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ON INFINITE REAL TRACE RATIONAL LANGUAGES OF MAXIMUM TOPOLOGICAL COMPLEXITY

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We consider the set $\mathbb{R}^\omega(\Gamma, D)$ of infinite real traces, over a dependence alphabet (Γ, D) with no isolated letter, equipped with the topology induced by the prefix metric. We prove that all rational languages of infinite real traces are analytic sets. We also reprove that there exist some rational languages of infinite real traces that are analytic but non-Borel sets; in fact, these sets are even $\mathbf{\Sigma}_1^1$-complete, hence have maximum possible topological complexity. For this purpose, we give an example of a $\mathbf{\Sigma}_1^1$-complete language that is fundamentally different from the known example of a $\mathbf{\Sigma}_1^1$-complete infinitary rational relation given by Finkel (2003). Bibliography: 35 titles.

1. Introduction

Trace monoids were first considered by Cartier and Foata in the context of studying combinatorial problems [5]. Then Mazurkiewicz introduced finite traces as a semantic model for concurrent systems [25]. Since then, traces have been much investigated by various authors, and they have been extended to infinite traces, to model systems that may not terminate; see the handbook [6] and its chapter about infinite traces [16] for many results and references.

In particular, **real traces** have been studied by Gastin, Petit, and Zielonka, who characterized in [17] the two important families of recognizable and rational languages of real traces over a dependence alphabet (Γ, D), in connection with rational languages of finite or infinite words.

Several metrics have been defined on the set $\mathbb{R}(\Gamma, D)$ of real traces over (Γ, D); in particular, the prefix metric defined by Kwiatkowska [21] and the Foata normal form metric defined by Bonnizzoni, Mauri, and Pighizzini [4]. Kummetz and Kuske stated in [20] that for *finite* dependence alphabets, these two metrics define the same topology on $\mathbb{R}(\Gamma, D)$. Moreover, if we consider only **infinite real traces** over a dependence alphabet (Γ, D) without isolated letter, then the topological subspace $\mathbb{R}^\omega(\Gamma, D) = \mathbb{R}(\Gamma, D) - \mathbb{M}(\Gamma, D)$ of $\mathbb{R}(\Gamma, D)$ (where $\mathbb{M}(\Gamma, D)$ is the set of finite traces over (Γ, D)) is homeomorphic to the Cantor set 2^ω, or, equivalently, to any set Σ^ω of infinite words, over a finite alphabet Σ, equipped with the product of the discrete topology on Σ [20, 33, 29].

Starting from open subsets of the topological space $\mathbb{R}^\omega(\Gamma, D)$, we can define the hierarchy of Borel sets by successive operations of countable intersections and countable unions. Furthermore, it is well known that there exist some subsets of the Cantor set, and hence also some subsets of $\mathbb{R}^\omega(\Gamma, D)$, that are not Borel. There is another hierarchy beyond the Borel one, called the projective hierarchy.

It is then natural to try to locate classical languages of infinite real traces with regard to these hierarchies, and this question is posed by Lescow and Thomas in [24] (for the general case of infinite labelled partial orders like traces). In the case of infinite words, McNaughton's theorem implies that every ω-regular language is a boolean combination of $\mathbf{\Pi}_2^0$-sets, hence a $\mathbf{\Delta}_3^0 = (\mathbf{\Pi}_3^0 \cap \mathbf{\Sigma}_3^0)$-set. Landweber [22] first studied the topological properties of ω-regular languages and characterized ω-regular languages in each of the Borel classes $\mathbf{\Sigma}_1^0$, $\mathbf{\Pi}_1^0$, $\mathbf{\Sigma}_2^0$, $\mathbf{\Pi}_2^0$. In this paper, we study the topological complexity of **rational languages of infinite real traces**.

We show below that all rational languages of infinite real traces are analytic sets and that there exist some rational languages of infinite real traces that are analytic but non-Borel sets; in fact, these sets are even $\mathbf{\Sigma}_1^1$-complete, hence have maximum possible topological complexity, so that we obtain a partial answer to the question of comparing the topological complexity of rational languages of infinite words and of infinite traces [24].

The first author recently showed in [10] that there exists a $\mathbf{\Sigma}_1^1$-complete infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$, where Σ_1 and Σ_2 are two finite alphabets having at least two letters. We could have used this result to prove that there exists a $\mathbf{\Sigma}_1^1$-complete rational language of infinite real traces $L \subseteq \mathbb{R}^\omega(\Gamma, D)$, whenever (Γ, D) is a dependence alphabet and $\Gamma \supseteq \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are two independent dependence cliques having at least two letters. This can be done by considering the natural embedding $i : \Sigma_1^\omega \times \Sigma_2^\omega \to \mathbb{R}^\omega(\Gamma, D)$. The language

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$R' = i(R)$ is then $\mathbf{\Sigma}_1^1$-complete. But this way the language R' would in fact have the structure of an infinitary rational relation.

On the other hand, the $\mathbf{\Sigma}_1^1$-complete language $\mathcal{L}$ presented in this paper is a new example, whose structure is radically different from that of R'. In particular, $\mathcal{L}$ does not contain any $\mathbf{\Sigma}_1^1$-complete language of infinite traces having the structure of an infinitary rational relation.

This is of importance because in trace theory the structure of dependence alphabets is very significant: some results are known to be true for some dependence alphabets and false for other dependence alphabets (see, for example, [18]).

Moreover, we think that the presentation of this new example is also of interest for the following reasons. The proof given in this paper is self-contained and stated in the general context of traces. The problem is exposed in this general context, and we use general properties of traces instead of particular properties of infinitary rational relations. We use the characterization of rational languages of infinite real traces given by Theorem 2.1 of Gastin, Petit, and Zielonka, which states a connection between rational languages of infinite words and rational languages of infinite real traces, instead of the notion of Büchi transducer.

