Hamiltonian $H^\varepsilon(t, x, y, p, 0)$ are continuous with respect to ε and are continuously extended onto the interval $[0, 1]$.

Introduce the "upper" H^ε_+ and "lower" H^ε_- Hamiltonians:

$$H^\varepsilon_+(t, x, p) = \max_{s_+ \in S'_+} \min \Big\{ \langle f, p \rangle - r : (f, r) \in \mathrm{co}\, \mathrm{proj}_{x,z}\, M^\varepsilon_+\big(t, x, Y^\varepsilon_+(t, x, s_+), s_+\big) \Big\}, \tag{10.11}$$
$$H^\varepsilon_-(t, x, p) = \max_{s_- \in S'_-} \min \Big\{ \langle f, p \rangle - r : (f, r) \in \mathrm{co}\, \mathrm{proj}_{x,z}\, M^\varepsilon_-\big(t, x, Y^\varepsilon_-(t, x, s_-), s_-\big) \Big\}, \tag{10.12}$$

where $\mathrm{proj}_{x,z}\, M^\varepsilon_\pm$ are the projections of the sets $M^\varepsilon_\pm$ defined in the space of variables $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ onto the subspace $\mathbb{R}^n \times \mathbb{R}$ of the variables (x, z).

(A$^\varepsilon$10) Let the inequality

$$|H^\varepsilon_-(t, x, p) - H^\varepsilon_+(t, x, p)| \leq \alpha(\varepsilon) \tag{10.13}$$

hold for any $(t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $\varepsilon \in (0, 1]$ and let $\alpha(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Introduce the notation

$$H^0(t, x, p) = \lim_{\varepsilon \downarrow 0} H^\varepsilon_+(t, x, p) = \lim_{\varepsilon \downarrow 0} H^\varepsilon_-(t, x, p); \tag{10.14}$$

$H^0(t, x, p)$ will play the role of the Hamiltonian in the limit unperturbed problem $\mathbf{P}^0$ called the asymptotics:

$$\frac{\partial u(t, x)}{\partial t} + H^0(t, x, D_x u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \tag{10.15}$$
$$u(T, x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{10.16}$$

(A$^\varepsilon$11) Assume that for any $(t, x) \in [0, T] \times \mathbb{R}^n$ and $s_+ \in S'_+$ and $s_- \in S'_-$, the following condition holds:

$$Y^0_+(t, x, s_+) \cap Y^0_-(t, x, s_-) \neq \emptyset. \tag{10.17}$$

The main result of Chap. III is the following sufficient conditions for the convergence of $u^\varepsilon(t, x, y)$ to $u(t, x)$ as $\varepsilon \to 0$.

Theorem III.1. *Let conditions* **(A$^\varepsilon$1)**–**(A$^\varepsilon$11)** *for the singularly perturbed Cauchy problem* $\mathbf{P}^\varepsilon$ (9.3)–(9.4), $\varepsilon \in (0, 1]$, *hold. Then the minimax solutions* $u^\varepsilon(t, x, y)$ *of the problem converge to the minimax solution* $u(t, x)$ *of the unperturbed Cauchy problem* $\mathbf{P}^0$ (10.15)–(10.16) *as* $\varepsilon \to 0$:

$$u(t, x) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x, y), \quad (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k. \tag{10.18}$$

The convergence is uniform on any compact set $D \times (D^0 \cup D_0) : D \in [0, T] \times \mathbb{R}^n,\ D^0 \cup D_0 \in \mathbb{R}^k$.

10.2. Comments. We make the following remarks on the sufficient conditions $(\mathbf{A}^\varepsilon\mathbf{1})$–$(\mathbf{A}^\varepsilon\mathbf{11})$.

Remark III.3. Assumption $(\mathbf{A}^\varepsilon\mathbf{5})$ on the existence of the Lipschitz continuous, multi-valued mappings $(t,x,y) \mapsto M^\varepsilon(t,x,y,s')$ determining characteristic complexes is not exotic under standard requirements $(\mathbf{A}^\varepsilon\mathbf{3})$ and $(\mathbf{A}^\varepsilon\mathbf{4})$. One can also reject the requirement that input data of the considered problem be Lipschitz continuous with respect to t. In addition, it is not necessary for M^ε and Y^ε to be Lipschiz continuous in t. To show the validity of these remarks, one can consider the following complexes: the sets of parameters are

$$S_\pm = \mathbb{R}^n \times \mathbb{R}^k \ni s_\pm = (p,q)$$

and the multi-valued mappings

$$M^\varepsilon(t,x,y,s_\pm) = \Big\{(f,h,g) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} : \\ \|f\| \le \lambda^\varepsilon(x,y),\ \|h\| \le \lambda^\varepsilon(x,y),\ g = \langle f,p\rangle + \frac{1}{\varepsilon}\langle h,q\rangle - H^\varepsilon(t,x,y,p,q)\Big\}, \quad (10.19)$$

where $\lambda^\varepsilon(x,y) = \mu^\varepsilon(1+\|x\|+\|y\|)$ (see condition $(\mathbf{A}^\varepsilon\mathbf{3})$).

Remark III.4. According to assumptions $(\mathbf{A}^\varepsilon\mathbf{5})$ and $(\mathbf{A}^\varepsilon\mathbf{9})$, the regular parts of input data of the problem $\mathbf{P}^\varepsilon$ are continuous with respect to the parameter ε. This implies the convergence of the following sets in the Hausdorff metric as $\varepsilon \downarrow 0$:

$$\hat{M}^\varepsilon_+(t,x,s_+) = \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_+(t,x,Y^\varepsilon_+(t,x,s_+),s_+) \mapsto \hat{M}^0_+(t,x,s_+), \quad (10.20)$$

$$\hat{M}^\varepsilon_-(t,x,s_-) = \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_-(t,x,Y^\varepsilon_-(t,x,s_-),s_-) \mapsto \hat{M}^0_-(t,x,s_-). \quad (10.21)$$

The convergence takes place for all $(t,x) \in [0,T]\times\mathbb{R}^n$ and $s_\pm \in S'_\pm$.

Remark III.4 and assumption $(\mathbf{A}^\varepsilon\mathbf{7})$ imply that the multi-valued mappings

$$(t,x) \mapsto \hat{M}^0_\pm(t,x,s_\pm)$$

are convex and compact for all $s_\pm \in S'_\pm$; they also satisfy the Lipschitz condition with constant

$$L^0 = \lim_{\varepsilon\downarrow o} L^\varepsilon(1+K^\varepsilon) > 0.$$

It follows from condition $(\mathbf{A}^\varepsilon\mathbf{10})$ that the complexes $(S'_+,\hat{M}^0_+)$ and $(S'_-,\hat{M}^0_-)$ are the upper and lower characteristic complexes for the limit unperturbed Cauchy problem $\mathbf{P}^0$ (10.15), (10.16), respectively, where the Hamiltonian $H^0(t,x,p)$ has the form

$$H^0(t,x,p) = \max_{s_+\in S'_+} \min\big\{\langle f,p\rangle - g : (f,g) \in \hat{M}^0_+(t,x,s_+)\big\} \\ = \min_{s_-\in S'_-} \max\big\{\langle f,p\rangle - g : (f,g) \in \hat{M}^0_-(t,x,s_-)\big\}. \quad (10.22)$$

Remark III.5. According to the definitions of upper and lower characteristic complexes and (10.19), we can set in conditions $(\mathbf{A}^\varepsilon\mathbf{8})$–$(\mathbf{A}^\varepsilon\mathbf{10})$:

- $S'_+ = \{(p,0) = s_+\} \in \mathbb{R}^n \times \mathbb{R}^k$,

$$\hat{M}^\varepsilon_+ = \hat{M}^\varepsilon_+(t,x,s_+) = \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_+(t,x,Y^\varepsilon_+(t,x,(p,0)),(p,0)) \\ = \Big\{(f,g) \in \mathbb{R}^n\times\mathbb{R}^k : \|f\| \le \mu^\varepsilon(1+\chi^\varepsilon)(1+\|x\|), \\ g \in \langle f,p\rangle - \operatorname{co} H^\varepsilon\big(t,x,Y^\varepsilon_+(t,x,(p,0)),p,0\big)\Big\}$$

as an upper characteristic complex for the upper Hamiltonian $H^\varepsilon_+(t,x,p)$,

- $S'_- = \{(p,0) = s_-\} \in \mathbb{R}^n \times \mathbb{R}^k$,

$$\begin{aligned}\hat{M}^\varepsilon_- = \hat{M}^\varepsilon_-(t,x,s_-) &= \operatorname{co} pr_{x,z} M^\varepsilon_-(t,x,Y^\varepsilon_-(t,x,(p,0)),(p,0)) \\ &= \Big\{(f,g) \in \mathbb{R}^n \times \mathbb{R}^k : \|f\| \le \mu^\varepsilon(1+\chi^\varepsilon)(1+\|x\|), \\ &\qquad g \in \langle f,p\rangle - \operatorname{co} H^\varepsilon\big(t,x,Y^\varepsilon_-(t,x,(p,0)),p,0\big)\Big\}\end{aligned}$$

as a lower characteristic complex for the lower Hamiltonian $H^\varepsilon_-(t,x,p)$ (10.12).

Remark III.6. For the Hamiltonian $H^0(t,x,p)$ (10.14), we can consider the complexes

- $S'_+ = \{(p,0) = s_+\} \in \mathbb{R}^n \times \mathbb{R}^k$,

$$\begin{aligned}\hat{M}^0_+ = \hat{M}^0_+(t,x,s_+) &= \operatorname{co}\operatorname{proj}_{x,z} M^0_+\big(t,x,Y^0_+(t,x,(p,0)),(p,0)\big) \\ &= \Big\{(f,g) \in \mathbb{R}^n \times \mathbb{R}^k : \|f\| \le \mu^0(1+\chi^0)(1+\|x\|), \\ &\qquad g \in \langle f,p\rangle - \operatorname{co} H^0\big(t,x,Y^0_+(t,x,(p,0)),p,0\big)\Big\};\end{aligned}$$

- $S'_- = \{(p,0) = s_-\} \in \mathbb{R}^n \times \mathbb{R}^k$,

$$\begin{aligned}\hat{M}^0_- = \hat{M}^0_-(t,x,s_-) &= \operatorname{co}\operatorname{proj}_{x,z} M^0_-\big(t,x,Y^0_-(t,x,(p,0)),(p,0)\big) \\ &= \Big\{(f,g) \in \mathbb{R}^n \times \mathbb{R}^k : \|f\| \le \mu^0(1+\chi^0)(1+\|x\|), \\ &\qquad g \in \langle f,p\rangle - \operatorname{co} H^0\big(t,x,Y^0_-(t,x,(p,0)),p,0\big)\Big\}.\end{aligned}$$

In this case, the well-known properties of characteristic complexes of the form (10.19) [236], Remark III.5, and conditions **(A$^\varepsilon$7)**–**(A$^\varepsilon$10)** imply the validity of condition **(A$^\varepsilon$11)**.

Taking into account the form of characteristic complexes, it is possible to present the following form of the limit unperturbed Hamiltonian H^0 (10.22):

$$H^0(t,x,p) = \max_{s_+ \in S'_+} \min_{y^* \in Y^0(t,x,s_+)} H^0(t,x,y^*,p,0) = \min_{s_- \in S'_-} \max_{y_* \in Y^0(t,x,s_-)} H^0(t,x,y_*,p,0), \tag{10.23}$$

where $H^0(t,x,y,p,0) = \lim\limits_{\varepsilon \downarrow 0} H^\varepsilon(t,x,y,p,0)$.

11. Proof of Sufficient Conditions for the Convergence

11.1. Preliminaries. The following fact of the theory of differential inclusions (see [14, 36, 75]) will be useful to prove Theorem III.1 below.

Let

$$[t_0,T] \times \mathbb{R}^n \times \mathbb{R} \mapsto 2^{\mathbb{R}^n \times \mathbb{R}} : (t,x,z) \mapsto F_i(t,x,z) \subset \mathbb{R}^n \times \mathbb{R}, \quad i = 1,2,$$

be two upper semicontinuous, multi-valued mappings with convex, compact, nonempty value sets. Fix $x_i^0 \in \mathbb{R}^n$ and $z_i^0 \in \mathbb{R}$, $i = 1,2$, and consider the differential inclusions

$$\begin{gathered}(\dot{x}_i(t), \dot{z}_i(t)) \in F_i(t, x_i(t), z_i(t)), \quad t \in [t_0,T], \\ (x_i(t_0), z_i(t_0)) = (x_i^0, z_i^0), \quad i = 1,2.\end{gathered} \tag{11.1}$$

The set of all solutions $(x_i(\cdot), z_i(\cdot))$ of the ith differential inclusion (11.1) is denoted by $\mathrm{Sol}_i(t_0, x_i^0, z_i^0)$. The following assertion holds (see [75]).

Assertion III.1. *For any solution* $(x_1(\cdot), z_1(\cdot)) \in \mathrm{Sol}_1(t_0, x_1^0, z_1^0)$*, there is a solution* $(x_2(\cdot), z_2(\cdot)) \in \mathrm{Sol}_2(t_0, x_2^0, z_2^0)$ *satisfying the estimate*

$$\|w_1(t) - w_2(t)\| \le \|w_1(t_0) - w_2(t_0)\| + \int_{t_0}^{t} \operatorname{dist}\Big(F_1\big(\tau, x_1(\tau), z_1(\tau)\big), F_2\big(\tau, x_2(\tau), z_2(\tau)\big)\Big) d\tau \tag{11.2}$$

for all $t \in [t_0, T]$*,* $w = x$*, and* $w = z$*.*

11.2. Proof of Theorem III.1. Note that most of the further constructions will deal with *upper* characteristic complexes and the corresponding attracting sets defined in condition $(\mathbf{A}^\varepsilon\mathbf{8})$. One can obtain similar constructions and theorems to *lower* characteristic complexes after the following replacements:

- subscripts "+" by subscripts "−,"
- inequality signs $\ge$ by $\le$,
- the sets $\operatorname{epi} v$ by the sets $\operatorname{hypo} w$.

Consider compact sets D, D^0, and D_0, complexes $(S'_\pm, M^\varepsilon_\pm)$, and the corresponding generalized characteristics

$$(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s_\pm), \quad s_\pm \in S'_\pm,$$

with initial conditions $(t_0, x_0) \in D$, $y_0 \in D^0$ or $y_0 \in D_0$, $z_0 \in \mathbb{R}$. We also consider also the corresponding attracting sets (*attractors*) $Y^\varepsilon_\pm$, which are defined in condition $(\mathbf{A}^\varepsilon\mathbf{8})$.

Recall that the dynamics of the generalized characteristics $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ are described by the differential inclusions

$$\begin{gathered}\big(\dot x^\varepsilon(t), \varepsilon \dot y^\varepsilon(t), \dot z^\varepsilon(t)\big) \in M^\varepsilon_\pm\big(t, x^\varepsilon(t), y^\varepsilon(t), s_\pm\big),\\ \big(x^\varepsilon(t_0), y^\varepsilon(t_0), z^\varepsilon(t_0)\big) = (x_0, y_0, z_0).\end{gathered} \tag{11.3}$$

Consider a solution $(x^\varepsilon_\pm(\cdot), z^\varepsilon_\pm(\cdot))$ of the differential inclusion

$$\begin{gathered}\big(\dot x^\varepsilon_\pm(t), \dot z^\varepsilon_\pm(t)\big) \in \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_\pm\big(t, x^\varepsilon_\pm(t), Y^\varepsilon_\pm(t, x^\varepsilon_\pm(t), s_\pm), s_\pm\big),\\ \big(x^\varepsilon_\pm(t_0), z^\varepsilon_\pm(t_0)\big) = (x_0, z_0).\end{gathered} \tag{11.4}$$

Let $\varepsilon \in [0,1]$. Denote the set of all solutions

$$\big(x^\varepsilon_\pm(\cdot), z^\varepsilon_\pm(\cdot)\big) : [t_0, T] \mapsto \mathbb{R}^n \times \mathbb{R}$$

of the differential inclusion (11.4) by $\mathrm{Sol}^\varepsilon_\pm(t_0, x_0, z_0, s_\pm)$.

Lemma III.1. *For any compact sets* D*,* D_0*, and* D^0 *introduced in* $(\mathbf{A}^\varepsilon\mathbf{8})$*, there are functions*

$$(0,1] \to \mathbb{R}_+ \times \mathbb{R}_+ : \varepsilon \mapsto (\delta(\varepsilon), \rho(\varepsilon))$$

satisfying the following conditions:

- $\delta(\varepsilon) \downarrow 0$*,* $\rho(\varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$*.*
- *For any* $(t_0, x_0) \in D$*,* $y_0 \in D_0 \cup D^0$*,* $z_0 \in \mathbb{R}$*,* $s' = s_\pm \in S'_\pm$*,* $\varepsilon \in (0, \varepsilon_1]$*, where* $T - t_0 > \alpha(\varepsilon_1)$*, and for all solutions*

  $$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s')$$

  satisfying the inequalities

  $$\begin{aligned} z^\varepsilon(t) &\ge u^\varepsilon(t, x^\varepsilon(t), y^\varepsilon(t)) \quad \text{if} \quad s' = s_+,\\ z^\varepsilon(t) &\le u^\varepsilon(t, x^\varepsilon(t), y^\varepsilon(t)) \quad \text{if} \quad s' = s_-,\end{aligned} \tag{11.5}$$

  there are solutions $(x_\varepsilon(t), z_\varepsilon(t)) = (x^\varepsilon_\pm(t), z^\varepsilon_\pm(t))$

  $$(x^\varepsilon_\pm(\cdot), z^\varepsilon_\pm(\cdot)) \in \mathrm{Sol}^\varepsilon_\pm(t_0, x_0, z_0, s_\pm),$$

satisfying the estimates

$$\|x^\varepsilon(t) - x_\varepsilon(t)\| \le \rho(\varepsilon), \quad |z^\varepsilon(t) - z_\varepsilon(t)| \le \rho(\varepsilon) \tag{11.6}$$

for all $t \in [t_0 + \delta(\varepsilon), T]$.

Proof. Let $(t_0, x_0, y_0) \in D \times D_0$, $z_0 \in \mathbb{R}^1$, $\varepsilon \in (0, 1]$, and $s_+ \in S'_+$. Consider the attractors $Y^\varepsilon_+(t, x, s_+)$ for the fast components of generalized characteristics corresponding to the complexes (S'_+, M^ε_+) (see $(\mathbf{A}^\varepsilon\mathbf{8})$). Choose a characteristic $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s_+)$, $s_+ \in S'_+$, i.e.,

$$\begin{gathered}\big(\dot x^\varepsilon(t), \varepsilon \dot y^\varepsilon(t), \dot z^\varepsilon(t)\big) \in M^\varepsilon_+\big(t, x^\varepsilon(t), y^\varepsilon(t), s_+\big),\\ \big(x^\varepsilon(t_0), y^\varepsilon(t_0), z^\varepsilon(t_0)\big) = (x_0, y_0, z_0).\end{gathered} \tag{11.7}$$

The multi-valued mappings

$$(t, x, y) \mapsto M^\varepsilon_+(t, x, y, s_+), \quad (t, x) \mapsto Y^\varepsilon_+(t, x, s_+) \quad s_+ \in S'_+,$$

are Lipschiz continuous in the Hausdorff metric by conditions $(\mathbf{A}^\varepsilon\mathbf{5})$ and $(\mathbf{A}^\varepsilon\mathbf{7})$. Therefore, the following inclusion holds for all $t \in [t_0, t_0 + \delta(\varepsilon)]$:

$$M^\varepsilon_+\big(t, x^\varepsilon(t), y^\varepsilon(t), s_+\big) \subset M^\varepsilon_+\big(t, x^\varepsilon(t), Y^\varepsilon_+(t, x^\varepsilon(t), s_+), s_+\big) + B^{\rho_1}_{n+k+1}. \tag{11.8}$$

According to conditions $(\mathbf{A}^\varepsilon\mathbf{8})$ (10.9), the quantity ρ_1 in (11.8) satisfies the inequality

$$\rho_1 \le r^\varepsilon \cdot \mathrm{dist}\,\big(y^\varepsilon(t), Y^\varepsilon_+(t, x^\varepsilon(t), s_+)\big) \le r^\varepsilon \cdot d^\varepsilon_0, \tag{11.9}$$

where $r^\varepsilon = r^\varepsilon(B)$, and the set B has the form

$$B = \big(D_1 \times (D_0 \cup D^0) + B^{d^\varepsilon_0}_k\big) \cdot \exp\left(\frac{\mu^\varepsilon T}{1 + d^\varepsilon_0}\right),$$

where B^d_m is the closed ball of radius d in the space $\mathbb{R}^k$.

It follows from (11.8) and (11.9) that the dynamics of the components $x^\varepsilon(\cdot)$ and $z^\varepsilon(\cdot)$ of generalized characteristics (11.7) can be described by the differential inclusion

$$\begin{gathered}\big(\dot x^\varepsilon(t), \dot z^\varepsilon(t)\big) \in \mathrm{co}\,\mathrm{proj}_{x,z}\, M^\varepsilon_+\Big(t, x^\varepsilon(t), Y^\varepsilon_+\big(t, x^\varepsilon(t), s_+\big), s_+\Big) + B^{\rho_1}_{n+1},\\ \big(x^\varepsilon(t_0), z^\varepsilon(t_0)\big) = (x_0, z_0).\end{gathered} \tag{11.10}$$

Now we consider a solution

$$\big(x^\varepsilon_+(\cdot), z^\varepsilon_+(\cdot)\big) \in \mathrm{Sol}^\varepsilon_+(t_0, x_0, z_0, s_+)$$

of the differential inclusion

$$\begin{gathered}\big(\dot x^\varepsilon_+(t), \dot z^\varepsilon_+(t)\big) \in \mathrm{co}\,\mathrm{proj}_{x,z}\, M^\varepsilon_+\Big(t, x^\varepsilon_+(t), Y^\varepsilon_+\big(t, x^\varepsilon_+(t), s_+\big), s_+\Big),\\ \big(x^\varepsilon_+(t_0), z^\varepsilon_+(t_0)\big) = (x_0, z_0).\end{gathered} \tag{11.11}$$

Using conditions $(\mathbf{A}^\varepsilon\mathbf{5})$ and $(\mathbf{A}^\varepsilon\mathbf{7})$ about the Lipschitz continuity in the Hausdorff metrics of the multi-valued mappings $M^\varepsilon_+(\cdot)$ and $Y^\varepsilon_+(\cdot)$, one can obtain the following relations similar to (11.8) and (11.9):

$$\begin{aligned}\hat M^\varepsilon_1(t) = \mathrm{co}\,\mathrm{proj}_{x,z}\, M^\varepsilon_+\Big(t, x^\varepsilon_+(t), Y^\varepsilon_+\big(t, x^\varepsilon_+(t), s_+\big), s_+\Big)&\\ \subset \mathrm{co}\,\mathrm{proj}_{x,z}\, M^\varepsilon_+\Big(t, x^\varepsilon(t), Y^\varepsilon_+\big(t, x^\varepsilon(t), s_+\big), s_+\Big) + B^{\rho_2}_{n+1} = \hat M^\varepsilon_2(t) + B^{\rho_2}_{n+1},&\end{aligned} \tag{11.12}$$

where

$$\begin{aligned}\rho_2 \le L^\varepsilon\Big[\big\|x^\varepsilon(t) - x^\varepsilon_+(t)\big\| + \mathrm{dist}\,\Big(Y^\varepsilon_+\big(t, x^\varepsilon(t), s_+\big), Y^\varepsilon_+\big(t, x^\varepsilon_+(t), s_+\big)\Big)\Big]&\\ \le L^\varepsilon(1 + K^\varepsilon)\big(\|x^\varepsilon(t) - x^\varepsilon_+(t)\|\big).&\end{aligned} \tag{11.13}$$

Now we estimate the distance between a solution $(x^\varepsilon(\cdot), z^\varepsilon(\cdot))$ of differential inclusion (11.10) and solutions $(x^\varepsilon_+(\cdot), z^\varepsilon_+(\cdot))$ of differential inclusion (11.11). For this purpose, we apply Assertion III.1. Taking (11.8), (11.12), and (11.13) into account, we have

$$\begin{gathered}
\big(x_1(\cdot), z_1(\cdot)\big) = \big(x^\varepsilon(\cdot), z^\varepsilon(\cdot)\big), \quad \big(x_2(\cdot), z_2(\cdot)\big) = \big(x^\varepsilon_+(\cdot), z^\varepsilon_+(\cdot)\big),\\
F_1(t,x,z) = \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_+\big(t,x,Y^\varepsilon_+(t,x,s_+),s_+\big) + B^{\rho_1}_{n+1},\\
F_2(t,x,z) = \operatorname{co}\operatorname{proj}_{x,z} M^\varepsilon_+\big(t,x,Y^\varepsilon_+(t,x,s_+),s_+\big),\\
F_1\big(t,x^\varepsilon(t),z^\varepsilon(t)\big) = \hat{M}^\varepsilon_2(t) + B^{\rho_1}_{n+1},\\
F_2\big(t,x^\varepsilon_+(t),z^\varepsilon_+(t)\big) = \hat{M}^\varepsilon_1(t) \subset \hat{M}^\varepsilon_2(t) + B^{\rho_2}_{n+1}
\end{gathered}$$

Hence, we can choose $x^\varepsilon_+(\cdot), z^\varepsilon_+(\cdot) \in \operatorname{Sol}^\varepsilon_+(t_0,x_0,z_0,s_+)$ such that

$$\begin{aligned}
\|x^\varepsilon(t) - x^\varepsilon_+(t)\| &\le \int\limits_{t_0}^{t} \operatorname{dist}\Big(F_1\big(t,x^\varepsilon(t),z^\varepsilon(t)\big), F_2\big(t,x^\varepsilon_+(t),z^\varepsilon_+(t)\big)\Big)d\tau\\
&\le \int\limits_{t_0}^{t} \operatorname{dist}\big(\hat{M}^\varepsilon_2(\tau) + B^{\rho_1}_{n+1}, \hat{M}^\varepsilon_2(\tau) + B^{\rho_2}_{n+1}\big)d\tau \le \int\limits_{t_0}^{t} C_1\big\|x^\varepsilon(\tau) - x^\varepsilon_+(\tau)\big\| + C_2 d\tau,
\end{aligned}$$

where $C_1 > 0$ and $C_2 \ge 0$ are constants.

Applying the Gronwall lemma [20, 300] and inequality (11.9), we see that the following inequality holds on the interval $[t_0, t_0 + \delta(\varepsilon)]$:

$$\big\|x^\varepsilon(t) - x^\varepsilon_+(t)\big\| \le \frac{C_2}{C_1}\Big(e^{C_1\delta(\varepsilon)} - 1\Big) = \varphi(\varepsilon),$$

where $C_1 = L^\varepsilon(1+K^\varepsilon) > 0$, $C_2 = \rho_1 > 0$, and $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence, one can estimate the difference of components z of solutions of differential inclusions (11.10) and (11.11) on the interval $[t_0, t_0 + \delta(\varepsilon)]$ as follows:

$$\begin{aligned}
\big|z^\varepsilon(t) - z^\varepsilon_+(t)\big| &\le \int\limits_{t_0}^{t} \operatorname{dist}\Big(F_1\big(t,x^\varepsilon(t),z^\varepsilon(t)\big), F_2\big(t,x^\varepsilon_+(t),z^\varepsilon_+(t)\big)\Big)d\tau\\
&\le \int\limits_{t_0}^{t} C_1\big\|x^\varepsilon(\tau) - x^\varepsilon_+(\tau)\big\| + C_2 d\tau \le \big(C_1\varphi(\varepsilon) + C_2\big)\cdot\delta(\varepsilon) = \varphi_1(\varepsilon).
\end{aligned}$$

Using condition $(\mathbf{A}^\varepsilon\mathbf{8})$ and (10.10), we obtain that ρ_1 is equal to zero in differential inclusion (11.10) describing the dynamics $x^\varepsilon(t), z^\varepsilon(t)$ on the interval $t \in [t_0 + \delta(\varepsilon), T]$. Now, by Assertion III.1, we can choose a solution $x^\varepsilon_+(t), z^\varepsilon_+(t)$ of differential inclusion (11.11) satisfying the inclusions on the interval $[t_0 + \delta(\varepsilon), T]$:

$$\begin{aligned}
\big\|x^\varepsilon(t) - x^\varepsilon_+(t)\big\| &\le \big\|x^\varepsilon(t_0+\delta(\varepsilon)) - x^\varepsilon_+(t_0+\delta(\varepsilon))\big\|\\
&\quad + \int\limits_{t_0+\delta(\varepsilon)}^{t} \operatorname{dist}\Big(F_1\big(t,x^\varepsilon(t),z^\varepsilon(t)\big), F_2\big(t,x^\varepsilon_+(t),z^\varepsilon_+(t)\big)\Big)d\tau
\end{aligned}$$

$$\leq \left\|x^\varepsilon(t_0+\delta(\varepsilon)) - x_+^\varepsilon(t_0+\delta(\varepsilon))\right\| + \int\limits_{t_0+\delta(\varepsilon)}^{t} \operatorname{dist}\left(\hat M_2^\varepsilon(\tau), \hat M_2^\varepsilon(\tau) + B_{n+1}^{\rho_2}\right) d\tau$$

$$\leq \left\|x^\varepsilon(t_0+\delta(\varepsilon)) - x_+^\varepsilon(t_0+\delta(\varepsilon))\right\| + \int\limits_{t_0+\delta(\varepsilon)}^{t} C_1 \left\|x^\varepsilon(\tau) - x_+^\varepsilon(\tau)\right\| d\tau.$$

Applying the Gronwall lemma, we obtain the estimate

$$\left\|x^\varepsilon(t) - x_+^\varepsilon(t)\right\| \leq \left\|x^\varepsilon(t_0+\delta(\varepsilon)) - x_+^\varepsilon(t_0+\delta(\varepsilon))\right\| \cdot e^{C_1(t-t_0+\delta(\varepsilon))} \leq \varphi(\varepsilon)\cdot e^{C_1 T} = \psi(\varepsilon), \tag{11.14}$$

where $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

A similar difference for components z on the interval $t \in [t_0+\delta(\varepsilon), T]$ can be estimated as follows:

$$\left|z^\varepsilon(t) - z_+^\varepsilon(t)\right| \leq \left|z^\varepsilon(t_0+\delta(\varepsilon)) - z_+^\varepsilon(t_0+\delta(\varepsilon))\right|$$

$$+ \int\limits_{t_0+\delta(\varepsilon)}^{t} \operatorname{dist}\Big(F_1\big(t, x^\varepsilon(t), z^\varepsilon(t)\big), F_2\big(t, x_+^\varepsilon(t), z_+^\varepsilon(t)\big)\Big) d\tau \leq\leq \varphi_1(\varepsilon) + \int\limits_{t_0+\delta(\varepsilon)}^{t} C_1 \left\|x^\varepsilon(\tau) - x_+^\varepsilon(\tau)\right\| d\tau,$$

i.e.,

$$|z^\varepsilon(t) - z_+^\varepsilon(t)| \leq \varphi_1(\varepsilon) + C_1 T \psi(\varepsilon) = \psi_1(\varepsilon), \tag{11.15}$$

where $\psi_1(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

The estimates and the remark above about the validity of similar reasonings and conclusions also for lower characteristic complexes in condition **($\mathbf{A}^\varepsilon$8)** imply that $\rho(\varepsilon)$ in estimate (11.6) can be chosen equal to $\psi_1(\varepsilon)$, which is defined in estimates (11.14) and (11.15). Lemma III.1 is proved. □

Remark III.7. Consider the attainability set $G_\varepsilon(t_0, \tau, x_0, y_0, z_0, s')$ for system (11.7) at the moment $t = \tau$ starting at the initial point (t_0, x_0, y_0, z_0), for $s' \in S'_\pm$:

$$G_\varepsilon\big(t_0, \tau, x_0, y_0, z_0, s'\big) = \Big\{\forall\big(x^\varepsilon(\tau), y^\varepsilon(\tau), z^\varepsilon(\tau)\big) :$$
$$\big(\dot x^\varepsilon(t), \varepsilon \dot y^\varepsilon(t), \dot z^\varepsilon(t)\big) \in M_+^\varepsilon\big(t, x^\varepsilon(t), y^\varepsilon(t), s_+\big),\ t \in [t_0, \tau],$$
$$\big(x^\varepsilon(t_0), y^\varepsilon(t_0), z^\varepsilon(t_0)\big) = (x_0, y_0, z_0)\Big\}.$$

Denote by $\hat G_\varepsilon^\rho(t_0, \tau, x_0, z_0, s')$ the closed ρ-neighborhood of the attainability set $\hat G_\varepsilon$ for system (11.11):

$$\hat G_\varepsilon(t_0, \tau, x_0, z_0, s') = \Big\{\forall\big(x_+^\varepsilon(\tau), z_+^\varepsilon(\tau)\big) :$$
$$\big(\dot x_+^\varepsilon(t), \dot z_+^\varepsilon(t)\big) \in \operatorname{co}\operatorname{proj}_{x,z} M_+^\varepsilon\big(t, x_+^\varepsilon(t), Y_+^\varepsilon\big(t, x_+^\varepsilon(t), s_+\big), s_+\big),\ t \in [t_0, \tau],$$
$$\big(x_+^\varepsilon(t_0), z_+^\varepsilon(t_0)\big) = (x_0, z_0).$$

Then condition (11.6) can be rewritten as follows:

$$\operatorname{proj}_{x,z} G_\varepsilon\big(t_0, \tau, x_0, y_0, z_0, s'\big) \cap \hat G_\varepsilon^\rho\big(t_0, \tau, x_0, z_0, s'\big) \neq \emptyset,$$

where $\rho = \rho(\varepsilon)$ is the same for all $(t_0, x_0) \in D$, $y_0 \in D_0 \cup D^0$, $z_0 \in \mathbb{R}$, and $s' = s_+ \in S'_\pm$; it is defined by the relation $\rho = \rho(\varepsilon) = \psi_1(\varepsilon)$ (see Lemma III.1).

Introduce the functions

$$\begin{aligned} v_\varepsilon^+(t, x, s_+) &= \min_{y \in Y_+^\varepsilon(t,x,s_+) + B_k^\varepsilon} v^\varepsilon(t, x, y), \\ w_\varepsilon^-(t, x, s_-) &= \max_{y \in Y_-^\varepsilon(t,x,s_-) + B_k^\varepsilon} w^\varepsilon(t, x, y), \end{aligned} \tag{11.16}$$

where $v^\varepsilon(t,x,y)$ is an upper minimax solution of the singularly perturbed problem $\mathbf{P}^\varepsilon$ and $w^\varepsilon(t,x,y)$ is a lower minimax solution of the problem $\mathbf{P}^\varepsilon$.

Constructions similar to (11.16) and the appropriate technique of weak limits as $\varepsilon \to 0$ were suggested by Barles and Perthame in [24]. Now, the technique has many applications in the theory of viscosity solutions (see [80]).

Lemma III.2. *For any $(t_0,x_0) \in D$, $y_0 \in D_0 \cup D^0$, $s_\pm \in S'_\pm$, $z^0_+ \geq v^+_\varepsilon(t_0,x_0,s_+)$, $z^0_- \leq w^-_\varepsilon(t_0,x_0,s_-)$, $\tau \in [t_0+\delta(\varepsilon),T]$, $\rho = \rho(\varepsilon)$, where $(\delta(\varepsilon),\rho(\varepsilon))$ are chosen in accordance with Lemma III.1, there are points*

$$(x^*_+,z^*_+) \in \hat{G}^{\rho(\varepsilon)}_\varepsilon(t_0,\tau,x_0,z^0_+,s_+), \quad (x^*_-,z^*_-) \in \hat{G}^{\rho(\varepsilon)}_\varepsilon(t_0,\tau,x_0,z^0_-,s_-)$$

satisfying the inclusions

$$(\tau,x^*_+,z^*_+) \in \operatorname{epi} v^+_\varepsilon, \quad (\tau,x^*_-,z^*_-) \in \operatorname{hypo} w^-_\varepsilon.$$

Proof. Let us fix values of parameters $s_+ \in S_+$ and $s_- \in S_-$ and choose points $y^0_\pm \in Y^\varepsilon_\pm(t_0,x_0,s_\pm) + B^\varepsilon_k$ according to the conditions

$$v^\varepsilon(t_0,x_0,y^0_+) = \min_{y \in Y^\varepsilon_+(t_0,x_0,s_+)+B^\varepsilon_k} v^\varepsilon(t_0,x_0,y) = v^+_\varepsilon(t_0,x_0,s_+),$$

$$w^\varepsilon(t_0,x_0,y^0_-) = \min_{y \in Y^\varepsilon_-(t_0,x_0,s_-)+B^\varepsilon_k} w^\varepsilon(t_0,x_0,y) = w^-_\varepsilon(t_0,x_0,s_-).$$

Assume that $z^0_+ \geq v^+_\varepsilon(t_0,x_0,s_+)$. Therefore, $(t_0,x_0,y^0_+,z^0_+) \in \operatorname{epi} v^\varepsilon$. Similarly, it follows from assumption $z^0_- \leq w^-_\varepsilon(t_0,x_0,s_-)$ that $(t_0,x_0,y^0_-,z^0_-) \in \operatorname{hypo} w^\varepsilon$, where the function v^ε is an upper solution of the problem $\mathbf{P}^\varepsilon$ $(s_+ \in S'_+)$ and the function w^ε is a lower solution to the problem $\mathbf{P}^\varepsilon$ $(s_- \in S'_-)$. Hence, there are points

$$(x^*_\pm,y^*_\pm,z^*_\pm) \in G_\varepsilon(t_0,\tau,x_0,y^0_\pm,z^0_\pm,s_\pm)$$

satisfying the inequalities

$$z^*_+ \geq v^\varepsilon(\tau,x^*_+,y^*_+), \quad z^*_- \leq w^\varepsilon(\tau,x^*_-,y^*_-). \tag{11.17}$$

Estimates (11.14) and (11.15) and Lemma III.1 imply that

$$(x^*_\pm,z^*_\pm) \in \hat{G}^{\rho(\varepsilon)}_\varepsilon(t_0,\tau,x_0,z^0_\pm,s_\pm).$$

According to condition (10.10) in condition **($\mathbf{A}^\varepsilon$8)**, we have

$$y^*_\pm \in Y^\varepsilon_\pm(\tau,x^*_\pm,s_\pm) + B^\varepsilon_k \quad \text{for all} \quad \tau \in [t_0+\delta(\varepsilon),\theta].$$

Hence, by the definitions of the functions $v^+_\varepsilon(\cdot)$ and $w^-_\varepsilon(\cdot)$, we obtain the estimates

$$v^\varepsilon(\tau,x^*_+,y^*_+) \geq v^+_\varepsilon(\tau,x^*_+,s_+), \quad w^\varepsilon(\tau,x^*_-,y^*_-) \leq w^-_\varepsilon(\tau,x^*_-,s_-).$$

These inequalities and (11.17) imply

$$z^*_+ \geq v^+_\varepsilon(\tau,x^*_+,s_+),$$

i.e., $(\tau,x^*_+,z^*_+) \in \operatorname{epi} v^+_\varepsilon$, and

$$z^*_- \leq w^-_\varepsilon(\tau,x^*_-,s_-),$$

i.e., $(\tau,x^*_-,z^*_-) \in \operatorname{hypo} w^-_\varepsilon$. Lemma III.2 is proved. ☐

Lemma III.3. *The function*

$$v^\natural(t,x) = \inf_{s_+ \in S'_+} \liminf_{\substack{\varepsilon \downarrow 0 \\ (t',x') \to (t,x)}} v^+_\varepsilon(t',x,s_+) \tag{11.18}$$

is an upper solution of the problem $\mathbf{P}^0$ and the function

$$w^\natural(t,x) = \sup_{s_- \in S'_-} \limsup_{\substack{\varepsilon \downarrow 0 \\ (t',x') \to (t,x)}} w^-_\varepsilon(t',x',s_-) \tag{11.19}$$

is a lower solution of the problem $\mathbf{P}^0$.