We also prove that all rational languages of infinite real traces are analytic. Our proof is not difficult, but it is original, using the Baire space ω^ω and the characterization of rational languages of infinite real traces given by Theorem 2.1: every rational language of infinite traces is a finite union of sets of the form $R.S^\omega$, where S and R are rational monoalphabetic languages of finite traces. This proof is quite different from usual proofs in the theory of ω-languages. It uses a connection between the topological complexity of the ω-powers of languages of finite traces (i.e., of languages of the form S^ω, where S is a language of finite traces) and the topological complexity of rational languages of infinite traces. The closure of the class of analytic sets under countable unions also plays an important role in the proof.

We expect that the inverse way could also be fruitful: it seems to us that there should exist context-free ω-languages and infinitary rational relations of high transfinite Borel rank. We think that we could use this fact to show that there exist ω-powers of finitary languages of high transfinite Borel rank. Note that the question of the topological complexity of ω-powers (of languages of finite words) has been raised by several authors [28, 30, 33, 34], and some new results have recently been proved [7, 8, 14, 23].

The paper is organized as follows. In Sec. 2, we recall the notions of words and traces. In Sec. 3, we recall the definitions of Borel and analytic sets, and in Sec. 4, we prove our main results.

2. Words and traces

Let us now introduce notation for words. For a finite alphabet Σ, a nonempty *finite word* over Σ is a finite sequence of letters $x = a_1a_2 \ldots a_n$, where $a_i \in \Sigma$ for every $i \in [1; n]$. We denote by $x(i) = a_i$ the ith letter of x and set $x[i] = x(1) \ldots x(i)$ for $i \leq n$. The *length* of x is $|x| = n$. The empty word, which has no letters, is denoted by ε; its length is 0. The set of nonempty finite words over Σ is denoted by Σ^+, and $\Sigma^\star = \Sigma^+ \cup \{\varepsilon\}$ is the set of finite words over Σ. A (finitary) *language* L over Σ is a subset of $\Sigma^\star$. The usual concatenation product of u and v is denoted by $u.v$ or just uv. For $V \subseteq \Sigma^\star$, we set $V^\star = \{v_1 \ldots v_n \mid n \geq 1 \text{ and } v_i \in V \text{ for every } i \in [1; n]\} \cup \{\varepsilon\}$.

The first infinite ordinal is ω. An *ω-word* over Σ is an ω-sequence $a_1a_2 \ldots a_n \ldots$, where $a_i \in \Sigma$ for every $i \geq 1$. If σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$, and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ is a finite word of length n, a prefix of σ. The set of ω-words over an alphabet Σ is denoted by Σ^ω. An *ω-language* over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid u_i \in V \text{ for every } i \geq 1\}$ is the *ω-power* of V. The concatenation product extends to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that $(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

The prefix relation is denoted by $\sqsubseteq$: we say that a finite word u is a *prefix* of a finite word v (respectively, an infinite word v) and write $u \sqsubseteq v$ if and only if there exists a finite word w (respectively, an infinite word w) such that $v = u.w$. We denote by $\Sigma^\infty = \Sigma^\star \cup \Sigma^\omega$ the set of finite or infinite words over Σ.

We first introduce traces as dependence graphs [6, 16, 20]. A *dependence relation* D over an alphabet Γ is a reflexive and symmetric relation on Γ. Its complement $I_D = (\Gamma \times \Gamma) - D$ is the *independence relation* induced by D; the relation I_D is irreflexive and symmetric. A *dependence alphabet* (Γ, D) is formed by a finite alphabet Γ and a dependence relation $D \subseteq \Gamma \times \Gamma$.

A dependence graph $[V, E, \lambda]$ over a dependence alphabet (Γ, D) is the isomorphism class of a node-labelled graph (V, E, λ) such that (V, E) is a directed acyclic graph, V is at most countably infinite, and $\lambda : V \to \Gamma$ is a function that associates a label $\lambda(a)$ to each node $a \in V$ so that

(1) for any $v, w \in V$, $(\lambda(v), \lambda(w)) \in D \leftrightarrow (v = w \text{ or } (v, w) \in E \text{ or } (w, v) \in E)$;

(2) the reflexive and transitive closure $E^\star$ of the edge relation E is well founded, i.e., there is no infinite strictly decreasing sequence of vertices.

Note that, since in this definition (V, E) is acyclic, $E^\star$ is a partial order on V.

The empty trace has no vertices and will be denoted by ε as in the case of words.

As usually, the concatenation of two dependence graphs $g_1 = [V_1, E_1, \lambda_1]$ and $g_2 = [V_2, E_2, \lambda_2]$, where we may assume without loss of generality that V_1 and V_2 are disjoint, is the dependence graph $g_1.g_2 = [V, E, \lambda]$ such that $V = V_1 \cup V_2$, $E = E_1 \cup E_2 \cup \{(v_1, v_2) \in V_1 \times V_2 \mid (\lambda_1(v_1), \lambda_2(v_2)) \in D\}$, and $\lambda = \lambda_1 \cup \lambda_2$.

The *alphabet* $\mathrm{alph}(t)$ of a trace $t = [V, E, \lambda]$ is the set $\lambda(V)$. The *alphabet at infinity* of t is the set $\mathrm{alphin}f(t) = \{a \in \Gamma \mid \lambda^{-1}[a] \text{ is infinite}\}$ of all $a \in \mathrm{alph}(t)$ occurring infinitely often in t.

The set $\mathbb{M}(\Gamma, D)$ of finite traces over (Γ, D) is the set of traces having finitely many vertices. For $t \in \mathbb{M}(\Gamma, D)$, the *length* of t, denoted by $|t|$, is the number of vertices of t.

The star operation $T \to T^\star$ and the operation $T \to T^\omega$ are naturally extended to subsets T of $\mathbb{M}(\Gamma, D)$:

$$T^\star = \{t_1.t_2 \ldots t_n \mid n \geq 1 \quad \text{and} \quad t_i \in T \quad \text{for every} \quad i \in [1; n]\} \cup \{\varepsilon\},$$
$$T^\omega = \{t_1.t_2 \ldots t_n \ldots \mid t_i \in T \quad \text{for every} \quad i\}.$$

A *real trace* over (Γ, D) is a dependence graph $[V, E, \lambda]$ such that for all $v \in V$ the set $\{u \in V \mid (u, v) \in E^\star\}$ is finite. The set of real traces over (Γ, D) is denoted by $\mathbb{R}(\Gamma, D)$, and the set $\mathbb{R}(\Gamma, D) - \mathbb{M}(\Gamma, D)$ of **infinite** real traces is denoted by $\mathbb{R}^\omega(\Gamma, D)$.

The prefix order over words can be extended to real traces in the following way. For any $s, t \in \mathbb{R}(\Gamma, D)$, let $s \sqsubseteq t$ if and only if there exists $z \in \mathbb{R}(\Gamma, D)$ such that $s.z = t$ if and only if s is a downward-closed subgraph of t. The corresponding suffix z is then unique.

Real traces may also be viewed as equivalence classes of (finite or infinite) words. Let (Γ, D) be a dependence alphabet, and let $\varphi : \Gamma^\infty \to \mathbb{R}(\Gamma, D)$ be the mapping defined by $\varphi(a) = [\{x\}, \emptyset, x \to a]$ for each $a \in \Gamma$, and $\varphi(a_1.a_2 \ldots) = \varphi(a_1).\varphi(a_2) \ldots$ for each $a_1.a_2 \ldots$ in Γ^∞. Note that if there is some (a, b) in I_D, then the mapping is not injective, because, for instance, $\varphi(ab) = \varphi(ba)$. One can define an equivalence relation $\sim_I$ on Γ^∞ as follows: for any $u, v \in \Gamma^\infty$, let $u \sim_I v$ if and only if $\varphi(u) = \varphi(v)$. Then φ induces a surjective morphism from the free monoid $\Gamma^\star$ onto the monoid of finite dependence graphs $\mathbb{M}(\Gamma, D) = \Gamma^\star / \sim$. And the set $\varphi(\Gamma^\infty) = \Gamma^\infty / \sim$ is the set of real traces $\mathbb{R}(\Gamma, D)$.

The empty trace is the image $\varphi(\varepsilon)$ of the empty word; it is still denoted by ε.

We assume the reader to be familiar with the theory of formal languages and ω-regular languages; see [35, 33, 29] for many results and references. We recall that ω-regular languages are accepted by Büchi automata and that the class of ω-regular languages is the omega-Kleene closure of the class of regular finitary languages.

The family of rational real trace languages over (Γ, D) is the smallest family that contains the empty set and all the singletons $\{[\{x\}, \emptyset, x \to a]\}$ for $a \in \Gamma$, and that is closed under finite union, concatenation product, $\star$-iteration, and ω-iteration on real traces.

Let us now recall the following characterization of rational languages of infinite real traces (there also exists a version for finite *or* infinite traces, see [17]). A real trace language R is said to be *monoalphabetic* if $\mathrm{alph}(s) = \mathrm{alph}(t)$ for all $s, t \in R$.

Theorem 2.1 [17]**.** *Let $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ be a language of infinite real traces over a dependence alphabet (Γ, D). The following assertions are equivalent:*

(1) *T is rational;*
(2) *T is a finite union of sets of the form $R.S^\omega$, where S and R are rational monoalphabetic languages of finite traces over (Γ, D) and $\varepsilon \notin S$;*
(3) *$T = \varphi(L)$ for some ω-regular language $L \subseteq \Gamma^\omega$.*

3. Topology

We assume the reader to be familiar with the basic notions of topology, which may be found in [19, 24, 33, 29].

On the set Σ^ω of infinite words over a finite alphabet Σ, there is a natural metric, which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$, let $d(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$, where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)$th letter of u is different from the $(n+1)$th letter of v. This metric induces on Σ^ω the usual Cantor topology for which open subsets of Σ^ω are of the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$. (Note that this prefix metric can be extended to the set Σ^∞ of finite or infinite words over the alphabet Σ.)

Kwiatkowska [21] extended the prefix metric to real traces, by defining first for all $s, t \in \mathbb{R}(\Gamma, D)$ with $s \neq t$,

$$l_{\text{pref}}(s,t) = \sup\{n \in \mathbb{N} \mid r \sqsubseteq s \leftrightarrow r \sqsubseteq t \text{ for all } r \in \mathbb{M}(\Gamma, D) \text{ with } |r| \leq n\},$$

and then setting $d_{\text{pref}}(s,t) = 2^{-l_{\text{pref}(s,t)}}$.

Note that in this paper we consider *infinite* traces, and Kwiatkowska defined the prefix metric over *finite or infinite* traces, as could also be done in the case of words. If $D = \Gamma \times \Gamma$, then the prefix metric on infinite real traces over (Γ, D) coincides with that from the preceding definition in the case of infinite words over Γ.

Given a dependence alphabet (Γ, D), a letter $a \in \Gamma$ is said to be *isolated* if a is independent from all other letters of Γ, i.e., $(a, b) \in I_D$ for every $b \in \Gamma - \{a\}$.

From now on we suppose that a dependence alphabet has no isolated letter. Then the set $\mathbb{R}^\omega(\Gamma, D)$ of infinite real traces over (Γ, D), equipped with the topology induced by the prefix metric, is homeomorphic to the Cantor set $\{0,1\}^\omega$, and hence also to Σ^ω for every finite alphabet Σ having at least two letters [20].

Borel subsets of the Cantor set (and hence also of the topological spaces Σ^ω or $\mathbb{R}^\omega(\Gamma, D)$) form a strict infinite hierarchy, the Borel hierarchy, which is defined starting from open sets by successive operations of countable unions and countable intersections. We give the definition in the case of Σ^ω, the definition being similar in the case of $\mathbb{R}^\omega(\Gamma, D)$. Then we recall some well-known properties of Borel sets.