Proof. Let us consider a set $D \in \operatorname{comp}([0,T] \times \mathbb{R}^n)$, $\operatorname{int} D \neq \emptyset$, and sets $D^0 \cup D_0 \subset \operatorname{comp} \mathbb{R}^k$ satisfying condition **($\mathbf{A}^\varepsilon$8)**. Let complexes

$$(S'_+, M^\varepsilon_+) \in \mathcal{C}^\uparrow(H^\varepsilon), \quad (S'_-, M^\varepsilon_-) \in \mathcal{C}^\downarrow(H^\varepsilon) \quad \text{for} \quad \varepsilon > 0,$$
$$(S'_+, \hat{M}^0_+) \in \mathcal{C}^\uparrow(H^0), \quad (S'_-, \hat{M}^0_-) \in \mathcal{C}^\downarrow(H^0)$$

satisfy conditions (10.20) and (10.21) in Remark III.4.

For any $\varepsilon > 0$ and $z^0_\pm \in Z_0 \subset \operatorname{comp} \mathbb{R}$, the proof of Lemma III.1 implies that there are compact sets $\hat{D}_\pm \subset \operatorname{comp} \mathbb{R}^n$ satisfying the following relations:

$$\hat{D}_\pm \supset \operatorname{cl} \Bigg\{ \bigcup_{\substack{s_\pm \in S'_\pm \\ (t_0,x_0) \in D \\ y_0 \in D^0 \cup D_0}} \operatorname{proj}_x G_\varepsilon\big(t_0, T, x_0, y_0, z^0_\pm, s_\pm\big) \Bigg\},$$
$$\hat{D}_\pm \supset \operatorname{cl} \Bigg\{ \bigcup_{\substack{s_\pm \in S'_\pm \\ (t_0,x_0) \in D}} \operatorname{proj}_x \hat{G}_0\big(t_0, T, x_0, z^0_\pm, s_\pm\big) \Bigg\}.$$

According to the definitions of the functions $v^\natural$ and $w^\natural$, these functions are locally bounded:

$$\min_{x \in \hat{D}_+} \sigma(x) \le v^\natural(t,x) \le \max_{x \in \hat{D}_+} \sigma(x), \quad (t,x) \in D,$$
$$\min_{x \in \hat{D}_-} \sigma(x) \le w^\natural(t,x) \le \max_{x \in \hat{D}_-} \sigma(x), \quad (t,x) \in D,$$

and the boundary condition

$$v^\natural(T,x) = w^\natural(T,x) = \sigma(x), \quad x \in \mathbb{R}^n,$$

holds. One can easily prove that the function $v^\natural$ is lower semicontinuous and the function $w^\natural$ is upper semicontinuous.

First, for definiteness, let us prove this lemma for the function $v^\natural$ (11.18).

Choose arbitrarily $(t_0, x_0) \in \operatorname{int} D$, $z^0_+ \ge v^\natural(t_0, x_0)$, $\tau \in (t_0, T]$, and $s_+ \in S'_+$. Show that there is a trajectory

$$(x(\cdot), z(\cdot)) \in \operatorname{Sol}^0_+(t_0, x_0, z^0_+, s_+)$$

of differential inclusion (11.4), where $\varepsilon = 0$, satisfying the inclusion $(\tau, x(\tau), z(\tau)) \in \operatorname{epi} v^\natural$, i.e. (in other words),

$$\{\tau\} \times \hat{G}_0(t_0, \tau, x_0, z^0_+, s_+) \cap \operatorname{epi} v^\natural \neq \emptyset, \tag{11.20}$$

where $\hat{G}_0(\dots, s_+)$ is the attainability set of the characteristic inclusion corresponding to $s_+ \in S'_+$ and M^0_+.

First, we consider the case where $z^0_+ > v^\natural(t_0, x_0)$.

By (11.18), there exist a value of parameter $s'_+ \in S'_+$ and a sequence $\{\varepsilon_k, t_k, x_k\}_{k=1}^\infty$ satisfying the relations

$$z^0_+ \ge v^+_{\varepsilon_k}(t_k, x_k, s'_+) = v^\varepsilon(t_k, x_k, y'_k),$$
$$y'_k \in Y^{\varepsilon_k}_+(t_k, x_k, s'_+) + B^{\varepsilon_k},$$
$$\lim_{k \to \infty} \varepsilon_k = 0, \quad \lim_{k \to \infty} (t_k, x_k) = (t_0, x_0).$$

Assume that $(t_k, x_k) \in \operatorname{int} D$ and $t_k + \delta(\varepsilon) < \tau$ for all $k = 1, 2, \dots$. Hence, $y'_k \in D^0$ for all $k = 1, 2, \dots$.

Using the scheme of the proof and the results of Lemma III.2, one can obtain that for any fixed value of parameter $s_+ \in S_+$, there is a sequence

$$(x^k_+, z^k_+) \in \operatorname{proj}_{x,z} G_{\varepsilon_k}(t_k, \tau, x_k, y'_k, z^0_+, s_+)$$

satisfying the inclusions

$$(x_+^k, z_+^k) \in \hat{G}_{\varepsilon_k}^{\rho^k}(t_k, \tau, x_k, z_+^0, s_+), \quad (\tau, x_+^k, z_+^k) \in \operatorname{epi} v_{\varepsilon_k}^+,$$

where, by Lemma III.1, we have

$$\rho^k = \rho(\varepsilon_k) \to 0 \quad \text{as} \quad k \to \infty$$

uniformly with respect to $s_+ \in S_+$. Without loss of generality, we assume that the limit

$$\lim_{k\to\infty} (x_+^k, z_+^k) = (x_+^*, z_+^*).$$

exists. Note that

$$(x_+^*, z_+^*) \in \hat{G}_0(t_0, \tau, x_0, z_+^0, s_+).$$

Since

$$z_+^k \geq v_{\varepsilon_k}^+(\tau, x_+^k, s_+),$$

the definition of the function $v^\natural$ implies that

$$z_+^* \geq v^\natural(\tau, x_+^*).$$

Therefore, the relation (11.20) is proved for any value of the parameter $s_+ \in S_+$ and $z_+^0 > v^\natural(t_0, x_0)$.

In the case where $z_+^0 \geq v^\natural(t_0, x_0)$, we consider a sequence

$$z_k = z_+^0 + \frac{1}{k}.$$

Since $z_k > v^\natural(t_0, x_0)$, we have

$$\{\tau\} \times \hat{G}_0(t_0, \tau, x_0, z_k, s_+) \cap \operatorname{epi} v^\natural \neq \emptyset, \quad s_+ \in S_+.$$

Let us pass to the limit as $t \to \infty$ taking into account the fact that the set $\operatorname{epi} v^\natural$ is closed and the mapping

$$z \mapsto \hat{G}_0(t_0, \tau, x_0, z, s_+)$$

is upper semicontinuous. Again, we see that relation (11.20) is valid.

Consider $\varepsilon > 0$, a lower solution w^ε of the problem $\mathbf{P}^\varepsilon$, and the definition of the function $w^\natural$. The proof of the second part of the present lemma is similar to the proof of relation (11.20): it is obtained by the replacement of the subscripts "+" by "−," upper complexes by lower complexes, the inequality signs "≥" by "≤," and the sets $\operatorname{epi} v$ by $\operatorname{hypo} w$.

Since the mapping

$$(t_0, x_0) \mapsto \hat{G}_0(t_0, T, x_0, z_\pm^0, s_\pm)$$

is upper semicontinuous and the sets $\hat{G}_0$, $\operatorname{epi} v$, and $\operatorname{hypo} w$ are closed, it follows that relations (11.20) hold for all $(t_0, x_0) \in D$.

Since the set D is arbitrary, Lemma III.3 is proved. □

Let $u^\varepsilon(t, x, y)$, $\varepsilon > 0$, be a minimax solution of the problem $\mathbf{P}^\varepsilon$. We set $v^\varepsilon = w^\varepsilon = u^\varepsilon$ in (11.16), (11.18), and (11.19). Define the corresponding functions $v^\natural$ and $w^\natural$. According to the definitions, we have $w^\natural \geq v^\natural$. On the other hand, since $v^\natural$ is an upper solution and $w^\natural$ is a lower solution of the problem $\mathbf{P}^0$, these solutions satisfy the inequality (see [238]) $w^\natural \leq v^\natural$. Therefore,

$$v^\natural = w^\natural = u,$$

where u is the minimax solution to the problem $\mathbf{P}^0$. (Recall that a minimax solution is defined as a function which is equal to an upper solution and a lower solution, simultaneously.)

Hence, we have obtained the formula

$$u(t,x) = \sup_{s_- \in S_-} \limsup_{\substack{\varepsilon \downarrow 0 \\ (t',x') \to (t,x)}} \max_{y \in Y^\varepsilon_-(t',x',s_-)+B^\varepsilon_k} u^\varepsilon(t',x',y)$$
$$= \inf_{s_+ \in S_+} \liminf_{\substack{\varepsilon \downarrow 0 \\ (t',x') \to (t,x)}} \min_{y \in Y^\varepsilon_-(t',x',s_+)+B^\varepsilon_k} u^\varepsilon(t',x',y).$$

This implies (10.18), i.e.,

$$u(t,x) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y), \quad (t,x) \in D, \quad y \in (D^0 \cup D_0).$$

The estimates obtained in Lemma III.1 allow one to conclude that this convergence is uniform on compact sets $D \subset [0,T] \times \mathbb{R}^n$ and $D_0 \cup D^0 \subset \mathbb{R}^k$ chosen according to $(\mathbf{A}^\varepsilon\mathbf{8})$. Theorem III.1 is proved. □

12. Examples

Example III.1. Consider a singularly perturbed Cauchy problem $\mathbf{P}^\varepsilon$, where the sufficient conditions presented in Theorem III.1 are satisfied. In this problem, the Hamiltonian H^ε has the form

$$H^\varepsilon\left(t,x,y_1,y_2,s,\frac{1}{\varepsilon}\zeta_1,\frac{1}{\varepsilon}\zeta_2\right) = \langle f(t,x,y_1,y_2),s\rangle + g(t,x,y_1,y_2)$$
$$+\frac{1}{\varepsilon}\Big[k_1(y_1)\cdot\zeta_1 + k_2(y_2)\cdot\zeta_2 + \min_{\alpha\in A}(\zeta_1\cdot\alpha) + \max_{\beta\in B}(\zeta_2\cdot\beta)\Big], \tag{12.1}$$

the phase variables are $t \in [0,T]$, $x \in \mathbb{R}^2$, $y = (y_1,y_2) \in \mathbb{R}^2$, the impulse variable are $s \in \mathbb{R}^2$, $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^2$, and

$$k_1(y_1) = \begin{cases} -y_1, & \text{if } y_1 \ge 0, \\ -2y_1, & \text{if } y_1 < 0, \end{cases} \qquad k_2(y_2) = \begin{cases} -4y_2, & \text{if } y_2 \ge 0, \\ -y_2, & \text{if } y_2 < 0. \end{cases}$$

One can consider the differential game $\mathbf{DG}^\varepsilon$, where the Isaacs equation has Hamiltonian (12.1). In this problem, x is the slow variable and y is the fast variable. The dynamics is described as follows:

$$\dot{x} = f(t,x,y_1,y_2), \quad \varepsilon\dot{y}_1 = k_1(y_1) + \alpha, \quad \varepsilon\dot{y}_2 = k_2(y_2) + \beta.$$

The restrictions on values of control parameters $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ have the form

$$\alpha \in A = [\alpha_0,\alpha^0] \ni 0, \quad \beta \in B = [\beta_0,\beta^0] \ni 0,$$

and the cost functional of the Bolza type (3.2) is given:

$$I^\varepsilon_{t_0,x_0,y_0}\big(x(\cdot),y(\cdot),\alpha(\cdot),\beta(\cdot)\big) = \sigma\big(x(T;t_0,x_0,y_0)\big)$$
$$+\int\limits_{t_0}^{T} g\Big(t,x(t),y_1\big(t;t_0,y_{10},\alpha(\cdot)\big),y_2\big(t;t_0,y_{20},\beta(\cdot)\big)\Big)dt.$$

Assume that the functions $f(\cdot)$, $g(\cdot)$, and $\sigma(\cdot)$ are continuous and the functions $f(\cdot)$ and $g(\cdot)$ are *Lipschitz* continuous with respect to the variables t, x, y_1, and y_2 with constant $L_* > 0$. Assume also that the Isaacs condition holds.

The upper and lower characteristic complexes can be chosen as follows:

$$S'_+ = B, \quad s_+ = \beta, \quad S'_- = A, \quad s_- = \alpha,$$
$$M^\varepsilon_+\big(t,x,y_1,y_2,\beta\big) = \mathrm{co}\left\{f\big(t,x,y_1,y_2\big),\ \frac{1}{\varepsilon}[k_1(y_1)+A],\ \frac{1}{\varepsilon}[k_2(y_2)+\beta],\ -g\big(t,x,y_1,y_2\big)\right\},$$
$$M^\varepsilon_-(t,x,y_1,y_2,\alpha) = \mathrm{co}\left\{f\big(t,x,y_1,y_2\big),\frac{1}{\varepsilon}[k_1(y_1)+\alpha],\frac{1}{\varepsilon}[k_2(y_2)+B],\ -g\big(t,x,y_1,y_2\big)\right\}.$$

The corresponding attractors $Y^\varepsilon_+ = Y_+$ and $Y^\varepsilon_- = Y_-$ have the form

$$Y_+ = Y^\varepsilon_+(t,x,\beta) = \left\{ (y_1,y_2) : y_1 \in \bigcup_{\alpha\in A} \xi_1(\alpha)\alpha,\ y_2 = \xi_2(\beta)\beta \right\}, \tag{12.2}$$

$$Y^- = Y^\varepsilon_-(t,x,\alpha) = \left\{ (y_1,y_2) : y_2 \in \bigcup_{\beta\in B} \xi_2(\beta)\beta,\ y_1 = \xi_1(\alpha)\alpha \right\}, \tag{12.3}$$

where

$$\xi_1(\alpha) = \begin{cases} 1, & \text{if } \alpha \ge 0, \\ 1/2, & \text{if } \alpha \le 0, \end{cases} \qquad \xi_2(\beta) = \begin{cases} 1/4, & \text{if } \beta \ge 0, \\ 1, & \text{if } \beta \le 0. \end{cases}$$

Let $r^\varepsilon_\beta[t]$ be the distance between the fast variable $y^\varepsilon_+(t)$ and the attractor $Y^\varepsilon_+(t, x^\varepsilon_+(t), \beta)$ (see (12.2)). Estimate the speed of variation of the function $r^\varepsilon_\beta[t]$, $\beta \in B$, along solutions $y^\varepsilon_+(t) = (y_1(t), y_2(t))$ of the differential inclusions

$$\varepsilon \dot{y}_1(t) \in [k_1(y_1(t)) + A], \quad \varepsilon \dot{y}_2(t) = [k_2(y_2(t)) + \beta].$$

Let $r^\varepsilon_\alpha[t]$ be the distance between the fast variable $y^\varepsilon_-(t)$ and the attractor $Y^\varepsilon_-(t, x^\varepsilon_-(t), \alpha)$ (see (12.3)). Estimate the speed of variation of the function $r^\varepsilon_\alpha[t]$, $\alpha \in A$, along solutions $y^\varepsilon_-(t) = (y_1(t), y_2(t))$ of the differential inclusions

$$\varepsilon \dot{y}_1(t) \in [k_1(y_1(t)) + \alpha], \quad \varepsilon \dot{y}_2(t) = [k_2(y_2(t)) + B].$$

The following values in condition $(\mathbf{A}^\varepsilon\mathbf{8})$ can be chosen:

$$C^\varepsilon = C = 1 + |\alpha^0 - \alpha_0| + |\beta^0 - \beta_0|,$$
$$\delta(\varepsilon) = \varepsilon \cdot \big(1 - \varepsilon \cdot (\operatorname{diam}(D_0 \cup D^0) + C)\big),$$
$$\delta(\varepsilon) < \varepsilon < \min\left\{ \frac{1}{\operatorname{diam}(D_0 \cup D^0) + C},\ T - t_0,\ 1 \right\},$$
$$\delta(\varepsilon) \downarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

According to Theorem III.1, the Hamiltonian H^0 of the limit unperturbed Cauchy problem $\mathbf{P}^0$ can be presented by the formula

$$\begin{aligned} H^0(t,x,s) &= \min_{p\in P} \max_{q\in Q} \Big[\big\langle f(t,x,p,q), s \big\rangle + g(t,x,p,q) \Big] \\ &= \max_{q\in Q} \min_{p\in P} \Big[\big\langle f(t,x,p,q), s \big\rangle + g(t,x,p,q) \Big], \end{aligned} \tag{12.4}$$

where the compact sets P and Q are defined as follows:

$$P = \bigcup_{\alpha\in A} \xi_1(\alpha)\alpha, \quad Q = \bigcup_{\beta\in B} \xi_2(\beta)\beta. \tag{12.5}$$

The upper characteristic complex for the unperturbed problem $\mathbf{P}^0$ can be chosen as follows:

$$s_+ = \xi_2(\beta)\beta = \hat{q}, \quad S_+ = Q,$$
$$M^0_+(t,x,\hat{q}) = \operatorname{co}\Big\{ \big(f(t,x,p,\hat{q}),\ -g(t,x,p,\hat{q})\big) : p \in P \Big\}.$$

The lower characteristic complex for the unperturbed problem $\mathbf{P}^0$ can be chosen as follows:

$$s_- = \xi_1(\alpha)\alpha = \hat{p}, \quad S_- = P,$$
$$M^0_-(t,x,\hat{p}) = \operatorname{co}\Big\{ \big(f(t,x,\hat{p},q),\ -g(t,x,\hat{p},q)\big) : q \in Q \Big\}.$$

The unperturbed problem $\mathbf{P}^0$ can be considered as the Cauchy problem for the Isaacs equation with the Hamiltonian H^0 (12.4) and the boundary condition

$$u(T,x)=\sigma(x),\quad x\in\mathbb{R}^2,$$

corresponding to the following unperturbed differential game $\mathbf{DG}^0$. The unperturbed dynamics is described by the equation

$$\dot{x}=f(t,x,p,q),$$

where $t\in[0,T]$, $x\in\mathbb{R}^2$ are the phase variables, control parameters $(p,q)\in\mathbb{R}\times\mathbb{R}$ are restricted,

$$p\in P,\quad q\in Q,$$

and the compact sets P and Q are defined in (12.5). The limit cost functional has the form

$$I^0_{t_0,x_0}\big(x(\cdot),p(\cdot),q(\cdot)\big)=\sigma\big(x(T;t_0,x_0,p(\cdot),q(\cdot))\big)+\int\limits_{t_0}^{T}g\big(t,x(t),p(t),p(t)\big)dt.$$

Example III.2. Consider a singularly perturbed Cauchy problem $\mathbf{P}^\varepsilon$, where the sufficient conditions presented in Theorem III.1 do not hold (see also [84]). In this problem, the Hamiltonian H^ε has the form

$$\begin{aligned}H^\varepsilon\left(t,x,y_1,y_2,s,\frac{1}{\varepsilon}\zeta_1,\frac{1}{\varepsilon}\zeta_2\right)&=-(y_1)^2\cdot s+\frac{1}{\varepsilon}(\zeta_1\cdot y_2)\\-\frac{1}{\varepsilon}\zeta_2(\omega^2y_1+k\cdot y_2)&+\min_{|u|\le 1}\left[u^2\cdot s+\frac{1}{\varepsilon}(\zeta_2\cdot u)\right];\end{aligned}\tag{12.6}$$

the phase variables are $t\in[0,T]$, $x\in\mathbb{R}^1$, $y=(y_1,y_2)\in\mathbb{R}^2$, and the impulse variables are $s\in\mathbb{R}^1$, $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^2$.

One can consider the optimal control problem $\mathbf{OCP}^\varepsilon$, where the Bellman equation has the Hamiltonian (12.6). In this problem, x is the slow variable and y is the fast variable. The dynamics is described by the system

$$\dot{x}=-(y_1)^2+u^2,\tag{12.7}$$

$$\varepsilon\dot{y_1}=y_2,\quad \varepsilon\dot{y_2}=-\omega^2\cdot y_1-k\cdot y_2+u.\tag{12.8}$$

The restriction on values of control parameter $u\in\mathbb{R}^1$ has the form

$$|u|\le 1.$$

Let the cost functional be of Mayer type, i.e., the integrand in (3.2) equals zero. Hence, the minimized functional is

$$I^\varepsilon_{t_0,x_0,y_0}\big(x(\cdot),y(\cdot),u(\cdot)\big)=\sigma\Big(x\big(T;t_0,x_0,y_0,u(\cdot)\big),y\big(T;t_0,x_0,y_0,u(\cdot)\big)\Big),$$

where the terminal cost function has the form

$$\sigma(x,y)=x,\quad x\in\mathbb{R},\quad y\in\mathbb{R}^2.$$

Let the constants ω and k in (12.8) satisfy the inequalities

$$\omega>1,\quad k\omega<1.$$

The upper complex independent of the parameters s' can be chosen as follows:

$$S'_+=\emptyset,\quad M^\varepsilon_+(t,x,y_1,y_2)=\operatorname{co}\big\{-(y_1)^2+u^2;\ y_2;-\omega^2\cdot y_1-k\cdot y_2+u:u\in P\big\}.\tag{12.9}$$

According to constructions (10.5), the corresponding attractor $Y^\varepsilon_+=Y_+$ is

$$Y_+=Y^\varepsilon_+(t,x,)=\left\{(y_1,y_2):y_2=0;\ y_1\in\frac{[-1,1]}{\omega^2}\right\}.\tag{12.10}$$

Let us consider the admissible control

$$u^\varepsilon(t) = \cos\left(\frac{\omega}{\varepsilon}t\right). \tag{12.11}$$

Consider motions $(y_1^\varepsilon(t), y_2^\varepsilon(t))$ of the fast subsystem (12.8) under control (12.11). One can verify that the periodic motions have the form

$$y_1^\varepsilon(t) = \frac{1}{\omega k}\sin\left(\frac{\omega}{\varepsilon}t\right), \quad y_2^\varepsilon(t) = \frac{1}{k}\cos\left(\frac{\omega}{\varepsilon}t\right).$$

Note that the motions $(x^\varepsilon(t), y_1^\varepsilon(t), y_2^\varepsilon(t))$ of system (12.7), (12.8) under control (12.11) can be considered as upper generalized characteristics of the Bellman equation, where the Hamiltonian has the form (12.6). The motions are also solutions of the upper characteristic differential inclusion corresponding to complex (12.9).

The periodic motion $y_2^\varepsilon(t)$ does not tend to zero as $\varepsilon \to 0$. Hence, the fast components $(y_1^\varepsilon(t), y_2^\varepsilon(t))$ of the upper generalized characteristics do not tend to the attractor Y_+ (12.10). Therefore, relation (10.10) in the sufficient condition $(\mathbf{A}^\varepsilon\mathbf{8})$ does not hold.

Note that advanced constructions of asymptotics for dynamical optimization problems for periodic systems were suggested by Gaitsgory.

Chapter IV

APPLICATIONS OF THE GENERALIZED METHOD OF CHARACTERISTICS TO DIFFERENTIAL GAMES WITH FAST AND SLOW MOTIONS

The results obtained in Chap. III can be applied to many areas of research, in particular, to problems of optimal control and differential games with "fast" (singular) and "slow" (regular) motions. Chapter IV contains applications to differential games.

The key role in the theory of differential games belongs to concepts of the value function of the game [133, 135]. For any initial phase state, the value function defines equilibrium values of optimal guaranteed results for two antagonistic players. The value function is also a basic element in designs of optimal feedbacks. It is known [235] that the value function of a differential game coincides with the minimax solution of the corresponding Hamilton–Jacobi–Isaacs equation.

In this chapter, questions on the convergence of the value functions of singularly perturbed differential games with the Bolza cost functional are studied, as the parameter of singularity tends to zero. This means the convergence as the speeds of fast phase variables tend to infinity. Effective sufficient conditions for convergence of the value functions are obtained. The limit unperturbed differential games (asymptotics) [270] are described, where the value functions coincide with the limit of the value functions of the considered singularly perturbed differential games. To describe the asymptotics, the specific form of the Isaacs equation and the generalized reduction technique (see Chap. III and [193, 281, 295]) are used. Applications of the theory of minimax solutions provide constructions of unperturbed dynamics and the corresponding cost functional for the asymptotics in the subspace of slow (regular) phase variables.

13. Feedback Differential Games $\mathbf{G}^\varepsilon$

Mathematical models of dynamical systems with "fast" and "slow" motions frequently arise in various applied and theoretical problems of mechanics, physics, engineering, biology, economics, etc.

In this chapter, singularly perturbed differential games $\mathbf{G}^\varepsilon$ are considered, where the dynamics of the system under controls of two antagonistic players is described by the following equations:

$$\dot{x} = f^\varepsilon(t, x, y, u, v), \quad \varepsilon\dot{y} = h^\varepsilon(t, x, y, u, v, \alpha, \beta), \tag{13.1}$$

where $\varepsilon > 0$ is a small singularity parameter, $t \in [0,T]$ is time, (x,y) is the phase vector in $\mathbb{R}^n \times \mathbb{R}^k$, x is the *slow* (*regular*) phase variable, and y is the *fast* (*singular*) variable. The words "slow" and "fast" emphasize the difference between dynamics of variables x and y. The speed $\dot{y}$ has order $1/\varepsilon$. Therefore, the variable y can change as fast as desired as $\varepsilon \to 0$.

Assume that the compact sets of admissible values of the first player controls (u,α) and the second player controls (v,β) are known, namely,

$$\begin{aligned} &u \in P \in \operatorname{comp} \mathbb{R}^{m1}, \quad \alpha \in A \in \operatorname{comp} \mathbb{R}^{r1}, \\ &v \in Q \in \operatorname{comp} \mathbb{R}^{m2}, \quad \beta \in B \in \operatorname{comp} \mathbb{R}^{r2}. \end{aligned} \tag{13.2}$$

The terminal time moment T is fixed. Let us consider the Bolza cost functional

$$I^{\varepsilon}_{t,x,y}\big(x(\cdot), y(\cdot), u(\cdot), \alpha(\cdot), v(\cdot), \beta(\cdot)\big) = \sigma^{\varepsilon}(x(T)) + \int_t^T g^{\varepsilon}\big(\tau, x(\tau), y(\tau), u(\tau), v(\tau)\big) d\tau, \tag{13.3}$$

where $(x(\cdot), y(\cdot)) : [t,T] \to \mathbb{R}^n \times \mathbb{R}^k$ are trajectories of Eq. (13.1) starting at a point $(x(t), y(t)) = (x,y)$, $t \in [0,T]$, under measurable controls

$$\big(u(\cdot), v(\cdot), \alpha(\cdot), \beta(\cdot)\big) : [t,T] \to P \times Q \times A \times B.$$

The purpose of the first player is to minimize the cost functional and the second player is interested in maximization of I^{ε}.

For any $\varepsilon > 0$, assume that there is an equilibrium, i.e., there exists the value function of the feedback differential game $\mathbf{G}^{\varepsilon}$ (13.1)–(13.3) [133, 135]:

$$[0,T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} : (t,x,y) \mapsto \operatorname{Val}^{\varepsilon}(t,x,y).$$

The definition of this notion is cited below in Sec. 14.1.

It is known that the value function $\operatorname{Val}^{\varepsilon}(t,x,y)$ of the differential game $\mathbf{G}^{\varepsilon}$ coincides with the minimax (see [235, 236, 238]) and/or viscosity (see [59, 60]) generalized solutions to the following Cauchy problem $\mathbf{P}^{\varepsilon}$:

$$\frac{\partial \operatorname{Val}^{\varepsilon}}{\partial t} + H^{\varepsilon}\left(t, x, y, D_x \operatorname{Val}^{\varepsilon}, \frac{1}{\varepsilon} D_y \operatorname{Val}^{\varepsilon}\right) = 0, \quad (t,x,y) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^k, \tag{13.4}$$

$$\operatorname{Val}^{\varepsilon}(T,x,y) = \sigma^{\varepsilon}(x), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^k, \tag{13.5}$$

where the vectors $p = D_x \operatorname{Val}^{\varepsilon} \in \mathbb{R}^n$ and $q = D_y \operatorname{Val}^{\varepsilon} \in \mathbb{R}^k$ are gradients of the function $\operatorname{Val}^{\varepsilon}(t,x,y)$ with respect to x and y, i.e.,

$$D_x \operatorname{Val}^{\varepsilon} = \left(\frac{\partial \operatorname{Val}^{\varepsilon}}{\partial x_1}, \dots, \frac{\partial \operatorname{Val}^{\varepsilon}}{\partial x_n}\right), \quad D_y \operatorname{Val}^{\varepsilon} = \left(\frac{\partial \operatorname{Val}^{\varepsilon}}{\partial y_1}, \dots, \frac{\partial \operatorname{Val}^{\varepsilon}}{\partial y_k}\right),$$

and $H^{\varepsilon}(t,x,y,p,\frac{1}{\varepsilon}q)$ is the Hamiltonian of the problem.

Note that the Hamiltonian has the following specific form:

$$\begin{aligned} &H^{\varepsilon}(t,x,y,p,\frac{1}{\varepsilon}q) \\ &= \min_{\substack{u\in P \\ \alpha \in A}} \max_{\substack{v \in Q \\ \beta \in B}} \left[\big\langle f^{\varepsilon}(t,x,y,u,v), p\big\rangle + g^{\varepsilon}(t,x,y,u,v) + \frac{1}{\varepsilon}\big\langle h^{\varepsilon}(t,x,y,u,v,\alpha,\beta), q\big\rangle\right] \\ &= \max_{\substack{v \in Q \\ \beta \in B}} \min_{\substack{u\in P \\ \alpha \in A}} \left[\big\langle f^{\varepsilon}(t,x,y,u,v), p\big\rangle + g^{\varepsilon}(t,x,y,u,v) + \frac{1}{\varepsilon}\big\langle h^{\varepsilon}(t,x,y,u,v,\alpha,\beta), q\big\rangle\right]. \end{aligned} \tag{13.6}$$

The equilibrium condition (13.6) is called the Isaacs condition. Equation (13.4) is called the Isaacs equation [108] for the differential game $\mathbf{G}^{\varepsilon}$ (13.1)–(13.3). This equation is a *singularly perturbed* partial differential equation of the Hamilton–Jacobi type, since it contains terms with coefficients $1/\varepsilon$, where ε is a small parameter of singularity.

The purpose of Chap. IV is the justification of effective sufficient conditions for the convergence of $\mathrm{Val}^\varepsilon(t,x,y)$, and construction of the limit Hamilton–Jacobi equation for (13.4) as $\varepsilon \downarrow 0$. It is important to describe asymptotics, namely, limit differential games, where the corresponding Isaacs equation coincides with the limit Hamilton–Jacobi equation, and the minimax and/or viscosity solution of the Hamilton–Jacobi equation coincides with the limit of the value functions $\mathrm{Val}^\varepsilon(t,x,y)$.

14. Formalizations

14.1. Value function of the feedback differential game $\mathbf{G}^\varepsilon$. Fix $\varepsilon \in (0,1)$.

Recall the definition of the value function of the feedback differential game $\mathbf{G}^\varepsilon$ (13.1)–(13.3) (see [133, 135]), where admissible feedback controls (positional strategies) for two players are arbitrary functions of the form:

- $[0,T]\times\mathbb{R}^n\times\mathbb{R}^k \ni (t,x,y) \mapsto (U(t,x,y),\ \alpha(t,x,y)) \in P\times A \subset \mathbb{R}^{m1}\times\mathbb{R}^{r1}$ is a feedback for the first player minimizing the cost functional I^ε, and
- $[0,T]\times\mathbb{R}^n\times\mathbb{R}^k \ni (t,x,y) \mapsto (V(t,x,y),\ \beta(t,x,y)) \in Q\times B \subset \mathbb{R}^{m2}\times\mathbb{R}^{r2}$ is a feedback for the second player maximizing I^ε.

Consider two strategies (probably, discontinuous):

$$[0,T]\times\mathbb{R}^n\times\mathbb{R}^k \ni (t,x,y) \mapsto (U(t,x,y),\alpha(t,x,y)),$$
$$[0,T]\times\mathbb{R}^n\times\mathbb{R}^k \ni (t,x,y) \mapsto (V(t,x,y),\beta(t,x,y)).$$

Choose $t_0 \in [0,T]$ and a partition

$$\Delta := \{t_0 = \tau_0 < \cdots < \tau_i < \cdots < \tau_{N+1} = T\}$$

of the interval $[t_0,T]$ with diameter

$$\operatorname{diam}\Delta := \max\{(\tau_{i+1}-\tau_i) : 0 \le i \le N\}.$$

Definition IV.1. A *step-by-step motion* of system (13.1)

$$x_\Delta(\cdot) = x_\Delta\big(\cdot; t_0, x_0, y_0, U, \alpha, v(\cdot), \beta(\cdot)\big) : [t_0,T] \to \mathbb{R}^n,$$
$$y_\Delta(\cdot) = x_\Delta\big(\cdot; t_0, x_0, y_0, U, \alpha, v(\cdot), \beta(\cdot)\big) : [0,T] \to \mathbb{R}^k,$$

under a feedback of the first player $(t,x,y) \mapsto (U(t,x,y),\alpha(t,x,y))$ discrete realized for Δ and under a measurable control of the opponent $(v(\cdot),\beta(\cdot)) : [t_0,T] \to Q\times B$ is an Euler polygon (an Euler solution) of system (13.1). For any subinterval $[\tau_i,\tau_{i+1})$, $i = 0,1,\ldots,N-1$, it is a trajectory of system (13.1) under the piecewise constant control of the first player:

$$U_\Delta[t] = u_i = U\big(\tau_i, x_\Delta(\tau_i), y_\Delta(\tau_i)\big),$$
$$\alpha_\Delta[t] = \alpha_i = \alpha\big(\tau_i, x_\Delta(\tau_i), y_\Delta(\tau_i)\big),$$

i.e.,

$$\dot{x}_\Delta(t) = f^\varepsilon\big(t, x_\Delta(t), y_\Delta(t), u_i, v(t)\big), \qquad t \in [\tau_i,\tau_{i+1});$$
$$\varepsilon\dot{y} = h^\varepsilon\big(t, x_\Delta(t), y_\Delta(t), u_i, v(t), \alpha_i, \beta(t)\big), \qquad t \in [\tau_i,\tau_{i+1})$$

the initial state is $x_\Delta(t_0) = x_0$, $y_\Delta(t_0) = y_0$.

Definition IV.2. A *solution* of system (13.1)

$$\big(x(\cdot), y(\cdot)\big) = \big(x(\cdot; t_0, x_0, y_0, U, \alpha), y(\cdot; t_0, x_0, y_0, U, \alpha)\big) : [t_0,T] \to \mathbb{R}^n\times\mathbb{R}^k$$

generated by a feedback of the first player $(t, x, y) \mapsto (U(t, x, y), \alpha(t, x, y))$ is the uniform limit of a sequence of step-by-step motions

$$\big\{(x_{\Delta_k}(\cdot), y_{\Delta_k}(\cdot))\big\}_1^\infty \quad \text{as} \quad \operatorname{diam}\Delta_k \to 0,$$

where the Euler polygons $\big(x_{\Delta_k}(\cdot), y_{\Delta_k}(\cdot)\big)$ depend also on measurable controls of the opponent $(v_k(\cdot), \beta_k(\cdot)) : [t_0, T] \to Q \times B,\ k = 1, 2, \ldots$.

Step-by-step motions and solutions of system (13.1) generated by a feedback of the second player $(t, x, y) \mapsto (V(t, x, y), \beta(t, x, y))$ are defined similarly.

Definition IV.3. A *solution* of system (13.1)

$$\big(x(\cdot), y(\cdot)\big) = \big(x(\cdot; t_0, x_0, y_0, U, \alpha), y(\cdot; t_0, x_0, y_0, U, \alpha)\big) : [t_0, T] \to \mathbb{R}^n \times \mathbb{R}^k$$

generated by a feedback of the second player $(t, x, y) \mapsto (V(t, x, y),\ \beta(t, x, y))$ is the uniform limit of a sequence of step-by-step motions

$$\big\{(x_{\Delta_k}(\cdot), y_{\Delta_k}(\cdot))\big\}_1^\infty \quad \text{as} \quad \operatorname{diam}\Delta_k \to 0,$$

where the Euler polygons $(x_{\Delta_k}(\cdot), y_{\Delta-k}(\cdot))$ depend also on measurable controls of the opponent $(u_k(\cdot), \alpha_k(\cdot)) : [t_0, T] \to Q \times B,\ k = 1, 2, \ldots$.

Denote by $\operatorname{Sol}(t_0, x_0, y_0, U, \alpha)$ the set of all solutions $(x(\cdot), y(\cdot))$ of system (13.1) starting at an initial point (t_0, x_0, y_0) under a feedback $(t, x, y) \mapsto (U(t, x, y), \alpha(t, x, y))$ of the first player.

Similarly, denote by $\operatorname{Sol}(t_0, x_0, y_0, V, \beta)$ the set of all solutions $(x(\cdot), y(\cdot))$ of system (13.1) starting at an initial point (t_0, x_0, y_0) under a feedback $(t, x, y) \to (V(t, x, y), \beta(t, x, y))$ of the second player.

Definition IV.4. The *guaranteed result* $\Gamma_I(t_0, x_0, y_0, U, \alpha)$ of the first player under a feedback $(t, x, y) \mapsto (U(t, x, y),\ \alpha(t, x, y))$ at an initial point (t_0, x_0, y_0) in the game $\mathbf{G}^\varepsilon$ is defined as follows:

$$\Gamma_I\big(t_0, x_0, y_0, U, \alpha\big) = \limsup_{\substack{\operatorname{diam}\Delta_k \to 0 \\ k\to\infty}} I^\varepsilon_{t_0,x_0,y_0}\big(x_{\Delta_k}(\cdot), y_{\Delta_k}(\cdot), U_{\Delta_k}[\cdot], \alpha_{\Delta_k}[\cdot], v_k(\cdot), \beta_k(\cdot)\big).$$

The *guaranteed result* $\Gamma_{II}(t_0, x_0, y_0, V, \beta)$ of the second player under a feedback $(t, x, y) \mapsto (V(t, x, y), \beta(t, x, y))$ at an initial point (t_0, x_0, y_0) is defined as follows:

$$\Gamma_{II}(t_0, x_0, y_0, V, \beta) = \liminf_{\substack{\operatorname{diam}\Delta_k \to 0 \\ k\to\infty}} I^\varepsilon_{t_0,x_0,y_0}\big(x_{\Delta_k}(\cdot), y_{\Delta_k}(\cdot), u_k(\cdot), \alpha_k(\cdot), V_{\Delta_k}[\cdot], \beta_{\Delta_k}[\cdot]\big).$$

Definition IV.5. The *optimal guaranteed result* $\Gamma_I^0(t_0, x_0, y_0)$ of the first player at an initial point (t_0, x_0, y_0) in the game $\mathbf{G}^\varepsilon$ is defined by the relation

$$\Gamma_I^0(t_0, x_0, y_0) = \sup_{(U,\alpha)} \Gamma_I(t_0, x_0, y_0, U, \alpha),$$

where the supremum is taken over all admissible feedbacks $(t, x, y) \mapsto (U(t, x, y), \alpha(t, x, y))$.

Similarly, the *optimal guaranteed result* $\Gamma_{II}^0(t_0, x_0, y_0)$ of the second player at an initial point (t_0, x_0, y_0) in the game $\mathbf{G}^\varepsilon$ is defined by the relation

$$\Gamma_{II}^0(t_0, x_0, y_0) = \inf_{(V,\beta)} \Gamma_{II}(t_0, x_0, y_0, V, \beta),$$

where the infimum is taken over all admissible feedbacks $(t, x, y) \mapsto (V(t, x, y), \beta(t, x, y))$.

Definition IV.6. The equilibrium value

$$\operatorname{Val}^\varepsilon(t_0, x_0, y_0) = \Gamma_I^0(t_0, x_0, y_0) = \Gamma_{II}^0(t_0, x_0, y_0)$$

is called the *value of the differential game* $\mathbf{G}^\varepsilon$ (13.1)–(13.3) at a point (t_0, x_0, y_0).