Definition 3.1. *The classes $\mathbf{\Sigma}^0_n$ and Π^0_n of the Borel hierarchy on the topological space Σ^ω are defined as follows:*
- *$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω;*
- *Π^0_1 is the class of closed subsets of Σ^ω, i.e., the complements of open subsets of Σ^ω;*

And for any integer $n \geq 1$,
- *$\mathbf{\Sigma}^0_{n+1}$ is the class of countable unions of Π^0_n-subsets of Σ^ω;*
- *Π^0_{n+1} is the class of countable intersections of $\mathbf{\Sigma}^0_n$-subsets of Σ^ω.*

The Borel hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma}^0_\alpha$ and Π^0_α, for a non-null countable ordinal α, are defined in the following way:
- *$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω belonging to $\cup_{\gamma<\alpha}\Pi^0_\gamma$.*
- *Π^0_α is the class of countable intersections of subsets of Σ^ω belonging to $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.*

Theorem 3.2.

(a) $\mathbf{\Sigma}^0_\alpha \cup \Pi^0_\alpha \subsetneq \mathbf{\Sigma}^0_{\alpha+1} \cap \Pi^0_{\alpha+1}$ *for each countable ordinal $\alpha \geq 1$.*
(b) $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma = \cup_{\gamma<\alpha}\Pi^0_\gamma \subsetneq \mathbf{\Sigma}^0_\alpha \cap \Pi^0_\alpha$ *for each countable limit ordinal α.*
(c) *A set $W \subseteq \Sigma^\omega$ is in the class $\mathbf{\Sigma}^0_\alpha$ if and only if its complement is in the class Π^0_α.*
(d) $\mathbf{\Sigma}^0_\alpha - \Pi^0_\alpha \neq \emptyset$ *and* $\Pi^0_\alpha - \mathbf{\Sigma}^0_\alpha \neq \emptyset$ *for every countable ordinal $\alpha \geq 1$.*

We say that a subset of Σ^ω is a *Borel set of rank* α, for a countable ordinal α, if and only if it is in $\mathbf{\Sigma}^0_\alpha \cup \Pi^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \Pi^0_\gamma)$.

Let us recall the characterization of Π^0_2-subsets of Σ^ω, involving the δ-limit W^δ of a finitary language W. For $W \subseteq \Sigma^\star$ and $\sigma \in \Sigma^\omega$, we have $\sigma \in W^\delta$ if and only if σ has infinitely many prefixes in W, i.e., $W^\delta = \{\sigma \in \Sigma^\omega / \exists^\omega i$ such that $\sigma[i] \in W\}$, see [33].

Proposition 3.3. *A subset L of Σ^ω is a Π^0_2-subset of Σ^ω if and only if there exists a set $W \subseteq \Sigma^\star$ such that $L = W^\delta$.*

Example 3.4. Let $\Sigma = \{0, 1\}$, and let $\mathcal{A} = (0^\star.1)^\omega \subseteq \Sigma^\omega$ be the set of ω-words over the alphabet Σ with infinitely many occurrences of the letter 1. It is well known that $\mathcal{A}$ is a Π^0_2-subset of Σ^ω, because $\mathcal{A} = ((0^\star.1)^+)^\delta$.

There are some subsets of the Cantor set (and hence also of the topological spaces Σ^ω or $\mathbb{R}^\omega(\Gamma, D)$) that are not Borel sets. There exists another hierarchy beyond the Borel one, called the *projective hierarchy.* Projective sets are defined starting from Borel sets by successive operations of projection and complementation. In this paper, we will need only the first class of the projective hierarchy: the class $\mathbf{\Sigma}^1_1$ of **analytic** sets. A set $A \subseteq \Sigma^\omega$ is *analytic* if and only if there exists a Borel set $B \subseteq (\Sigma \times Y)^\omega$, where Y is a finite alphabet, such that $x \in A \Leftrightarrow$ there exists $y \in Y^\omega$ such that $(x, y) \in B$, where $(x, y) \in (\Sigma \times Y)^\omega$ is defined by $(x, y)(i) = (x(i), y(i))$ for all integers $i \geq 1$.

Analytic sets are also characterized as continuous images of the Baire space ω^ω, which is the set of infinite sequences of nonnegative integers. It may be regarded as the set of infinite words over the infinite alphabet $\omega = \{0, 1, 2, \dots\}$. The topology of the Baire space is then defined by the prefix metric, which is just the extension of the prefix metric defined above to the case of an infinite alphabet.

Then a set $A \subseteq \Sigma^\omega$ (respectively, $A \subseteq \mathbb{R}^\omega(\Gamma, D)$) is analytic if and only if there exists a continuous function $f : \omega^\omega \to \Sigma^\omega$ (respectively, $f : \omega^\omega \to \mathbb{R}^\omega(\Gamma, D)$) such that $f(\omega^\omega) = A$.

A $\mathbf{\Sigma}^0_\alpha$- (respectively, Π^0_α-, $\mathbf{\Sigma}^1_1$-) complete set is a $\mathbf{\Sigma}^0_\alpha$- (respectively, Π^0_α-, $\mathbf{\Sigma}^1_1$-) set that is, in a sense, a set of highest topological complexity among the $\mathbf{\Sigma}^0_\alpha$- (respectively, Π^0_α-, $\mathbf{\Sigma}^1_1$-) sets. This notion is defined via reductions by continuous functions. More precisely, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$- (respectively, Π^0_α-, $\mathbf{\Sigma}^1_1$-) *complete* set if for any set $E \subseteq Y^\omega$ (where Y is a finite alphabet), $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \Pi^0_\alpha$, $\mathbf{\Sigma}^1_1$) if and only if there exists a continuous function f such that $E = f^{-1}(F)$. $\mathbf{\Sigma}^0_n$- (respectively, Π^0_n-) complete sets, for an integer $n \geq 1$, are thoroughly characterized in [32].

The ω-regular language $\mathcal{A} = (0^\star.1)^\omega$ from Example 3.4 is a well-known example of a Π^0_2-complete set.

4. Rational languages of infinite traces

Now we want to investigate the topological complexity of rational languages of infinite real traces. At the first step, we will give an upper bound on this complexity, showing that all rational languages $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ are analytic sets.

We would like to use the characterization of rational languages $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ given in item 3 of Theorem 2.1: $T = \varphi(L)$ for some ω-regular language $L \subseteq \Gamma^\omega$. Indeed, every ω-regular language is a Borel set (of rank at most 3), and a continuous image of a Borel set is an analytic set. Unfortunately, the mapping φ is not continuous, as the following example shows. Let $(a, b) \in I_D$, and let $x_n \in \Gamma^\omega$ be defined by $x_n = a^n b a^\omega$ for each integer $n \geq 1$. Then the sequence $(x_n)_{n\geq 1}$ is convergent in Γ^ω, and its limit is equal to a^ω. But the sequence $(\varphi(x_n))_{n\geq 1}$ is constant in $\mathbb{R}^\omega(\Gamma, D)$, because $\varphi(x_n) = \varphi(ba^\omega)$ for all $n \geq 1$. Thus the sequence $(\varphi(x_n))_{n\geq 1}$ is convergent, but its limit is equal to $\varphi(ba^\omega)$, which is different from $\varphi(a^\omega)$.

We will use the characterization of rational languages $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ given in item 2 of Theorem 2.1: T is a finite union of sets of the form $R.S^\omega$, where S and R are rational monoalphabetic languages of finite traces over (Γ, D) and $\varepsilon \notin S$.

We first consider such rational languages in the simple form S^ω, where S is a monoalphabetic language of finite traces over (Γ, D) that does not contain the empty trace. The set S is at most countable, so it can be finite or countably infinite. In the first case, card(S)=p and we can fix an enumeration of S by a bijective function $\psi : \{0, 1, 2, \ldots, p-1\} \to S$, and in the second case, we can fix an enumeration of S by a bijective function $\psi : \omega = \{0, 1, 2, \ldots\} \to S$.

Now let H be the function from $\{0, 1, 2, \ldots, p-1\}^\omega$ (in the first case) or from ω^ω (in the second case) into $\mathbb{R}^\omega(\Gamma, D)$ defined by the formula

$$H(n_1 n_2 \ldots n_i \ldots) = \psi(n_1).\psi(n_2) \ldots \psi(n_i) \ldots$$

for all sequences $n_1 n_2 \ldots n_i \ldots$ in $\{0, 1, 2, \ldots, p-1\}^\omega$ (in the first case) or in ω^ω (in the second case). Then $H(\{0, 1, 2, \ldots, p-1\}^\omega) = S^\omega$ (in the first case) or $H(\omega^\omega) = S^\omega$ (in the second case).

It is easy to see that H is a continuous function. The crucial point is that S is a **monoalphabetic** language, i.e., there exists $\Gamma' \subseteq \Gamma$ such that alph$(s) = \Gamma'$ for all $s \in S$. Let $\mathcal{N} = (n_i)_{i\geq 1}$ and $\mathcal{M} = (m_i)_{i\geq 1}$ be two infinite sequences of integers in $\{0, 1, 2, \ldots, p-1\}^\omega$ or in ω^ω such that $n_i = m_i$ for all $i \leq k$. Then $r \sqsubseteq H(\mathcal{N}) \leftrightarrow r \sqsubseteq H(\mathcal{M})$ (at least) for all $r \in \mathcal{M}(\Gamma, D)$ with $|r| \leq k$. Thus $l_{\mathrm{pref}(H(\mathcal{N}), H(\mathcal{M}))} \geq k$ and $d_{\mathrm{pref}}(H(\mathcal{N}), H(\mathcal{M})) = 2^{-l_{\mathrm{pref}(H(\mathcal{N}), H(\mathcal{M}))}} \leq 2^{-k}$. This implies that the function H is continuous (and even uniformly continuous).

If S is finite, then the set S^ω is a continuous image of the compact set $\{0, 1, 2, \ldots, p-1\}^\omega$; thus it is a closed, and hence analytic, subset of $\mathbb{R}^\omega(\Gamma, D)$.

If S is infinite, then the set S^ω is a continuous image of the Baire space ω^ω; thus it is an analytic set.

Now let $R \subseteq \mathbb{M}(\Gamma, D)$ be a language of finite traces. For $r \in R$, let $\theta_r : \mathbb{R}^\omega(\Gamma, D) \to \mathbb{R}^\omega(\Gamma, D)$ be the function defined by $\theta_r(t) = r.t$ for all $t \in \mathbb{R}^\omega(\Gamma, D)$. It is easy to see that this function is continuous. Then $r.S^\omega = \theta_r(S^\omega)$ is an analytic set, because the image of an analytic set under a continuous function is still an analytic set. The language $R.S^\omega = \bigcup_{r\in R} r.S^\omega$ is a countable union of analytic sets (because R is countable), but the class of analytic subsets of $\mathbb{R}^\omega(\Gamma, D)$ is closed under countable unions, thus $R.S^\omega$ is an analytic set.

A rational language $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ is a finite union of sets of the form $R.S^\omega$, where S and R are rational monoalphabetic languages of finite traces over (Γ, D) and $\varepsilon \notin S$. Then this language is an analytic set (as a finite union).

Note that we have not used the fact that S and R are rational, so the above proof can be applied to finite unions of sets of the form $R.S^\omega$, where S is a monoalphabetic language of finite traces, and we obtain the following result.

Proposition 4.1. *Let (Γ, D) be a dependence alphabet without isolated letter, and let S_i, R_i, $1 \leq i \leq n$, be languages of finite traces over (Γ, D), where, for all i, S_i does not contain the empty trace and is monoalphabetic. Then the language of infinite traces*

$$T = \bigcup_{1 \leq i \leq n} R_i.S_i^\omega$$

is an analytic set. In particular, every rational language $T \subseteq \mathbb{R}^\omega(\Gamma, D)$ is an analytic set.