The mapping $[0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} : (t_0, x_0, y_0) \mapsto \operatorname{Val}^\varepsilon(t_0, x_0, y_0)$ is called the *value function* of the positional differential game $\mathbf{G}^\varepsilon$ (see [133, 135]).

14.2. Characteristic complexes for the Cauchy problem $\mathbf{P}^\varepsilon$. It was mentioned in Chap. III that the definition of minimax solutions of singularly perturbed Hamilton–Jacobi equations differs from the standard definition presented in Sec. 2.1 (see also [235, 236, 238]). The specificity of modified definitions is the existence of singularly perturbed (fast) components of generalized characteristics.

Let us emphasize that Definitions III.1 and III.2 of upper, lower, and minimax solutions introduced in Chap. III are axiomatic. They are independent of the choice of complexes

$$(S, M^\varepsilon) \in C(H^\varepsilon), \quad (S_+, M_+^\varepsilon) \in C^\uparrow(H^\varepsilon), \quad (S_-, M_-^\varepsilon) \in C^\downarrow(H^\varepsilon))$$

used in their construction. Thus, these definitions provide freedom of choice of characteristic complexes and we can choose these complexes in the most convenient way, adequate to concrete problems in the theory and applications of minimax solutions.

This opportunity is used in the research presented below, namely, in the proof of sufficient conditions for convergence of the value functions of singularly perturbed differential games and in the description of the limit unperturbed game. This material is substantially based on the structure of the following form of complexes for the Cauchy problem $\mathbf{P}^\varepsilon$ (13.4)–(13.6):

$$\begin{gathered} S_+ = Q \times B \ni s_+ = (v^*, \beta^*), \\ M_+^\varepsilon(t,x,y,s_+) = \mathrm{co}\Big\{ f^\varepsilon(t,x,y,P,v^*),\ h^\varepsilon(t,x,y,P,v^*,A,\beta^*),\ -g^\varepsilon(t,x,y,P,v^*) \Big\}, \end{gathered} \tag{14.1}$$

$$\begin{gathered} S_- = P \times A \ni s_- = (u_*, \alpha_*), \\ M_-^\varepsilon(t,x,y,s_-) = \mathrm{co}\Big\{ f^\varepsilon(t,x,y,u_*,Q),\ h^\varepsilon(t,x,y,u_*,Q,\alpha_*,B),\ -g^\varepsilon(t,x,y,u_*,Q) \Big\}. \end{gathered} \tag{14.2}$$

Complexes (14.1) and (14.2) are widely applied in the theory of feedback differential games (see [133, 135]) and in research into the so-called *u-stability* and *v-stability* properties for the value function. As was mentioned above, the value function of the differential game $\mathbf{G}^\varepsilon$ is a unique continuous minimax solution of the boundary-value Cauchy problem (13.4), (13.5) for the Hamilton–Jacobi–Isaacs equation, where the singularly perturbed Hamiltonian has the form (13.6).

15. Assumptions and the Formulation of the Main Result

Assume that input data of the singularly perturbed differential game $\mathbf{G}^\varepsilon$ (13.1)–(13.3) satisfy the following conditions.

(B1) The functions $\sigma^\varepsilon(x)$, $f^\varepsilon(t,x,y,u,v)$, $g^\varepsilon(t,x,y,u,v)$, and $h^\varepsilon(t,x,y,u,v,\alpha,\beta)$ are defined and continuous with respect to all variables and the parameter ε on the sets

$$\varepsilon \in [0,1], \quad t \in [0,T], \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^k, \quad u \in P, \quad v \in Q, \quad \alpha \in A, \quad \beta \in B.$$

(B2) The following condition of sublinear growth holds:

$$\|f^\varepsilon(t,x,y,u,v)\| + \|h^\varepsilon(t,x,y,u,v,\alpha,\beta)\| + |g^\varepsilon(t,x,y,u,v)| \le \lambda^\varepsilon(x,y),$$

where $\lambda^\varepsilon(x,y) = \mu^\varepsilon(1 + \|x\| + \|y\|)$ and $\mu^\varepsilon > 0$ is a constant.

(B3) On any compact set $\bar{D} \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^k$, the functions $f^\varepsilon(t,x,y,u,v)$, $g^\varepsilon(t,x,y,u,v)$, and $h^\varepsilon(t,x,y,u,v,\alpha,\beta)$ satisfy the Lipschitz condition with respect to the variables (t,x,y) with constants $L^\varepsilon = L^\varepsilon(\bar{D}) > 0$ uniform with respect to $(u,v,\alpha,\beta) \in P \times Q \times A \times B$.

(B4) For any $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^k$ and $(p,q) \in \mathbb{R}^n \times \mathbb{R}^k$, the Isaacs condition (13.6) holds.

It is known (see [133, 135]) that conditions **(B1)**–**(B4)** guarantee the existence of the value function $\mathrm{Val}^\varepsilon(t,x,y)$ of the game $\mathbf{G}^\varepsilon$ for any fixed $\varepsilon \in (0,1]$. To provide the convergence of the value functions $\mathrm{Val}^\varepsilon(t,x,y)$ as $\varepsilon \to 0$, we introduce the following additional conditions and constructions.

Further, we use characteristic complexes (14.1) and (14.2), where the values of controls of players play the role of parameters s_+ and s_-, i.e.,

$$s_+ = (v^*, \beta^*) \in Q \times B = S_+, \quad s_- = (u_*, \alpha_*) \in P \times A = S_-.$$

In the subspace $\mathbb{R}^k$ of the fast variables, we define the following sets Y^ε, Y^ε_+, and Y^ε_- of *"fast" roots* $\bar{y}$ for the singular part of Hamiltonians:

$$Y^\varepsilon = Y^\varepsilon(t, x, u, v, \alpha, \beta) = \big\{\bar{y} \in \mathbb{R}^k : h^\varepsilon(t, x, \bar{y}, u, v, \alpha, \beta) = 0\big\}, \tag{15.1}$$

$$\forall s_+ = (v^*, \beta^*): \quad Y^\varepsilon_+ = Y^\varepsilon_+(t, x, s_+) = \bigcup_{\substack{u \in P \\ \alpha \in A}} Y^\varepsilon(t, x, u, v^*, \alpha, \beta^*), \tag{15.2}$$

$$\forall s_- = (u_*, \alpha_*): \quad Y^\varepsilon_- = Y^\varepsilon_-(t, x, s_-) = \bigcup_{\substack{v \in Q \\ \beta \in B}} Y^\varepsilon(t, x, u_*, v, \alpha_*, \beta). \tag{15.3}$$

Introduce the following assumptions.

(B5) For any $(t, x) \in [0, T] \times \mathbb{R}^n$, $u \in P$, $v \in Q$, $\alpha \in A$, and $\beta \in B$, the sets of "fast" roots are nonempty:

$$Y^\varepsilon(t, x, u, v, \alpha, \beta) = \big\{\bar{y} \in \mathbb{R}^k : h^\varepsilon(t, x, \bar{y}, u, v, \alpha, \beta) = 0\big\} \neq \emptyset.$$

(B6) For any $(t, x) \in [0, T] \times \mathbb{R}^n$ and $s_\pm \in S_\pm$, the sets $Y^\varepsilon_\pm(t, x, s_\pm)$ are *bounded*, i.e.,

$$\|y\| \le \chi^\varepsilon(1 + \|x\|), \quad y \in Y^\varepsilon_\pm(t, x, s_\pm),$$

where χ^ε is a constant, $\chi^\varepsilon \in (0, \mu^\varepsilon]$.

(B7) For any $(t', x') \in [0, T] \times \mathbb{R}^n$, $(t'', x'') \in [0, T] \times \mathbb{R}^n$ and $s_\pm \in S_\pm$, the following Lipschitz conditions holds:

$$\operatorname{dist}\big(Y^\varepsilon_\pm(t', x', s_\pm), Y^\varepsilon_\pm(t'', x'', s_\pm)\big) \le \mathcal{K}^\varepsilon\big(|t' - t''| + \|x' - x''\|\big),$$

where $\mathcal{K}^\varepsilon > 0$ is a constant and $\operatorname{dist}(Y^1, Y^2)$ is the Hausdorff distance between sets Y^1 and Y^2.

Consider compact sets $D \subset [0, T] \times \mathbb{R}^n$ and $D^0_\pm \subset \mathbb{R}^k$ of initial states $(t_0, x_0) \in D$ and $y^0_\pm \in D^0_\pm$, which satisfy the inclusion

$$D^0_+ \cup D^0_- \supset \bigcup_{\varepsilon \in [0,1]} \bigcup_{(t_0, x_0) \in D^1} \bigcup_{s^* \in S^*} Y^\varepsilon(t_0, x_0, s^*) + B^\varepsilon_k,$$

where $D^1 = D + B_{n+1}$, B_{n+1} is the closed unit ball in the space $\mathbb{R}^{n+1}$, and B^ε_k is the closed ball of radius ε in the space $\mathbb{R}^k$. For any $(t_0, x_0) \in D$, $y^0_\pm \in D^0_\pm$, and $z_0 \in \mathbb{R}$, we consider the sets $\operatorname{Sol}^\varepsilon(t_0, x_0, y^0_\pm, z_0, s_\pm)$ of generalized characteristics $(x^\varepsilon_\pm(\cdot), y^\varepsilon_\pm(\cdot), z^\varepsilon_\pm(\cdot))$ corresponding to the complexes $(S_\pm, M^\varepsilon_\pm)$ (see (14.1) and (14.2)). To provide the exponential convergence for the fast components $y^\varepsilon_\pm(\cdot)$ of the characteristics to the corresponding sets of attractivity $Y^\varepsilon_\pm$, assume that the following conditions hold.

(B8) For the sets D and $D^0_\pm$, there are numbers $\kappa^\varepsilon_\pm = \kappa^\varepsilon_\pm(D, D^0_\pm) > 0$ such that

$$\max_{\substack{u \in P \\ \alpha \in A}} \Big\langle y - y^0,\ h^\varepsilon\big(t, x, y, u, v^*, \alpha, \beta^*\big) - h^\varepsilon\big(t, x, y^0, u, v^*, \alpha, \beta^*\big)\Big\rangle \le -\kappa^\varepsilon_+ \operatorname{dist}^2\big(y, Y^\varepsilon_+(t, x, s_+)\big)$$

for all values of parameters $(v^*, \beta^*) = s_+ \in S_+ = Q \times B$ and all points $(t, x) \in D^1 \cdot \varphi^\varepsilon_+$, $y^0 \in Y^\varepsilon_+(t, x, s_+)$, and $y \in (D^0_+ \cup B^{\varphi^\varepsilon_+}_k)$. Similarly,

$$\max_{\substack{v \in Q \\ \beta \in B}} \Big\langle y - y_0,\ h^\varepsilon\big(t, x, y, u_*, v, \alpha_*, \beta\big) - h^\varepsilon\big(t, x, y_0, u_*, v, \alpha_*, \beta\big)\Big\rangle \le -\kappa^\varepsilon \operatorname{dist}^2\big(y, Y^\varepsilon_-(t, x, s_*)\big)$$

for all values of parameters $(u_*, \alpha_*) = s_- \in S_- = P \times A$ and all points $(t, x) \in D^1 \cdot \varphi_-^\varepsilon$, $y_0 \in Y_-^\varepsilon(t, x, s_-)$, and $y \in (D_-^0 \cup B_k^{\varphi_-^\varepsilon})$, where

$$\varphi_\pm^\varepsilon = \varphi_\pm^\varepsilon(D_\pm^0) = 2\mathcal{K}^\varepsilon(1 + \mathcal{M}^\varepsilon \cdot \exp \mathcal{M}^\varepsilon) + \left(\frac{\mathcal{M}^\varepsilon}{T} + L^\varepsilon d_\pm^0\right) \times \left[T + \frac{T^2}{2} \cdot \exp \mathcal{M}^\varepsilon\right],$$

$$d_\pm^0 = \operatorname{diam} D_\pm^0, \quad \mathcal{M}^\varepsilon = \mu^\varepsilon(1 + \chi^\varepsilon)T,$$

and $\operatorname{diam} D^0$ is the diameter of the set $D^0 \in \operatorname{comp} \mathbb{R}^k$, i.e.,

$$\operatorname{diam} D' = \max_{\substack{w' \in D' \\ w'' \in D'}} \|w' - w''\|.$$

(B9) The above-mentioned constants μ^ε, L^ε, χ^ε, $\mathcal{K}^\varepsilon$, and κ^ε continuously depend on the parameter $\varepsilon \in [0, 1]$.

Introduce the *upper Hamiltonian* H_+^ε and the *lower Hamiltonian* H_-^ε by the formulas

$$H_+^\varepsilon(t, x, s) = \max_{s_+ = (v^*, \beta^*) \in S_+} \quad \min_{(f,g) \in F_+^\varepsilon(t,x,s_+)} [\langle f, s \rangle + g],$$

where

$$F_+^\varepsilon(t, x, s_+) = \operatorname{co}\left\{f^\varepsilon\big(t, x, Y_+^\varepsilon(t, x, s_+), P, v^*\big),\ g^\varepsilon\big(t, x, Y_+^\varepsilon(t, x, s_+), P, v^*\big)\right\},$$

and

$$H_-^\varepsilon(t, x, s) = \min_{s_- = (u_*, \alpha_*) \in S_-} \quad \max_{(f,g) \in F_-^\varepsilon(t,x,s_-)} [\langle f, s \rangle + g],$$

where

$$F_-^\varepsilon(t, x, s_-) = \operatorname{co}\left\{f^\varepsilon\big(t, x, Y_-^\varepsilon(t, x, s_-), u_*, Q\big),\ g^\varepsilon\big(t, x, Y_-^\varepsilon(t, x, s_-), u_*, Q\big)\right\}.$$

Assume the following.

(B10) For any $(t, x, s) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, the inequality

$$|H_+^\varepsilon(t, x, s) - H_-^\varepsilon(t, x, s)| \le \delta(\varepsilon)$$

holds, where $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Define $H^0(t, x, s)$ by the relation

$$H^0(t, x, s) = \lim_{\varepsilon \downarrow 0} H_+^\varepsilon(t, x, s) = \lim_{\varepsilon \downarrow 0} H_-^\varepsilon(t, x, s).$$

It will play the role of the Hamiltonian in the limit unperturbed boundary problem $\mathbf{P}^0$:

$$\frac{\partial \operatorname{Val}^0(t, x)}{\partial t} + H^0\big(t, x, D_x \operatorname{Val}^0(t, x)\big) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \tag{15.4}$$

with the boundary condition

$$\operatorname{Val}^0(T, x) = \sigma^0(x) = \lim_{\varepsilon \downarrow 0} \sigma^\varepsilon(x), \quad x \in \mathbb{R}^n. \tag{15.5}$$

The main result of the present chapter is the following theorem.

Theorem IV.1. *Let assumptions* **(B1)–(B10)** *for the differential game* $\mathbf{G}^\varepsilon$ (13.1)–(13.3) *hold. Then the value function* $\operatorname{Val}^\varepsilon(t, x, y)$ *converges to the minimax solution* $\operatorname{Val}^0(t, x)$ *of the problem* (15.4), (15.5) *uniformly on any compact set*

$$\bar{D} = D \times (D_+^0 \cup D_-^0) \subset [0, T] \times \mathbb{R}^n \times \mathbb{R}^k$$

as $\varepsilon \to 0$. The limit $\mathrm{Val}^0(t,x)$ *coincides with the optimal guaranteed results of players* $\Gamma^0_I(t,x)$, $\Gamma^0_{II}(t,x)$ *in the limit differential games* $\mathbf{G}^0_I$ *and* $\mathbf{G}^0_{II}$*. The cost functional in the games has the form*

$$I^0_{t_0,x_0}\big(x(\cdot),u(\cdot),v(\cdot),y(\cdot)\big) = \sigma^0(x(T)) + \int_{t_0}^{T} g^0\big(\tau,x(\tau),y(\tau),u(\tau),v(\tau)\big)d\tau, \tag{15.6}$$

where $(x(\cdot)) : [t_0,T] \to \mathbb{R}^n$ *is a trajectory of the equation*

$$\dot{x}(t) = f^0\big(t,x(t),y(t),u(t),v(t)\big), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \quad x(t_0) = x_0, \quad t_0 \in [0,T], \tag{15.7}$$

generated by measurable controls of players in the following way.

- *In the game* $\mathbf{G}^0_I$*, the first player has a control* $(u(\cdot),\alpha(\cdot)) : [0,T] \to P \times A$*, the second player has a control* $(v(\cdot),y(\cdot)) : [0,T] \to Q \times Y^0_-$*, and the restriction*

$$y(t) \in Y^0_- = \bigcup_{\beta \in B} Y^0\big(t,x(t),u(t),v(t),\alpha(t),\beta\big) \tag{15.8}$$

  holds for almost all $t \in [t_0,T]$*. The value* $\Gamma^0_I(t,x)$ *means the optimal guaranteed result for the first player in the game* $\mathbf{G}^0_I$*.*
- *In the game* $\mathbf{G}^0_{II}$*, the second player has a control* $(v(\cdot),\beta(\cdot)) : [0,T] \to Q \times B)$*, the first player has a control* $(u(\cdot),y(\cdot)) : [0,T] \to P \times Y^0_+)$*, and the restriction*

$$y(t) \in Y^0_+ = \bigcup_{\alpha \in A} Y^0\big(t,x(t),u(t),v(t),\alpha,\beta(t)\big) \tag{15.9}$$

  holds for almost all $t \in [t_0,T]$*. The value* $\Gamma^0_{II}(t,x)$ *means the optimal guaranteed result for the second player in the game* $\mathbf{G}^0_{II}$*.*

The functions $f^0(\cdot)$, $g^0(\cdot)$, and $\sigma^0(\cdot)$ in (15.6) and (15.7) are obtained by passage to the limit in the corresponding input data for the singularly perturbed differential game $\mathbf{G}^\varepsilon$ as $\varepsilon \to 0$. The set $Y^0(t,x,u,v,\alpha,\beta)$ is the limit of the set $Y^\varepsilon(t,x,u,v,\alpha,\beta)$ in the Hausdorff metric as $\varepsilon \downarrow 0$.

Note that the limit unperturbed games $\mathbf{G}^0_i$, $i = I, II$ (see (15.6), (15.7)), are nonstandard, since their dynamics are defined in the reduced phase space of slow motions, and the influence of players is enhanced with the help of controls $y(\cdot) : [0,T] \to \mathbb{R}^k$. However, the controls must satisfy the nonstationary geometrical restrictions $y(t) \in Y^0_\pm$, and the sets $Y^0_\pm \subset \mathbb{R}^k$ depend on both the current phase state of the system $(t,x(t))$ and the control of the opponent.

The limit unperturbed games $\mathbf{G}^0_i$, $i = I, II$ (see (15.6)–(15.7)), are called *asymptotics* for the singularly perturbed differential games $\mathbf{G}^\varepsilon$ (13.1)–(13.3). A unique minimax solution of the limit Cauchy problem $\mathbf{P}^0$ for the unperturbed Hamilton–Jacobi equation (15.4), (15.5) coincides with both the optimal guaranteed result of the first player $\Gamma^0_I(t,x)$ in the game $\mathbf{G}^0_I$ and the optimal guaranteed result of the second player $\Gamma^0_{II}(t,x)$ in the game $\mathbf{G}^0_{II}$.

Thus, the limit Hamilton–Jacobi equation (15.4) can be interpreted as the Isaacs equation for the asymptotic games $\mathbf{G}^0_i$, $i = I, II$ (see (15.6)–(15.7)).

16. Sufficient Convergence Conditions for Value Functions in Singularly Perturbed Differential Games

This section contains the proof of Theorem IV.1 on sufficient conditions for the convergence of the value functions of singularly perturbed differential games and on the structure of asymptotics. The proof is based on results presented in Chap. III and also in [263, 265], where the convergence of the minimax solutions of nonlinear singularly perturbed Hamilton–Jacobi equations (13.4) is studied.

16.1. Properties of the sets Y^ε_+ and Y^ε_-. It is easy to see that conditions **(B1)**, **(B6)**, **(B7)**, and **(B10)** imply the compactness of the sets Y^ε, Y^ε_+, and Y^ε_- for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and

$$s_+ = (v^*, \beta^*) \in S_+ = Q \times B, \quad s_- = (u_*, \alpha_*) \in S_- = P \times A.$$

There is also the following convergence in the Hausdorff metric:

$$Y^\varepsilon(t,x,u,v,\alpha,\beta) \to Y^0(t,x,u,v,\alpha,\beta), \quad Y^\varepsilon_+(t,x,s_+) \to Y^0_+(t,x,s_+), \quad Y^\varepsilon_-(t,x,s_-) \to Y^0_-(t,x,s_-)$$

as $\varepsilon \downarrow 0$. The limit sets

$$Y^0(t,x,u,v,\alpha,\beta), \quad Y^0_+(t,x,s_+), \quad Y^0_-(t,x,s_-)$$

are also compact. The multi-valued mappings

$$(t,x) \mapsto Y^0(t,x,u_*,v^*,\alpha_*,\beta^*), \quad (t,x) \mapsto Y^0_+(t,x,s_+), \quad (t,x) \mapsto Y^0_-(t,x,s_-)$$

are Lipschitz continuous uniformly with respect to all parameters

$$s_+ = (v^*, \beta^*) \in Q \times B = S_+, \quad s_- = (u_*, \alpha_*) \in P \times A = S_-.$$

Consider a continuous, bounded function $x(\cdot) : [0,T] \to \mathbb{R}^n$ and fix values of parameters $s_+ \in S_+$, $s_- \in S_-$, and $\varepsilon > 0$. Condition **(B8)** implies [55, 238] that

- the sets $Y^\varepsilon_+(t,x(t),s_+) + B^\varepsilon_k$ are strongly invariant with respect to the corresponding differential inclusions
$$\varepsilon \dot y^0(t) \in \operatorname{co}\Big\{h^\varepsilon\big(t,x(t),y^0(t),P,v^*,A,\beta^*\big)\Big\};$$
- the sets $Y^\varepsilon_-(t,x(t),s_-) + B^\varepsilon_k$ are strongly invariant with respect to the corresponding differential inclusions
$$\varepsilon \dot y_0(t) \in \operatorname{co}\Big\{h^\varepsilon\big(t,x(t),y_0(t),u_*,Q,\alpha_*,B\big)\Big\}.$$

For definiteness, all considerations below are given for the *upper* sets Y^ε_+ and *upper* characteristic differential inclusions (14.1). One can also obtain similar conclusions for the *lower* sets Y^ε_- and *lower* characteristic inclusions (14.2) when similar reasonings are used.

Let

$$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) : [t_0,T] \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$$

be a solution of the lower characteristic inclusion (14.1) corresponding to the fixed value of parameter $s_+ = (v^*, \beta^*) \in S_+$ and to the initial state $(x^\varepsilon(t_0), y^\varepsilon(t_0), z^\varepsilon(t_0)) = (x_0, y_0, z_0)$, i.e.,

$$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) \in \operatorname{Sol}(t_0, x_0, y_0, z_0, s_+).$$

Estimate $\operatorname{dist}^2(y^\varepsilon(t), Y^\varepsilon_+(t, x^\varepsilon(t), s_+))$. For this, we choose $y_0 \notin Y^\varepsilon_+(t_0, x_0, s_+)$ and define a vector $\tilde y_0$ by the relations

$$\tilde y_0 \in Y^\varepsilon_+(t_0,x_0,s_+), \quad \|y_0 - \tilde y_0\| = \operatorname{dist}\big(y_0, Y^\varepsilon_+(t_0,x_0,s_+)\big) = \tilde d_0 > 0.$$

Consider a solution of the differential inclusion

$$\varepsilon \dot y^0(t) \in \operatorname{co}\Big\{h^\varepsilon\big(t,x^\varepsilon(t),y^0(t),P,v^*,A,\beta^*\big)\Big\}, \quad y^0(t_0) = \tilde y_0. \tag{16.1}$$

The strong invariance of the sets $Y^\varepsilon_+(t,x^\varepsilon(t),s_+)$ with respect to inclusions (16.1) implies that $y^0(t) \in Y^\varepsilon_+(t,x^\varepsilon(t),s_+)$ for all $t \in [t_0,T]$.

According to (14.1), the dynamics of the fast variable $y^\varepsilon(t)$ is described as follows:

$$\varepsilon \dot y^\varepsilon(t) \in \operatorname{co}\Big\{h^\varepsilon\big(t,x^\varepsilon(t),y^\varepsilon(t),P,v^*,A,\beta^*\big)\Big\}, \quad y^\varepsilon(t_0) = y_0. \tag{16.2}$$

We use the fact (see [300]) that trajectories of the differential inclusions (16.2) and (16.1) can be described by generalized controls

$$\mathbf{u}_t(du),\ \mathbf{a}_t(d\alpha) : [t_0,T] \ni t \to \operatorname{rpm}(P) \times \operatorname{rpm}(A).$$

Recall that the controls are measurable functions on $[0,T]$ with values in the sets of all regular probability measures defined on the sets P and A of admissible values of controls of players. Recall that the following representations hold:

$$\varepsilon\dot{y}^\varepsilon(t) = \int\limits_P \int\limits_A h^\varepsilon\big(t, x^\varepsilon(t), y^\varepsilon(t), u, v^*, \alpha, \beta^*\big)\mathbf{u}_t^\varepsilon(du)\cdot\mathbf{a}_t^\varepsilon(d\alpha),$$
$$\varepsilon\dot{y}^0(t) = \int\limits_P \int\limits_A h^\varepsilon\big(t, x^\varepsilon(t), y^0(t), u, v^*, \alpha, \beta^*\big)\mathbf{u}_t^0(du)\cdot\mathbf{a}_t^0(d\alpha).$$

Consider trajectories $y^\varepsilon(t)$ and $y^0(t)$ generated by the same generalized controls, i.e.,

$$\mathbf{u}_t^\varepsilon(du) = \mathbf{u}_t^0(du), \quad \mathbf{a}_t^\varepsilon(d\alpha) = \mathbf{a}_t^0(d\alpha), \quad t \in [t_0, T].$$

The following estimates hold:

$$\operatorname{dist}\big(y^\varepsilon(t), Y_+^\varepsilon(t, x^\varepsilon(t), s_+)\big) \le \|y^\varepsilon(t) - y^0(t)\|,$$
$$\frac{d\|y^\varepsilon(t) - y^0(t)\|^2}{dt} = 2\left\langle y^\varepsilon(t) - y^0(t),\ \frac{dy^\varepsilon(t)}{dt} - \frac{dy^0(t)}{dt}\right\rangle \le \frac{2}{\varepsilon}\max_{u\in P}\big\langle y^\varepsilon(t) - y^0(t),\ k^\varepsilon(t) - k_0^\varepsilon(t)\big\rangle,$$
$$k^\varepsilon(t) = k^\varepsilon\big(t, x^\varepsilon(t), y^\varepsilon(t), u, v^*\big), \quad k_0^\varepsilon(t) = k^\varepsilon\big(t, x^\varepsilon(t), y^0(t), u, v^*\big).$$

Using these relations and condition **(B8)**, one can obtain the estimates

$$\begin{gathered}\operatorname{dist}^2\Big(y^\varepsilon(t),\ Y_+^\varepsilon\big(t, x^\varepsilon(t), s_+\big)\Big) \le \tilde{d}_0^2 - \int\limits_{t_0}^{t} \frac{2\kappa^\varepsilon}{\varepsilon}\operatorname{dist}\Big(y^\varepsilon(\tau),\ Y_+^\varepsilon\big(\tau, x^\varepsilon(\tau), s_+\big)\Big)d\tau, \\ \operatorname{dist}\Big(y^\varepsilon(t), Y_+^\varepsilon\big(t, x^\varepsilon(t), s_+\big)\Big) \le \tilde{d}_0\exp\left(-\frac{2\kappa^\varepsilon}{\varepsilon}(t - t_0)\right) \le \tilde{d}_0.\end{gathered} \tag{16.3}$$

Thus, the fast components $y^\varepsilon(t)$ of the upper characteristics (16.2) exponentially tend to the corresponding upper sets $Y_+^\varepsilon(t, x^\varepsilon(t), s_+)$ as $\varepsilon \to 0$ for any $s_+ \in S_+$. A similar convergence for the fast components of the lower characteristics to the corresponding lower sets Y_-^ε also holds.

Therefore, one can consider the sets $Y_\pm^\varepsilon + B_k^\varepsilon$ (see conditions **(B8)**–**(B10)**) as the attractors for the fast components $y_\pm^\varepsilon(\cdot)$ of the corresponding generalized characteristics. Conditions **(B8)** and estimates (16.3) imply that $\tilde{d}_0$-neighborhoods of the attractors are strongly invariant with respect to the corresponding characteristic inclusions for all $\tilde{d}_0 > 0$.

16.2. Proof of the main result. To prove Theorem IV.1, we show that conditions **(B1)**–**(B11)** imply the validity of sufficient conditions **(A$^\varepsilon$1)**–**(A$^\varepsilon$11)** presented in Theorem III.1.

Obviously, conditions **(A$^\varepsilon$1)**–**(A$^\varepsilon$4)** are implied by conditions **(B1)**–**(B4)** and definition (13.6) of the Hamiltonian $H^\varepsilon(t, x, y, p, \frac{1}{\varepsilon}q)$. The Lipschitz constants $\lambda^\varepsilon(x, y)$ and $L^\varepsilon = L^\varepsilon(t, x)$ in these conditions coincide.

It is easy to see that condition **(B3)** implies condition **(A$^\varepsilon$5)** on the Lipschitz continuity in the Hausdorff metric of the mappings

$$(t, x, y) \mapsto M_\pm^\varepsilon(t, x, y, s_\pm), \quad s_\pm \in S_\pm,$$

of the form (14.1)–(14.2).

Conditions **(B6)** and **(B7)** coincide with conditions **(A$^\varepsilon$6)** and **(A$^\varepsilon$7)**, respectively.

As was shown in Sec. 16.1, estimates (16.3), uniform for all $(t_0, x_0) \in D$, $y_0 \in D_\pm^0$, and $z_0 \in Z^0$, imply the validity of conditions (10.9) and (10.10) (see Chap. III). This means the validity of condition **(A$^\varepsilon$8)** for solutions

$$\big(x_\pm^\varepsilon, y_\pm^\varepsilon, z_\pm^\varepsilon\big) \in \operatorname{Sol}\big(t_0, x_0, y_0, z_0, s_\pm\big), \quad s_\pm \in S_\pm,$$

satisfying the inequalities

$$z_+^\varepsilon(t) \ge u^\varepsilon\big(t, x_+^\varepsilon(t), y_+^\varepsilon(t)\big) \quad \forall t \ge t_0,$$
$$z_-^\varepsilon(t) \le u^\varepsilon\big(t, x_-^\varepsilon(t), y_-^\varepsilon(t)\big) \quad \forall t \ge t_0.$$

Moreover, this means that estimates (16.3) and conditions (10.9) and (10.10) (see Chap. III) hold for *all* solutions $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \mathrm{Sol}(t_0, x_0, y_0, z_0, s_\pm)$.

Obviously, conditions **(B9)** and **(B10)** coincide with conditions **(A$^\varepsilon$9)** and **(A$^\varepsilon$10)**, respectively.

Finally, conditions **(B1)** and **(B5)** and the definitions of the sets $Y_+^\varepsilon(t, x, s_+)$ (10.5) and $Y_-^\varepsilon(t, x, s_-)$ (10.6) imply the validity of condition **(A$^\varepsilon$11)**, i.e.,

$$Y_+^0(t, x, s_+) \cap Y_-^0(t, x, s_-) \ne \emptyset$$

for all $s_+ \in S_+$, $s_- \in S_-$, and $(t, x) \in [0, T] \times \mathbb{R}^n$.

Thus, the validity of the conclusions of Theorem IV.1 follows now from Theorem III.1. Still, because of the specificity of characteristic complexes (14.1) and (14.2), it is necessary to modify the formulations and improve the estimates of Lemmas III.1–III.3. The modified proof of Lemma III.1 and formulations of Lemmas IV.2 and IV.3 are presented below.

Choose $\varepsilon \in (0, 1]$, an initial state $(t_0, x_0, y_0) \in D \times D_0$, $z_0 \in \mathbb{R}^1$, complexes (S, M_+^ε) defined by relation (14.1), and sets $Y_+^\varepsilon(t, x, s_+)$, $s_+ \in S_+$, of the form (10.5).

Fix a solution

$$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) \in \mathrm{Sol}^\varepsilon(t_0, x_0, y_0, z_0, s_+), \quad s_+ \in S_+.$$

According to the definition, the differential inclusion

$$\big(\dot{x}^\varepsilon(t), \varepsilon\dot{y}^\varepsilon(t), \dot{z}^\varepsilon(t)\big) \in M_+^\varepsilon\big(t, x^\varepsilon(t), y^\varepsilon(t), s_+\big)$$

holds for almost all $t \in [t_0, T]$, and the following boundary condition is satisfied:

$$\big(x^\varepsilon(t_0), y^\varepsilon(t_0), z^\varepsilon(t_0)\big) = (x_0, y_0, z_0).$$

Using the fast component $y^\varepsilon(\cdot) : [t_0, T] \to \mathbb{R}^k$ of the solution, we construct the following multi-valued mapping:

$$(t, x) \mapsto Y_{0+}^\varepsilon(t, x, s_+) = \Big\{y_0 \in Y_+^\varepsilon(t, x, s_+) : \\ \|y^\varepsilon(t) - y_0\| = \mathrm{dist}\,\big(y^\varepsilon(t),\ Y_+^\varepsilon(t, x, s_+)\big)\Big\} \subset Y_+^\varepsilon(t, x, s_+). \quad (16.4)$$

It is easy to verify that the multi-valued mapping $(t, x) \mapsto Y_{0+}^\varepsilon(t, x, s_+)$ has compact value set and is upper semicontinuous for any $s_+ \in S_+$. Hence, the mapping

$$(t, x) \mapsto \mathrm{co}\Big\{f^\varepsilon\big(t, x, Y_{0+}^\varepsilon(t, x, s_+), P, v^*\big),\ -g^\varepsilon\big(t, x, Y_{0+}^\varepsilon(t, x, s_+), P, v^*\big)\Big\} \quad (16.5)$$

inherits the same properties.

Consider the differential inclusion corresponding to mapping (16.5):

$$\big(\dot{x}_0^\varepsilon(t), \dot{z}_0^\varepsilon(t)\big) \in \mathrm{co}\Big\{f^\varepsilon\big(t, x_0^\varepsilon(t), Y_{0+}^\varepsilon\big(t, x_0^\varepsilon(t), s_+\big), P, v^*\big),\ -g^\varepsilon\big(t, x_0^\varepsilon(t), Y_{0+}^\varepsilon\big(t, x_0^\varepsilon(t), s_+\big), P, v^*\big)\Big\} \quad (16.6)$$

with the initial condition

$$x_0^\varepsilon(t_0) = x_0, \quad z_0^\varepsilon(t_0) = z_0. \quad (16.7)$$

According to the theory of differential inclusions (see [75]), there exists a solution of the differential inclusion (16.6), (16.7) defined on $[t_0, T]$. Denote by $\mathrm{Sol}_{0+}^\varepsilon(t_0, x_0, z_0, s_+)$ the set of all solutions $(x_0^\varepsilon(\cdot), z_0^\varepsilon(\cdot))$ of inclusion (16.6) for the fixed parameter $s_+ \in S_+$, which start at the initial state (16.7). The symbol $\mathrm{Sol}_+^\varepsilon(t_0, x_0, z_0, s_+)$ denotes the set of all solutions $(x_\varepsilon(\cdot), z_\varepsilon(\cdot))$ of the differential inclusion

$$\big(\dot{x}_\varepsilon(t), \dot{z}_\varepsilon(t)\big) \in \mathrm{co}\Big\{f^\varepsilon\big(t, x_\varepsilon(t), Y_+^\varepsilon\big(t, x_\varepsilon(t), s_+\big), P, v^*\big),\ -g^\varepsilon\big(t, x_\varepsilon(t), Y_+^\varepsilon\big(t, x_\varepsilon(t), s_+\big), P, v^*\big)\Big\} \quad (16.8)$$

with the initial condition

$$x_\varepsilon(t_0) = x_0, \quad z_\varepsilon(t_0) = z_0. \tag{16.9}$$

Obviously, the inclusion

$$\mathrm{Sol}^\varepsilon_{0+}(t_0, x_0, z_0, s_+) \subset \mathrm{Sol}^\varepsilon_+(t_0, x_0, z_0, s_+)$$

holds.

Consider the trajectory chosen above

$$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) \in \mathrm{Sol}^\varepsilon(t_0, x_0, y_0, z_0, s_+)$$

and $t \in [t_0, T]$. Estimate the distance between the point $(x^\varepsilon(t), z^\varepsilon(t))$ and the point $(x_\varepsilon(t),\ z_\varepsilon(t))$ on a trajectory

$$\big(x_\varepsilon(\cdot), z_\varepsilon(\cdot)\big) \in \mathrm{Sol}^\varepsilon_+(t_0, x_0, z_0, s_+),$$

which is the nearest trajectory to $(x^\varepsilon(\cdot),\ z^\varepsilon(\cdot))$.

By the definitions of these points, one can see that the mentioned distance is not greater than the distance between $(x^\varepsilon(t), z^\varepsilon(t))$ and the point $(x^\varepsilon_0(t), z^\varepsilon_0(t))$ on a trajectory

$$\big(x^\varepsilon_0(\cdot), z^\varepsilon_0(\cdot)\big) \in \mathrm{Sol}^\varepsilon_{0+}(t_0, x_0, z_0, s_+),$$

which is the nearest trajectory to $(x^\varepsilon(\cdot),\ z^\varepsilon(\cdot))$.

By Assertion III.1, we obtain the following estimates:

$$\begin{aligned}\big\|x^\varepsilon(t) - x_\varepsilon(t)\big\| &\le \big\|x^\varepsilon(t) - x^\varepsilon_0(t)\big\| \le \int_{t_0}^{t} \big\|\dot x^\varepsilon(\tau) - \dot x^\varepsilon_0(\tau)\big\| d\tau \\ &\le \int_{t_0}^{t} \mathrm{dist}\Big(\operatorname{co} f^\varepsilon\big(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), P, v^*\big),\ \operatorname{co} f^\varepsilon\big(\tau, x^\varepsilon_0(\tau), Y^\varepsilon_0\big(\tau, x^\varepsilon_0(\tau), s_+\big), P, v^*\big)\Big) d\tau \\ &\le \int_{t_0}^{t} \max_{u\in P} \Big\| f^\varepsilon\big(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), u, v^*\big) - f^\varepsilon\big(\tau, x^\varepsilon_0(\tau), y^\varepsilon_0(\tau), u, v^*\big)\Big\| d\tau, \end{aligned} \tag{16.10}$$

$$\begin{aligned}\big|z^\varepsilon(t) - z_\varepsilon(t)\big| &\le \big|z^\varepsilon(t) - z^\varepsilon_0(t)\big| \le \int_{t_0}^{t} \big|\dot z^\varepsilon(\tau) - \dot z^\varepsilon_0(\tau)\big| d\tau \\ &\le \int_{t_0}^{t} \max_{u\in P} \Big| g^\varepsilon\big(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), u, v^*\big) - g^\varepsilon\big(\tau, x^\varepsilon_0(\tau), y^\varepsilon_0(\tau), u, v^*\big)\Big| d\tau, \end{aligned} \tag{16.11}$$

where

$$y^\varepsilon_0(\cdot) : [t_0, \tau] \ni t \mapsto y^\varepsilon_0(t) \in Y^\varepsilon_{0+}\big(t, x^\varepsilon_0(t), s_+\big) \subset D_0$$

is a measurable function defined by (16.4) and satisfying the relation

$$\big\|y^\varepsilon(t) - y^\varepsilon_0(t)\big\| = \mathrm{dist}\,\big(y^\varepsilon(t), Y^\varepsilon_+(t, x^\varepsilon_0(t), s_+)\big).$$

Using **(B7)** and the properties of dist, we continue inequalities (16.10):

$$\left\|x^\varepsilon(t) - x_0^\varepsilon(t)\right\| \le \int\limits_{t_0}^{t} L^\varepsilon\Big\{\left\|x^\varepsilon(\tau) - x_0^\varepsilon(\tau)\right\| + \left\|y^\varepsilon(\tau) - y_0^\varepsilon(\tau)\right\|\Big\}d\tau$$

$$\le \int\limits_{t_0}^{t} L^\varepsilon\Big\{\left\|x^\varepsilon(\tau) - x_0^\varepsilon(\tau)\right\| + \operatorname{dist}\big(y^\varepsilon(\tau),\ Y_+^\varepsilon\big(\tau, x^\varepsilon(\tau), s_+\big)\big)$$

$$+ \operatorname{dist}\big(Y_+^\varepsilon(\tau, x^\varepsilon(\tau), s_+),\ Y_+^\varepsilon(\tau, x_0^\varepsilon(\tau), s_+)\big)\Big\}d\tau$$

$$\le \int\limits_{t_0}^{t} L^\varepsilon\Big\{(1 + K^\varepsilon)\left\|x^\varepsilon(\tau) - x_0^\varepsilon(\tau)\right\| + \operatorname{dist}\big(y^\varepsilon(\tau), Y_+^\varepsilon\big(\tau, x^\varepsilon(\tau), s_+\big)\big)\Big\}d\tau.$$

To complete estimates (16.10), we use the exponential estimate (16.3) for the distance between the fast components of generalized characteristics and the corresponding attractors, and the Gronwall inequality (see [300]):

$$\left\|x^\varepsilon(t) - x_0^\varepsilon(t)\right\| \le \int\limits_{t_0}^{\theta} L^\varepsilon(1 + K^\varepsilon)\left\|x^\varepsilon(\tau) - x_0^\varepsilon(\tau)\right\|d\tau$$

$$+ \int\limits_{t_0}^{\theta} \exp\left(-\frac{\kappa^\varepsilon}{\varepsilon}(\tau - t_0)\right)\tilde{d}_0 d\tau \le \int\limits_{t_0}^{\theta} L^\varepsilon(1 + K^\varepsilon)\left\|x^\varepsilon(\tau) - x_0^\varepsilon(\tau)\right\|d\tau + \varepsilon\frac{d^0}{\kappa^\varepsilon};$$

therefore,

$$\left\|x^\varepsilon(t) - x_0^\varepsilon(t)\right\| \le N^\varepsilon \cdot \varepsilon = \rho(\varepsilon), \tag{16.12}$$

where

$$N^\varepsilon = \frac{d^0}{\kappa^\varepsilon}\Big[1 + L^\varepsilon(1 + K^\varepsilon)(T - t_0)\cdot\exp\big(L^\varepsilon(1 + K^\varepsilon)(T - t_0)\big)\Big], \quad d^0 = \operatorname{diam}\{D_+^0 \cup D_-^0\}.$$

Obviously, $\rho(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Similar reasonings prove estimates (16.12) for the difference $|z^\varepsilon(t) - z_0^\varepsilon|$.

Estimates above for the upper complexes and similar estimates for the lower characteristic complexes imply the following assertion.

Lemma IV.1. *For any compact sets D, D_+^0, and D_-^0 chosen by condition* **(B8)**, *there are numbers $\delta^0 > 0$ and $\rho^0 > 0$ and mappings*

$$(0, 1] \to (0, \delta^0] \times (0, \rho^0] : \varepsilon \mapsto (\delta(\varepsilon), \rho(\varepsilon))$$

that satisfy the following requirements:

$$\delta(\varepsilon) \downarrow 0, \quad \rho(\varepsilon) \downarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0,$$

and for any initial state $(t_0, x_0) \in D$, $y_0 \in D_\pm^0$, $z_0 \in \mathbb{R}$, $s_\pm \in S_\pm$, $\varepsilon \in (0, 1]$, and a trajectory

$$\big(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)\big) \in \operatorname{Sol}^\varepsilon(t_0, x_0, y_0, z_0, s_\pm),$$

one can choose a trajectory

$$\big(x_\varepsilon(\cdot), z_\varepsilon(\cdot)\big) \in \operatorname{Sol}_\pm^\varepsilon(t_0, x_0, z_0, s_\pm)$$

such that the following estimates hold:

$$\left\|x^\varepsilon(\tau) - x_\varepsilon(\tau)\right\| \le \rho(\varepsilon), \quad \left\|z^\varepsilon(\tau) - z_\varepsilon(\tau)\right\| \le \rho(\varepsilon), \quad \operatorname{dist}\Big(y^\varepsilon(\tau), Y_\pm^\varepsilon\big(\tau, x^\varepsilon(\tau), s_\pm\big)\Big) \le \varepsilon$$

for all $\tau \in [t_0 + \delta(\varepsilon), T]$.

Remark IV.1. The values of the mapping $\delta(\varepsilon)$ in Lemma IV.1 can be chosen as follows:

$$\delta(\varepsilon) = -\frac{\varepsilon}{\kappa^2} \ln \frac{\varepsilon}{d^0}.$$

The values of mapping $\rho(\varepsilon)$ are defined by relations (16.12).

Note that Lemma IV.1 is similar to Lemma III.1.

Note also that differential inclusions (16.8) satisfy the definition of upper characteristic inclusions in the "upper" perturbed Cauchy problem $\hat{\mathbf{P}}_+^\varepsilon$ of the type (15.4), (15.5), where the Hamiltonian $\hat{H}_+^\varepsilon$ has the form

$$\hat{H}_+^\varepsilon(t,x,s) = \max_{s_+\in S_+} \min_{(f,g)\in F_+^\varepsilon(t,x,s_+)} [\langle s, f\rangle - g]$$

and

$$F_+^\varepsilon(t,x,s_+) = \mathrm{co}\Big\{ f^\varepsilon\big(t,x,Y_+^\varepsilon(t,x,s_+),P,v^*\big),\ -g^\varepsilon\big(t,x,Y_+^\varepsilon(t,x,s_+),P,v^*\big)\Big\}.$$

Similarly, the construction of the Hamiltonian $\hat{H}_-^\varepsilon$, i.e.,

$$\hat{H}_-^\varepsilon(t,x,s) = \min_{s_-\in S_-} \min_{(f,g)\in F_-^\varepsilon(t,x,s_-)} [\langle s, f\rangle - g],$$

where

$$F_-^\varepsilon(t,x,s_-) = \mathrm{co}\Big\{ f^\varepsilon\big(t,x,Y_-^\varepsilon(t,x,s_-),u_*,Q\big),\ -g^\varepsilon\big(t,x,Y_-^\varepsilon(t,x,s_-),u_*,Q\big)\Big\},$$

provides an opportunuty to consider the complexes (S_-, M_-^ε):

$$S_- = P \times A \ni s_- = (u^*, \alpha^*),$$

$$M_-^\varepsilon = M_-^\varepsilon(t,x,s_-) = \mathrm{co}\Big\{ f^\varepsilon\big(t,x_\varepsilon(t),Y_-^\varepsilon(t,x_\varepsilon(t),s_-),u^*,Q\big),\ -g^\varepsilon\big(t,x_\varepsilon(t),Y_-^\varepsilon(t,x_\varepsilon(t),s_-),u^*,Q\big)\Big\},$$

$$Y_-^\varepsilon(t,x,s_-) = \bigcup_{v\in Q,\beta\in B} Y^\varepsilon(t,x,u_*,v,\alpha_*,\beta), \quad s_- = (u_*,\alpha_*),$$

as lower characteristic complexes for the corresponding "lower" perturbed Cauchy problem $\hat{\mathbf{P}}_-^\varepsilon$ of the type (15.4), (15.5) with the Hamiltonian $\hat{H}_-^\varepsilon$.

According to the definitions of minimax solutions $\mathrm{Val}^\varepsilon(t,x,y)$ of the singularly perturbed Isaacs equations (13.4)–(13.6), the epigraphs and the hypographs of the solutions are weakly invariant with respect to the characteristic inclusions corresponding to the complexes (14.1) and (14.2). Using the weak invariance, one can follow the general scheme of the proof given in the previous chapter to obtain the following results.

Lemma IV.2. *The function*

$$w^{\sharp\uparrow}(t,x) = \inf_{s_+\in S_+} \liminf_{\substack{\varepsilon\downarrow 0\\ (t',x')\to(t,x)}} \min_{y'\in Y_+^\varepsilon(t',x',s_+)+B_k^\varepsilon} w^\varepsilon(t',x',y')$$

is an upper minimax solution of the "upper" limit Cauchy problem $\mathbf{P}_+^0$ of the type (15.4), (15.5), *where the Hamiltonian H_+^0 has the presentation*

$$H_+^0(t,x,s) = \max_{s_+\in S_+} \min_{(f,g)\in F_+^0(t,x,s_+)} [\langle s, f\rangle - g] \tag{16.13}$$

and

$$F_+^0(t,x,s_+) = \mathrm{co}\Big\{ f^0\big(t,x,Y_+^0(t,x,s_+),P,v^*\big),\ -g^0\big(t,x,Y_+^0(t,x,s_+),P,v^*\big)\Big\}.$$

Lemma IV.3. *The function*

$$w^{\sharp\downarrow}(t,x) = \sup_{s_-\in S_-} \limsup_{\substack{\varepsilon\downarrow 0\\ (t',x')\to(t,x)}} \max_{y\in Y_-^\varepsilon(t,x,s_-)+B_k^\varepsilon} w_\varepsilon^{0\downarrow}(t',x')$$

is a lower minimax solution of the "lower" limit Cauchy problem $\mathbf{P}_-^0$ *of the type* (15.4), (15.5), *where the Hamiltonian* H_-^0 *has the representation*

$$H_-^0(t,x,s) = \min_{s_-\in S_-} \max \min_{(f,g)\in F_-^0(t,x,s_-)} [\langle s,f\rangle - g] \tag{16.14}$$

and

$$F_-^0(t,x,s_-) = \mathrm{co}\Big\{f^0\big(t,x,Y_-^0(t,x,s_-),u_*,Q\big),\ -g^0(t,x,Y_-^0(t,x,s_-),u_*,Q)\Big\}.$$

Condition **(B10)** implies the equality

$$w^{\sharp\uparrow} = w^{\sharp\downarrow} = \mathrm{Val}^0\,. \tag{16.15}$$

This means that Val^0 is a minimax solution of the problem $\mathbf{P}^0$ (15.4), (15.5), where the Hamiltonian is defined as follows:

$$H^0(t,x,s) = H_+^0(t,x,s) = H_-^0(t,x,s). \tag{16.16}$$

Uniform estimate (16.12) and equality (16.15) imply that

$$\lim_{\varepsilon\downarrow 0} \mathrm{Val}^\varepsilon(t,x,y) = \mathrm{Val}^0(t,x);$$

this convergence is uniform on any compact set $D\times(D_+^0\cup D_-^0)$. The proof of Theorem IV.1 is complete.

Note that the limit unperturbed differential games $\mathbf{G}_i^0$, $i=I,II$ (see (15.6)–(15.9)), are reconstructed in accordance with the form of the limit Hamiltonian $H^0(t,x,s)$ (16.13), (16.14), (16.16) and the definition of characteristic complexes corresponding to the Hamiltonian. According to the theory of minimax solutions of Hamilton–Jacobi–Isaacs equations, the minimax solution $\mathrm{Val}^0(t,x)$ of the limit problem $\mathbf{P}^0$ coincides with the optimal guaranteed result of the first player Γ_I^0 in the game G_I^0 and with the optimal guaranteed result of the second player Γ_{II}^0 in the game G_{II}^0. The Hamiltonians in the games $\mathbf{G}_i^0$, $i=I,II$, coincide, and they are equal to $H^0(t,x,s)$. Thus, one can say that the games $\mathbf{G}_i^0$, $i=I,II$, are equivalent relative to the guaranteed results.

17. Example

Consider the following singularly perturbed differential game, where the sufficient conditions presented in Theorem IV.1 are satisfied:

$$\dot{x} = f(t,x,y),\quad \varepsilon\dot{y} = k(y)+\xi(t,x,\alpha,\beta),\quad (t,x,y)\in[t_0,T]\times\mathbb{R}\times\mathbb{R},$$
$$x(t_0)=x_0\in\mathbb{R},\quad y(t_0)=y_0\in\mathbb{R},\quad t_0\in[0,T],$$

where the Lipschitz continuous function $y\to k(y):\mathbb{R}\to\mathbb{R}$ has the form

$$k(y) = \begin{cases} -y & \text{if } y\ge 0,\\ -2y & \text{if } y\le 0.\end{cases}$$

Values of controls of the first and second players (α and β, respectively) are constrained by the restrictions

$$\alpha\in A\subset\mathbb{R},\quad \beta\in B\subset\mathbb{R},$$

where A and B are compact sets.

The terminal time moment T is fixed. The cost functional has the form

$$I^{\varepsilon}_{t_0,x_0,y_0}\big(x(\cdot),y(\cdot)\big) = \sigma(x(T)) + \int_{t_0}^{T} g\big(\tau,x(\tau),y(\tau)\big)d\tau.$$

In the considered game, the first player tries to minimize the cost functional and the second player, on the contrary, tries to maximize the functional.

Assume that the functions $f(\cdot)$, $g(\cdot)$, $\xi(\cdot)$, and $\sigma(\cdot)$ are continuous and the functions $f(\cdot)$, $\xi(\cdot)$,, and $g(\cdot)$ are *Lipschitz continuous* with respect to the variables t, x, and y with a constant $L_* > 0$. Assume also that the Isaacs condition

$$\min_{\alpha\in A}\max_{\beta\in B}\langle s,\xi(t,x,\alpha,\beta)\rangle = \max_{\beta\in B}\min_{\alpha\in A}\langle s,\xi(t,x,\alpha,\beta)\rangle$$

holds for all $(t,x,s) \in [0,T]\times\mathbb{R}\times\mathbb{R}$.

According to constructions considered in this chapter, choose characteristic complexes for the perturbed problem as follows:

$$s_+ = \beta, \quad S_+ = B, \quad s_- = \alpha, \quad S_- = A, \tag{17.1}$$

$$M^{\varepsilon}_+(t,x,y,\beta) = \mathrm{co}\left\{\left(f(t,x,y),\ \frac{1}{\varepsilon}\cdot(\xi-y),\ -g(t,x,y)\right) : \xi\in\mathrm{co}\,\xi(t,x,A,\beta)\right\}, \tag{17.2}$$

$$M^{\varepsilon}_-(t,x,y,\alpha) = \mathrm{co}\left\{\left(f(t,x,y),\ \frac{1}{\varepsilon}\cdot(\xi-y),\ -g(t,x,y)\right) : \xi\in\mathrm{co}\,\xi(t,x,\alpha,B)\right\}. \tag{17.3}$$

The sets of "fast roots" $Y(t,x,\alpha,\beta)$ are defined by the formula

$$Y(t,x,\alpha,\beta) = \varphi(\xi)\xi(t,x,\alpha,\beta),$$

where the function $\xi\mapsto\varphi(\xi)$ is defined as follows:

$$\varphi(\xi) = \begin{cases} 1 & \text{for } \ \xi\geq 0,\\ 1/2 & \text{for } \xi\leq 0.\end{cases}$$

The attractors $Y^{\varepsilon}_{\pm}$ for complexes (17.1)–(17.2) have the form

$$Y^{\varepsilon}_+(t,x,s_+) = Y(t,x,\beta) + B^{\varepsilon}, \quad s_+ = \beta,$$
$$Y^{\varepsilon}_-(t,x,s_-) = Y(t,x,\alpha) + B^{\varepsilon}, \quad s_- = \alpha,$$

where B^{ε} is the closed ball of radius ε in the space $\mathbb{R}$.

In the limit unperturbed problem, one can choose characteristic complexes as follows:

$$s_+ = \beta, \quad S_+ = B, \quad s_- = \alpha, \quad S_- = A,$$
$$M^0_+(t,x,\beta) = \mathrm{co}\Big\{\big(f(t,x,\xi),\ -g(t,x,\xi)\big) : \xi\in Y(t,x,\beta)\Big\},$$
$$M^0_-(t,x,\alpha) = \mathrm{co}\Big\{\big(f(t,x,\xi),\ -g(t,x,\xi)\big) : \xi\in Y(t,x,\alpha)\Big\},$$

where

$$Y(t,x,\beta) = \varphi(\xi)\bigcup_{\alpha\in A}\xi(t,x,\alpha,\beta), \quad Y(t,x,\alpha) = \varphi(\xi)\bigcup_{\beta\in B}\xi(t,x,\alpha,\beta).$$

The limit unperturbed differential games $\mathbf{G}^0_i$, $i = I, II$, can be considered in the framework of the following differential game $\mathbf{G}^0$:

$$\dot{x} = f\big(t,x,Y(t,x,\alpha,\beta)\big) = f\big(t,x,\varphi(\xi)\xi(t,x,\alpha,\beta)\big)$$

with the cost functional

$$I^0_{t_0,x_0}\big(x(\cdot),\alpha(\cdot),\beta(\cdot)\big) = \sigma(x(T)) + \int\limits_{t_0}^{T} g\big(\tau, x(\tau), Y(t,x,\alpha,\beta)\big)d\tau$$
$$= \sigma(x(T)) + \int\limits_{t_0}^{T} g\big((\tau, x(\tau), \varphi(\xi)\xi(t,x,\alpha,\beta)\big)d\tau. \tag{17.4}$$

Admissible controls of the players are measurable functions

$$[0,T] \to A : t \mapsto \alpha(t) \qquad [0,T] \to B : t \mapsto \beta(t). \tag{17.5}$$

The first player has the control $\alpha(\cdot)$ and tries to minimize the cost functional (17.4) and the second player has the control $\beta(\cdot)$ and tries to minimize (17.4).

According to Theorem IV.1, assume that the Isaacs condition

$$\min_{\alpha\in A}\max_{\beta\in B}\Big[\big\langle s, f\big(t,x,\varphi(\xi)\xi(t,x,\alpha,\beta)\big)\big\rangle + g\big(t,x,\varphi(\xi)\xi(t,x,\alpha,\beta)\big)\Big]$$
$$= \max_{\beta\in B}\min_{\alpha\in A}\Big[\big\langle s, f\big(t,x,\varphi(\xi)\xi(t,x,\alpha,\beta)\big)\big\rangle + g\big(t,x,\varphi(\xi)\xi(t,x,\alpha,\beta)\big)\Big].$$

also holds for the limit unperturbed game.

Chapter V

GENERALIZED METHOD OF CHARACTERISTICS IN THEORY OF MINIMAX SOLUTIONS FOR QUASI-LINEAR PARABOLIC EQUATIONS

In Chap. V, we develop the concept of generalized minimax solutions of quasi-linear parabolic equations of the Hamilton–Jacobi–Isaacs type:

$$\frac{\partial\rho(t,x)}{\partial t} + H\left(t,x,\frac{\partial\rho(t,x)}{\partial x}\right) + \sum_{i,j=1}^{n} a_{ij}(t)\frac{\partial^2\rho(t,x)}{\partial x_i\partial x_j} = 0.$$

Such equations arise in the study of stochastic differential games. Investigations of value functions have a great importance for the construction of optimal feedbacks for these games. It is known [76, 79, 146, 168] that the value function of a diffusion game satisfies the above-mentioned parabolic equation at regular points, where the function is sufficient smooth; this second-order partial differential equation is called the Isaacs equation or the main equation of the theory of differential games.

For control processes of diffusion type with a nondegenerate noise and a cost functional of the Mayer type, the value function is smooth. The corresponding Hamilton–Jacobi–Isaacs equation is a nondegenerate parabolic equation, which has the unique smooth classical solution of a boundary-value problem corresponding to the cost functional [192]. The solution coincides with the value function of the considered diffusion game. Thus, in this case, the Hamilton–Jacobi–Isaacs equation defines the value function uniquely.

However, when considering a diffusion control process with degenerate noise, as a rule, the corresponding boundary-value problem for the Hamilton–Jacobi–Isaacs equation of parabolic type has no classical solutions. The value function is defined but it is nonsmooth; it does not satisfy the Hamilton–Jacobi–Isaacs equation on a set of zero measure, and there are many functions satisfying the Hamilton–Jacobi–Isaacs equation almost everywhere. Therefore, we need to improve the definition of generalized solutions in the theory of quasi-linear parabolic partial differential equations to provide uniqueness for a given boundary-value problem and coincidence with the value function for a diffusion differential game.

In Chap. V, we develop the concept of generalized (minimax) solutions of quasi-linear second-order partial differential equations of the Hamilton–Jacobi–Isaacs type by using the generalized method of characteristics.

Stochastic diffusion differential games with terminal cost functionals are considered. Main ideas, statements, methods, and results of research into stochastic games can be found in [29, 76]. The results presented in Chap. V are obtained in the framework of another (positional) statement suggested by Krasovskii [133, 136, 238].

Notions of generalized stochastic derivatives are introduced below. Generalized stochastic derivatives play an important role in necessary and sufficient infinitesimal conditions for the value function of a diffusion differential game. Namely, a pair of differential inequalities is obtained, which contains such derivatives. For deterministic games (without a noise), these inequalities turn out to be the known differential inequalities in terms of the directional Dini semiderivatives [235, 238] (see, e.g., (5.4)). The inequalities are transformed to the Isaacs equation at regular points, where the value function is smooth.

The pair of inequalities for stochastic derivatives underlies the definition of minimax solutions of Hamilton–Jacobi–Isaacs equations. We obtain that a generalized solution of a given boundary problem for a quasi-linear parabolic partial differential equation of the Hamilton–Jacobi–Isaacs type exists, is unique, and coincides with the value function of a stochastic diffusion differential game. The definition of minimax solutions is equivalent to the definition of viscosity solution of Hamilton–Jacobi–Isaacs equations (see [73, 168]).

Also, a class of continuous functions is considered below, where all functions are differentiable with respect to a part of the variables. Formulas for generalized stochastic derivatives of the functions are obtained. These formulas are used for obtaining a more precise form of the quasi-linear parabolic Hamilton–Jacobi–Isaacs equation for diffusion differential games, where noise and controls of players affect a part of the phase variables of a controlled process, simultaneously.

18. Value Functions of a Stochastic Diffusion Differential Game and Its Properties. Generalized Stochastic Derivatives

18.1. Formalization of a positional stochastic differential game. Consider the following diffusion game. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ be a probability space, where $\{\mathcal{F}_s\}$, $s \geq 0$, is a nondecreasing family of σ-algebras of subsets Ω. Let W_s be an m-dimensional standard Wiener process, which is a $\mathcal{F}_s$-martingale. Consider a diffusion control process ξ_s, which is described by the Ito stochastic differential equation [31, 170]:

$$\xi_s = x_0 + \int_0^r f(t_0 + s, \xi_s, u_s, v_s)ds + \int_0^r \psi(t_0 + s)dW_s, \qquad (t_0, x_0) \in \mathbb{T}' \times \mathbb{R}^n, \quad r \in [0, T - t_0], \tag{18.1}$$

where $\mathbb{T}' = [0, T]$ is the fixed interval of real time on which the game is considered. Functions $f(\cdot) : \mathbb{T}' \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ and $\psi(\cdot) : \mathbb{T}' \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ are defined. Denote by $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ the space of all continuous linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$, and by u_s and v_s progressively measurable processes with values in the given compact sets $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$, respectively. The process u_s is called the control of the first player and the process v_s is called the control of the second player. Note that the real time $t \in \mathbb{T}'$ is replaced by $s = t - T_0 \in [0, T - t_0]$ in (18.1). The solution of the stochastic equation (18.1) is understood in the strong sense [31, 170].

Assume that the functions $\psi(\cdot) : \mathbb{T}' \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and $f(\cdot) : \mathbb{T}' \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ are continuous and satisfy the following conditions:

$$\big|\psi_{ij}(t_1) - \psi_{ij}(t_2)\big| \le L_1 |t_1 - t_2|^\alpha, \quad i, j \in \overline{1, n}, \quad (t_1, t_2) \in \mathbb{T}' \times \mathbb{T}', \tag{18.2}$$

$$\max_{\substack{t_0 \le t \le T \\ i,j \in \overline{1,n}}} |\psi_{ij}(t)| \le F_1, \tag{18.3}$$

where $\psi_{ij}(t)$ is an element of the diffusion matrix $\psi(t)$; the constants F_1, L_1, and α are positive, and $\alpha > 1/2$;

$$\sup_{(t,x,u,v) \in \mathbb{T}' \times \mathbb{R}^n \times P \times Q} \big\|f(t, x, u, v)\big\| \le F_2, \tag{18.4}$$

$$\big\|f(t, x_1, u, v) - f(t, x_2, u, v)\big\| \le L_2 \left\|x_1 - x_2\right\|, \tag{18.5}$$

where $F_2 > 0$ and $L_2 > 0$ are constants, $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, $(t, u, v) \in \mathbb{T}' \times P \times Q$,

$$\sup_{\substack{|t_1 - t_2| \le \delta \\ \|x_1 - x_2\| \le \delta \\ \|u_1 - u_2\| \le \delta \\ \|v_1 - v_2\| \le \delta}} \big\|f(t_1, x_1, u_1, v_1) - f(t_2, x_2, u_2, v_2)\big\| \le \beta(\delta), \qquad \beta(\delta) \to 0 \quad \text{as} \quad \delta \downarrow 0, \tag{18.6}$$

where $(t_1, t_2) \in \mathbb{T}' \times \mathbb{T}'$, $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, $(u_1, u_2) \in P \times P$, $(v_1, v_2) \in Q \times Q$, and

$$\min_{u \in P} \max_{v \in Q} \big\langle s, f(t, x, u, v)\big\rangle = \max_{v \in Q} \min_{u \in P} \big\langle s, f(t, x, u, v)\big\rangle = H(t, x, s), \tag{18.7}$$

where $(t, x, s) \in \mathbb{T}' \times \mathbb{R}^n \times \mathbb{R}^n$.

The value $H(t, x, s)$ (18.7) is called the Hamiltonian of the diffusion process (18.1).

Let us consider a cost functional $\gamma_* = \gamma_*(\xi_s)$ estimating the quality of the control process ξ_s as follows:

$$\gamma_*(\xi_s) = E\{\gamma(\xi_{T - t_0})\}, \tag{18.8}$$

where $E\{\cdot\}$ is the mean value of the random variable $\gamma(\xi_{T-t_0}(\omega))$ and T is the fixed terminal time for the considered process. The function $\gamma : \mathbb{R}^n \to \mathbb{R}$ satisfies the condition

$$\big|\gamma(x_1) - \gamma(x_2)\big| \le L_3 \big\|x_1 - x_2\big\|, \tag{18.9}$$

where $L_3 > 0$ is a constant and $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$.

The first player tries to minimize the value γ_* by choosing the control u and the second player tries to maximize γ_* by choosing the control v.

Let us consider the positional formalization [136] of the diffusion differential game (18.1), (18.8) for a class of positional strategies called feedbacks $U(t, x)$ and $V(t, x)$, which are *Borel measurable* functions $U(\cdot) : \mathbb{T}' \times \mathbb{R}^n \to P$ and $V(\cdot) : \mathbb{T}' \times \mathbb{R}^n \to Q$.

Let $(t_0, x_0) \in \mathbb{T}' \times \mathbb{R}^n$ be an initial state of the process, and

$$\Delta = \{0 = \tau_0 < \tau_1 < \cdots < \tau_{k+1} = T - t_0\}$$

be a partition of the τ-time interval $[0, T - t_0]$, which is obtained from the real t-time interval of the game by the substitution $\tau = t - t_0$. Consider a progressively measurable process $v_{(\cdot)} : [0, T - t_0] \times \Omega$ and a feedback $U : \mathbb{T}' \times \mathbb{R}^n \to P$. Denote by $\xi_r(t_0, x_0, U, v_{(\cdot)}, \Delta)$ the random process ξ_r, $r \in [0, T - t_0]$ described by the stochastic equation

$$\xi_r = \xi_{\tau_i} + \int\limits_{\tau_i}^{r} f\big(t_0 + s, \xi_s, u_{\tau_i}, v_s\big)ds + \int\limits_{\tau_i}^{r} \psi(t_0 + s)dW_s,$$
$$r \in [\tau_i, \tau_{i+1}), \quad i = 0, 1, \dots, k,$$
$$u_{\tau_i} = U(\tau_i + t_0, \xi_{\tau_i}) \text{ (}\mathbb{P}\text{-a.e.)},$$
$$\xi_{\tau_0} = x_0 \text{ (}\mathbb{P}\text{-a.e.)}. \tag{18.10}$$

where the notation ($\mathbb{P}$-a.e.) means that the corresponding relation holds almost everywhere on Ω relative to the measure $\mathbb{P}$. Recall that $\operatorname{diam}\Delta = \max\limits_{0 \le i \le k}(\tau_{i+1} - \tau_i)$.

The guaranteed result Γ_1 for the strategy U at the initial state (t_0, x_0) is defined by the following relation:

$$\Gamma_1(t_0, x_0, U) = \limsup_{\operatorname{diam}\Delta \downarrow 0} \sup_{v(\cdot)} E\Big\{\gamma\big(\xi_{T-t_0}(t_0, x_0, U, v_{(\cdot)}, \Delta)\big)\Big\}. \tag{18.11}$$

The optimal guaranteed result of the first player ρ_1 in the class of strategies $U\,(t, x)$ is defined by the relation

$$\rho_1(t_0, x_0) = \inf_U \Gamma_1(t_0, x_0, U). \tag{18.12}$$

Similarly, replacing u by v in (18.10), one can define the guaranteed result Γ_2 of the second player for a strategy $V\,(t, x)$ as follows:

$$\Gamma_2(t_0, x_0, V) = \liminf_{\operatorname{diam}\Delta \downarrow 0} \inf_{v(\cdot)} E\Big\{\gamma\big(\xi_{T-t_0}(t_0, x_0, V, u_{(\cdot)}, \Delta)\big)\Big\}. \tag{18.13}$$

The optimal guaranteed result for the second player is defined by the relation

$$\rho_2(t_0, x_0) = \sup_V \Gamma_2(t_0, x_0, V). \tag{18.14}$$

Assertion V.1 (see [136]). *For any initial position* $(t_0, x_0) \in \mathbb{T}' \times \mathbb{R}^n$, *there exists the value* $\rho_0(t_0, x_0)$ *of the diffusion differential game* (18.1), (18.8), *i.e.,*

$$\rho_1(t_0, x_0) = \rho_2(t_0, x_0) = \rho_0(t_0, x_0). \tag{18.15}$$

The mapping $\rho_0 : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}$ defined by relation (18.15) is called the *value function* of the diffusion game (18.1), (18.8). Note that this function coincides with the value function defined in the framework of another known formalization suggested by Fleming [76], where the value of a diffusion game is obtained by taking the limit of the minorant and majorant discrete games, as the steps of discretizations tend to zero.

18.2. Generalized program controls and stochastic processes under controls. Similarly to the constructions in Sec. 3.3, let us consider the set

$$A_I = \big\{\forall \alpha_{(\cdot)} : s \mapsto \alpha_s : [0, T] \mapsto \operatorname{rpm}(P) \text{ is measurable}\big\} \tag{18.16}$$

of all generalized program controls of the first player, whose elements are measurable mappings defined on the interval $[0, T]$ with values in the set $\operatorname{rpm}(P)$ of all regular probability measures defined on P. On the set $\operatorname{rpm}(P)$, we consider the weak norm generating a topology equivalent to the weak-* topology in the space $C^*(P)$, which is the conjugate space to the space of continuous scalar functions on P. Mappings $s \mapsto \alpha_s$ can be identified with continuous linear functionals defined on the space $\mathcal{B}$ of the Carathéodory functions, i.e., with elements of the space $\mathcal{B}^*$ conjugated to $\mathcal{B}$. As is known [300]), A_I is a compact set in $\mathcal{B}^*$ with respect to the weak-norm topology equivalent to the weak-* topology in the space $\mathcal{B}^*$.

Therefore, the following assertion holds.

Assertion V.2. *The set* A_I (18.16) *of all generalized program controls* α_s *is a metric compact set.*

We prove the following theorem.

Theorem V.1. *For any* $(t_*, x_*, \alpha, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times A_I \times Q$, *there is a unique process* $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ *satisfying the equation*

$$\xi_r = x_* + \int_0^r ds \int_P f\big(t_* + s, \xi_s, u, v_*\big)\alpha_s(du) + \int_0^r \psi(t_* + s)dW_s, \qquad r \in [0, T - t_*], \quad (t_*, x_*) \in T \times \mathbb{R}^n, \quad v_* \in Q, \tag{18.17}$$

where W_s *is an* m*-dimensional Wiener process, and the properties of the functions* $f(\cdot) : \mathbb{T}' \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ *and* $\psi(\cdot) : \mathbb{T}' \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ *are described in Sec.* 18.1.

Proof. The scheme of proof repeats similar reasonings in the standard proofs of theorems on the existence and uniqueness of solutions of stochastic differential equations, which are exposed in many textbooks on the theory of stochastic processes [31, 170].

Uniqueness. Let ξ_r^1 and ξ_r^2 be two solutions of Eq. (18.17). Then

$$\xi_r^1 = \xi_r^2 \quad \mathbb{P}\text{-a.e. on } \Omega, \quad \forall r \in [0, T - t_*]. \tag{18.18}$$

In fact, according to (18.17), we have

$$\xi_r^1 - \xi_r^2 = \int_0^r ds \int_P \Big[f\big(t_* + s, \xi_s^1, u, v_*\big) - f\big(t_* + s, \xi_s^2, u, v_*\big)\Big]\alpha_s(du) + \int_0^r \psi(t_* + s)dW_s - \int_0^r \psi(t_* + s)dW_s. \tag{18.19}$$

The Lipschitz condition (18.5) and (18.19) imply the estimate

$$\big\|\xi_r^1 - \xi_r^2\big\| \le L_2 \int_0^r \big\|\xi_s^1 - \xi_s^2\big\|ds \quad (\mathbb{P}\text{-a.e.}). \tag{18.20}$$

By the Gronwall lemma, we obtain from (18.20) the inequalities

$$0 \le \big\|\xi_r^1 - \xi_r^2\big\| \le 0 \quad (\mathbb{P}\text{-a.e.}). \tag{18.21}$$

Using the continuity of the processes ξ_r^1 and ξ_r^2 in r, we obtain from (18.21)

$$\big\|\xi_r^1 - \xi_r^2\big\| = 0 \quad (\mathbb{P}\text{-a.e.}) \quad \forall r \in [0, T - t_*], \tag{18.22}$$

which is equivalent to condition (18.18).

Existence. Construct the following iterative procedure:

$$y_r^{(0)} = x_*, \qquad y_r^{(n)} = x_* + \int_0^r ds \int_P f\big(t_* + s, y_s^{(n-1)}, u, v_*\big)\alpha_s(du) + \int_0^r \psi(t_* + s)dW_s, \quad n = 1, 2, \ldots. \tag{18.23}$$

Note that for all $n = 0, 1, 2, \dots$, $r \in [0, T - t_*]$, and almost all $\omega \in \Omega$, the trajectories $r \to y_r^{(n)}(\omega)$ of the random process $y_s^{(n)}$ exist and are continuous. Random variables $\omega \mapsto y_r^{(n)}(\omega)$ are measurable with respect to σ-algebra $\mathcal{F}_r$ (see [31, 170]).

Using estimates of the type (18.19), (18.20), one can obtain the inequality for the variables $y_r^{(n)}$ and $y_r^{(n+1)}$:

$$\left\| y_r^{(n+1)} - y_r^{(n)} \right\| \le L_2 \int_0^r \left\| y_s^{(n)} - y_s^{(n-1)} \right\| ds \quad (\mathbb{P}\text{-a.e.}). \tag{18.24}$$

Using estimate (18.24) recurrently, we obtain

$$\left\| y_r^{(n+1)} - y_r^{(n)} \right\| \le L_2^n \int_0^r \frac{(r-s)^{n-1}}{(n-1)!} \left\| y_s^{(1)} - x_* \right\| ds$$

$$\le L_2^n \int_0^r \frac{(r-s)^{n-1}}{(n-1)!} \left[\left\| \int_0^s d\tau \int_P f\big(t_* + \tau, x_*, u, v_*\big) d_\tau(du) \right\| + \left\| \int_0^s \psi(t_* + s) dW_s \right\| \right].$$

By condition (18.4), the Fubini theorem, and the Cauchy inequality, this implies

$$E\left\{ \left\| y_r^{(n+1)} - y_r^{(n)} \right\| \right\} \le L_2^n \int_0^r \frac{(r-s)^{n-1}}{(n-1)!} \left[s \cdot F_2 + + \sqrt{E\left\{ \left(\int_0^s \psi(t_* + \tau) dW_\tau \right)^2 \right\}} \right] ds.$$

Using the properties of stochastic integrals, we obtain

$$E\left\{ \left\| y_r^{(n+1)} - y_r^{(n)} \right\| \right\} \le L_2^n \int_0^r \frac{(r-s)^{n-1}}{(n-1)!} \left[F_2 s + \sqrt{2s \cdot \operatorname{tr} A} \right] ds$$

$$\le L_2^n \left[F_2 r + \sqrt{2r \cdot \operatorname{tr} A} \right] \int_0^r \frac{(r-s)^{n-1}}{(n-1)!} ds = \left[F_2 r + \sqrt{2r \cdot \operatorname{tr} A} \right] \frac{(L_2 r)^n}{n!},$$

where $\operatorname{tr} A = \max\limits_{0 \le t \le T} |\operatorname{tr} A(t)|$ and $\operatorname{tr} A(t)$ is the trace of the matrix $A(t)$.

Moreover, this and (18.24) imply

$$E\left\{ \sup_{0 \le s \le r} \left\| y_s^{(n+1)} - y_s^{(n)} \right\| \right\} \le \left[F_2 r + \sqrt{2r \cdot \operatorname{tr} A} \right] \frac{(L_2 r)^n}{n!} \le C_1 \frac{(L_2 r)^n}{n!},$$

where

$$C_1 = F_2 T + \sqrt{2T \cdot \operatorname{tr} A}.$$

By the Chebyshev inequality, for any $\varepsilon \in (0, 1)$, we have

$$\mathbb{P}\left\{ \sup_{0 \le s \le r} \left\| y_s^{(n+1)} - y_s^{(n)} \right\| > \frac{\varepsilon}{n^2} \right\} \le \frac{E\left\{ \sup\limits_{0 \le s \le r} \left\| y_s^{(n+1)} - y_s^{(n)} \right\| \right\}}{\varepsilon / n^2} \le \frac{C_1}{\varepsilon} \cdot \frac{(L_2 r)^n}{n!} n^2. \tag{18.25}$$

Using (18.25), we obtain the estimate

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \sup_{0 \le s \le r} \left\| y_s^{(n+1)} - y_s^{(n)} \right\| > \frac{\varepsilon}{n^2} \right\} \le \frac{C_1}{\varepsilon} \sum_{n=1}^{\infty} \frac{(L_2 r)^n}{n!} n^2 \le \frac{C_1 \cdot C_0}{\varepsilon} < \infty, \tag{18.26}$$

where

$$C_0 = \sum_{n=1}^{\infty} (L_2 T)^n \frac{n^2}{n!} < \infty.$$

Consider the sets

$$\begin{aligned} B_m^{\varepsilon} &= \left\{ \omega \in \Omega : \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\| > \frac{\varepsilon}{m^2} \right\}, \quad m = 1, 2, \ldots, \\ B^{\varepsilon} &= \bigcap_{n=1}^{\infty} \bigcup_{m\ge n} B_m^{\varepsilon} = \left\{ \omega \in \Omega : \forall n\ \exists m \ge n \quad \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\| > \frac{\varepsilon}{m^2} \right\}. \end{aligned} \tag{18.27}$$

According to the Borel–Cantelli lemma, it follows from (18.25)–(18.27)

$$P(B^{\varepsilon}) = 0, \quad \varepsilon \in (0, 1). \tag{18.28}$$

Consider the sequence

$$\left\{ \varepsilon_l = \frac{6}{\pi^2 l} \right\}, \quad l = 1, 2, \ldots.$$

For $\varepsilon = \varepsilon_l$, it follows from (18.28) that

$$\sum_{l=1}^{\infty} P(B^{\varepsilon_l}) = 0. \tag{18.29}$$

Then for the set

$$B_1 = \bigcup_{l=1}^{\infty} B^{\varepsilon_l} = \left\{ \omega \in \Omega : \exists \varepsilon_l\ \forall n\ \exists m \ge n \quad \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\| > \frac{\varepsilon_l}{m^2} \right\}, \tag{18.30}$$

we obtain from (18.29)

$$0 \le \mathbb{P}(B_1) = \mathbb{P}\left(\bigcup_{l=1}^{\infty} B^{\varepsilon_l} \right) \le \sum_{l=1}^{\infty} \mathbb{P}(B^{\varepsilon_l}) = 0. \tag{18.31}$$

Let us prove the following inclusion:

$$B_2 = \left\{ \omega : \lim_{n\to\infty} \sum_{m=n}^{\infty} \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\| > 0 \right\} \subset B_1. \tag{18.32}$$

If $\omega \in B_2$, then there exist numbers $l^0 > 0$ and $N < \infty$ such that the inequality

$$\sum_{m=n}^{\infty} \sup_{0\le s\le r} \left\| y_s^{(m+1)}(\omega) - y_s^{m}(\omega) \right\| > \frac{1}{l^0} \tag{18.33}$$

holds for any $n \ge N$. Hence, for any n, there exists $m \ge n$ such that the inequality

$$\sup_{0\le s\le r} \left\| y_s^{(m+1)}(\omega) - y_s^{(m)}(\omega) \right\| > \frac{\varepsilon_{l^0}}{m^2} \tag{18.34}$$

holds. Otherwise, for some $n \ge N$, we have

$$\sum_{m=n}^{\infty} \sup_{0\le s\le r} \left\| y_s^{(m+1)}(\omega) - y_s^{(m)}(\omega) \right\| \le \varepsilon_{l^0} \sum_{m=n}^{\infty} \frac{1}{m^2} \le \varepsilon_{l^0} \frac{\pi^2}{6} = \frac{6}{\pi^2 l_0} \frac{\pi^2}{6} = \frac{1}{l^0},$$

which contradicts (18.33). Thus, for $\omega \in B_2$, we see that $\omega \in B_1$.