In order to prove the existence of a $\mathbf{\Sigma}_1^1$-complete rational language of infinite traces, we will use results about languages of infinite binary trees whose nodes are labelled from a finite alphabet Σ having at least two letters.

A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$, where r means "right" and l means "left." Then an infinite binary tree whose nodes are labelled from Σ may be viewed as a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled from Σ will be denoted by T_Σ^ω.

There is a natural topology on this set T_Σ^ω, which is determined by the following distance (see [24]). Let t and s be two distinct infinite trees in T_Σ^ω. Then the distance between t and s is $\frac{1}{2^n}$ where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^\star$ of length n. Open sets are then of the form $T_0.T_\Sigma^\omega$, where T_0 is a set of finite labelled trees and $T_0.T_\Sigma^\omega$ is the set of infinite binary trees that extend some finite labelled binary tree $t_0 \in T_0$; here t_0 is a sort of prefix, an "initial subtree" of a tree in $t_0.T_\Sigma^\omega$.

It is well known that the topological space T_Σ^ω is homeomorphic to the Cantor set, and hence also to the topological spaces Σ^ω or $\mathbb{R}^\omega(\Gamma, D)$.

The Borel hierarchy and the projective hierarchy on T_Σ^ω are defined starting from open sets as for the topological spaces Σ^ω or $\mathbb{R}^\omega(\Gamma, D)$.

Let t be a tree. A *branch* B of t is a subset of the set of nodes of t that is linearly ordered by the tree partial order $\sqsubseteq$ and closed under the prefix relation, i.e., if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$, then $x \in B$. A branch B of t is said to be *maximal* if no branch of t strictly contains B.

Let t be an infinite binary tree in T_Σ^ω. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i \geq 0}$ be the enumeration of the nodes of B that is strictly increasing in the prefix order. The infinite sequence of labels of the nodes of such a maximal branch B, i.e., the sequence $t(u_0)t(u_1)....t(u_n).....$, is called a *path*. It is an ω-word over the alphabet Σ.

For $L \subseteq \Sigma^\omega$, we denote by $\mathrm{Path}(L)$ the set of infinite trees t in T_Σ^ω such that t has at least one path in L.

It is well known that if $L \subseteq \Sigma^\omega$ is a Π_2^0-complete subset of Σ^ω (or a Borel set of higher complexity in the Borel hierarchy), then the set $\mathrm{Path}(L)$ is a $\mathbf{\Sigma}_1^1$-complete subset of T_Σ^ω (see [27, 31], [29, Exercise]).

In order to use this result, we first code trees labelled from Σ by infinite words over the finite alphabet $\Gamma = \Sigma \cup \Sigma' \cup \{A, B\}$, where $\Sigma' = \{a' \mid a \in \Sigma\}$ is a disjoint copy of the alphabet Σ and A, B are additional letters not in $\Sigma \cup \Sigma'$.

Now consider the set $\{l, r\}^\star$ of nodes of binary infinite trees. For each integer $n \geq 0$, let C_n be the set of words of length n in $\{l, r\}^\star$. Then $C_0 = \{\varepsilon\}$, $C_1 = \{l, r\}$, $C_2 = \{ll, lr, rl, rr\}$, and so on; C_n is the set of nodes that appear at the $(n+1)$th level of an infinite binary tree. The number of nodes of C_n is $\mathrm{card}(C_n) = 2^n$. Now consider the lexicographic order on C_n (assuming that l precedes r in this order). Then, in the enumeration of the nodes with regard to this order, the nodes of C_1 will be l, r; the nodes of C_3 will be $lll, llr, lrl, lrr, rll, rlr, rrl, rrr$. Let $u_1^n, \ldots, u_j^n, \ldots, u_{2^n}^n$ be this enumeration of C_n in the lexicographic order, and let $v_1^n, ..., v_j^n, ..., v_{2^n}^n$ be the enumeration of the elements of C_n in the reverse order. Then for all integers $n \geq 0$ and i, $1 \leq i \leq 2^n$, we have $v_i^n = u_{2^n+1-i}^n$.

For $t \in T_\Sigma^\omega$, let $U_n^t = t(u_1^n)t(u_2^n)\ldots t(u_{2^n}^n)$ be the finite word enumerating the labels of nodes of C_n in the lexicographic order, and let $V_n^t = t(v_1^n)t(v_2^n)\ldots t(v_{2^n}^n)$ be the reverse sequence. Let $V_n^{'t} = \psi(V_n^t)$, where ψ is the morphism from $\Sigma^\star$ into $\Sigma^{'\star}$ defined by $\psi(a) = a'$ for all $a \in \Sigma$. Then the code $g(t)$ of t is

$$g(t) = V_0^{'t}.A.U_1^t.B.V_2^{'t}.A.U_3^t.B.V_4^{'t}.A \ldots A.U_{2n+1}^t.B.V_{2n+2}^{'t}.A \ldots.$$

The ω-word $g(t)$ enumerates the labels of the nodes of the tree t that appear at successive levels $1, 2, 3, \ldots$. The (images under ψ of the) labels of nodes occuring at an odd level $2n+1$ are enumerated in the reverse lexicographic order by the sequence $V_{2n}^{'t}$, and the labels of nodes occuring at an even level $2n$ are enumerated in the lexicographic order by the sequence U_{2n-1}^t. The labels of nodes of distinct levels are alternatively separated by the letter A or B.

Now let (Γ, D) be a dependence alphabet, where $\Gamma = \Sigma \cup \Sigma' \cup \{A, B\}$ and the independence relation $I_D = \Gamma \times \Gamma - D$ is defined by

$$I_D = \Sigma \times (\{A\} \cup \Sigma') \bigcup (\{A\} \cup \Sigma') \times \Sigma \bigcup \Sigma' \times \{B\} \bigcup \{B\} \times \Sigma',$$

i.e., letters of Σ may commute only with A and with letters of Σ', while letters of Σ' may commute only with B and with letters of Σ, the letter A (respectively, B) may commute only with letters of Σ (respectively, with letters of Σ').

Now let $h : T_\Sigma^\omega \to \mathbb{R}^\omega(\Gamma, D)$ be the function defined by

$$h(t) = \varphi(g(t)) \quad \text{for every} \quad t \in T_\Sigma^\omega.$$

First we state the following result.

Lemma 4.2. *The above-defined function $h : T_\Sigma^\omega \to \mathbb{R}^\omega(\Gamma, D)$ is continuous.*

Proof. Let us note that in a segment

$$B.V_{2n}^{'t}.A.U_{2n+1}^t.B.V_{2n+2}^{'t}.A$$

of an ω-word $g(t)$ written as above, letters of U_{2n+1}^t may commute only with the preceding letter A and letters of $V_{2n}^{'t}$. In a similar manner, letters of $V_{2n+2}^{'t}$ may commute only with the preceding letter B and letters of U_{2n+1}^t.

Thus if two infinite binary trees $t, s \in T_\Sigma^\omega$ have the same labels on the first k levels ($k > 1$), then for all $r \in \mathbb{M}(\Gamma, D)$ such that

$$|r| \leq (k-1) + 1 + 2 + 2^2 + \ldots + 2^{k-2}$$

we have

$$r \sqsubseteq h(t) \leftrightarrow r \sqsubseteq h(s).$$

Thus

$$l_{\text{pref}(h(t),h(s))} \geq (k-1) + 1 + 2 + 2^2 + \ldots + 2^{k-2} \geq 2^{k-1}$$

and

$$d_{\text{pref}}((h(t), h(s)) = 2^{-l_{\text{pref}(h(t),h(s))}} \leq 2^{-2^{k-1}},$$

so we have proved that

$$d(t,s) \leq 2^{-k} \to d_{\text{pref}}((h(t), h(s)) \leq 2^{-2^{k-1}} \quad \text{for every} \quad t, s \in T_\Sigma^\omega.$$

Hence the function h is continuous (and even uniformly continuous). $\square$

Now let $\mathcal{R} \subseteq \Sigma^\omega$ be a regular ω-language. Given $\mathcal{R}$, we are going to define a language of infinite real traces $\mathcal{L}$ over the dependence alphabet (Γ, D) defined above. Then we will prove that $\mathcal{L}$ is rational and that $\text{Path}(\mathcal{R}) = h^{-1}(\mathcal{L})$.

Let us first define $\mathcal{L}$ as the set of infinite traces $\varphi(\sigma)$ such that $\sigma \in \Gamma^\omega$ may be written in the following form:

$$\sigma = x(1).u_1.A.v_1.x(2).u_2.B.v_2.x(3).u_3.A\ldots$$

$$\ldots A.v_{2n+1}.x(2n+2).u_{2n+2}.B.v_{2n+2}.x(2n+3).u_{2n+3}.A\ldots,$$

where for all integers $i \geq 0$,

$$x(2i+2) \in \Sigma \text{ and } x(2i+1) \in \Sigma',$$

$$u_{2i+2}, v_{2i+1} \in \Sigma^\star \text{ and } u_{2i+1}, v_{2i+2} \in \Sigma^{'\star},$$

$$|v_i| = 2|u_i| \text{ or } |v_i| = 2|u_i| + 1,$$

and the ω-word $x = \psi^{-1}(x(1))x(2)\psi^{-1}(x(3))\ldots x(2n)\psi^{-1}(x(2n+1))x(2n+2)\ldots$ is in $\mathcal{R}$.

Lemma 4.3. *The above-defined language $\mathcal{L}$ of infinite real traces is rational.*

Proof. Every ω-word

$$\sigma = x(1).u_1.A.v_1.x(2).u_2.B.v_2.x(3).u_3.A\ldots$$

$$\ldots A.v_{2n+1}.x(2n+2).u_{2n+2}.B.v_{2n+2}.x(2n+3).u_{2n+3}.A\ldots$$

written as above is equivalent, modulo the equivalence relation $\sim_{I_D}$ over infinite words in Γ^ω, to the infinite word

$$\sigma' = x(1).u_1.v_1.A.x(2).u_2.v_2.B.x(3).u_3.v_3.A\ldots$$

$$\dots A.x(2n+2).u_{2n+2}.v_{2n+2}.B.x(2n+3).u_{2n+3}.v_{2n+3}.A\dots.$$

But letters of Σ may commute also with letters of Σ', and for all integers i,

$$|v_i| = 2|u_i| \quad \text{or} \quad |v_i| = 2|u_i| + 1$$

by the definition of $\mathcal{L}$. Thus every ω-word σ written as above is also equivalent, modulo $\sim_{I_D}$, to an infinite word in Γ^ω of the form

$$\sigma'' = x(1).W_1.A.x(2).W_2.B.x(3).W_3.A\dots$$
$$\dots A.x(2n+2).W_{2n+2}.B.x(2n+3).W_{2n+3}.A\dots,$$

where for all integers $i \geq 0$,

$$W_{2i+1} \in (\Sigma'\Sigma^2)^\star.(\Sigma \cup \{\varepsilon\}), \quad W_{2i+2} \in (\Sigma\Sigma'^2)^\star.(\Sigma' \cup \{\varepsilon\}).$$

Let L be the ω-language over the alphabet Γ formed by all ω-words σ'' such that

$$x = \psi^{-1}(x(1))x(2)\psi^{-1}(x(3))\dots x(2n)\psi^{-1}(x(2n+1))x(2n+2)\dots$$

is in $\mathcal{R}$. It is easy to see that L is an ω-regular language. Moreover, $\mathcal{L} = \varphi(L)$, thus we can infer from Theorem 2.1 that $\mathcal{L}$ is a rational language of infinite real traces. □

Now we are going to prove the following result.