Further, the inequalities

$$0 \le \lim_{n\to\infty} \sup_{0\le s\le r} \left\| y_s^{(n)} - y_s \right\| \le \lim_{n\to\infty} \sup_{0\le s\le r} \left\| \sum_{m=n}^{\infty} y_s^{(m+1)} - y_s^{(m)} \right\| \le \lim_{n\to\infty} \sum_{m=n}^{\infty} \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\|$$

and inclusions (18.32) imply

$$
\begin{aligned}
B_3 &= \left\{\omega : \lim_{n\to\infty} \sup_{0\le s\le r} \left\| y_s - y_s^{(n)} \right\| > 0\right\} \\
&= \left\{\omega : \lim_{n\to\infty} \sup_{0\le s\le r} \left\| \sum_{m=n}^{\infty} y_s^{(m+1)} - y_s^{(m)} \right\| > 0\right\} \\
&\subset \left\{\omega : \lim_{n\to\infty} \sum_{m=n}^{\infty} \sup_{0\le s\le r} \left\| y_s^{(m+1)} - y_s^{(m)} \right\| > 0\right\} \subset B_1.
\end{aligned}
$$

This and (18.31) imply

$$0 \le \mathbb{P}(B_3) \le \mathbb{P}(B_1) = 0.$$

This means that the uniform convergence

$$y_s^{(n)} \to y_s = x_* + \sum_{m=1}^{\infty} \left[y_s^{(m+1)} - y_s^{(m)} \right]$$

holds on the interval $[0, r]$ for almost all $\omega \in \Omega$. The continuity of the random processes $y_s^{(n)}$ implies the continuity of the process y_s.

By the Lebesque theorem, one can pass to the limit as $n \to \infty$ in Eq. (18.23) and obtain that the process y_s satisfies the stochastic equation

$$y_r = x_* + \int_o^r ds \int_P f\big(t_* + s, y_s, u, v_*\big)\alpha_s(du) + \int_0^r \psi(t_* + s)dW_s, \quad r \in [0, T - t_*].$$

It was mentioned above that for any fixed $(t_*, x_*, \alpha, v_*, r)$, the constructed random variables $y_r^n(\omega)$ are measurable with respect to σ-algebra $\mathcal{F}_r$. Hence, the pointwise limit $y_r(\omega)$ is also measurable with respect to the σ-algebra.

Thus, the existence of a solution $\xi_r = y_r$ of Eq. (18.17) is proved. □

Theorem V.2. *The solution $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ of Eq.* (18.17) *has the following properties*:

(i) *the mapping $\omega \mapsto \xi_r(t_*, x_*, \alpha, v_*)(\omega)$ is measurable with respect to $\mathcal{F}_r$ for any fixed $(t_*, x_*, \alpha, v_*, r)$;*

(ii) *the mapping*

$$T \times \mathbb{R}^n \times A_I \times Q \times [0, T - t_*] \to \mathbb{R}^n : (t_*, x_*, \alpha, v_*, r) \mapsto \xi_r(t_*, x_*, \alpha, v_*)(\omega)$$

is continuous for almost all $\omega \in \Omega$.

Proof. Property (i) is proved in Theorem V.1. To prove property (ii), we perform the following estimates.

Let $J_\omega(\delta) : [0, T] \to \mathbb{R}^n$ be a trajectory of the stochastic integral $\int_0^\delta dW_s$ corresponding to an event $\omega \in \Omega$. By properties of the Wiener process W_s, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\max_{0\le\delta\le T} \big\| J_\omega(\delta) \big\| \le F_\omega < \infty,$$

$$\sup_{\substack{\delta^1,\delta^2\in[0,T] \\ |\delta^1-\delta^2|=\Delta}} \big\| J_\omega(\delta^1) - J_\omega(\delta^2) \big\| \le \beta_\omega(\Delta), \quad \beta_\omega(\Delta) \to 0 \ \text{ as } \Delta \to 0.$$

Let $\alpha^i, \alpha^* \in A_I$ (see (18.16)),

$$v_i, v^* \in Q, \quad 0 \le r_* < r_i \le T,$$
$$(t_*, t_0) \in T \times \mathbb{R}^n, \quad (t_i, x_i) \in T \times \mathbb{R}^n, \quad t_i \ge t_0.$$

For $\xi^i_{r_i} = \xi_{r_i}(t_i, x_i, \alpha^i, v_*)$ and $\xi^*_{r*} = \xi_{r*}(t_*, x_*, \alpha^*, v_*)$, the following representations hold:

$$\begin{aligned}
\xi^i_{r_i} = x_i + \int_0^{r_i} ds \int_P f\big(t_i + s, \xi^i_s, u, v_i\big)\alpha^i_s(du) + \int_0^{r_i} \psi(t_i + s) dW_s \\
= x_i + \int_{t_i - t_*}^{r*} ds \int_P f\big(t_* + s, \xi^i_s, u, v_i\big)\alpha^i_s(du) + \int_0^{r_* - (t_i - t_*)} \psi(t_i + s) dW_s \\
+ \int_0^{r_i - r_*} ds \int_P f\big(t_* + r_* + s, \xi^i_s, u, v_i\big)\alpha^i_s(du) + \int_0^{(r_i - r_*) - (t_i - t_*)} \psi(t_* + r_* + s) dW_s,
\end{aligned} \tag{18.35}$$

$$\begin{aligned}
\xi^*_{r*} = x_* + \int_0^{r_*} ds \int_P f\big(t_* + s, \xi^*_s, u, v_*\big)\alpha^*_s(du) + \int_0^{r*} \psi(t_* + s) dW_s \\
= x_* + \int_0^{t_i - t_*} ds \int_P f\big(t_* + s, \xi^*_s, u, v_*\big)\alpha^*_s(du) + \int_0^{t_i - t_*} \psi(t_* + s) dW_s \\
+ \int_{t_i - t_*}^{r_*} ds \int_P f\big(t_* + s, \xi^*_s, u, v_*\big)\alpha^*_s(du) + \int_0^{r_* - (t_i - t_*)} \psi(t_i + s) dW_s.
\end{aligned} \tag{18.36}$$

Introduce the notation

$$\Phi_*[\alpha^i - \alpha^*] = \int_{t_i - t_*}^{r_*} ds \int_P f\big(t_* + s, \xi^*_s, u, v_*\big)[\alpha^i - \alpha^*](du).$$

For $\mathbb{P}$-a.e. $\omega \in \Omega$, one can obtain the estimate

$$\begin{aligned}
\big\|\xi^i_{r_i}(\omega) - \xi^*_{r_i}(\omega)\big\| \le \|x_i - x_*\| + F_2|r_i - r_*| \\
+ F_1 n^2 \cdot \beta_\omega\big(|r_i - r_*| + |t_i - t_*|\big) + F_2|t_i - t_*| + F_1 n^2 \cdot \beta_\omega\big(|t_i - t_*|\big) \\
+ \left\| \int_{t_i - t_*}^{r*} ds \left[\int_P f\big(t_* + s, \xi^i_s, u, v_i\big)\alpha^i_s(du) - \int_P f\big(t_* + s, \xi^*_s, u, v_*\big)\alpha^*_s(du) \right] \right\| \\
\le \|x_i - x_*\| + F_2 \cdot \big(|t_i - t_*| + |r_i - r_*|\big) + 2F_1 n^2 \cdot \beta_\omega\big(|t_i - t_*| + |r_i - r_*|\big) \\
+ T \cdot \beta(v_i - v_*) + L_2 \int_{t_i - t_*}^{r*} |\xi^i_s - \xi^*_s| ds + \Phi_*[\alpha^i - \alpha^*].
\end{aligned} \tag{18.37}$$

Applying the Gronwall lemma and the estimate

$$\big\|\xi^i_{r_i}(\omega) - \xi^*_{r*}(\omega)\big\| \le \big\|\xi^i_{r_i}(\omega) - \xi^*_{r_i}(\omega)\big\| + \big(F_2 + F_1 n^2 \cdot \beta_\omega(|r_i - r_*|)\big),$$

we obtain from (18.18) the inequality

$$\big\|\xi^i_{r_i}(\omega) - \xi^*_{r*}(\omega)\big\| \le e^{L_2 \cdot (|t_* + r_* - t_i|)} \chi_\omega\big(|t_i - t_*|,\ |r_i - r_*|,\ \|x_i - x_*\|,\ \|v_i - v_*\|,\ \Phi_*[\alpha^i - \alpha^*]\big), \tag{18.38}$$

where

$$\chi_\omega\big(|t_i - t_*|,\ |r_i - r_*|,\ \|x_i - x_*\|,\ \|v_i - v_*\|,\ \Phi_*[\alpha^i - \alpha^*]\big) \to 0 \tag{18.39}$$

as

$$t_i \downarrow t_*, \quad r_i \downarrow r_*, \quad x_i \to x_*, \quad v_i \to v_*, \quad \alpha^i \xrightarrow{\omega} \alpha^*. \tag{18.40}$$

Estimate (18.37) remains valid for $r_i < r_*$ and $t_i < t_*$ after appropriate corrections in formulas (18.35)–(18.36). Therefore, estimates (18.38)–(18.40) still hold if $t_i \to t_*$ and $r_i \to r_*$ in (18.40) instead of $t_i \downarrow t_*$ and $r_i \downarrow r_*$. Theorem V.2 is proved. ☐

18.3. Properties of generalized program controls and random processes under controls. In this section, we study the properties of generalized program controls and random processes generated by the controls; they will be used in the proof of Theorem V.3. All lemmas are proved for generalized program controls of the first player. Similar assertions for generalized program controls of the second player and random processes generated by the controls can be obtained by replacing in all reasonings below v_* by u_*, $\alpha \in A_I$ by $\alpha \in A_{II}$; the symbol min by the symbol max; the set $F_1(t_*, x_*, v_*)$ (18.60) by the set $F_2(t_*, x_*, u_*)$ (18.82); in formulas (18.77) and (18.78), the symbol $\overline{\xi}$ by the symbol ξ^*; and in formula (18.81), the symbol ξ^* by the symbol $\overline{\xi}$. In this section, a number of useful assertions from the theory of random processes is presented without proof but there are references to sources where the proofs can be found.

Lemma V.1 (see [170, Lemma 1.5])**.** *Let $\tau(\cdot)$ be a Markov moment relative to $\{\mathcal{F}_s\}$ and $\mathcal{F}_\tau$ be the σ-algebra of subsets in Ω generated by this moment. Then $\tau(\cdot)$ is measurable relative to $\mathcal{F}_\tau$. If $\tau^{(1)}(\cdot)$ and $\tau^{(2)}(\cdot)$ are two Markov moments and $\tau^1(\omega) \le \tau^2(\omega)$ $\mathbb{P}$-a.e., then $\mathcal{F}_{\tau^{(1)}} \subset \mathcal{F}_{\tau^{(2)}}$.*

Lemma V.2 (see [170, Corollary of Lemma 1.8])**.** *Let $(\varphi_s, \mathcal{F}_s)$, $s \in T$, be a continuous from the right (or from the left) random process. Then $\varphi_\tau = \varphi_{\tau(\omega)}(\omega)$ is a random variable measurable relative to $\mathcal{F}_\tau$.*

Lemma V.3. *Let $\tau^{(1)} : \Omega \mapsto [0, T - t_*]$ be a Markov moment relative to a nondecreasing system of σ-algebras $\{\mathcal{F}_s\}$, $\mathcal{F}_{\tau^{(1)}}$ be a σ-algebra on Ω generated by $\tau^{(1)}$, and*

$$\Omega \to [0, T] : \omega \mapsto \delta[\omega] \tag{18.41}$$

be a $\mathcal{F}_{\tau^{(1)}}$-measurable mapping. Then the random function

$$\omega \mapsto \tau^0(\omega) = \tau^1(\omega) + \delta[\omega] \tag{18.42}$$

is a Markov moment relative to $\{\mathcal{F}_s\}$.

Proof. The proof of this lemma is similar to the proof of [170, Lemma 1.3] and follows from the relation

$$\begin{aligned} \Big\{\omega : \tau^{(1)}(\omega) + \delta[\omega] \le s\Big\} = \Big\{\omega : \tau^{(1)}(\omega) = 0,\ \delta[\omega] = s\Big\} \cup \big\{\tau^1(\omega) = s,\ \delta[\omega] = 0\big\} \\ \cup \Bigg(\bigcup_{\substack{a+b<s \\ a,b\ge 0}} \Big\{\omega : \tau^{(1)}(\omega) < a,\ \delta[\omega] < b\Big\} \Bigg), \end{aligned} \tag{18.43}$$

where a and b are rational numbers. According to definitions of $\mathcal{F}_\tau$ and $\{\mathcal{F}_s\}$ and [170, Lemma 1.1], we have

$$\begin{gathered} \Big\{\omega : \tau^{(1)}(\omega) = 0,\ \delta[\omega] = s\Big\} \in \mathcal{F}_0 \subset \mathcal{F}_s, \\ \Big\{\omega : \delta[\omega] = 0,\ \tau^{(1)}(\omega) = s\Big\} \cap \Big\{\omega : \tau^{(1)} \le s\Big\} = \Big\{\omega : \delta\,[\omega] = 0,\ \tau^{(1)}\,(\omega) = s\Big\} \in \mathcal{F}_s, \\ \Big\{\omega : \tau^{(1)}(\omega) < a,\ \delta[\omega] < b\Big\} \in \mathcal{F}_a \subset \mathcal{F}_s \end{gathered} \tag{18.44}$$

for $s \ge 0$. For any $s \in [0, T]$, we also have

$$\Big\{\omega : \tau^{(0)}(\omega) \le s\Big\} \in \mathcal{F}_s, \tag{18.45}$$

i.e., $\tau^0(\omega) = \tau^{(1)}(\omega) + \delta[\omega]$ is a Markov moment relative to $\{\mathcal{F}_s\}$ and $\tau^0(\omega) \ge \tau^{(1)}(\omega)$ for $\omega \in \Omega\mathbb{P}$-a.e.. ☐

Lemma V.4. *Let $\tau^{(1)}(\cdot)$ and $\tau^{(2)}(\cdot)$ be Markov moments relative to $\{\mathcal{F}_s\}$ and $\tau^{(1)}(\omega) \le \tau^{(2)}(\omega)$ $\mathbb{P}$-a.e.. Let $\alpha^{(i)}[\omega] : \Omega \to A_I$, $i = 1, 2$, be measurable mappings relative to σ-algebras $\mathcal{F}_{\tau^{(i)}}$, respectively. Then the mapping*

$$\omega \mapsto \alpha^0[\omega] = \begin{cases} \alpha_s^{(1)}[\omega] & \text{for } 0 \le s \le \tau^{(1)}(\omega), \\ \alpha_s^{(2)}[\omega] & \text{for } \tau^{(1)}(\omega) \le s \le T \end{cases} \tag{18.46}$$

is measurable relative to σ-algebra $\mathcal{F}_{\tau^{(2)}}$.

Proof. According to the definition of the σ-algebra $\mathcal{F}_{\tau^{(2)}}$, it suffices to show that

$$\Big\{\omega : \alpha^0[\omega] \in \mathcal{A}\Big\} \cap \Big\{\omega : \tau^{(2)}(\omega) \le t\Big\} \in \mathcal{F}_t \tag{18.47}$$

for any closed subset $\mathcal{A}$ of the set A_I and any $t \in \mathbb{T}'$.

Let us construct the following multi-valued mappings on the interval $[0, T]$ with values in the set of all closed subsets of metric compact set A_I:

$$t \mapsto \mathcal{A}_t^1 = \left\{ \alpha \in A_I \;\middle|\; \begin{array}{l} \alpha_s = \alpha_s^0[\omega],\ 0 \le s \le t,\ \omega \in \Omega, \\ \alpha_s : [t, T] \mapsto \mathrm{rpm}(P) \text{ is measurable if } \alpha^0[\omega] \in \mathcal{A}, \\ \text{otherwise } \mathcal{A}_t^1 = \emptyset \end{array} \right\}, \tag{18.48}$$

$$t \mapsto \mathcal{A}_t^2 = \left\{ \alpha \in A_I \;\middle|\; \begin{array}{l} \alpha_s = \alpha_s^0[\omega],\ t \le s \le T,\ \omega \in \Omega, \\ \alpha_s : [0, t] \mapsto \mathrm{rpm}(P) \text{ is measurable if } \alpha^0[\omega] \in \mathcal{A}, \\ \text{otherwise } \mathcal{A}_t^2 = \emptyset. \end{array} \right\}. \tag{18.49}$$

The mapping $t \mapsto \mathcal{A}_t^1$ is continuous from the right (and upper semicontinuous) and the mapping $t \mapsto \mathcal{A}_t^2$ is continuous from the left (and upper semicontinuous) relative to the Hausdorff metric. Hence, by Lemmas V.1 and V.2, the mappings

$$\omega \mapsto \mathcal{A}^{(1)}_{\tau^{(1)}(\omega)}, \quad \omega \mapsto \mathcal{A}^{(2)}_{\tau^{(1)}(\omega)} \tag{18.50}$$

are $\mathcal{F}_{\tau^{(1)}}$-measurable and $\mathcal{F}_{\tau^{(2)}}$-measurable. The measurability and the definitions of $\alpha^{(i)}[\omega]$, $i = 1, 2$, imply

$$\Big\{\omega : \alpha^{(1)}[\omega] \in \mathcal{A}^1_{\tau^{(1)}(\omega)}\Big\} \subset \mathcal{F}_{\tau^{(1)}} \subset \mathcal{F}_{\tau^{(2)}}, \quad \Big\{\omega : \alpha^{(2)}[\omega] \in \mathcal{A}^2_{\tau^{(1)}(\omega}\Big\} \subset \mathcal{F}_{\tau^{(2)}}. \tag{18.51}$$

Therefore, (18.51), the relations

$$\Big\{\omega : \alpha^{(0)}[\omega] \in \mathcal{A}\Big\} = \Big\{\omega : \alpha^{(1)}[\omega] \in \mathcal{A}^1_{\tau^{(1)}(\omega)}\Big\} \cap \Big\{\omega : \alpha^{(2)}[\omega] \in \mathcal{A}^2_{\tau^{(1)}(\omega}\Big\}, \tag{18.52}$$

and the definition of the σ-algebra $\mathcal{F}_{\tau^{(2)}}$ imply that condition (18.47) holds. □

Lemma V.5. *Let $\rho : \mathbb{T}' \times \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function, $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ be a solution of the stochastic equation (18.17), $\tau(\cdot)$ be a Markov moment relative to $\{\mathcal{F}_s\}$, and $\mathcal{F}_\tau$ be a σ-algebra on Ω generated by $\tau(\cdot)$. Then there exists a mapping $\Omega \to A_I : \omega \mapsto \alpha^*[\omega]$, which is measurable relative to $\mathcal{F}_\tau$, and the equality*

$$\rho\Big(t_* + \tau(\omega), \xi_{\tau(\omega)}\big(t_*, x_*, \alpha^*[\omega], v_*\big)(\omega)\Big) = \min_{\alpha \in A_I} \rho\Big(t_* + \tau(\omega), \xi_{\tau(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big) \tag{18.53}$$

holds for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. According to Assertion V.2 and Lemma V.2, the functions

$$\rho(\alpha; \cdot) : \omega \mapsto \rho\Big(t_* + \tau(\omega), \xi_{\tau(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big),$$

$$\rho[\cdot] : \omega \mapsto \min_{\alpha \in A_I} \rho\Big(t_* + \tau(\omega), \xi_{\tau(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big)$$

are $\mathcal{F}_\tau$-measurable for any fixed

$$(t_*, x_*, \alpha, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times A_I \times Q, \quad (t_*, x_*, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times Q,$$

respectively. For any fixed $(t_*, x_*, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times Q$ and almost all $\omega \in \Omega$, the function

$$\rho(\cdot, \omega) : A_I \to \mathbb{R}^n, \quad \alpha \mapsto \rho\Big(t_* + \tau(\omega), \xi_{\tau(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big) \tag{18.54}$$

is continuous. We extend the function $\rho(\alpha, \omega)$ to provide the continuity of this function for all $\omega \in \Omega$. For example, one can set it equal to a constant for all points, where it is not defined, i.e., on $A_I \times E$, $E \in \mathcal{F}_\tau$, $\mathbb{P}(E) = 0$. Let us also achieve that the function $\omega \mapsto \rho(\alpha, \omega)$ is measurable relative to $\mathcal{F}_\tau$ for all $\alpha \in A_I$.

After this, one can use Corollary 6.3 and Theorem 7.2 in [103] and obtain that for the multi-valued mapping

$$\omega \mapsto A^*(\omega) = \begin{cases} \alpha^* \in A_I : \rho(\alpha^*, \omega) - \rho[\omega] = 0 & \text{for } \omega \in \Omega \setminus E, \\ A_I, & \text{for } \omega \in E, \end{cases} \tag{18.55}$$

an $\mathcal{F}_\tau$-measurable selector

$$\Omega \to A_I, \quad \omega \mapsto \alpha^*[\omega], \qquad \alpha^*[\omega] \in A^*(\omega), \quad \omega \in \Omega, \tag{18.56}$$

exists and definition (18.55) implies (18.53). Lemma V.5 is proved. □

Corollary V.1. *Let $\rho : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}^1$ be a continuous function and $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ be a solution of the stochastic equation* (18.17). *Then for any $r \in (0, T - t_*)$, there is a mapping $\Omega \to A_I : \omega \mapsto \alpha^*[\omega]$, which is measurable relative to $\mathcal{F}_r$ and the equality*

$$\rho\Big(t_* + r,\ \xi_r = \xi_r\big(t_*, x_*, \alpha^*[\omega], v_*\big)\Big) = \min_{\alpha \in A_I} \rho\Big(t_* + r, \xi_\tau\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big) \tag{18.57}$$

holds for almost all $\omega \in \Omega$.

Proof. To prove this assertion, one can choose $\tau(\omega) \equiv r$. In this case, the functions $\omega \mapsto \rho(\alpha, \omega)$ and $\omega \mapsto \rho[\omega]$ are $\mathcal{F}_r$-measurable and the selector $\omega \mapsto \alpha^*[\omega]$ (see (18.56)) obtained by the scheme in the proof of Lemma V.5 is also measurable relative to $\mathcal{F}_r$. □

Lemma V.6. *Let $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ be a solution of the stochastic equation* (18.17). *Consider the superposition $\xi_r[\omega] = \xi_r(t_*, x_*, \alpha[\omega],\ v_*)(\omega)$, where $\Omega \to A_I : \omega \mapsto \alpha[\omega]$ is a mapping measurable relative to σ-algebras $\mathcal{F}_r$. Then*

$$E\big\{\|\xi_r[\omega] - x_*\|\big\} < \infty. \tag{18.58}$$

Proof. First, note that Theorem V.2 implies that the superposition $\omega \mapsto \xi_r[\omega]$ of the mappings

$$(\alpha, \omega) \mapsto \xi_r(t_*, x_*, \alpha, v_*)(\omega), \quad \omega \mapsto \alpha[\omega]$$

is measurable relative to $\mathcal{F}_r$. The following representation holds:

$$\xi_r[\omega] = x_* + f_*[\omega] \cdot r + \sqrt{r}\psi(t_*)\eta(\omega) + \xi_r^{(1)}(\omega) + \xi_r^{(2)}(\omega) + \xi_r^{(3)}(\omega). \tag{18.59}$$

Indeed,

$$\xi_r[\omega] = x_* + \int_0^r ds \int_P f\big(t_* + s, \xi_s[\omega], u, v_*\big)\alpha_s[\omega](du) + \int_0^r \psi(t_* + s) dW_s(\omega)$$

$$= x_* + \int_0^r ds \int_P f\big(t_*, x_*, u, v_*\big)\alpha_s[\omega](du) + \psi(t_*) \int_0^r dW_s(\omega)$$

$$+\int_0^r ds \int_P \Big[f\big(t_*+s,x_*,u,v_*\big)-f\big(t_*,x_*,u,v_*\big)\Big]\alpha_s[\omega](du)$$

$$+\int_0^r ds \int_P \Big[f\big(t_*+s,\xi_s[\omega],u,v_*\big)-f\big(t_*+s,x_*,u,v_*\big)\Big]\alpha_s[\omega](du)$$

$$+\int_0^r \big[\psi(t_*+s)-\psi(t_*)\big]dW_s\,(\omega)$$

$$= x_* + f_*[\omega]\cdot r + \sqrt{r}\psi(t_*)\eta(\omega) + \xi_r^{(1)}(\omega) + \xi_r^{(2)}(\omega) + \xi_r^{(3)}(\omega),$$

where

$$f_*[\omega] \in \mathrm{co}\,\big\{f(t_*,x_*,u,v_*) : u \in P\big\} = F_1(t_*,x_*,v_*), \tag{18.60}$$

$$\sqrt{r}\cdot\eta(\omega) = \int_0^r dW_s(\omega). \tag{18.61}$$

By the properties of the Wiener process W_s, the random variable $\eta_{(\omega)}$ is the m-dimensional normalized Gauss variable with independent components,

$$\xi_r^{(1)}(\omega) = \int_0^r ds \int_P \Big[f\big(t_*+s,x_*,u,v_*\big)-f\big(t_*,x_*,u,v_*\big)\Big]\alpha_s[\omega](du), \tag{18.62}$$

$$\xi_r^{(2)}(\omega) = \int_0^r ds \int_P \Big[f\big(t_*+s,\xi_s[\omega],u,v_*\big)-f\big(t_*+s,x_*,u,v_*\big)\Big]\alpha_s[\omega](du), \tag{18.63}$$

$$\xi_r^{(3)}(\omega) = \int_0^r \big[\psi(t_*+s)-\psi(t)\big]dW_s. \tag{18.64}$$

Using conditions (18.3) and (18.4), we obtain the following estimates:

$$\Big\|\xi_r^{(1)}\Big\| \le \int_0^r ds \int_P \Big\|f\big(t_*+s,x_*,u,v_*\big)-f\big(t_*,x_*,u,v_*\big)\Big\|\alpha_s[\omega](du) \le \beta(r)r, \tag{18.65}$$

$$\Big\|\xi_r^{(2)}\Big\| \le \int_0^r ds \int_P \Big\|f\big(t_*+s,\xi_s[\omega],u,v_*\big)-f\big(t_*+s,x_*,u,v_*\big)\Big\|\alpha_s[\omega](du)$$

$$\le L_2\cdot\int_0^r \big\|\xi_s[\omega]-x_*\big\|ds, \tag{18.66}$$

$$\Big\|\xi_r^{(3)}\Big\| \le L_1\cdot r^{\alpha+\frac{1}{2}}\cdot\sqrt{n\sum_{i=1}^m \eta_i^2(\omega)}, \qquad E\left\{\Big\|\xi_r^{(3)}\Big\|\right\} \le L_1\cdot r^{\alpha+\frac{1}{2}}\cdot\sqrt{n\cdot m}. \tag{18.67}$$

Thus, we obtain from (18.59)–(18.67), (18.3), and (18.4) the estimate

$$E\Big\{\big\|\xi_{r(\omega)}-x_*\big\|\Big\} \le K_2 r + \sqrt{rmn} + \beta(r)r + L_2\int_0^r E\Big\{\big\|\xi_s[\omega]-x_*\big\|\Big\}ds + L_1\cdot r^{\alpha}\sqrt{rmn}. \tag{18.68}$$

We set

$$K = K_2 T + K_1 \sqrt{mn} + \max_{r\in[0,T]} \beta(r) \cdot \sqrt{T} + L_1 \cdot T^\alpha \sqrt{rm} < \infty. \tag{18.69}$$

Using the Gronwall lemma, we obtain from (18.68) the estimate

$$E\Big\{ \big\| \xi_s[\omega] - x_* \big\| \Big\} \le \sqrt{r} K \exp L_2 r < \infty. \tag{18.70}$$

Lemma V.6 is proved. □

Consider the class Lip of functions $\rho : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the Lipschitz conditions with positive constants $L_\rho < \infty$, namely,

$$\big|\rho(t_1, x_1) - \rho(t_2, x_2)\big| < L_\rho \big(|t_1 - t_2| + \|x_1 - x_2\|\big) \quad \forall t_1, t_2 \in [0,T] = \mathbb{T}', \quad x_1, x_2 \in \mathbb{R}^n. \tag{18.71}$$

Lemma V.7. *Let $\rho \in$ Lip and $\xi_r = \xi_r(t_*, x_*, \alpha, v_*)$ be a solution of the stochastic equation* (18.17)*. Then for any $(t_*, x_*) \in \mathbb{T}' \times \mathbb{R}^n$, $r \in (0, T - t_*]$, and $v_* \in Q$, the estimate*

$$\left| E\left\{ \min_{\alpha\in A_I} \rho\big(t_* + r, \xi_r(t_*, x_*, \alpha, v_*)\big) \right\} - E\left\{ \min_{f \in F_1(t_*, x_*, v_*)} \rho\big(t_* + r, x_* + f \cdot r + \sqrt{r}\psi(t_*)\eta\big) \right\} \right| \le r \cdot \zeta(r) \tag{18.72}$$

holds, where

$$\zeta(r) \to 0 \quad \textit{as} \quad r \to 0, \tag{18.73}$$

The class A_I is defined by (18.16) *and the compact set $F_1(t_*, x_*, v_*)$ is defined by* (18.60)*.*

Proof. Let the equality

$$E\left\{ \min_{f \in F_1(t_*, x_*, v_*)} \rho\big(t_* + r, x_* + f \cdot r + \sqrt{r}\psi(t_*)\eta\big) \right\} = E\Big\{ \rho\big(t_* + r, x_* + f^0[\omega] \cdot r + \sqrt{r}\psi(t_*)\eta(\omega)\big) \Big\} \tag{18.74}$$

hold for $\rho \in$ Lip, where the $\mathcal{F}_r$-measurable selector $f^0[\cdot] : \Omega \mapsto F_1(t_*, x_*, v_*)$ is chosen by the scheme in the proof of Lemma V.5. According to the Carathéodory theorem (see [300, Theorem I.6.2]), for the selector in the convex compact set $F_1(t_*, x_*, v_*)$ (see (18.60)), there exists a measure $\alpha^0[\omega] \in \mathrm{rpm}(P)$, $\omega \in \Omega$, such that the representation

$$f^0[\omega] = \int_P f\big(t_*, x_*, u, v_*\big) \alpha^0[\omega](du) \tag{18.75}$$

holds and the mapping $\Omega \to \mathrm{rpm}(\mathbb{P}) : \omega \mapsto \alpha^0[\omega]$ is $\mathcal{F}_r$-measurable [103, 300].

Let the mapping $\omega \mapsto \alpha^*[\omega] \in A_I$ be chosen according to Corollary V.1 and the mapping $\omega \mapsto \overline{\alpha}[\omega] \in A_I$ satisfy the equalities

$$\overline{\alpha}_s[\omega] = \alpha^0_s[\omega], \quad 0 \le s \le T - t_*. \tag{18.76}$$

Using the representations (18.59)–(18.64), Lemma V.6, and condition (18.57), one can obtain the following estimate:

$$\begin{aligned} E\left\{ \min_{\alpha\in A_I} \rho\Big(t_* + r, \xi_r\big(t_*, x_*, \alpha, v_*\big)\Big)(\omega) \right\} &= E\left\{ \rho\left(t_* + r, x_* + f^*[\omega] \cdot r + \sqrt{r}\psi(t_*)\eta(\omega) + \xi_r^{(1)*} + \xi_r^{(2)*} + \xi_r^{(3)*}\right) \right\} \\ &\le E\left\{ \rho\left(t_* + r, x_* + \overline{f}[\omega] \cdot r + \sqrt{r}\psi(t_*)\eta(\omega) + \overline{\xi}_r^{(1)} + \overline{\xi}_r^{(2)} + \overline{\xi}_r^{(3)}\right) \right\}, \end{aligned} \tag{18.77}$$

where the asterisk and the bar mean that the corresponding value are calculated for $\alpha^*[\omega]$ or for $\overline{\alpha}[\omega]$, respectively.

It follows from (18.43) and condition (18.71) that the following inequalities hold:

$$
\begin{aligned}
E&\left\{\min_{\alpha\in A_I}\rho\Big(t_*+r,\xi_r\big(t_*,x_*,\alpha,v_*\big)\Big)(\omega)\right\}\\
&\qquad - E\left\{\min_{f\in F_1(t_*,x_*,v_*)}\rho\Big(t_*+r,x_*+f\cdot r+\sqrt{r}\psi(t_*)\eta(\omega)\Big)\right\}\\
&\le E\left\{\rho\left(t_*+r,x_*+\overline{f}[\omega]r+\sqrt{r}\psi(t_*)\eta(\omega)+\overline{\xi}_r^{(1)}(\omega)+\overline{\xi}_r^{(2)}(\omega)+\overline{\xi}_r^{(3)}(\omega)\right)\right\}\\
&\qquad - E\left\{\rho\Big(t_*+r,x_*+\overline{f}[\omega]r+\sqrt{r}\psi(t_*)\eta(\omega)\Big)\right\}\\
&\le L_\rho\cdot E\left\{\left\|\overline{\xi}_r^{(1)}\right\|+\left\|\overline{\xi}_r^{(2)}\right\|+\left\|\overline{\xi}_r^{(3)}\right\|\right\}. \qquad (18.78)
\end{aligned}
$$

Using conditions (18.65)–(18.67), (18.71), and (18.3), this estimate can be continued:

$$
E\left\{\left\|\overline{\xi}_r^{(1)}\right\|+\left\|\overline{\xi}_r^{(2)}\right\|+\left\|\overline{\xi}_r^{(3)}\right\|\right\}\le r\left[\beta(r)+L_2\sqrt{r}\cdot K\exp\left(L_2r\right)+L_1r^{\alpha-\frac{1}{2}}\sqrt{nm}\right]. \qquad (18.79)
$$

We set

$$
\zeta(r)=L_\rho\cdot\left[\beta(r)+L_2\sqrt{r}\cdot K\exp\left(L_2r\right)+L_1r^{\alpha-\frac{1}{2}}\sqrt{nm}\right] \qquad (18.80)
$$

and obtain one of inequalities (18.72) and condition (18.71) from (18.78)–(18.80).

Similar estimates can be used to verify the second inequality in (18.72) taking into account (18.74), (18.77), and (18.71), namely,

$$
\begin{aligned}
E&\left\{\min_{f\in F_1(t_*,x_*,v_*)}\rho\Big(t_*+r,x_*+r\cdot f+\sqrt{r}\psi(t_*)\eta(\omega)\Big)\right\}\\
&\qquad - E\left\{\min_{\alpha\in A_I}\rho\Big(t_*+r,\xi_r\big(t_*,x_*,\alpha,v_*\big)\Big)(\omega)\right\}\\
&= E\left\{\rho\Big(t_*+r,x_*+r\cdot f^0[\omega]+\sqrt{r}\psi(t_*)\eta(\omega)\Big)\right\}\\
&\qquad - E\left\{\rho\Big(t_*+r,x_*+r\cdot f^*[\omega]+\sqrt{r}\psi(t_*)\eta(\omega)+\xi_r^{(1)*}+\xi_r^{(2)*}+\xi_r^{(3)*}\Big)\right\}\\
&\le E\left\{\rho\Big(t_*+r,x_*+f^*[\omega]\cdot r+\sqrt{r}\psi(t_*)\eta(\omega)\Big)\right\}\\
&\qquad - E\left\{\rho\Big(t_*+r,x_*+f^*[\omega]\cdot r+\sqrt{r}\psi(t_*)\eta(\omega)+\xi_r^{(1)*}+\xi_r^{(2)*}+\xi_r^{(3)*}\Big)\right\}\\
&\le L_\rho\cdot E\left\{\left\|\overline{\xi}_r^{(1)*}\right\|+\left\|\overline{\xi}_r^{(2)*}\right\|+\left\|\overline{\xi}_r^{(3)*}\right\|\right\}\le r\cdot\zeta(r), \qquad (18.81)
\end{aligned}
$$

where $\zeta(r)$ has the form (18.80). Lemma V.7 is proved. □

To prove similar assertions for generalized program controls of the second player and random control processes generated by the controls, one must use the following constructions:

- the set $F_2(t_*,x_*,u_*)$ of the form

$$
F_2(t_*,x_*,u_*)=\operatorname{co}\left\{f(t_*,x_*,u_*,v):v\in Q\right\} \qquad (18.82)
$$

  instead of the set $F_1(t_*,x_*,v_*)$ (18.60);
- the set

$$
A_{II}=\left\{\forall\alpha_{(\cdot)}:\mathbb{T}'\to\operatorname{rpm}(Q):s\mapsto\alpha_s\text{ is measurable}\right\} \qquad (18.83)
$$

  of generalized program controls of the second player;

- the random control process $\xi_r = \xi_r(t_*, x_*, u_*, \alpha)$ generated by the control $\alpha \in A_{II}$ as a solution of the stochastic equation

$$\xi_r = x_* + \int\limits_0^r ds \int\limits_P f\big(t_* + s, \xi_s, u_*, v\big)\alpha_s(du) + \int\limits_0^r \psi(t_* + s)dW_s,$$
$$r \in [0, T - t_*], \quad (t_*, x_*, u_*) \in \mathbb{T}' \times \mathbb{R}^n \times P. \tag{18.84}$$

Remarks on replacements in formulas and necessary corrections in proof of similar results for the second player are at the beginning of this section.

18.4. Stochastic stability properties for continuous functions. In this section, the notions of stochastic *u-stability* and *v-stability* for continuous functions (see [136]) are introduced. The definitions are used in Chap. V to obtain an infinitesimal form (19.8) and (19.9) of the notion and to apply the last form to a development of the concept of minimax solutions of boundary-value problems for quasi-linear parabolic Hamilton–Jacobi–Isaacs equations. It is possible to state other equivalent formulations.

Note that Definitions V.1 and V.2 of the properties of stochastic u-stability and v-stability have a deterministic analogue involving tools of the theory of differential inclusions [133, 135, 235, 238]. There are generalized program controls (18.16) and (18.83) and stochastic processes generated by the controls (18.17) and (18.84) in the basis of the notions of stochastic stability.

Recall the definitions of stability for a continuous function $\rho(t, x)$ relative to stochastic processes (18.17) and (18.84).