Lemma 4.4. *For $\mathcal{L}$ defined as above for a ω-language $\mathcal{R}$, we have* $\text{Path}(\mathcal{R}) = h^{-1}(\mathcal{L})$, *i.e.,* $h(t) \in \mathcal{L} \longleftrightarrow t \in \text{Path}(\mathcal{R})$ *for every* $t \in T_\Sigma^\omega$.

Proof. Assume that $h(t) \in \mathcal{L}$ for some $t \in T_\Sigma^\omega$. Then $h(t) = \varphi(g(t)) = \varphi(\sigma)$, where $\sigma \in \Gamma^\omega$ can be written in the following form:

$$\sigma = x(1).u_1.A.v_1.x(2).u_2.B.v_2.x(3).u_3.A\dots$$
$$\dots A.v_{2n+1}.x(2n+2).u_{2n+2}.B.v_{2n+2}.x(2n+3).u_{2n+3}.A\dots,$$

where for all integers $i \geq 0$,

$$x(2i+2) \in \Sigma \quad \text{and} \quad x(2i+1) \in \Sigma',$$
$$u_{2i+2}, v_{2i+1} \in \Sigma^\star \quad \text{and} \quad u_{2i+1}, v_{2i+2} \in \Sigma'^\star,$$
$$|v_i| = 2|u_i| \quad \text{or} \quad |v_i| = 2|u_i| + 1,$$

and the ω-word

$$x = \psi^{-1}(x(1))x(2)\psi^{-1}(x(3))\dots x(2n)\psi^{-1}(x(2n+1))x(2n+2)\dots$$

is in $\mathcal{R}$.

Then it is easy to see that $\sigma = g(t)$, because of the definition of σ, of $g(t)$, and of the independence relation I_D on Γ. Thus $\psi^{-1}(x(1)) = t(v_1^0)$ and $u_1 = \varepsilon$; then $|v_1| = 2|u_1| = 0$ or $|v_1| = 2|u_1| + 1 = 1$. If $|v_1| = 0$, then $x(2) = t(u_1^1)$, and if $|v_1| = 1$, then $x(2) = t(u_2^1)$. Thus the choice of $|v_1| = 2|u_1|$ or of $|v_1| = 2|u_1| + 1$ implies that $x(2)$ is the label of the left or the right successor of the root node $v_1^0 = \varepsilon$.

This phenomenon will happen for the next levels. The choice of $|v_i| = 2|u_i|$ or of $|v_i| = 2|u_i| + 1$ determines one of the two successor nodes of a node at level i (whose label is $x(i)$ if i is even, or $\psi^{-1}(x(i))$ if i is odd), and then the label of this successor is $\psi^{-1}(x(i+1))$ if i is even, or $x(i+1)$ if i is odd.

Thus successive choices determine a branch of t, and the labels of the nodes of this branch (changing only $x(2n+1)$ in $\psi^{-1}(x(2n+1))$) form a path

$$x = \psi^{-1}(x(1))x(2)\psi^{-1}(x(3))\dots x(2n)\psi^{-1}(x(2n+1))x(2n+2)\dots,$$

which is in $\mathcal{R}$. Then $t \in \text{Path}(\mathcal{R})$.

Conversely, it is easy to see that if $t \in \text{Path}(\mathcal{R})$, then the infinite word $g(t)$ can be written as a word σ in the above form. Then $h(t) = \varphi(g(t)) = \varphi(\sigma)$ is in $\mathcal{L}$. □

Now we can state the following theorem.

Theorem 4.5. *There exist $\mathbf{\Sigma}_1^1$-complete, hence non-Borel, rational languages of infinite real traces.*

Proof. Assume that $\mathcal{R} \subseteq \Sigma^\omega$ is a Π_2^0-complete ω-regular language. Let $\mathcal{L} \subseteq \mathbb{R}^\omega(\Gamma, D)$ be defined as above. Then $\mathcal{L}$ is a rational language of infinite real traces by Lemma 4.3. Hence $\mathcal{L}$ is an analytic subset of $\mathbb{R}^\omega(\Gamma, D)$ by Proposition 4.1.

But $\text{Path}(\mathcal{R})$ is a $\mathbf{\Sigma}_1^1$-complete set, and $\text{Path}(\mathcal{R}) = h^{-1}(\mathcal{L})$ by Lemma 4.4, thus $\mathcal{L}$ is also $\mathbf{\Sigma}_1^1$-complete. In particular, $\mathcal{L}$ is not a Borel set. □

5. Concluding remarks

The existence of a $\mathbf{\Sigma}_1^1$-complete infinitary rational relation was used to obtain many undecidability results in [11], and the existence of a $\mathbf{\Sigma}_1^1$-complete context-free ω-language led to other undecidability results in [8, 9, 12]. In particular, the topological complexity and the degree of ambiguity of an infinitary rational relation or of a context-free ω-language are highly undecidable.

In a similar way, the existence of a $\mathbf{\Sigma}_1^1$-complete recognizable language of infinite pictures, proved in [3] by Altenbernd, Thomas, and Wöhrle, was used in [13] to prove many undecidability results, giving, in particular, the answer to some open questions of [3].

Topological arguments following from the existence of $\mathbf{\Sigma}_1^1$-complete rational languages of infinite real traces can also be used to prove similar undecidability results for languages of infinite traces.

In [12], some links were established between the existence of a $\mathbf{\Sigma}_1^1$-complete ω-language of the form V^ω and the number of decompositions of ω-words of V^ω into words of V. We think that such facts could be useful in the combinatorics of traces. The code problem for traces is important in trace theory, and several questions are still open (see [18]). The analog of the notion of ω-code and the study of the number of decompositions of infinite traces of V^ω, where V is a set of finite traces, into an infinite product of traces of V, is also an important subject related to practical applications and to the notion of ambiguity (see [1, 2] for related results in the case of words). We think that topological arguments could be useful in this research area, and the existence of several $\mathbf{\Sigma}_1^1$-complete languages of infinite traces having different structures could be useful for different dependence alphabets.

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