Definition V.1. A continuous function $\rho : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}$ is said to be *u-stable* if the inequality

$$E\left\{\min_{\alpha \in A_I} \rho\Big(t_* + r, \xi_r\big(t_*, x_*, \alpha, v_*\big)\Big)\right\} \le \rho(t_*, x_*) \tag{18.85}$$

holds for all $(t_*, x_*, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times Q$ and $r \in [0, T - t_*]$.

As follows from property (ii) in Theorem V.2 and Lemma V.1, the minimum in calculations of the mean value in inequality (18.85) is achieved for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, condition (18.85) means that for any given $(t_*, x_*, v_*) \in \mathbb{T}' \times \mathbb{R}^n \times Q$, $r \in [0, T - t_*]$, and $\omega \in \Omega$, there exists a generalized program control of the first player $\alpha[\omega] \in A_I$ such that the mean value of the random variable

$$\rho\big[\xi_r[\omega]\big] = \rho\Big(t_* + r, \xi_r\big(t_*, x_*, \alpha[\omega], v_*\big)\Big)(\omega)$$

is not greater than $\rho(t_*, x_*)$ for trajectories $\xi_r[\omega]$ of the stochastic process (18.17) generated by the control.

Definition V.2. A continuous function $\rho : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}$ is said to be *v-stable* if the inequality

$$E\left\{\max_{\alpha \in A_{II}} \rho\Big(t_* + r, \xi_r\big(t_*, x_*, u_*, \alpha, \big)\Big)\right\} \ge \rho(t_*, x_*) \tag{18.86}$$

holds for all $(t_*, x_*, u_*) \in \mathbb{T}' \times \mathbb{R}^n \times P$ and $r \in [0, T - t_*]$.

Assertion V.3 (see [136])**.** *A continuous function* $\rho(\cdot) : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}$ *coincides with the value function* $\rho_0(\cdot)$ *for the differential game* (18.1), (18.8) *if and only if it is u-stable and v-stable simultaneously and satisfies the boundary condition*

$$\rho(T, x) = \gamma(x), \quad x \in \mathbb{R}^n. \tag{18.87}$$

18.5. Generalized stochastic derivatives. It is known [76, 136] that the value function $\rho_0(t,x)$ (18.15) of the diffusion differential game (18.1), (18.8) under consideration is an element of the class Lip if assumptions (18.3)–(18.7) and (18.9) hold. According to the Rademacher theorem, a function of class Lip can be nondifferentiable on a subset of $\mathbb{T}' \times \mathbb{R}^n$ of zero measure. The following notions of generalized stochastic derivatives are suggested for the infinitesimal analysis of such functions at each point of the strip $(0,T)\times\mathbb{R}^n$.

Let $\rho \in$ Lip, $(t,x)\in(0,T)\times\mathbb{R}^n$, F be a compact set in $\mathbb{R}^n$, the random variable $\eta = (\eta_1,\dots,\eta_m)$ be an m-dimensional Gauss normalized variable with independent components, and $\psi : \mathbb{T}' \to L(\mathbb{R}^m,\mathbb{R}^n)$ be a diffusion matrix satisfying conditions (18.3).

Definition V.3. We introduce the following terminology: $\dfrac{\widetilde{d}^-\rho(t,x)}{(F,\psi)}$ is called the *lower generalized stochastic derivative* and $\dfrac{\widetilde{d}^+\rho(t,x)}{(F,\psi)}$ is called the *upper generalized stochastic derivative* of a function $\rho\in$ Lip at a point (t,x) with respect to the set F and the diffusion matrix $\psi=\psi(t)$:

$$\frac{\widetilde{d}^-\rho(t,x)}{(F,\psi)} = \liminf_{\delta\downarrow 0}\frac{1}{\delta}\left[E\left\{\min_{f\in F}\rho\left(t+\delta, x+\delta f+\sqrt{\delta}\psi(t)\eta\right)\right\}-\rho(t,x)\right], \tag{18.88}$$

$$\frac{\widetilde{d}^+\rho(t,x)}{(F,\psi)} = \limsup_{\delta\downarrow 0}\frac{1}{\delta}\left[E\left\{\max_{f\in F}\rho\left(t+\delta, x+\delta f+\sqrt{\delta}\psi(t)\eta\right)\right\}-\rho(t,x)\right]. \tag{18.89}$$

Note that in the case where $\psi(t)=0$ and the set F is a singleton $\{f\}$, formulas (18.88) and (18.89) define the lower and upper Dini semiderivatives for the function $\rho(\cdot)\in$ Lip at a point (t,x) in the direction $(1,f)$ (cf. Definition I.8, Sec. 2.3).

Consider a point (t,x), where the function $\rho(\cdot)$ has the first derivative in t and the first and second derivatives in x_i. It is easy to verify that the following equalities hold at this point:

$$\frac{\widetilde{d}^-\rho(t,x)}{(F,\psi(t))} = \frac{\partial\rho(t,x)}{\partial t} + \min_{f\in F}\left\langle\frac{\partial\rho(t,x)}{\partial x}, f\right\rangle + \sum_{i,j=1}^n a_{ij}(t)\frac{\partial^2\rho(t,x)}{\partial x_i\partial x_j}, \tag{18.90}$$

$$\frac{\widetilde{d}^+\rho(t,x)}{(F,\psi(t))} = \frac{\partial\rho(t,x)}{\partial t} + \max_{f\in F}\left\langle\frac{\partial\rho(t,x)}{\partial x}, f\right\rangle + \sum_{i,j=1}^n a_{ij}(t)\frac{\partial^2\rho(t,x)}{\partial x_i\partial x_j}, \tag{18.91}$$

where the $(n\times n)$-matrix $A(t)=(a_{ij}(t))$ has the form

$$A(t)=\frac{1}{2}\psi(t)\psi^\top(t).$$

One can develop the calculus for stochastic derivatives. In particular, in Sec. 20, we obtain formulas for generalized stochastic derivatives for a class of functions differentiable in a part of the variables.

19. Parabolic Hamilton–Jacobi–Isaacs Equations and Their Minimax Solutions in Terms of Generalized Stochastic Derivatives

19.1. Isaacs equation for the value function of a stochastic differential game. It is well known (see, e.g., [29, 79, 146, 168]) that the value function of a diffusion differential game (18.1), (18.8) is Lipschitz continuous and, at points of smoothness, satisfies the following quasi-linear partial differential equation of parabolic type called the Isaacs equation:

$$\frac{\partial\rho_0(t,x)}{\partial t} + H\left(t,x,\frac{\partial\rho_0(t,x)}{\partial x}\right) + \sum_{i,j=1}^n a_{ij}(t)\frac{\partial^2\rho_0(t,x)}{\partial x_i\partial x_j} = 0, \tag{19.1}$$

where the function $H(t, x, s)$ is the Hamiltonian (18.7) of process (18.1):

$$H(t,x,s) = \min_{u\in P}\max_{v\in Q}\big\langle s, f(t,x,u,v)\big\rangle = \max_{v\in Q}\min_{u\in P}\big\langle s, f(t,x,u,v)\big\rangle.$$

Therefore, Eq. (19.1) has the type of the Hamilton–Jacobi–Isaacs equation. The symbol $\dfrac{\partial\rho_0}{\partial x}$ means the gradient

$$\frac{\partial\rho_0}{\partial x} = \left(\frac{\partial\rho_0}{\partial x_1},\dots,\frac{\partial\rho_0}{\partial x_n}\right),$$

and the $(n\times n)$-matrix $A(t) = (a_{ij}(t))$ has the form

$$A(t) = \frac{1}{2}\psi(t)\psi^\top(t). \tag{19.2}$$

Obviously, the value function satisfies the boundary condition

$$\rho_0(T,x) = \gamma(x), \quad x\in\mathbb{R}^n. \tag{19.3}$$

19.2. Minimax solution of the boundary-value problem (19.1)–(19.3)**.** Consider the boundary-value problem (19.1)–(19.3) under assumptions (18.2)–(18.7) and (18.9).

If the matrix $A(t)$ is positive definite, then the boundary-value problem (19.1)–(19.3) has a unique classical solution [146, 192]. Therefore, problem (19.1)–(19.3) completely defines the value function of the diffusion differential game (18.1), (18.8) since the value function is equal to the smooth solution. In particular, this takes place if the noise in the control process is nondegenerate, i.e., $m = n$, and the diffusion matrix $\psi(t)$ in (18.1) is nondegenerate for all $t\in T$.

In the case where (19.1) is a degenerate equation of parabolic type, the boundary-value problem (19.1)–(19.3) has no classical solutions. Similarly to the deterministic case (see Sec. 2.4, conditions **(U2)** and **(L2)**), we introduce the notion of a minimax solution of the boundary-value problem (19.1)–(19.3) in terms of generalized stochastic derivatives. According to the theory of minimax solutions, it is possible to consider Definition V.4 as an infinitesimal form of the generalized method of characteristics for the quasi-linear parabolic Hamilton–Jacobi–Isaacs equation.

Definition V.4. A function $\rho(\cdot) : [0,T]\times\mathbb{R}^n \to \mathbb{R}$, $\rho\in \mathrm{Lip}$, is called a *minimax solution* of problem (19.1)–(19.3) if the boundary condition

$$\rho(T,x) = \gamma(x), \quad x\in\mathbb{R}^n, \tag{19.4}$$

and the inequalities

$$\max_{v\in Q}\frac{\widetilde{d}^-\rho(t,x)}{(F_1(t,x,v),\psi(t))} \le 0, \qquad \min_{u\in P}\frac{\widetilde{d}^+\rho(t,x)}{(F_2(t,x,u),\psi(t))} \ge 0 \tag{19.5}$$

hold for all $(t,x)\in(0,T)\times\mathbb{R}^n$, where

$$F_1(t,x,v) = \mathrm{co}\big\{f(t,x,u,v) : u\in P\big\}, \qquad F_2(t,x,u) = \mathrm{co}\big\{f(t,x,u,v) : v\in Q\big\}. \tag{19.6}$$

In Sec. 19.3, we prove Theorem V.4 on the existence and uniqueness of a minimax solution of the boundary-value problem (19.1)–(19.3) under assumptions (18.2)–(18.7) and (18.9). It was obtained as a consequence of Theorem V.3 on stochastic stability properties of the value function of the diffusion differential game (18.1), (18.8).

19.3. The infinitesimal form of stability conditions. The basic result of this section is the following assertion.

Theorem V.3. *A function $\rho(t,x) \in \mathrm{Lip}$ is the value function of the stochastic differential game* (18.1), (18.8) *if and only if the following conditions hold*:

$$\rho(T,x) = \gamma(x), \quad x \in \mathbb{R}^n, \tag{19.7}$$

$$\max_{v \in Q} \frac{\widetilde{d}^- \rho(t,x)}{(F_1(t,x,v),\psi)} \le 0, \tag{19.8}$$

$$\min_{u \in P} \frac{\widetilde{d}^+ \rho(t,x)}{(F_2(t,x,u),\psi)} \ge 0, \tag{19.9}$$

where

$$F_1(t,x,v) = \mathrm{co}\,\big\{f(t,x,u,v) : v \in Q\big\}, \qquad F_2(t,x,u) = \mathrm{co}\,\big\{f(t,x,u,v) : v \in P\big\},$$

$\psi = \psi(t)$ is the diffusion matrix in Eq. (18.1) *connected with the Hamilton–Jacobi–Isaacs equation* (19.1) *by relation* (19.2).

Proof. As follows from Assertion V.3, to prove Theorem V.3, it suffices to establish the equivalence of conditions (19.8), (19.9) and conditions (18.85), (18.86) of u-stability and v-stability, respectively.

First, we prove that condition (18.85) of u-stability implies condition (19.8).

Indeed, for any fixed $(t,x,v) \in [0,T] \times \mathbb{R}^n \times Q$ and $r \in [0, T-t]$, condition (18.85) and Lemma V.7 imply the estimate

$$\left[E\left\{\min_{f \in F_1(t,x,v)} \rho\Big(t+r, x + r \cdot f + \sqrt{r}\psi(t)\eta\Big)\right\} - \rho(t,x)\right] r^{-1}$$
$$\le \left[E\left\{\min_{\alpha \in A_I} \rho\Big(t+r, \xi_r\big(t,x,\alpha,v\big)\Big)(\omega)\right\} - \rho(t,x)\right] r^{-1} + \zeta(r) \le \zeta(r).$$

Passing to the limit in both sides of this inequality as $r \downarrow 0$ and using (18.73) and (18.88), one can obtain the inequality

$$\frac{\widetilde{d}^- \rho(t,x)}{(F_1(t,x,v),\psi(t))} = \liminf_{r \downarrow 0} \frac{E\left\{\min\limits_{f \in F_1(t,x,v)} \rho\big(t+r, x+r\cdot f + \sqrt{r}\psi(t)\eta\big)\right\} - \rho(t,x)}{r} \le 0. \tag{19.10}$$

Since $v \in Q$ is arbitrary, (19.10) implies (19.8).

Now we prove the implication (19.8) $\Longrightarrow$ (18.85).

Let a function $\rho \in \mathrm{Lip}$ satisfy condition (19.8). Choose a positive number ε and construct the function

$$\rho^\varepsilon(t,x) = \rho(t,x) - (T-t)\cdot\varepsilon. \tag{19.11}$$

It is easy to verify that the function ρ^ε satisfies the inequality

$$\max_{v \in Q} \frac{\widetilde{d}^- \rho(t,x)}{(F_1(t,x,v),\psi(t))} \le -\varepsilon. \tag{19.12}$$

Let us prove that the function ρ^ε satisfies the stability condition (18.85) at any arbitrary point $(t,x,v) \in [0,T) \times \mathbb{R}^n \times Q$ for any number $r \in (0, T-t_*]$. For this purpose, we apply the Zorn lemma [114].

Consider a partially ordered set S whose elements are Markov moments $\tau(\cdot)$ relative to the family $\{\mathcal{F}_s\}$ of σ-algebras. The moments satisfy the inequalities

$$0 \le \tau(\omega) \le r \quad \mathbb{P}\text{-a.e.}, \tag{19.13}$$

$$E\left\{\min_{\alpha \in A_I} \rho^\varepsilon\Big(t+\tau, \xi_\tau\big(t_*,x_*,\alpha,v_*\big)\Big)\right\} \le \rho^\varepsilon(t_*,x_*). \tag{19.14}$$

Obviously, the set S is nonempty since

$$\tau_0(\omega) \equiv 0 \quad \Rightarrow \quad \tau_0 \in S.$$

Condition (19.12) and Lemma V.7 imply also that there exist $\tau_C(\omega) \equiv C > 0$ such that $\tau_C(\omega) \in S$, where C is a constant close to zero.

For any pair of elements (τ_1, τ_2) of the set S, there is an order relation $\tau_1 \geq \tau_2$ (respectively, $\tau_1 > \tau_2$) such that

$$\tau_1(\omega) \geq \tau_2(\omega) \quad (\text{respectively }, \tau_1(\omega) > \tau_2(\omega)) \quad \mathbb{P}\text{-a.e.} \tag{19.15}$$

Consider a linearly ordered subset $\widetilde{S} \subset S$. For any element $\tau \in \widetilde{S}$, we define the number $b = b(\tau)$ as follows:

$$\tau \mapsto b = b(\tau) = E\{\tau\} \leq r = E\{\tau \equiv r\}. \tag{19.16}$$

It follows from conditions (19.13) and (19.16) that $\widetilde{S}$ is a bounded set. The set of mean values for elements of $\widetilde{S}$ has the least upper bound $b_* \leq r$. Also, there is a sequence

$$b_i = b(\tau_i), \quad i = 1, 2, \dots,$$

such that

$$\tau_i \in \widetilde{S}, \quad i = 1, 2, \dots, \quad b_i \uparrow b_* \quad \text{as} \quad i \to \infty. \tag{19.17}$$

Using the linear order in the set $\widetilde{S}$ and the linearity of the operation $E\{\cdot\}$, one can easily obtain that elements of a sequence $\{\tau_i\}$ corresponding to the monotone increasing sequence $\{b_i\}$ (19.17) satisfy the following order relation:

$$\tau_{i+1} > \tau_i, \quad i = 1, 2, \dots. \tag{19.18}$$

This implies that the sequences $\{\tau_i(\omega)\}$, $i = 1, 2, \dots$, are monotone nondecreasing for almost all $\omega \in \Omega$. They are bounded from the above by the number r. This means that there exist the limit

$$\lim_{i\to\infty} \tau_i(\omega) = \tau_*(\omega) = \sup_{i=1,2,\dots} \{\tau_i(\omega)\}. \tag{19.19}$$

According to [170, Lemma 1.4], $\tau_* = \tau_*(\omega)$ is also a Markov moment relative to $\{\mathcal{F}_s\}$. According to the Lebesque theorem, one can pass to the limit inside the operation of calculation of the following mean values:

$$E\{\tau_i\} = b_i \leq b_* \leq r, \quad E\left\{\min_{\alpha\in A_I} \rho^\varepsilon\Big(t_* + \tau_i, \xi_{\tau_i}\big(t_*, x_*, \alpha, v_*\big)\Big)\right\} \leq \rho^\varepsilon(t_*, x_*)$$

as $i \to \infty$. Therefore, we obtain the following limit relations for $\tau_*(\omega)$:

$$E\{\tau_i(\cdot)\} \leq E\{\tau_*(\cdot)\} = b_* \leq r, \quad E\left\{\min_{\alpha\in A_I} \rho^\varepsilon\Big(t_* + \tau_*(\cdot), \xi_{\tau_*(\cdot)}\big(t_*, x_*, \alpha, v_*\big)\Big)\right\} \leq \rho^\varepsilon(t_*, x_*),$$

i.e.,

$$\tau_* \in S \qquad \tau_* \geq \tau \quad \forall \tau \in \widetilde{S},$$

This means that τ_* is an upper boundary of the linearly ordered set $\widetilde{S}$.

The subset $\widetilde{S} \subset S$ is chosen arbitrarily. Hence, according to the Zorn lemma [114], there exists a maximal element $\tau_{\max}$ in the set S such that

$$\forall \tau \geq \tau_{\max} \quad \tau \in S \quad \Rightarrow \quad \tau(\omega) = \tau_{\max}(\omega) \quad (\mathbb{P}\text{-a.e.}). \tag{19.20}$$

The purpose of the further reasonings is to prove that

$$\tau_{\max} = r \quad (\mathbb{P}\text{-a.e.}). \tag{19.21}$$

Assume that there is a set $\Omega_* \in \mathcal{F}_{\tau_{\max}}$ such that

$$\mathbb{P}(\Omega_*) > 0, \quad \tau_{\max}(\omega) < r \quad \text{for} \quad \omega \in \Omega_* \subset \Omega. \tag{19.22}$$

Let us obtain a contradiction to (19.20).

Construct the multi-valued mapping

$$B(\cdot) : [t_*, T] \times \mathbb{R}^n \to [0, T - t_*]$$

by the following rule:

$$B(t,x) = \left\{ \delta \in [0,T] : E\left\{ \min_{\alpha \in A_I} \rho^\varepsilon\Big(t+\delta, \xi_\delta(t,x,\alpha,v_*)\Big) \right\} \le \rho^\varepsilon(t,x) \right\}. \tag{19.23}$$

Obviously, $0 \in B(t,x)$ for any (t,x).

Let us show that for any $(t,x) \in [t_*, t_* + r) \times \mathbb{R}^n$, the set $B(t,x)$ contains elements $\delta > 0$.

For $\varepsilon > 0$, choose a number $\beta > 0$ such that

$$\zeta(\delta) < \frac{\varepsilon}{4} \quad \text{for all} \quad \delta \le \beta, \tag{19.24}$$

where $\zeta(\cdot)$ is a function satisfying conditions (18.72) and (18.73).

Inequality (19.12) implies that one can choose a number $\delta \in (0, \beta]$ such that

$$E\left\{ \min_{f \in F_1(t,x,v_*)} \rho^\varepsilon\Big(t+\delta, x + \delta \cdot f + \sqrt{\delta}\psi(t)\eta\Big) \right\} - \rho^\varepsilon(t,x) \le -\varepsilon\delta + \frac{\varepsilon}{4}\delta. \tag{19.25}$$

One can see that the definition (19.11) of $\rho^\varepsilon(t,x)$ implies Lemma V.7 for the function $\rho^\varepsilon(t,x)$. Hence, relations (18.72), (19.24), and (19.25) imply the estimate

$$E\left\{ \min_{\alpha \in A_I} \rho^\varepsilon\Big(t+\delta, \xi_\delta\big(t,x,\alpha,v_*\big)\Big) \right\} - \rho^\varepsilon(t,x) \le -\varepsilon\delta + \frac{\varepsilon}{4}\delta + \frac{\varepsilon}{4}\delta \le -\frac{\varepsilon}{2}\delta < 0.$$

Thus, the number $\delta > 0$ chosen from conditions (19.24) and (19.25) belongs to $B(t,x)$.

According to property (ii) of Theorem V.2 and the continuity of the function ρ^ε, the function

$$(t,x,\delta) \to E\left\{ \min_{\alpha \in A} \rho^\varepsilon\Big(t+\delta, \xi_\delta\big(t,x,\alpha,v_*\big)\Big) \right\}$$

is also continuous; this implies that the multi-valued mapping $(t,x) \mapsto B(t,x)$ has compact value set and is upper semicontinuous.

Hence (see [103, 300]), there exists a measurable on $[0,T] \times \mathbb{R}^n$ selector $\delta(t,x) \in B(t,x)$ satisfying the relations

$$\delta(t,x) \in (0, t_* + r - t] \quad \text{for} \quad t \in [t_*, t_* + r]\,, \tag{19.26}$$

$$\delta(t_* + r, x) = 0. \tag{19.27}$$

For the Markov moment $\tau_{\max}(\cdot)$ (19.20) and the measurable mapping $\alpha^*[\cdot] : \Omega \to A_I$ chosen by Lemma V.5, we construct the $\mathcal{F}_{\tau_{\max}}$-measurable mapping

$$\omega \mapsto \delta[\omega] = \delta\left(t_* + \tau_{\max}(\omega), \xi_{\tau_{\max}(\omega)}\big(t_*, x_*, \alpha^*[\omega], v_*\big)(\omega)\right). \tag{19.28}$$

It follows from the definition of the set Ω_* (19.22) and condition (19.27) that

$$\delta[\omega] = 0 \;\; \text{for} \;\; \omega \in \Omega \setminus \Omega_*, \qquad \delta[\omega] > 0 \;\; \text{for} \;\; \omega \in \Omega_*, \quad \delta[\omega] \le r - \tau_{\max}(\omega). \tag{19.29}$$

According to Lemma V.3, the random variable

$$\tau^0(\omega) = \tau_{\max}(\omega) + \delta[\omega] \tag{19.30}$$

is a Markov moment relative to $\{\mathcal{F}_s\}$. It follows from (19.29) and (19.30) that

$$\tau^0 > \tau_{\max}, \quad \tau^0(\cdot) \le r. \tag{19.31}$$

To obtain a contradiction to (19.20), we show that $\tau^0 \in S$, i.e., it satisfies inequality (19.14).

Consider a mapping $\alpha^0[\cdot] : \Omega \to A_I$ measurable relative to $\mathcal{F}_{\tau^0}$, which is constructed according to Lemma V.5 and satisfies the equality

$$\rho\Big(t_* + \tau^0(\omega), \xi_{\tau^0(\omega)}\big(t_*, x_*, \alpha^0[\omega], v_*\big)(\omega)\Big) = \min_{\alpha \in A_I} \rho\Big(t_* + \tau^0(\omega), \xi_{\tau^0(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big)$$

for almost all $\omega \in \Omega$. Any other mapping $\alpha[\cdot] : \Omega \to A_I$ measurable relative to $\mathcal{F}_{\tau^0}$ satisfies the inequality

$$E\left\{\rho\Big(t_* + \tau^0, \xi_{\tau^0}\big(t_*, x_*, \alpha^0[\cdot], v_*\big)(\omega)\Big)\right\} \le E\left\{\rho\Big(t_* + \tau^0, \xi_{\tau^0}\big(t_*, x_*, \alpha[\cdot], v_*\big)\Big)\right\}. \tag{19.32}$$

Construct the mapping $\alpha^{00}[\cdot] : \Omega \to A_I$ measurable relative to $\mathcal{F}_{\tau^0}$ as follows. First, according to Lemma V.5, construct the mapping

$$\alpha_1[\cdot] : \Omega \to A_I$$

measurable relative to $\mathcal{F}_{\tau \max}$ and satisfying the equality

$$\rho\Big(t_* + \tau_{\max}(\omega), \xi_{\tau_{\max}(\omega)}\big(t_*, x_*, \alpha_1[\omega], v_*\big)(\omega)\Big) = \min_{\alpha \in A_I} \rho\Big(t_* + \tau_{\max}(\omega), \xi_{\tau_{\max}(\omega)}\big(t_*, x_*, \alpha, v_*\big)(\omega)\Big) \tag{19.33}$$

for almost all $\omega \in \Omega$.

As follows from Theorem V.2 and Lemma V.2, the mappings

$$\omega \mapsto t[\omega] = t_* + \tau_{\max}(\omega), \quad \omega \mapsto x[\omega] = \xi_{\tau_{\max}(\omega)}\big(t_*, x_*, \alpha_1[\omega], v_*\big)(\omega)$$

are $\mathcal{F}_{\tau \max}$-measurable. Hence, the mapping

$$A_I \times [0, r] \times \Omega \to \mathbb{R}^n, \quad (\alpha, \delta, \omega) \mapsto \xi_\delta\big(t[\omega], x[\omega], \alpha, v_*\big)(\omega) \tag{19.34}$$

is $\mathcal{F}_{\tau \max}$-measurable in ω for any α and δ. The mapping is continuous with respect to α and δ for almost all $\omega \in \Omega$. For $\delta = \delta[\omega]$ (19.28), it follows from Lemma V.2 and relation (19.34) that the mapping

$$A_I \times \Omega \to \mathbb{R}^n, \quad (\alpha, \omega) \mapsto \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha, v_*\big)(\omega) \tag{19.35}$$

is continuous in α for almost all $\omega \in \Omega$. The mapping is $\mathcal{F}_{\tau \max}$-measurable in ω for any fixed $\alpha \in A_I$.

According to Lemma V.1 and condition (19.31), the inclusion

$$\mathcal{F}_{\tau \max} \subset \mathcal{F}_{\tau^0}$$

holds and the Markov moment $\tau^0(\cdot)$ (19.30) is measurable relative to $\mathcal{F}_{\tau^0}$. Hence, the superposition

$$(\alpha, \omega) \mapsto \rho^\varepsilon\Big(t_* + \tau^0(\omega), \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha, v_*\big)(\omega)\Big)$$

is continuous in α for almost all $\omega \in \Omega$ and the variable is $\mathcal{F}_{\tau^0}$-measurable for any fixed $\alpha \in A_I$.

Applying reasonings similar to the proof of Lemma V.5, construct the mapping

$$\Omega \to A_I, \quad \omega \mapsto \alpha_2[\omega],$$

which is measurable relative to $\mathcal{F}_{\tau^0}$ and satisfies the equality

$$\rho^\varepsilon\Big(t_* + \tau^0(\omega), \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big) = \min_{\alpha \in A_I} \rho^\varepsilon\Big(t[\omega] + \delta[\omega], \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha, v_*\big)(\omega)\Big) \tag{19.36}$$

for almost all $\omega \in \Omega$.

Now we construct the mapping

$$\alpha^{00}[\cdot] : \Omega \to A_I$$

from two mappings $\alpha_1[\cdot]$ (see (19.33)) and $\alpha_2[\cdot]$ (see (19.36)) as follows:

$$\alpha^{00}_t[\omega] = \begin{cases} \alpha_{1t}[\omega] & \text{for } \ 0 \le t \le t_* + \tau_{\max}(\omega), \\ \alpha_{2t}[\omega] & \text{for } \ t_* + \tau_{\max}(\omega) \le t \le T. \end{cases} \tag{19.37}$$

The mapping $\alpha^{00}[\cdot]$ is $\mathcal{F}_{\tau^0}$-measurable by Lemma V.3. Relations (19.32), (19.37), and (19.36) imply the estimate

$$E\left\{\rho^\varepsilon\Big(t_* + \tau^0, \xi_{\tau^0}\big(t_*, x_*, \alpha^0[\cdot], v_*\big)(\omega)\Big)\right\} \le E\left\{\rho^\varepsilon\Big(t_* + \tau^0, \xi_{\tau^0}\big(t_*, x_*, \alpha^{00}[\cdot], v_*\big)\Big)\right\}$$
$$= E\left\{\rho^\varepsilon\Big(t_* + \tau_{\max}(\omega) + \delta[\omega], \xi_{\delta[\omega]}\big(t_* + \tau_{\max}(\omega), x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big)\right\}. \quad (19.38)$$

Calculate the conditional mean value relative to σ-algebra $\mathcal{F}_{\tau\max} \subset \mathcal{F}_{\tau^0}$ for the $\mathcal{F}_{\tau^0}$-measurable variable:

$$\Omega \to \mathbb{R}, \quad \omega \mapsto \rho^\varepsilon\Big(t_* + \tau^0(\omega), \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big).$$

In other words, we construct the $\mathcal{F}_{\tau\max}$-measurable function

$$\omega \mapsto \widetilde{E}\left\{\rho^\varepsilon\Big(t_* + \tau_{\max}(\omega) + \delta[\omega], \xi_{\delta[\omega]}\big(t_* + \tau_{\max}(\omega), x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big)\right\}$$
$$= \widetilde{E}\left\{\rho^\varepsilon\Big(t[\omega] + \delta[\omega], \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big)\right\},$$

According to the Fubini theorem (see [31, 170]), this function satisfies the relation

$$\int_{\mathcal{B}} \rho^\varepsilon\Big(t_* + \tau^0(\omega), \xi_{\tau^0(\omega)}\big(t_*, x_*, \alpha^{00}[\omega], v_*\big)(\omega)\Big)\mathbb{P}(d\omega)$$
$$= \int_{\mathcal{B}} \widetilde{E}\left\{\rho^\varepsilon\Big(t[\omega] + \delta[\omega], \xi_{\delta[\omega]}\big(t[\omega], x[\omega], \alpha_2[\omega], v_*\big)(\omega)\Big)\right\}\mathbb{P}(d\omega). \quad (19.39)$$

on sets $\mathcal{B} \in \mathcal{F}_{\tau\max}$.

Using conditions (19.38) and (19.39) at $\mathcal{B} = \Omega$ and (19.36), (19.26)–(19.29), (19.23), (19.33), and (19.14), one can obtain the following final estimate:

$$E\left\{\min_{\alpha\in A_I} \rho^\varepsilon\big(t_* + \tau^0, \xi_{\tau^0}(t_*, x_*, \alpha, v_*)(\omega)\big)\right\} = E\Big\{\rho^\varepsilon\big(t_* + \tau^0, \xi_{\tau^0}(t_*, x_*, \alpha^0[\cdot], v_*)\big)\Big\}$$
$$\le E\Big\{\rho^\varepsilon\big(t_* + \tau^0, \xi_{\tau^0}(t_*, x_*, \alpha^{00}[\cdot], v_*)\big)\Big\} = E\left\{\widetilde{E}\left\{\min_{\alpha\in A_I} \rho^\varepsilon\big(t[\omega] + \delta[\omega], \xi_{\delta[\omega]}(t[\omega], x[\omega], \alpha, v_*)(\omega)\big)\right\}\right\}$$
$$\le E\Big\{\rho^\varepsilon\big(t[\omega], x[\omega]\big)\Big\} = E\left\{\rho^\varepsilon\Big(t_* + \tau_{\max}(\omega), \xi_{\tau_{\max}}(t_*, x_*, \alpha_1[\omega], v_*)(\omega)\Big)\right\}$$
$$= E\left\{\min_{\alpha\in A_I} \rho^\varepsilon\big(t_* + \tau_{\max}(\omega), \xi_{\tau_{\max}}(t_*, x_*, \alpha, v_*)(\omega)\big)\right\} \le \rho^\varepsilon(t_*, x_*).$$

Thus, the Markov moment τ^0 (19.30) belongs to the set S (19.14). According to (19.31), we have $\tau^0 > \tau_{\max}$, which contradicts condition (19.20) on the maximality of the element $\tau_{\max}$ in S.

Hence, the above assumption (19.22) is invalid and condition (19.21) holds, i.e.,

$$\tau_{\max} = r \quad (\mathbb{P}\text{-a.e.}).$$

Thus, we have proved that the function $\rho^\varepsilon(t, x)$ satisfies condition (18.85) at any point $(t_*, x_*, v_*) \in [0, T) \times \mathbb{R}^n \times Q$ for any given number $r \in (0, T - t_*]$, i.e.,

$$E\left\{\min_{\alpha\in A_I} \rho^\varepsilon\big(t_* + r, \xi_r(t_*, x_*, \alpha, v_*)(\omega)\big)\right\} \le \rho^\varepsilon(t_*, x_*).$$

Hence, the function $\rho(t, x)$ connected with $\rho^\varepsilon(t, x)$ by relation (19.11) satisfies the estimate

$$E\left\{\min_{\alpha\in A_I} \rho\big(t_* + r, \xi_r(t_*, x_*, \alpha, v_*)(\omega)\big)\right\} \le \rho(t_*, x_*) + r \cdot \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this estimate implies the u-stability condition for $\rho(t, x)$ (18.85), i.e., the inequality

$$E\left\{\min_{\alpha\in A_I} \rho\big(t_* + r, \xi_r(t_*, x_*, \alpha, v_*)(\omega)\big)\right\} \le \rho(t_*, x_*)$$

holds. Similarly, if we replace v_* by u_*, the set A_I by the set A_{II}, the operations min by the operation max, and the inequality signs $\leq$ by $\geq$, we obtain the proof of the equivalence of the v-stability condition (18.86) and condition (19.9). Theorem V.3 is proved. □

Assertion V.1, Theorem V.3, and Definition V.4 of the minimax solution of the boundary-value problem (19.1)–(19.3) obviously imply the following theorem.

Theorem V.4. *Let conditions* (18.2)–(18.7) *and* (18.9) *for the boundary-value problem* (19.1), (19.3) *hold. Then this problem has a unique minimax solution, which coincides with the value function of the diffusion differential game* (18.1), (18.8).

One more obvious consequence of Theorem V.3 is that one can formulate a definition of a minimax solution of the boundary-value problem (19.1)–(19.3) equivalent to Definition V.4, by using inequalities (18.85) and (18.86).

20. Generalized Stochastic Derivatives for Functions of Several Variables Differentiable with Respect to a Part of the Variables

In Sec. 19. we introduced the notions of generalized stochastic derivatives and described infinitesimal properties of the value function of a diffusion differential game with a terminal cost functional. These derivatives were introduced for the general case of Lipschitz continuous functions. However, in a wide class of diffusion games (see, e.g., [78]), the value function has partial derivatives in a part of the variables. Therefore, exact formulas of generalized stochastic derivatives are interesting for the subclass of functions differentiable with respect to a part of the variables.

20.1. Class of functions differentiable with respect to a part of the variables. Formulas for stochastic derivatives. Let $\operatorname{int}\mathbb{T}' = (0, T)$ be an open time interval, $\mathbb{T}' = [0, T]$, $t \in \mathbb{T}'$, and $x \in \mathbb{R}^n$ be an n-dimensional phase vector. Let us consider the class $\mathbf{K}$ of functions $c(\cdot) : \mathbb{T}' \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

- there exist constants $L_0 > 0$ and $c_0 > 0$ such that

$$\left|c(t^1, x^1) - c(t^2, x^2)\right| \leq L_0\left(|t^1 - t^2| + \|x^1 - x^2\|\right) \quad \forall t^1, t^2 \in \mathbb{T}', \quad \forall x^1, x^2 \in \mathbb{R}^n; \tag{20.1}$$

$$\sup_{(t,x)\in\mathbb{T}'\times\mathbb{R}^n} \left|c(t, x)\right| \leq c_0 < \infty; \tag{20.2}$$

- there exist constants $L_1 > 0$ and $\alpha > 0$ and a number ν, $0 \leq \nu < n$, such that partial derivatives $\dfrac{\partial c(t, x)}{\partial x_i}$ are defined for $i = \nu + 1, \dots, n$ and satisfy the Hölder condition:

$$\left|\frac{\partial c(t^1, x^1)}{\partial x_i} - \frac{\partial c(t^2, x^2)}{\partial x_i}\right| \leq L_1\Big(\left|t^1 - t^2\right|^{\alpha/2} + \left\|x^1 - x^2\right\|^{\alpha}\Big) \quad \forall t^1, t^2 \in \operatorname{int}\mathbb{T}', \quad \forall x^1, x^2 \in \mathbb{R}^n. \tag{20.3}$$

Let F be a compact set in $\mathbb{R}^n$, $\psi = (\psi_{ij})$ be a diffusion $(n\times m)$-matrix, $i \in \overline{1, n}$, $j \in \overline{1, m}$, $\eta = (\eta_1, \dots, \eta_m) : \Omega \to \mathbb{R}^m$ be a normalized Gauss m-dimensional random variable with independent components, and $(t, x) \in \mathbb{T}' \times \mathbb{R}^n$ be a phase point.

Similarly to Sec. 18.5, consider the definitions of the lower and upper generalized stochastic derivatives $\dfrac{\widetilde{d}^- c(t, x)}{(F, \psi)}$ and $\dfrac{\widetilde{d}^+ c(t, x)}{(F, \psi)}$ of a function $c(\cdot) \in \mathbf{K}$ at a point (t, x) with respect to the set $F \subset \mathbb{R}^n$ and the diffusion matrix $\psi \in \mathcal{L}[\mathbb{R}^m, \mathbb{R}^n]$:

$$\frac{\widetilde{d}^- c(t, x)}{(F, \psi)} = \liminf_{\delta\downarrow 0} \frac{1}{\delta}\left[E\left\{\min_{f\in F} c\big(t + \delta, x + \delta f + \sqrt{\delta}\psi\eta\big)\right\} - c(t, x)\right], \tag{20.4}$$

$$\frac{\widetilde{d}^{+}c(t,x)}{(F,\psi)} = \limsup_{\delta\downarrow 0} \delta^{-1}\left[E\left\{\max_{f\in F} c\big(t+\delta, x+\delta f+\sqrt{\delta}\psi\eta\big)\right\} - c(t,x)\right], \tag{20.5}$$

where $E\{\cdot\}$ is the mean value.

The generalized stochastic derivatives (20.4) and (20.5) are defined for functions $c(\cdot)$ satisfying conditions (20.1) and (20.2). Functions $c(\cdot) \in \mathbf{K}$ also satisfy the additional condition (20.3).

Let the set F have the following structure:

$$F = \operatorname{co} F, \quad \max_{f\in F} \|f\| = C_1 < \infty, \quad F = \overline{f} + \widetilde{F}, \tag{20.6}$$

$$\overline{f} = \big(\overline{f}_1, \ldots, \overline{f}_\nu, 0, \ldots, 0\big) \in \mathbb{R}^n, \tag{20.7}$$

$$\widetilde{F} = \left\{\tilde{f} = \big(0, \ldots, 0, \tilde{f}_{\nu+1}, \ldots, \tilde{f}_n\big)\right\} \subset \mathbb{R}^n, \tag{20.8}$$

where ν is defined in condition (20.3) and $\operatorname{co} F$ is the convex hull of the set F.

Formulas (20.4) and (20.5) of generalized stochastic derivatives have the following specific form for the subclass $\mathbf{K}$.

Theorem V.5. *For functions $c(\cdot) \in \mathbf{K}$ and sets $F \subset \mathbb{R}^n$ of the form* (20.6)–(20.8), *the following formulas hold*:

$$\frac{\widetilde{d}^{-}c(t,x)}{(F,\psi)} = \min_{\tilde{f}\in\widetilde{F}} \sum_{i=\nu+1}^{n} \frac{\partial c(t,x)}{\partial x_i}\tilde{f}_i + \frac{\widetilde{d}^{-}c(t,x)}{(\overline{f},\psi)}, \tag{20.9}$$

$$\frac{\widetilde{d}^{+}c(t,x)}{(F,\psi)} = \max_{\tilde{f}\in\widetilde{F}} \sum_{i=\nu+1}^{n} \frac{\partial c(t,x)}{\partial x_i}\tilde{f}_i + \frac{\widetilde{d}^{+}c(t,x)}{(\overline{f},\psi)}. \tag{20.10}$$

Note that formulas (20.9) and (20.10) are trivial consequences of the following Lemmas V.8 and V.9.

20.2. Proof of the formulas for generalized stochastic derivatives.

Lemma V.8. *Let a function $c(\cdot) \in \mathbf{K}$ and a set $F \subset \mathbb{R}^n$ have the structure* (20.6)–(20.8). *Then*

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \cdot E\left\{c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta\big) - c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta\big)\right\} = \sum_{i=\nu+1}^{n} \frac{\partial c(t,x)}{\partial x_i}\tilde{f}_i \tag{20.11}$$

for any $f = \overline{f} + \tilde{f} \in \mathbb{R}^n$.

Lemma V.9. *Let a function $c(\cdot) \in \mathbf{K}$ and a set $F \subset \mathbb{R}^n$ have the structure* (20.6)–(20.8). *Then*

$$\frac{\widetilde{d}^{-}c(t,x)}{(F,\psi)} = \min_{f\in F}\left[\liminf_{\delta\downarrow 0} \frac{1}{\delta}\left[E\left\{c\big(t+\delta, x+\delta f+\sqrt{\delta}\psi\eta\big)\right\} - c(t,x)\right]\right], \tag{20.12}$$

$$\frac{\widetilde{d}^{+}c(t,x)}{(F,\psi)} = \max_{f\in F}\left[\limsup_{\delta\downarrow 0} \frac{1}{\delta}\left[E\left\{c\big(t+\delta, x+\delta f+\sqrt{\delta}\psi\eta\big)\right\} - c(t,x)\right]\right]. \tag{20.13}$$

For simplicity of calculations in the proof of Lemmas V.8 and V.9, we introduce the following notation.

The symbol $\operatorname{cons}_i[h]_j$, where $0 \le i < j \le n$, and $h \in \mathbb{R}^n$ has the following sense:

$$\operatorname{cons}_i[h]_j = \big(0, \ldots, 0, h_{i+1}, \ldots, h_j, 0, \ldots, 0\big) \in \mathbb{R}^n.$$

For $j = i = n$ and $j = i = 0$, we set

$$\operatorname{cons}_n[h]_n = 0 \in \mathbb{R}^n, \quad \operatorname{cons}_0[h]_0 = 0 \in \mathbb{R}^n$$

and, in addition,

$$\operatorname{cons}[h]_j = \operatorname{cons}_0[h]_j, \quad j \in \overline{0,n}, \qquad \operatorname{cons}_i[h] = \operatorname{cons}_i[h]_n, \quad i \in \overline{0,n}.$$

Using this notation, one can rewrite conditions (20.7) and (20.8) in the following form:

$$\overline{f} = \mathrm{cons}[f]_\nu = \mathrm{const} \quad \text{for any} \quad f \in F, \qquad \widetilde{F} = \{\mathrm{cons}_\nu[f] : f \in F\}.$$

For a function $c(\cdot) \in \mathbf{K}$, we set

$$\mathrm{cons}_\nu\left[\frac{\partial c(t,x)}{\partial x}\right] = \left(0, \ldots, 0, \frac{\partial c(t,x)}{\partial x_{\nu+1}}, \ldots, \frac{\partial c(t,x)}{\partial x_n}\right).$$

Proof of Lemma V.8. For any fixed $\delta > 0$, introduce the notation

$$J_\delta^1 = \delta^{-1} E\left\{c\big(t+\delta, x+\delta(\overline{f}+\tilde{f}) + \sqrt{\delta}\psi\eta\big) - c\big(t+\delta, x+\delta\overline{f} + \sqrt{\delta}\psi\eta\big)\right\}. \tag{20.14}$$

Applying the formula of finite increments, we have

$$\begin{aligned}
J_\delta^1 = E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)+\theta(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} \\
= E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} \\
+ E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)+\theta(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right]\right.\right. \\
\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} \\
\le E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} + \varphi^{(1)}(\delta),
\end{aligned} \tag{20.15}$$

where $0 < \theta(\cdot) < 1$, $\nu < n$, and

$$\varphi^{(1)}(\delta) = L_1 \cdot (C_1)^2 \cdot \sqrt{(n-\nu)} \cdot \delta^\alpha, \qquad \varphi^{(1)}(\delta) \to 0 \quad \text{as} \quad \delta \downarrow 0. \tag{20.16}$$

The estimate for $\varphi^{(1)}(\delta)$ is obtained by using the Hölder condition (20.3) and condition (20.6), namely,

$$\begin{aligned}
E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)+\theta(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right]\right.\right. \\
\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} \\
\le E\left\{\left\| \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)+\theta(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right]\right.\right. \\
\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right]\right\|\right\} \cdot \|\tilde{f}\| \\
\le C_1 E\left\{\left(\sum_{i=\nu+1}^n (L_1)^2 \big|\theta(\cdot)\delta\cdot\tilde{f}_i\big|^{2\alpha}\right)^{1/2}\right\} \le C_1 L_1 \delta^\alpha \sqrt{\sum_{i=\nu+1}^n |\tilde{f}_i|^{2\alpha}} \\
\le C_1^{1+\alpha} L_1 \delta^\alpha (n-\nu)^{1/2} \le C_1^2 L_1 (n-\nu)^{1/2} \delta^\alpha = \varphi^{(1)}(\delta).
\end{aligned}$$

On the other hand, using similar reasonings, we have

$$
\begin{aligned}
-J^1_\delta &= E\left\{\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\cdot)-\theta'(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right], (-\tilde{f})\right\rangle\right\}\\
&= E\left\{\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], (-\tilde{f})\right\rangle\right\}\\
&+ E\left\{\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\cdot)-\theta'(\cdot)\delta\cdot\tilde{f}\big)}{\partial x}\right]\right.\right.\\
&\left.\left. - \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], (-\tilde{f})\right\rangle\right\}\\
&\le -E\left\{\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\cdot)\big)}{\partial x}\right], \tilde{f}\right\rangle\right\} + \varphi^{(1)}(\delta), \quad (20.17)
\end{aligned}
$$

where $0 < \theta'(\cdot) < 1$, $\nu < n$, and the estimate for $\varphi^{(1)}(\delta)$ is (20.16).

For almost all $\omega \in \Omega$, the following relations holds:

$$
\begin{aligned}
&\lim_{\delta\downarrow 0}\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta\overline{f}+\sqrt{\delta}\psi\eta(\omega)\big)}{\partial x}\right], \tilde{f}\right\rangle\\
&\quad= \lim_{\delta\downarrow 0}\left\langle \text{cons}_\nu\left[\frac{\partial c\big(t+\delta, x+\delta(\overline{f}+\tilde{f})+\sqrt{\delta}\psi\eta(\omega)\big)}{\partial x}\right], \tilde{f}\right\rangle\\
&\quad= \sum_{i=\nu+1}^{n}\tilde{f}_i\frac{\partial c(t,x)}{\partial x_i} = \sum_{i=\nu+1}^{n} f_i\frac{\partial c(t,x)}{\partial x_i},
\end{aligned}
$$

Hence, one can apply the Lebesgue theorem on limiting transition inside the operation $E\{\cdot\}$. Making δ tend to 0 in (20.14)–(20.17), we obtain relation (20.11). Lemma V.8 is proved. □

Proof of Lemma V.9. Let us prove Eq. (20.12). One can prove Eq. (20.13) by using similar estimates, where the operations min and lim inf are replaced by max and lim sup, respectively.

Fix $\delta > 0$. Since condition (20.1) holds and the set F is closed, there exists a measurable selector $f(\cdot) : \Omega \to F$ such that

$$
\min_{f\in F} c\left(t+\delta, x+\delta\cdot f+\sqrt{\delta}\psi\eta(\omega)\right) = c\left(t+\delta, x+\delta\cdot f(\omega)+\sqrt{\delta}\psi\eta(\omega)\right) \quad (20.18)
$$

for almost all $\omega \in \Omega$. Denote by $f^* = (f^*_1, \dots, f^*_n) \in \mathbb{R}^n$ a vector of the form

$$
f^* = E\{f(\cdot)\}. \quad (20.19)
$$

For this vector, conditions (20.6)–(20.8) imply

$$
f^* \in F, \quad \text{cons}[f^*]_\nu = \text{cons}[f(\omega)]_\nu = \overline{f}, \quad \omega \in \Omega. \quad (20.20)
$$

Using the formula of finite increments and conditions (20.18), (20.20), (20.3), and (20.6), we have

$$
\begin{aligned}
&E\left\{c\left(t+\delta, x+\delta f^*+\sqrt{\delta}\psi\eta(\cdot)\right) - \min_{f\in F} c\left(t+\delta, x+\delta f+\sqrt{\delta}\psi\eta(\cdot)\right)\right\}\\
&\quad= E\left\{c\left(t+\delta, x+\delta f^*+\sqrt{\delta}\psi\eta(\cdot)\right) - \min_{f\in F} c\left(t+\delta, x+\delta f(\cdot)+\sqrt{\delta}\psi\eta(\cdot)\right)\right\}\\
&\quad= E\left\{\left\langle \text{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f(\cdot)+\sqrt{\delta}\eta(\cdot)+\theta''(\cdot)(f^*-f(\cdot))\delta\right)\right], (f^*-f(\cdot))\cdot\delta\right\rangle\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \delta \cdot E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}(t+\delta, x+\delta f^*)\right], (f^* - f(\cdot))\right\rangle\right\} \\
&+ \delta \cdot E\left\{\left\langle\left(\mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f(\cdot)+\sqrt{\delta}\psi\eta(\cdot)+\theta''(\cdot)\big(f^*-f(\cdot)\big)\cdot\delta\right)\right]\right.\right.\right. \\
&\left.\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f^*+\sqrt{\delta}\eta(\cdot)+\theta''(\cdot)(f^*-f(\cdot))\delta\right)\right]\right), \big(f^*-f(\cdot)\big)\right\rangle\right\} \\
&+ \delta \cdot E\left\{\left\langle \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f^*+\sqrt{\delta}\psi\eta(\cdot)+\theta''(\cdot)\big(f^*-f(\cdot)\big)\cdot\delta\right)\right]\right.\right. \\
&\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}(t+\delta, x+\delta f^*)\right], \big(f^*-f(\cdot)\big)\right\rangle\right\} \\
&\le \delta \cdot \left\langle \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}(t+\delta, x+\delta f^*)\right], E\{f^*-f(\cdot)\}\right\rangle + \delta \cdot E\left\{\left(\sum_{i=\nu+1}^{n} L_1^2\big|\delta(f_i(\cdot)-f_i^*\big|^{2\alpha}\right)^{1/2}\right\}\cdot 2C_1 \\
&+ \delta \cdot E\left\{\left\|\mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f^*+\sqrt{\delta}\psi\eta(\cdot)+\theta''(\cdot)\big(f^*-f(\cdot)\big)\delta\right)\right]\right.\right. \\
&\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}(t+\delta, x+\delta f^*)\right]\right\|\right\}\cdot 2C_1 \le 0 + \delta\cdot\left[\varphi^{(2)}(\delta)+\varphi^{(3)}(\delta)\right], \qquad (20.21)
\end{aligned}
$$

where $0 < \theta''(\cdot) < 1$, $\nu < n$,

$$
\varphi^{(2)}(\delta) = E\left\{\left(\sum_{i=\nu+1}^{n}(L_1)^2\delta^{2\alpha}\cdot\big|f_i(\cdot)-f_i^*\big|^{2\alpha}\right)^{1/2}\right\}\cdot 2C_1 \le L_1(n-\nu)^{1/2}\cdot(2C_1)^{1+\alpha}\cdot\delta^{\alpha/2}, \qquad (20.22)
$$

and

$$
\varphi^{(2)}(\delta) \to 0 \quad \text{as} \quad \delta \downarrow 0.
$$

The estimate for $\varphi^{(3)}(\delta)$ in (20.21) is obtained as follows. First, using the Hölder condition (20.3) for $c(\cdot)$ and the boundedness of the set F (see (20.6)), we obtain from the Hölder inequality and an estimate for the square of the sum:

$$
\begin{aligned}
\varphi^{(3)}(\delta) &= E\left\{\left\|\mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}\left(t+\delta, x+\delta f^*+\sqrt{\delta}\psi\eta(\cdot)+\theta''(\cdot)\big(f(\cdot)-f^*\big)\delta\right)\right]\right.\right. \\
&\left.\left. - \mathrm{cons}_\nu\left[\frac{\partial c}{\partial x}(t+\delta, x+\delta f^*)\right]\right\|\right\} 2C_1 \\
&\le 2C_1\cdot E\left\{\left(\sum_{i=\nu+1}^{n} L_1^2\cdot\left|\sqrt{\delta}\sum_{j=1}^{m}\psi_{ij}\eta_j(\cdot)+\theta''(\cdot)\big(f_i(\cdot)-f_i^*\big)\delta\right|^{2\alpha}\right)^{1/2}\right\} \\
&\le 2C_1L_1\left(\sum_{i=\nu+1}^{n} E\left\{\left(2\delta\cdot\left|\sum_{j=1}^{m}\psi_{ij}\eta_j(\cdot)\right|^2+\delta^2\cdot 8C_1^2\right)^{\alpha}\right\}\right)^{1/2} \\
&= 2C_1L_1\cdot(2\delta)^{\alpha/2}\left(\sum_{i=\nu+1}^{n} E\left\{\left(\left|\sum_{j=1}^{m}\psi_{ij}\eta_j(\cdot)\right|^2+4\delta^2C_1^2\right)^{\alpha}\right\}\right)^{1/2} = J_\delta^*. \qquad (20.23)
\end{aligned}
$$

As follows from the properties of the Gauss variable $\eta(\cdot)$, the random variables

$$\xi_i(\cdot) = \sum_{j=1}^{m} \psi_{ij}\eta_j(\cdot), \quad i \in \overline{\nu+1, n},$$

are also Gauss variables with zero mean values and dispersions $\sum_{j=1}^{m} (\psi_{ij})^2$, respectively.

We set $b = 4\delta C_1^2$ and consider $\alpha < 1$. It is easy to see that

$$\begin{aligned} E\left\{(\xi_i^2(\cdot) + b)^\alpha\right\} &\le E\left\{\xi_i^2(\cdot) + b\right\} + \mathbb{P}\left\{\|\xi_i^2(\cdot) + b\| \le 1\right\} \\ &\le E\left\{\xi_i^2(\cdot)\right\} + b + \mathbb{P}\left\{-\infty < \xi_i^2(\cdot) < +\infty\right\} = \sum_{j=1}^{m} \psi_{ij}^2 + b + 1, \end{aligned} \tag{20.24}$$

where $\mathbb{P}\{\mathcal{A}\}$ is the probability of an event $\mathcal{A} \subset \Omega$.

Taking (20.24) into account, we continue inequalities (20.23) and obtain the estimate for $\varphi^{(3)}(\delta)$:

$$\begin{gathered} \varphi^{(3)}(\delta) \le \delta^{\alpha/2} \cdot 4C_1L_1 \left[\sum_{i=\nu+1}^{n} \sum_{j=1}^{m} \psi_{ij}^2 + (n-\nu)(4\delta C_1 + 1)\right]^{1/2}, \\ \varphi^{(3)}(\delta) \to 0 \quad \text{as} \quad \delta \downarrow 0. \end{gathered} \tag{20.25}$$

It follows from (20.18), (20.19), and (20.21) that the following inequalities hold:

$$\begin{aligned} &\frac{1}{\delta} \cdot \left[E\left\{\min_{f\in F} c\big(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} - c(t,x)\right] \\ &\qquad \le \frac{1}{\delta} \cdot \left[\min_{f\in F} E\left\{c\big(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} - c(t,x)\right] \\ &\qquad \le \frac{1}{\delta} \cdot \left[E\left\{c\big(t+\delta, x+\delta f^* + \sqrt{\delta}\psi\eta(\cdot)\big) - c(t,x)\right\}\right] \\ &\qquad = \frac{1}{\delta} \cdot \left[E\left\{\min_{f\in F} c\big(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} - c(t,x)\right] \\ &\qquad + \frac{1}{\delta} \cdot E\left\{c\big(t+\delta, x+\delta f^* + \sqrt{\delta}\psi\eta(\cdot)\big) - \min_{f\in F} c\big(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} \\ &\qquad \le \frac{1}{\delta} \cdot \left[E\left\{\min_{f\in F} c\big(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} - c(t,x)\right] + \varphi^{(2)}(\delta) + \varphi^{(3)}(\delta). \end{aligned} \tag{20.26}$$

Passing to the limit as $\delta \downarrow 0$ in inequalities (20.26), one can obtain from (20.4), (20.22), and (20.25):

$$\frac{\widetilde{d}^- c(t,x)}{(F,\psi)} = \liminf_{\delta\downarrow 0} \frac{1}{\delta} \cdot \left[\min_{f\in F} E\left\{c\left(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right]. \tag{20.27}$$

Note that the vector $f^* \in F$ of the form (20.19) is constructed for a fixed δ. Denote this vector by f_δ^*. By the Lipschitz condition (20.1), any two vectors $f^{(1)}$ and $f^{(2)}$ from F satisfy the estimate

$$\begin{gathered} -L_0 \cdot \delta\big\|f^{(1)} - f^{(2)}\big\| \le E\left\{c\big(t+\delta, x+\delta f^{(1)} + \sqrt{\delta}\psi\eta(\cdot)\big) - c\big(t+\delta, x+\delta f^{(2)} + \sqrt{\delta}\psi\eta(\cdot)\big)\right\} \\ \le L_0 \cdot \delta\big\|f^{(1)} - f^{(2)}\big\|. \end{gathered} \tag{20.28}$$

Using this fact, one can obtain from definition (20.27) and conditions (20.28) that there are converging sequences

$$\delta_i \downarrow 0, \quad f_{\delta_i}^* \to f_0^*,$$

where $f_0^* \in F$, and

$$\frac{\widetilde{d^-} c(t,x)}{(F,\psi)} = \lim_{\delta_i \downarrow 0} \frac{1}{\delta_i} \cdot \left[E\left\{ c\big(t+\delta_i x + \delta_i f^*_{\delta_i} + \sqrt{\delta_i}\psi\eta(\cdot)\big)\right\} - c(t,x)\right]$$
$$= \lim_{\delta_i \downarrow 0} \frac{1}{\delta_i} \cdot \left[E\left\{ c\big(t+\delta_i, x + \delta_i f^*_0 + \sqrt{\delta_i}\psi\eta(\cdot)\big)\right\} - c(t,x)\right]. \tag{20.29}$$

Using (20.29), we obtain the estimates

$$\inf_{f\in F} \liminf_{\delta\downarrow 0} \left[E\left\{ c\left(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta}$$
$$\le \liminf_{\delta\downarrow 0} \left[E\left\{ c\left(t+\delta, x+\delta f_0^* + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta}$$
$$\le \liminf_{\delta_i\downarrow 0} \left[E\left\{ c\left(t+\delta_i, x+\delta_i f_0^* + \sqrt{\delta_i}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta_i}$$
$$= \frac{\widetilde{d^-} c(t,x)}{(F,\psi)} = \liminf_{\delta\downarrow 0} \left[\min_{f\in F} E\left\{ c\left(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta}$$
$$\le \inf_{f\in F} \liminf_{\delta\downarrow 0} \left[E\left\{ c\left(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta}. \tag{20.30}$$

In addition, using (20.28), it is possible to prove that the mapping

$$f \to \liminf_{\delta\downarrow 0} \left[E\left\{ c\left(t+\delta, x+\delta f + \sqrt{\delta}\psi\eta(\cdot)\right)\right\} - c(t,x)\right] \frac{1}{\delta}$$

is Lipschitz continuous with a constant $L_0 > 0$. Hence, it follows from (20.30) that the operations lim inf and min in (20.3) commute. Therefore, formula (20.12) is proved. □

The following remark is a consequence of Theorem V.5.

Remark V.1. Let a function $c(\cdot) \in \mathbf{K}$ have additional properties, namely, the partial derivatives $\dfrac{\partial^2 c(t,x)}{\partial x_i \partial x_j}$, $i,j \in \overline{\nu+1,n}$, are continuous, elements of the matrix $\psi = (\psi_{ij})$, $i \in \overline{1,n}$, $j \in \overline{1,m}$, satisfy the relations

$$\psi_{ij} = 0, \quad i \in \overline{1,\nu}, \quad j \in \overline{1,m}, \tag{20.31}$$

and the set F satisfies conditions (20.6)–(20.8). Then, applying the Taylor formula, it is possible to show that formulas (20.9) and (20.10) for generalized stochastic derivatives can be written in the form

$$\frac{\widetilde{d^-} c(t,x)}{(F,\psi)} = \min_{\tilde f\in\widetilde F} \sum_{i=\nu+1}^{n} \frac{\partial c(t,x)}{\partial x_i} \cdot \tilde f_i$$
$$+ \frac{1}{2} \cdot \sum_{i,j=\nu+1}^{n} a_{ij} \frac{\partial^2 c(t,x)}{\partial x_i \partial x_j} + \liminf_{\delta\downarrow 0} \frac{\left[c\big(t+\delta, x+\delta\cdot\overline{f}\big) - c(t,x)\right]}{\delta}, \tag{20.32}$$

$$\frac{\widetilde{d^+} c(t,x)}{(F,\psi)} = \max_{\tilde f\in\widetilde F} \sum_{i=\nu+1}^{n} \frac{\partial c(t,x)}{\partial x_i} \cdot \tilde f_i$$
$$+ \frac{1}{2} \cdot \sum_{i,j=\nu+1}^{n} a_{ij} \frac{\partial^2 c(t,x)}{\partial x_i \partial x_j} + \limsup_{\delta\downarrow 0} \frac{\left[c\big(t+\delta, x+\delta\cdot\overline{f}\big) - c(t,x)\right]}{\delta}, \tag{20.33}$$

where

$$a_{ij} = \sum_{k=1}^{m} \psi_{ik}\psi_{jk}, \quad i,j,\in \overline{\nu+1,n}. \tag{20.34}$$

20.3. Application of the formulas for stochastic derivatives. Let us apply formulas (20.9), (20.10), and (20.32)–(20.34) for generalized stochastic derivatives to studying the value functions of the following class of diffusion differential games considered in [78].

Let a control diffusion process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be described by the equation

$$\xi_r = \xi_r\left(t_0, x_0, u_{(\cdot)}, v_{(\cdot)}\right) = x_0 + \int_0^r f\big(t_0 + s, \xi_s, u_s, v_s\big)ds + \int_0^r \psi(t_0 + s)dW_s, \quad r \geq 0, \tag{20.35}$$

where x_0 is an initial phase state (n-dimensional vector) given at an initial time moment t_0; ξ_r be the current phase state for the process; $\{\mathcal{F}_s\}$, $s \geq 0$, be a nondecreasing system of σ-algebras of subsets in the set Ω; W_s, $s \geq 0$, be a m-dimensional standard Wiener process relative to $\{\mathcal{F}_s\}$; $u_s : \Omega \to P \subset \mathbb{R}^p$ and $v_s : \Omega \to Q \subset \mathbb{R}^q$ be nonanticipating processes called the *control* and the *disturbance*, respectively; let the given sets of their values P and Q be compact; $0 \leq t_0 \leq T$ and T be the fixed terminal moment of the game.

Let the quality of the control process ξ_r be estimated by the quantity

$$\gamma^*(\xi_{(\cdot)}) = E\left\{\gamma\left(\xi_{T-t_0}(t_0, x_0, u_{(\cdot)}, v_{(\cdot)})\right)\right\}, \tag{20.36}$$

where $\gamma(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a given uniformly bounded and Lipschitz continuous function.

Assume that the function $f(\cdot) : [0,T] \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ is continuous, uniformly bounded, and uniformly Lipschitz continuous with respect to x. Let the $(n \times m)$-dimensional diffusion matrix

$$t \to \psi(t) = (\psi_{ij}(t)), \quad i \in \overline{1,n}, \quad j \in \overline{1,m}, \quad t \in [0,T],$$

and the drift vector-valued function $t \to f(t,x,u,v)$ satisfy the uniform Hölder condition. Assume also that there exists a number $\nu \in \overline{1,(n-1)}$ satisfying the conditions

$$f_i(t,x,u,v) = f_i(t,x); \qquad \psi_{ij}(t) = 0 \quad \text{for} \quad i \in \overline{1,\nu}, \quad j \in \overline{1,m}. \tag{20.37}$$

This means that a noise, a disturbance, and a control act on a part of the coordinates ($i = \nu+1, \ldots, n$) of process (20.35) simultaneously. Finally, assume that the Isaacs condition

$$\max_{v\in Q}\min_{u\in P}\sum_{i=\nu+1}^{n} s_i \cdot \tilde{f}_i(t,x,u,v) = \min_{u\in P}\max_{v\in Q}\sum_{i=\nu+1}^{n} s_i \cdot \tilde{f}_i(t,x,u,v) = H(t,x,\widetilde{s}) \tag{20.38}$$

holds for any $\widetilde{s} = \{0, \ldots, 0, s_{\nu+1}, \ldots, s_n\} \in \mathbb{R}^n$.

It was shown in [78] that the game (20.35), (20.36) has the value $c^0(t_0, x_0)$ for all initial positions (t_0, x_0). At regular points of smoothness, the value function $c^0(\cdot) \in \mathbf{K}$ satisfies the following quasi-linear parabolic equation of the Hamilton–Jacobi–Isaacs type:

$$\frac{\partial c^0(t,x)}{\partial t} + \max_{v\in Q}\min_{u\in P}\left\langle \frac{\partial c^0(t,x)}{\partial x}, f(t,x,u,v)\right\rangle + \frac{1}{2}\sum_{i,j=\nu+1}^{n} a_{ij}(t)\frac{\partial^2 c(t,x)}{\partial x_i \partial x_j} = 0, \tag{20.39}$$

where $a_{ij}(t)$ are constructed by using $\psi = (\psi_{ij}(t))$ in accordance with (20.34).

Differential inequalities (19.8) and (19.9) in Sec. 19 and formulas (20.32) and (20.34) give the following infinitesimal relations, which replace the Isaacs equation (20.39) and characterize the value function $c^0(t,x)$ on singular sets:

$$\begin{aligned} -\frac{d^+ c^0(t,x)}{(1, \overline{f}(t,x))} &\leq \max_{v\in Q}\min_{u\in P}\sum_{i=\nu+1}^{n}\frac{\partial c^0(t,x)}{\partial x_i}\cdot \tilde{f}_i(t,x,u,v) + \frac{1}{2}\sum_{i,j=\nu+1}^{n} a_{ij}(t)\frac{\partial^2 c^0(t,x)}{\partial x_i \partial x_j} \\ &= \min_{u\in P}\max_{v\in Q}\sum_{i=\nu+1}^{n}\frac{\partial c^0(t,x)}{\partial x_i}\cdot \tilde{f}_i(t,x,u,v) + \frac{1}{2}\sum_{i,j=\nu+1}^{n} a_{ij}(t)\frac{\partial^2 c^0(t,x)}{\partial x_i \partial x_j} \leq -\frac{d^- c^0(t,x)}{(1, \overline{f}(t,x))}, \end{aligned} \tag{20.40}$$

where the vector $\overline{f}(t,x) \in \mathbb{R}^n$ is equal to the uncontrolled and undisturbed part of the drift vector, namely,

$$\overline{f}_i(t,x) = \begin{cases} f_i(t,x) & \text{for} \quad i \in \overline{1,\nu}, \\ 0 & \text{for} \quad i \in \overline{\nu+1,n}, \end{cases}$$

where $\tilde{f}_i(t,x,u,v)$ are the controlled and disturbed components of the drift vector:

$$\tilde{f}_i(t,x,u,v) = f_i(t,x,u,v), \quad i \in \overline{\nu+1,n}.$$

One can consider relations (20.40) as a generalization of the quasi-linear parabolic Isaacs equation (19.1) for the stochastic game (20.35), (20.36).

REFERENCES

1. R. A. Adiatullina and A. M. Tarasyev, "An infinite time horizon differential game," *Prikl. Mat. Mekh.*, **51**, 531–537 (1987).
2. L. D. Akulenko, *Asymptotic Methods in Optimal Control* [in Russian], Nauka, Moscow (1987).
3. V. M. Alexeev, V. M. Tikhomirov, and S. V. Fomin, *Optimal Control* [in Russian], Nauka, Moscow (1979).
4. M. I. Alekseichik, "An advanced formalization of the basic notions to antagonistic differential games," *Mat. Anal. Prilozh.*, Rostov-on-Don State Univ., **7**, 191–199 (1975).
5. E. H. Albrecht, "Constructions of approximate solutions to quasi-linear differential games," *Proc. Steklov Inst. Math.*, Suppl. Issue 1, S24–S34 (2000).
6. E. H. Albrecht and G. S. Shelement'ev, *Lectures on the Theory of Stabilization* [in Russian], Ural State Univ., Sverdlovsk (1972).
7. B. I. Ananiev, "On minimax state estimates for multistage statistically uncertain systems," *Probl. Contr. Inform. Theory*, **18**, No. 1, 27–41 (1981).
8. V. I. Arnold, *Mathematical Methods of Classical Mechanics* [in Russian], Nauka, Moscow (1974).
9. V. I. Arnold, *Singularities of Caustics and Wave Fronts* [in Russian], Fazis, Moscow (1996).
10. Z. Artstein and V. Gaitsgory, "Tracking fast trajectories along a slow dynamics. A singular perturbations approach," *SIAM J. Contr. Optimiz.*, **35**, 1487–1507 (1997).
11. Z. Artstein and V. Gaitsgory, "The value function of singularly perturbed control systems," *Appl. Math. Optimiz.*, **41**, 425–445 (2000).
12. A. V. Arutyunov and S. M. Aseev, "The maximum principle in differential inclusions with phase restrictions," *Dokl. Ross. Akad. Nauk*, **334**, No. 2, 134–137 (1994).
13. J.-P. Aubin, *Viability Theory*, Birkhäuser, Boston (1991).
14. J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin–Heidelberg (1984).
15. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis* [Russian translation], Mir, Moscow (1988).
16. J.-P. Aubin and H. Frankowska, *Set Valued Analysis*, Birkhäuser, Boston, (1990)
17. N. S. Bakhvalov, *Average of Processes in Periodic Media* [in Russian], Nauka, Moscow (1984).
18. A. E. Barabanov and A. M. Ghulchak, "H-infinity optimisation problem with sign-indefinite quadratic form," *Systems Control Lett.*, **29**, 157–164 (1996).
19. N. E. Barabanov and R. Ortega, "Necessary and sufficient conditions for passivity of the Lugre friction model," *IEEE Trans. Autom. Contr.*, **1** (2000).
20. E. A. Barbashin, *Introduction to the Theory of Stability* [in Russian], Nauka, Moskow (1967).
21. M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*, Birkhäuser, Boston (1997).
22. M. Bardi and L. C. Evans, "On Hopf's formulas for solutions of Hamilton–Jacobi equations," *Nonlinear Analysis, Theory, Methods, Appl.*, **8**, No. 11, 1373–1381 (1984).

23. M. Bardi and M. Falcone, "An approximation scheme for the minimum time function," *SIAM J. Control Optimiz.*, **28**, 950–965 (1990).
24. G. Barles and B. Perthame, "Exit time problems in optimal control and vanishing viscosity solutions of Hamilton–Jacobi equations," *SIAM J. Control Optimiz.*, **26**, 1133–1148 (1988).
25. E. N. Barron and R. Jensen, "The Pontryagin maximum principle from dynamical programming and viscosity solutions to first-order partial differential equations," *Trans. Amer. Math. Soc.*, **298**, No. 2, 635–641 (1986).
26. E. N. Barron, L. C. Evans, and R. Jensen, "Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls," *J. Differ. Equations*, **53**, 213–233 (1984).
27. T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems*, Birkhäuser, Boston (1991).
28. V. D. Batukhtin, "The extremal aiming in a nonlinear approach game," *Dokl. Ross. Akad. Nauk*, **27**, No. 1 (1972).
29. R. Bellman, *Dynamic Programming*, Princeton Univ. Press, Princeton, New Jersey (1957).
30. R. Bellman and R. Kalaba,*Dynamic Programming and Modern Control Theory*, Academic Press, New York (1965).
31. A. Bensoussan and J. L. Lions, *Applications of Variational Inequalities in Stochastic Control*, North-Holland, Amsterdam–New York–Oxford (1982).
32. A. Bensoussan, *Perturbation Methods in Optimal Control*, Wiley-Gauthier, New York–Chichester (1988).
33. Yu. I. Berdyshev, "A qualitative analysis of attainability sets," *Kosmich. Issled.*, **34**, No. 2, 141–144 (1996).
34. L. D. Berkovitz, "A variational approach to differential games," in: *Adv. Game Theory*, Ann. Math. Stud., **52**, Princeton Univ. Press, Princeton (1964).
35. L. D. Berkovitz, "Optimal feedback controls," *SIAM J. Control Optimiz.*, **27**, 991–1006 (1989).
36. V. I. Blagodatskikh and A. F. Filippov, "Differential inclusions and optimal control problems," *Proc. Steklov Inst. Math.*, **169**, 194–252 (1985).
37. N. D. Botkin, M. A. Zarkh, V. N. Kein, V. S. Patsko, and V. L. Turova, "Differential games and control problems for an aircraft under windshear," *Izv. Ross. Akad. Nauk. Ser. Tekhn. Kibern.*, No. 1, 68–76 (1993).
38. V. G. Boltyanskii, *Mathematical Methods of Optimal Control* [in Russian], Nauka, Moscow (1966).
39. A. Bryson and Y.-C. Ho, *Applied Optimal Control Theory* [Russian translation], Mir, Moscow (1972).
40. S. A. Brykalov, "A conflict controlled system with a nonfixed end time moment," *Proc. Steklov Inst. Math.*, Suppl. Issue 2, S313–S319 (2000).
41. R. Bulirsh, E. Ners, H. J. Pesh, and O. von Stryk, *Combining direct and indirect methods in optimal control: angle maximisation of a hang glider*, Report Shcwerpunktprogr. DFG Anwendungsabgene Optimierung und Steuerung, No. 313, Math. Inst. Techn. Univ. München (1991).
42. P. Cannarsa and H. Frankowska, "Some characterization of optimal trajectories in control theory," *SIAM J. Control Optimiz.*, **29**, 1322–1347 (1991).
43. A. G. Chentsov, "On an approach game problem at a given time moment," *Mat. Sb.*, **99**, 394–420 (1976).
44. A. G. Chentsov, *Asymptotic Attainability*, Kluwer Academic, Dordrecht (1997).
45. A. G. Chentsov and V. E. Pak, "On the extension of the nonlinear problem of optimal control with nonstationary phase restrictions," *Nonlinear Analysis: Theory, Methods, Appl.* **26**, No. 2, 383–394 (1996).
46. F. L. Chernous'ko, L. D. Akulenko, and B. N. Sokolov, *Control of Oscillations* [in Russian], Nauka, Moscow (1980).

47. F. L. Chernous'ko and V. B. Kolmanovskii, *Optimal Control under Random Disturbances* [in Russian], Nauka, Moscow (1978).
48. F. L. Chernous'ko and A. A. Melikyan, *Game Problems of Control and Search* [in Russian], Nauka, Moscow (1978).
49. A. A. Chikrii, *Conflict Controlled Processes* [in Russian], Naukova Dumka, Kiev (1992).
50. S. V. Chistyakov, "On solutions of pursuit game problems," *Prikl. Mat. Mekh.*, **41**, 825–832 (1977).
51. F. H. Clarke, "Generalized gradients and applications," *Trans. Amer. Math. Soc.*, **205**, 246–262 (1975).
52. F. H. Clarke, *Optimization and Nonsmooth Analysis* [Russian translation], Nauka, Moscow (1988).
53. F. H. Clarke and Yu. S. Ledyaev, "New formulas for finite differences," *Dokl. Ross. Akad. Nauk*, **331**, No. 3, 275–277 (1962).
54. F. H. Clarke, Yu. S. Ledyaev, and R. Stern, "Proximal analysis and feedback constructions," *Proc. Steklov Inst. Math.*, Suppl. Issue **1**, S72–S89 (2000).
55. F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York (1997).
56. F. H. Clarke and R. Vinter, "The relationship between the maximum principle and dynamic programming," *SIAM J. Contr. Optimiz.* No. 5, 1291–1311 (1987).
57. E. D. Conway and E. Hopf, "Hamilton's theory and generalized solutions of the Hamilton–Jacobi equations," *Trans. Amer. Math. Soc.*, **13**, No. 2, 939–986 (1964).
58. M. G. Crandall, "A generalization of Peano's existence theorem and flow invariance," *Proc. Amer. Math. Soc.*, **36**, No. 1, 151–155 (1972).
59. M. G. Crandall and P. L. Lions, "Viscosity solutions of Hamilton–Jacobi equations," *Trans. Amer. Math. Soc.*, **277**, 1–42 (1983).
60. M. G. Crandall, L. C. Evans, and P. L. Lions, "Some properties of viscosity solutions of Hamilton–Jacobi equations," *Trans. Amer. Math. Soc.*, **282**, 487–502 (1984).
61. M. G. Crandall, H. Ishii, and P. L. Lions, "A user's guide to viscosity solutions," *Bull. Amer. Math. Soc.*, **27**, 1–67 (1992).
62. R. Courant and D. Hilbert, *Partial Differential Equations* [Russian translation], Mir, Moscow (1964).
63. A. R. Danilin, "Asymptotics of controls to a singular elliptic problem," *Dokl. Ross. Akad. Nauk*, **369**, No. 3, 305–308 (1999).
64. V. F. Demyanov and V. N. Malozemov, *An Introduction to Minimax* [in Russian], Nauka, Moscow (1972).
65. V. F. Demyanov and A. M. Rubinov. *Foundations of Nonsmooth Analysis and Quasi-differential Calculus* [in Russian], Nauka, Moscow (1990).
66. V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt (1995).
67. M. G. Dmitriev, "The theory of singular perturbations and some problems of optimal control," *Differ. Equations*, **21**, No. 10, 1693–1698 (1985).
68. A. Donchev, *Optimal Control Systems. Perturbations, Approximations and Analysis of Sensitivity* [Russian translation], Mir, Moscow (1987).
69. A. Ya. Dubovitzkii and A. A. Milyutin, "Problems on extremum under restrictions," *Dokl. Akad. Nauk SSSR*, **149**, No. 4, 759–762 (1963).
70. V. Ya. Dzhafarov, "On stability of the guaranteed result to a positional control problem," *Dokl. Acad. Nauk SSSR*, **285**, No. 1, 27–31 (1985).
71. R. Elliott, "Viscosity solutions and optimal control," in: *Pitman Res. Notes, Math. Ser.*, **165**, Boston (1987).

72. R. J. Elliott and N. J. Kalton, "The existence of value in differential games of pursuit and evasion," *J. Differ. Equations*, **12**, No. 3, 504–523 (1972).
73. L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math., **19**, Amer. Math. Soc., Providence, Rhode Island (1998).
74. A. F. Filippov, "On some questions in the theory of optimal control," *Vestn. Moscow State Univ. Ser. Math., Mech., Phys., Chem.*, No. 2, 25–32 (1959).
75. A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht (1988).
76. W. H. Fleming, "The convergence problem for differential game, II," *Adv. Game Theory*, Ann. Math. Stud., **52**, 195–210 (1964).
77. W. H. Fleming, "The Cauchy problem for degenerate parabolic equations," *J. Math. Mech.*, **13**, No. 6, 987–1008 (1964).
78. W. H. Fleming, "The Cauchy problem for a nonlinear first order differential equation," *J. Differ. Equations*, **5**, No. 3, 515–550 (1969).
79. W. H. Fleming and R. Rishel, *Deterministic and Stochastic Optimal Control* [Russian translation], Mir, Moscow (1978).
80. W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York (1993).
81. V. N. Fomin, A. M. Fradkov, and V. A. Yakubovich, *Adaptive Control of Dynamical Objects* [in Russian], Nauka, Moscow (1981).
82. A. Friedman, *Differential Games*, Wiley Interscience, New York (1971).
83. R. F. Gabasov and F. M. Kirillova, *Qualitative Theory of Optimal Processes* [in Russian], Nauka, Moscow (1971).
84. V. G. Gaitsgory, *Control of Systems with Fast and Slow Motions* [in Russian], Nauka, Moscow (1991).
85. V. G. Gaitsgory, "Suboptimal control of singularly perturbed systems and periodic optimization," *IEEE Trans. Automat. Contr.*, **38**, No. 6, 888–902 (1993).
86. V. G. Gaitsgory, "Limit Hamilton–Jacobi equations for singularly perturbed zero-sum differential games," *J. Math. Anal. Appl.*, **202**, 862–899 (1996).
87. R. V. Gamkrelidze, *Principles of Optimal Control* [in Russian] Tbilisi State Univ. (1975).
88. R. V. Gamkrelidze, A. A. Agrachev, and S. A. Vakhrameev, "Ordinary differential equations on vector fibers and chronological calculus," in: *Itogi Nauki i Tekhn. Ser. Sovr. Probl. Mat.*, **35**, All-Union Institute for Scientific and Technical Information, Moscow (1989), pp. 3–107.
89. V. L. Gasilov and V. B. Kostousov, "Methods of an extraction and a representation of information about geophysical fields," *Gyroscopes Navigation*, **4**, No. 15, 64–65 (1996).
90. I. M. Gel'fand, "Some problems to the theory of quasi-linear equations," *Usp. Mat. Nauk*, **14**, No. 2, 87–158 (1959).
91. E. I. Gerashchenko and S. M. Gerashchenko, *Method of Decomposition of Motions and Optimization of Nonlinear Systems* [in Russian], Nauka, Moscow (1975).
92. S. K. Godunov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1979).
93. A. Yu. Goritski and E. Yu. Panov, "Example of nonuniqueness of entropy solutions in the class of locally bounded functions," *Zh. Mat. Fiz.*, **6**, No. 4, 492–494 (1999).
94. N. L. Grigorenko, Yu. N. Kiselev, N. V. Lagunov, D. B. Silin, and N. G. Trin'ko, *Methods of Solving Differential Games. Mathematical Modelling* [in Russian], Moscow State Univ. (1993).
95. S. V. Grigoryeva, A. M. Tarasyev, A. A. Uspenskii, and V. N. Ushakov, "Constructions of the theory of differential games to solving the Hamilton–Jacobi equations," *Proc. Steklov Inst. Math.*, Suppl. Issue 2, S320–S336 (2000).

96. M. I. Gusev and A. B. Kurzhanskii, "On optimization of controlled systems with restrictions, I, II," *Differ. Equations*, **7**, No. 9, 1591–1602; No. 10, 1789–1800 (1971).
97. M. I. Gusev, "On the structure of optimal minimax estimates to problems of the guaranteed estimate," *Dokl. Ross. Akad. Nauk*, **322**, No. 5, 832–835 (1992).
98. Kh. G. Guseinov and V. Ya. Dzhafarov, "Left-hand side solutions of the Hamilton-Jacobi equations," *Proc. Steklov Inst. Math.*, Suppl. Issue 2, S337–S350 (2000).
99. Kh. G. Guseinov and V. N. Ushakov, "On constructions of differential inclusions with given properties," *Differ. Equations*, **36**, No. 4, 438–445 (2000).
100. Kh. G. Guseinov, A. I. Subbotin, and V. N. Ushakov, "Derivatives of multivalued mappings and their applications to game-theoretic control problems," *Probl. Control Inform. Theory*, **14**, No. 3, 1–14 (1985).
101. P. B. Gusyatnikov, *Theory of Differential Games* [in Russian], Moscow Phys.-Tekhn. Inst. (1982).
102. G. Haddad, "Monotone trajectories of differential inclusions and functional-differential inclusions with memory," *Isr. J. Math.*, **39**, 83–100 (1981).
103. C. J. Himmelberg, "Measurable relations," *Fundam. Math.*, **82**, No. 1, 53–72 (1975).
104. E. Hopf, "Generalized solutions of nonlinear equations of first order," *J. Math. Mech.*, **14**, 951–972 (1965).
105. A. M. Il'in, "Boundary layers," in: *Itogi Nauki i Tekhn. Ser. Fundam. Napr.*, **34**, All-Union Institute for Scientific and Technical Information, Moscow (1988).
106. A. M. Il'in, A. S. Kalashnikov, and O. A. Oleinik, "Second-order linear equations of parabolic type," *Usp. Mat. Nauk*, **17**, No. 3, 3–146 (1962).
107. A. D. Ioffe and V. M. Tikhomirov, *Theory of Extremal Problems* [in Russian], Nauka, Moscow (1974).
108. R. Isaacs, *Differential Games*, Wiley, New York (1965).
109. H. Ishii and S. Koike, "Remarks on elliptic singular perturbation problems," *Appl. Math. Optim.*, **23**, 1–75 (1991).
110. A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, New York (1995).
111. A. G. Ivanov, "The problem on brachistocrone in the central gravitation field," in: *Algorithms and Software of Parallel Calculations* [in Russian], **2**, Ekaterinburg (1998), pp. 95–109.
112. E. Kamke, *Handbook on First-Order Partial Differential Equations* [Russian translation], Nauka, Moscow (1966).
113. L. V. Kamneva, V. S. Patsko, and V. L. Turova, "Construction of the value function for game brachistochrone problem," in: *Proc. X Int. Symp. "Dynamical Games and Applications," July, 8–11, 2002, St. Petersburg State University, Russia*, **2**, St. Petersburg (2002), pp. 408–415.
114. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* [in Russian], Nauka, Moscow (1977).
115. I. Ya. Katz, *Method of Lyapunov Functions in Problems of Stability and Stabilization for Systems of Random Structure* [in Russian], Ekaterinburg (1998).
116. M. M. Khrustalev, "Necessary and sufficient optimality conditions in terms of the Bellman equation," *Dokl. Akad. Nauk SSSR*, **254**, 293–297 (1980).
117. A. V. Kim and V. G. Pimenov, "On applications of i-smooth analysis to numerical methods of solving finctional-differential equations," in: *Proc. Inst. Math. Mechanics, Ural Branch of the Russian Academy of Sciences*, **5**, Ekaterinburg (1998), pp. 119–142.
118. A. F. Kleimenov, *Nonantagonistic Positional Differential Games* [in Russian], Nauka, Ekaterinburg (1993).
119. A. I. Klimushev and N. N. Krasovskii, "Uniform asymptotic stability of systems of ordinary differential equations with a small parameter at derivatives," *Prikl. Mat. Mekh.*, **25**, 1011–1025 (1962).

120. P. V. Kokotovic, “Applications of singular perturbations techniques to control problems,” *SIAM J. Rev.*, **26**, No. 4, 501–510 (1984).
121. A. N. Kolmogorov and S. V. Fomin, *Elements of Theory of Functions and Functional Analysis* [in Russian], Nauka, Moscow (1968).
122. V. N. Kolokol’tsov and V. P. Maslov, *Idempotent Analysis and Applications to Optimal Control Problems* [in Russian], Nauka, Moscow (1994).
123. A. F. Kononenko, “On structure of an optimal strategy to dynamical control systems,” *Zh. Vychisl. Mat. Mat. Fiz.*, **20**, No. 5, 1105–1116 (1980).
124. A. I. Korotkii, “Dynamical modelling of parameters to hyperbolic systems,” *Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibern.*, No. 2, 154–164 (1991).
125. A. N. Krasovskii, “A differential game for the positional functional,” *Dokl. Akad. Nauk SSSR*, **253**, No. 6, 1303–1307 (1980).
126. A. N. Krasovskii and N. N. Krasovskii, *Control under Lack of Information*, Birkhäuser, Boston (1995).
127. N. N. Krasovskii, *Theory of Control of Motions* [in Russian], Nauka, Moscow (1968).
128. N. N. Krasovskii, *Game-Theoretic Problems on the Encounter of Motions* [in Russian], Nauka, Moscow (1970).
129. N. N. Krasovskii and N. Yu. Lukoyanov, “Equations of the Hamilton–Jacobi type in hereditary systems: Minimax solutions,” *Proc. Steklov Inst. Math.*, Suppl. Issue 1, S136–S154 (2000).
130. N. N. Krasovskii and V. M. Reshetov, “Approach-evasion problems for systems with a small parameter at derivatives,” *Prikl. Mat. Mekh.*, **38**, No. 5, 771–779 (1974).
131. N. N. Krasovskii and T. N. Reshetova, “On the program synthesis of a guaranteed control,” *Probl. Contr. Inform. Theory*, **17**, No. 6, 333–343 (1988).
132. N. N. Krasovskii and A. I. Subbotin, “An alternative to an approach game problem,” *Prikl. Mat. Mekh.*, **34**, 1005–1022 (1970).
133. N. N. Krasovskii and A. I. Subbotin, *Positional Differential Games* [in Russian], Nauka, Moscow (1974).
134. N. N. Krasovskii, *Control of Dynamical Systems* [in Russian], Nauka, Moscow (1985).
135. N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, New York (1988).
136. N. N. Krasovskii and V. E. Tretyakov, “A saddle point of a stochastic differential game,” *Dokl. Akad. Nauk SSSR*, **254**, No. 3, 534–539 (1980).
137. V. F. Krotov, “Global methods in optimal control theory,” in: *Advances in Nonlinear Dynamics and Control: A Report from Russia*, Progr. Systems =Control Theory, No. 17, Birkhäuser, Boston (1993), pp. 76–121.
138. V. F. Krotov and V. I. Gurman, *Methods and Problems of Optimal Control* [in Russian] Nauka, Moscow (1973).
139. S. N. Kruzhkov, “On methods of construction of generalized solutions of the Cauchy problem for a quasi-linear partial differential equation of first order,” *Usp. Mat. Nauk*, **20**, No. 6, 112–118 (1965).
140. S. N. Kruzhkov, “Generalized solutions of first-order nonlinear equations in several independent variables, I,” *Mat. Sb.*, **70**, No. 3, 394–415 (1966).
141. S. N. Kruzhkov and N. S. Petrosyan, “Asymptotic behavior of solutions of the Cauchy problem for nonlinear partial differential equations of first order,” *Usp. Mat. Nauk*, **42**, 3–40 (1987).
142. A. V. Kryazhimskii, “On the theory of positional differential games of approach and evasion,” *Dokl. Akad. Nauk SSSR*, **239**, No. 4, 779–782 (1987).
143. A. V. Kryazhimskii and Yu. S. Osipov, “On positional modelling of controls in dynamical systems,” *Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibern.*, No. 2, 51–60 (1983).

144. A. V. Kryazhimskii and Yu. S. Osipov, “On differential-evolutionary games,” *Proc. Steklov Inst. Math.*, **211**, 234–261 (1995).
145. A. V. Kryazhimskii and Yu. S. Osipov, “Extremal problems with separable graphs,” *Kibern. Syst. Anal.*, **2**, 32–55 (2002).
146. N. V. Krylov, *Controlled Diffusion Processes* [in Russian], Nauka, Moscow (1977).
147. S. I. Kumkov and V. S. Patsko, “Control of informational sets and pursuit problem,” in: *Ann. Int. Soc. Dynam. Games*, **3**, Birkhäuser, Boston (1995), pp. 191–206.
148. S. S. Kumkov and V. S. Patsko, “Maximal stable bridges to the Pontryagin control example,” *Vestn. Udmurt. State Univ.*, No. 1, 92–103 (2000).
149. S. I. Kumkov, V. S. Patsko, S. G. Pyatko, and A. A. Fedotov, “Construction of the stable bridge in a problem of aircraft guiding under wind disturbances,” in: *Proc. X Int. Symp. “Dynamical Games and Applications,” July 8–11, 2002, St. Petersburg State University*, **2**, St. Petersburg (2002), pp. 474–480.
150. A. B. Kurzhanskii, *Control and Observation under Conditions of Uncertainty* [in Russian], Nauka, Moscow (1979).
151. A. B. Kurzhanskii and T. F. Filippova, “On the method of singular perturbations to differential inclusions,” *Dokl. Akad. Nauk SSSR*, **321**, No. 3, 454–459 (1991).
152. A. B. Kurzhanski and T. F. Filippova, “On the theory of trajectory tubes – a mathematical formalism for uncertain dynamics, viability and control,” in: *Advances in Nonlinear Dynamics and Control: a Report from Russia*, Progr. Systems and Control Theory, **17**, Birkhäuser, Boston (1993), pp. 122–188.
153. A. B. Kurzhanskii and O. I. Nikonov, “Evolutionary equations for bundles of trajectories of designed control systems,” *Dokl. Ross. Akad. Nauk*, **333**, No. 5 (1993).
154. A. B. Kurzhanskii and I. F. Sivergina, “The dynamical programming method to inverse estimate problems for distributed systems,” *Dokl. Ross. Akad. Nauk*, **369**, No. 2, 161–166 (1998).
155. A. B. Kurzhanski and P. Varaiya, “On reachability under uncertainty,” *SIAM J. Control. Optimiz.*, **41**, No. 1, 181–216 (2002).
156. N. N. Kuznetsov and B. L. Rozhdestvenskii, Construction of a generalized solution of the Cauchy problem for a quasi-linear equation,” *Usp. Mat. Nauk*, **14**, No. 2, 211–215 (1959).
157. O. A. Ladyshenskaya and N. N. Uraltseva, *Linear and Quasi-linear Partial Differential Equations of Elliptic Type* [in Russian], Nauka, Moscow (1964).
158. O. A. Ladyshenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasi-linear Partial Differential Equations of Parabolic Type* [in Russian], Nauka, Moscow (1967).
159. A. S. Lakhtin and A. I. Subbotin, “Multivalued solutions of first-order partial differential equations,” *Mat. Sb.*, **189**, No. 6, 33–58 (1998).
160. A. S. Lakhtin and A. I. Subbotin, “Minimax and viscosity solutions to discontinuous first-order partial differential equations,” *Dokl. Ross. Akad. Nauk*, **359**, No. 4, 452–455 (1998).
161. P. Lax, “Hyperbolic systems of conservations laws, II,” *Commun. Pure Appl. Math.*, **10**, 537–566 (1957).
162. Yu. S. Ledyaev and E. F. Mishchenko, “Extremal problems in the theory of differential games,” *Proc. Steklov Inst. Math.*, **85**, 147–170 (1988).
163. G. Leitmann, *Introduction to the Theory of Optimal Control* [in Russian], Nauka, Moscow (1968).
164. G. Leitmann, “One approach to the control of uncertain dynamical systems,” *Appl. Math. Comput.*, **70**, 261–272 (1995).
165. K. Leichtveis, *Convex Sets* [in Russian] Nauka, Moscow (1985).
166. E. F. Lelikova, “On asymptotics of the fundamental solution to a parabolic partial differential equation of higher order,” *Dokl. Ross. Akad. Nauk*, **341**, No. 5, 532–537 (1995).

167. P. L. Lions, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman Research Notes, Math. Ser., **69**, Pitman, Boston (1982).
168. P. L. Lions, "Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations, 2," *Commun. Part. Differ. Equations*, **8**, No. 11, 1229–1276 (1983).
169. P. L. Lions and P. E. Souganidis, "Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs's equations," *SIAM J. Control Optimiz.*, **23**, No. 4, 566–583 (1985).
170. R. Sh. Lipzer and A. N. Shiryaev, *Statistics of Random Processes* [in Russian] Nauka, Moscow (1974).
171. M. I. Loginov, O. N. Sobolev, and G. S. Shelement'ev, *Introduction to Statistical Analysis* [in Russian], Ural State Univ. Ekaterinburg (1999).
172. N. Yu. Lukoyanov, "Minimax solutions of the Hamilton–Jacobi equations with a heredity," *Dokl. Ross. Akad. Nauk*, **371**, No. 2, 163–166 (2000).
173. V. I. Maksimov, "The principle of extrimal shift in a problem of solving operator equations," *Proc. Steklov Inst. Mat.*, Suppl. Issue 1, S163–S171 (2000).
174. O. A. Malafeyev, "Perfect equilibrium in noncooperative differential games," in: *Proc. X Int. Symp. "Dynamical Games and Applications," St.Petersburg State University, 2002*, **2**, St.Petersburg (2002), pp. 492–494.
175. V. V. Malyshev and A. I. Kibzun, *Analysis and Synthesis of High-Precision Controls of Aircraft* [in Russian], Moscow (1987).
176. V. P. Maslov and S. N. Samborskii, "Existence and uniqueness of solutions to the steady-state Hamilton–Jacobi–Bellman equations. A new approach," *Dokl. Ross. Akad. Nauk*, **324**, No. 6, 1143–1148 (1992).
177. A. A. Melikyan, "Singular characteristics first-order partial differential equations," *Dokl. Ross. Akad. Nauk*, **382**, No. 2, 203–217 (1996).
178. A. A. Melikyan, *Generalized Characteristics of First Order Partial Differential Equations: Applications in Optimal Control and Differential Games*, Birkkäuser Boston (1998).
179. A. A. Melikyan, "Equations of propagations of weak gaps of a solution to a variational problem," *Proc. Steklov Inst. Mat.*, Suppl. Issue 2, S446–S459 (2000).
180. L. I. Minchenko and O. F. Borisenko, *Differential Properties of Marginal Functions and Applications to Optimization Problems* [in Russian], Minsk (1992).
181. S. Mirică, "Extending Cauchy's method of characteristics for Hamilton–Jacobi equations," *Stud. Cerc. Mat.*, **37**, No. 6, 555–565 (1985).
182. E. F. Mishchenko, "Pursuit-evasion problems in the theory of differential games," *Izv. Akad. Nauk SSSR. Ser. Tekhn. Kibern.*, No. 5, 3–9 (1971).
183. E. F. Mishchenko and L. S. Pontryagin, "Periodic solutions of near discontinuous systems of ordinary differential equation," *Dokl. Alad. Nauk SSSR*, **102**, No. 5, 889–891 (1955).
184. E. F. Mishchenko and N. Kh. Rosov, *Differential Equations with a Small Parameter and Relaxational Oscillations* [in Russian], Nauka, Moscow (1975).
185. N. N. Moiseev, *Asymptotic Methods of Nonlinear Mechanics* [in Russian], Nauka, Moscow (1971).
186. B. Sh. Mordukhovich, *Methods of Appoximations to Optimization and Control Problems* [in Russian], Nauka, Moscow (1988).
187. A. A. Neznakhin and V. N. Ushakov, "A grid method to approximations of the kernel of viability of a differential inclusion," *Zh. Vychisl. Mat. Mat. Fiz.*, **41**, No. 6, 895–908 (2001).
188. M. S. Nikol'skii, "On the Pontryagin alternated integral," *Mat. Sb.*, **116**, No. 1, 136–144 (1981).
189. M. S. Nikol'skii and M. Aboubacar, "Some estimates of the reachable set for the controlled Van der Pol equation," *Proc. Steklov Inst. Math.*, Suppl. Issue 1, S172–S181 (2000).

190. O. A. Oleinik, “Discontinuous solutions to nonlinear differental equations,” *Usp. Mat. Nauk*, **12**, No. 3, 3–73 (1957).
191. O. A. Oleinik, “On construction of a generalized solution to the Cauchy problem for a quasi-linear first-order partial differential equation involving vanishing viscosity,” *Usp. Math. Nauk*, **14**, No. 2, 159–164 (1959).
192. O. A. Oleinik and S. N. Kruzhkov, “Quasi-linear parabolic partial differential equations of second order with several independent variables,” *Usp. Mat. Nauk*, **16**, No. 5, 115–155 (1961).
193. R. O'Malley, *Introduction to Singular Perturbations*, Academic Press, New York (1974).
194. R. Ortega, A. J. van der Schaft, and B. M. Maschke, Stabilization of Port-Controlled Hamiltonian Systems via Energy Balancing. in *Stability and Stabilization of Nonlinear Systems*. vol. 246. LNCIS. New York: Springer Verlag. 1999.
195. Yu. S. Osipov, “On the theory of differential games in systems with distributed parameters,” *Dokl. Akad. Nauk SSSR*, **223**, No. 6, 1314–1317 (1975).
196. Yu. S. Osipov, “Positional control in parabolic systems,” *Prikl. Mat. Mekh.*, **41**, No. 2 (1977).
197. A. I. Panasyuk and V. I. Panasyuk, *Asymptotic Magistral Optimization in Control Systems* [in Russian], Nauka, Moscow (1986).
198. T. Parthasarathy and T. Raghavan, *Some Topics in Two-Person Games*, Modern Analytic and Computational Methods in Science and Mathematics, **22**, Amer. Elsevier, New York (1971).
199. A. G. Pashkov and S. D. Terekhov, “Differential game of approach with two pursuers and one evader,” *J. Optimiz. Theory Appl.*, **55**, No. 2, 303–311 (1987).
200. V. S. Patsko and V. L. Turova, “Level sets of the value function in differential games with homicidal chauffeur dynamics,” *Int. Game Theory Rev.*, **3**, No. 1, 67–112 (2001).
201. A. A. Pervozvanskii and V. G. Gaitsgory, *Decomposition, Aggregation, and Approximation Optimization* [in Russian] Nauka, Moscow (1979).
202. A. A. Pervozvanski and V. G. Gaitsgory, *Theory of Suboptimal Solutions*, Kluwer Academic, Dordrecht (1988).
203. L. A. Petrosyan, *Pursuit Differential Games* [in Russian], Leningrad State Univ., Leningrad (1977).
204. L. A. Petrosyan and V. V. Zakharov, *Mathematical Models in Ecology* [in Russian], St.Petersburg State Univ., St.-Petersburg (1997).
205. N. N. Petrov, “On the existence of the value in a pursuit game,” *Dokl. Akad. Nauk SSSR*, **190**, No. 6, 621–624 (1970).
206. N. N. Petrov, *Game Theory* [in Russian], Udmurt. State Univ., Izhevsk (1997).
207. I. G. Petrovskii, *Lectures on the Theory of Ordinary Differential Equations* [in Russian], Nauka, Moscow (1964).
208. E. S. Polovinkin, *Elements of the Theory of Mulivalued Mappings* [in Russian], Moscow (1982).
209. B. T. Polyak, *An Introduction to Optimization* [in Russian], Nauka, Moscow (1983).
210. L. S. Pontryagin, *Ordinary Differential Equations* [in Russian], Nauka, Moscow (1965).
211. L. S. Pontryagin, “On linear differential games, 1, 2,” *Dokl. Akad. Nauk SSSR*, **174**, No. 6, 1278–1280; **175**, No. 4, 764–766 (1967).
212. L. S. Pontryagin, “Mathematical theory of optimal processes and differential games,” *Proc. Steklov Inst. Mat.*, **169**, 119–157 (1985).
213. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *Mathematical Theory of Optimal Processes* [in Russian], Nauka, Moscow (1961).
214. B. N. Pshenichnyi, “The structure of differential games,” *Dokl. Akad. Nauk SSSR*, **184**, No. 2, 285–287 (1969).
215. B. N. Pshenichnyi, *Convex Analysis and Extremal Problems* [in Russian], Nauka, Moscow (1980).

216. B. N. Pshenichnyi and V. V. Ostapenko, *Differential Games* [in Russian], Naukova Dumka, Kiev (1992).
217. R. T. Rockafellar, *Convex Analysis* [Russian translation], Mir, Moscow (1973).
218. R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*, Springer-Verlag, New York (1998).
219. R. T. Rockafellar and P. R. Wolenski, *Convexity and duality in Hamilton–Jacobi theory*, Report of Int. Inst. Appl. Syst. Anal. IR-98-057, Laxenburg (1998).
220. J. D. L. Rowland and R. B. Vinter, "Constructions of optimal feedback controls," *Systems Control Lett.*, **16**, 357–367 (1991).
221. E. Roxin, "The axiomatic approach in differential games," *J. Optimiz. Theory Appl.*, **3**, 153–163 (1969).
222. B. L. Rozhdestvenskii and N. N. Yanenko, *Systems of Quasi-Linear Equations and Applications to Gas Dynamics* [in Russian], Nauka, Moscow (1978).
223. L. I. Rozonoer, "The Pontryagin maximum principle in the theory of optimal systems," *Automat. Telemekh.*, **20**, No. 10–12 (1959).
224. L. Schwartz, *Analysis* [Russian translation], Mir, Moscow (1972).
225. A. N. Shiryaev, *Probability* [in Russian], Nauka, Moscow (1980).
226. A. F. Shorikov, *Minimax Estimates and Control of Discrete Dynamical Systems* [in Russian], Ural State Univ., Ekaterinburg (1997).
227. A. F. Sidorov, "On optimal unshocked compression of gas flows," *Dokl. Akad. Nauk SSSR*, **313**, No. 2, 283–287 (1990).
228. A. F. Sidorov, V. L. Gasilov, and A. P. Kukushkin, "Development of high-efficiency algorithms and software on the basis of parallel technologies," in: *Algorithms and Software of Parallel Calculations* [in Russian], Ekaterinburg (1995), pp. 3–20.
229. E. V. Sidorova and N. N. Subbotina, "An algorithm for calculating the value function in a linear differential game with a convex cost function," in: *Positional Control with Guaranteed Result* [in Russian], Sverdlovsk (1986), pp. 62–74.
230. S. L. Sobolev, *On Some Applications of Functional Analysis to Mathematical Physics* [in Russian] Nauka, Moscow (1988).
231. G. Sonnevend, *On constructing feedback control in differential games by use of central trajectories*, Report Schwerpunktprogr. DFG Anwendungsbezogene Optimierung und Steuerung, No. 385, Würzburg (1992).
232. P. E. Souganidis, "Max-min representations and product formulas for the viscosity solutions of Hamilton–Jacobi equations with applications to differential games," *Nonlinear Analysis. Theory, Meth. Appl.*, **9**, No. 3, 217–257 (1985).
233. V. V. Stepanov, *A Course in Differential Equations* [in Russian], Fizmatgiz, Moscow (1959).
234. J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions*, I, Springer-Verlag, New York (1970).
235. A. I. Subbotin, "On a generalization of the main equation to the theory of differential games," *Dokl. Akad. Nauk SSSR*, **254**, No. 2, 293–297 (1980).
236. A. I. Subbotin, *Minimax Inequalities and the Hamilton–Jacobi Equations* [in Russian], Nauka, Moscow (1991).
237. A. I. Subbotin, "Continuous and discontinuous solutions to boundary problems for first-order partial differential equations," *Dokl. Ross. Akad. Nauk*, **323**, No. 1, 30–34 (1992).
238. A. I. Subbotin, *Generalized Solutions of First-Order Partial Differential Equations: The Dynamical Optimization Perspective*, Birkhäuser, Boston (1995).
239. A. I. Subbotin, "Minimax solutions of first-order partial differential equations," *Usp. Mat. Nauk*, **51**, No. 2, 105–138 (1996).

240. A. I. Subbotin and A. G. Chentsov, *Optimization of the Guarantee in Control Problems* [in Russian], Nauka, Moscow (1981).
241. A. I. Subbotin and A. G. Chentsov, "An iteration procedure for constructing minimax and viscosity solutions to the Hamilton–Jacobi equations," *Dokl. Ross. Akad. Nauk*, **348**, No. 3, 45–48 (1996).
242. A. I. Subbotin and L. G. Shagalova, "A piecewise linear solution of the Cauchy problem for the Hamilton–Jacobi equation," *Dokl. Akad. Nauk*, **325**, No. 5, 932–936 (1992).
243. A. I. Subbotin and N. N. Subbotina, "Necessary and sufficient conditions for a piecewise smooth value of a differential game," *Dokl. Akad. Nauk SSSR*, **243**, No. 4, 829–865 (1978).
244. A. I. Subbotin and N. N. Subbotina, "Necessary and sufficient conditions for nonsmooth value of a differential game," in: *Problems of Dynamical Control* [in Russian], Sverdlovsk (1979).
245. A. I. Subbotin and N. N. Subbotina, "The optimal result function in a control problem," *Dokl. Akad. Nauk SSSR*, **266**, No. 2, 294–299 (1982).
246. A. I. Subbotin and N. N. Subbotina, "Properties of a differential game potential," *Prikl. Mat. Mekh.*, **46**, No. 2, 204–211 (1982).
247. A. I. Subbotin and N. N. Subbotina, "On justification of the method of dynamical programming to an optimal control problem," *Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibern.*, **2**, 24–32 (1983).
248. A. I. Subbotin and N. N. Subbotina, "Differentiability properties of the value finction in a differential game with an integral-terminal payoff," *Probl. Contr. Inform. Theory*, **12**, No. 3, 153–166 (1983).
249. A. I. Subbotin and N. N. Subbotina, "Piecewise smooth solutions of first order partial differential equations," *Dokl. Ross. Akad. Nauk*, **333**, No. 6, 705–707 (1993).
250. A. I. Subbotin, A. M. Tarasyev, and V. N. Ushakov, "Generalized characteristics of the Hamilton–Jacobi equations," *Izv. Akad. Nauk, Ser. Tekhn. Kibern.*, No. 1, 190–197 (1993).
251. A. I. Subbotin, A. M. Tarasyev, and V. N. Ushakov, "Generalized characteristics of Hamilton–Jacobi equations," *J. Comput. Systems Sci. Int.*, **32**, No. 2, 157–163 (1994).
252. N. N. Subbotina, "Universal optimal strategies in positional differential games," *Diff. Equations*, **19**, No. 11, 1377–1382 (1983).
253. N. N. Subbotina, "Some sufficient conditions for existence of universal strategies," in: *Investigations of Minimax Control Problems* [in Russian], Sverdlovsk (1985), pp. 72–81.
254. N. N. Subbotina, *Infinitesimal properties of the value function of a diffusion differential game* [in Russian], Inst. Math. Mech., Sverdlovsk, 1985, deposited at the All-Union Institute for Scientific and Technical Information, Moscow (1985), No. 7690-D19.11.85.
255. N. N. Subbotina, "Necessary and sufficient conditions for optimality of controls and trajectories," in: *Synthesis of Optimal Control in Game Systems* [in Russian], Sverdlovsk (1986).
256. N. N. Subbotina, "Generalized stochastic derivatives to functions of several variables differential in a part of the variables," in: *Control with Guaranteed Result* [in Russian], Sverdlovsk (1987), pp. 77–85.
257. N. N. Subbotina, *Necessary and sufficient optimality conditions in terms of the maximum principle and superdifferetial of the value function* [in Russian], Inst. Math. Mech., Sverdlovsk, 1988, deposited at the All-Union Institute for Scientific and Technical Information, Moscow (1988), No. 2898-B88.
258. N. N. Subbotina, "The maximum principle and the superdifferential of the value function," *Probl. Control Inform. Theory*, **18**, No. 3, 151–160 (1989).
259. N. N. Subbotina, "The Cauchy method of characteristics and generalized solutions of the Hamilton–Jacobi–Bellman equations," *Dokl. Akad. Nauk SSSR*, **320**, No. 3, 556–561 (1991).
260. N. N. Subbotina, *Constructing a generalized solution of the Hamilton–Jacobi–Bellman equation by the Cauchy method of characteristics* [in Russian], Inst. Math. Mech., Sverdlovsk (1991), deposited at the All-Union Institute for Scientific and Technical Information, Moscow (1991), No. 2571-B91.

261. N. N. Subbotina, “Unified optimality conditions to control problems,” in: *Proc. Inst. Math. Mech., Ural Branch Russ. Acad. Sci.* [in Russian], **1**, Ekaterinburg (1992), pp. 147–160.
262. N. N. Subbotina, *Necessary and sufficient optimality conditions in terms of characteristics of Hamilton–Jacobi–Bellman equation*, Report Schwerpunktprogr. DFG Anwendungsbezogene Optimierung und Steuerung, No. 393, Würzburg (1992).
263. N. N. Subbotina, “Asymptotic properties of minimax solutions to the Hamilton–Jacobi equations in differential games with fast and slow motions,” *Prikl. Mat. Mekh.*, **60**, No. 6, pp. 883–890 (1996).
264. N. N. Subbotina, “Asymptotics for singularly perturbed differential games,” in: *Proc. IFAC Conference “Nonsmooth and Discontinuous Problems of Control and Optimization, June 17–20, 1998, Chelyabinsk, Russia*, Elsevier Science, Oxford (1998), pp. 43–52.
265. N. N. Subbotina, “Asymptotics of the singularly perturbed Hamilton–Jacobi equations,” *Prikl. Mat. Mekh.*, **63**, No. 2, 220–230 (1999).
266. N. N. Subbotina, “Singular approximations of the minimax and viscosity solutions to the Hamilton–Jacobi equations,” *Proc. Steklov Institute of Mathematics*, Suppl. Issue 1, S210–S227 (2000).
267. N. N. Subbotina, “On structure of optimal feedbacks to control problems,” in: *Proc. 11 IFAC Int. Workshop “Control Applications of Optimization,” July 3–6, 2000, St.Petersburg State University*, St.-Petersburg (2000), pp. 254–255.
268. N. N. Subbotina, “On structure of optimal feedbacks to control problems,” in: *Proc. 11 IFAC Int. Workshop “Control Applications of Optimization,”* **1**, St. Petersburg State Univ. (2000), pp. 339–344.
269. N. N. Subbotina, “Generalized Cauchy characteristics to singularly perturbed Hamilton–Jacobi equations,” in: *Proc. Int. Conf. “Differential Equations and Dynamical Systems,” August 21–26, 2000, Suzdal*, Steklov Inst. Math., Vladimir State Univ. (2000), pp. 93–94.
270. N. N. Subbotina, “Asymptotics for singularly perturbed differential games,” *Game Theory Appl.*, **7**, 175–196 (2001).
271. N. N. Subbotina, “Near optimal feedbacks to mechanical systems with fast and slow motions,” in: *Proc. VIII Symp. on Theoretical and Applied Mechanics, August 23–29, 2001, Perm*, Ekaterinburg (2001), p. 546.
272. N. N. Subbotina, *Optimality conditions for feedbacks to control problems*, Inst. Math. Mech., Ekaterinburg (2002), deposited at the All-Union Institute for Scientific and Technical Information, Moscow (2002), No. 1212-B2002.
273. N. N. Subbotina, “Sufficient optimality conditions for feedbacks to control problems,” in: *Proc. X Int. Symp. “Dynamical Games and Applications,” July 8–11, 2002, St. Petersburg State University*, **2**, St. Petersburg (2002), pp. 829–834.
274. N. N. Subbotina, “The method of dynamical programming for a class of local Lipschitz functions,” *Dokl. Ross. Akad. Nauk*, **389**, No. 2, 169–172 (2003).
275. N. N. Subbotina and A. G. Chentsov, “On the existence of the Bellman function in a linear differential game,” in: *Proc. Inst. Math. Mech.* [in Russian], No. 26, Sverdlovsk (1979), pp. 80–86.
276. N. N. Subbotina, A. I. Subbotin, and V. E. Tretyakov, “Stochastic and deterministic control. Differential inclusions,” *Probl. Control Inform. Theory*, **14**, No. 6, P1–P15 (1985).
277. N. N. Subbotina, A. I. Subbotin, and V. E. Tretyakov, “Stochastic and deterministic control. Differential inequalities,” *Lect. Notes Control Inform.*, **81**, 728–737 (1987).
278. N. N. Subbotina and V. N. Ushakov, “On characteristic differential inclusions to unified differential games,” in: *Proc. IX Int. Symp. “Dynamical Games and Applications,” December 18–21, 2000, Adelaida, Australia*, Univ. of South Australia, Adelaide (2000), pp. 464–467,.
279. A. M. Tarasyev, “Approximation schemes for constructing minimax solutions to the Hamilton–Jacobi equations,” *Prikl. Mat. Mekh.*, **58**, No. 2, 22–36 (1994).

280. A. M. Tarasyev, A. A. Uspenskii, and V. N. Ushakov, "Approximation schemes and finite difference operators for constructing generalized solutions of the Hamilton–Jacobi equations," *Izv. Ross. Akad. Nauk, Ser. Tekhn. Kibern.*, No. 3, 173–185 (1994).
281. A. N. Tikhonov, "Systems of differential equations with a small parameter at derivatives," *Mat. Sb.*, **31**, No. 3, 575–586 (1952).
282. A. N. Tikhonov and A. A. Samarskii, "On discontinuous solutions of a quasi-linear first-order partial differential equation," *Dokl. Akad. Nauk SSSR*, **99**, No. 1, 27–30 (1954).
283. E. L. Tonkov, "Some questions in control of periodic motions," in: *Dynamics of Control Systems* [in Russian], Nauka, Novosibirsk (1979).
284. V. E. Tretyakov, "On the theory of stochastic differential games," *Dokl. Akad. Nauk SSSR*, **269**, No. 3, 1049–1053 (1983).
285. V. E. Tretyakov, I. V. Tzelishcheva, and G. I. Shishkin, "Optimal control in systems with incomplete and uncertain information," in: *Proc. Inst. Math. Mech.*, **2**, Ekaterinburg (1992), pp. 176–187.
286. V. A. Troitskii, *Optimal Oscillation Processes of Mechanical Systems* [in Russian], Leningrad (1976).
287. V. I. Ukhobotov, "Synthesis of guaranteed controls on the basis of an approximation scheme," *Proc. Steklov Math. Inst.*, Suppl. Issue 1, S254–S260 (2000).
288. V. N. Ushakov, "On a problem of constructing stable bridges to a pursuit-evasion differential game," *Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibern.*, No. 4, 29–36 (1980).
289. V. N. Ushakov, "On a question of stability in differential games," in: *Positional Control with Guaranteed Result* [in Russian], Sverdlovsk (1988), pp. 101–109.
290. V. N. Ushakov, "Constructions of solutions in differential games of pursuit-evasion. Differential inclusions and optimal control," *Lect. Notes Nonlin. Anal.*, **2**, 269–281 (1998).
291. V. N. Ushakov and A. P. Khripunov, "On constructing approximation solutions to game-theoretic control problems," *Prikl. Mat. Mekh.* **61**, No. 3, 413–421 (1997).
292. V. I. Utkin, *Sliding Behaviors and Applications to Systems with Changing Structure* [in Russian], Nauka, Moscow (1974).
293. V. A. Vakhrushev and V. N. Ushakov, "On simulations of control procedures with a guide," *Prikl. Mat. Mekh.*, **66**, No. 2, 228–238 (2002).
294. P. Varaiya, "On the existence of solutions to a differential game," *SIAM J. Control Optimiz.*, **5**, No. 1, 153–162 (1967).
295. A. B. Vasil'eva and V. F. Butuzov, *Asymptotic Expansions of Singularly Perturbed Equations* [in Russian] Nauka, Moscow (1973).
296. V. S. Vladimirov, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1976).
297. V. Veliov, "A generalization of the Tikhonov theorem for singularly perturbed differential inclusions," *J. Dynam. Control Systems*, **3**, 291–319 (1997).
298. V. Veliov, "Stability-like properties of differential inclusions," *Set-Valued Anal.*, **5**, No. 1, 73–88 (1997).
299. V. A. Vyazgin, "On justification of sufficient conditions with the help of the Weierstrass and Hamilton–Jacobi–Bellman methods," *Automat. Telemech.*, **4**, 31–37 (1984).
300. J. Warga, *Optimal Control of Differential and Functional Equations* [Russian translation], Nauka, Moscow (1977).
301. L. C. Young, *Lectures on the Calculus of Variations and Optimal Control Theory* [Russian translation], Mir, Moscow (1974).
302. S. T. Zavalishchin and A. N. Sesekin, *Impulse Processes. Models and Applications* [in Russian], Nauka, Moscow (1991).
303. M. I. Zelikin, *Optimal Control and Variational Calculus* [in Russian], Nauka, Moscow (1985).

304. L. F. Zelikina, “Universal manifolds and theorems on a magistral way to a class of optimal control problems,” *Dokl. Akad. Nauk SSSR*, **224**, No. 1 (1975).
305. O. A. Zhautykov, V. I. Zhukovskii, and S. Zharkynbaev, *Differential Games with Several Persons* [in Russian], Nauka, Alma-Ata (1988).
306. X.-Y. Zhou, “Maximum principle, dynamical programming, and their connection in deterministic controls,” *J. Optimiz. Theory Appl.*, **65**, 363–373 (1990).
307. V. I. Zubov, *Lectures on Control Theory* [in Russian], Nauka, Moscow (1975).

N. N. Subbotina
Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences
E-mail: `subb@uran.ru`