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A CHI-SQUARED TEST FOR THE GENERALIZED POWER WEIBULL FAMILY FOR THE HEAD-AND-NECK CANCER CENSORED DATA

M. Nikulin* **and F. Haghighi**† UDC 519.2

We propose a chi-squared type statistic to test the validity of the generalized power Weibull family based on the Head-and-Neck cancer censored data. Bibliography: 18 *titles.*

1. General information

1.1. Introduction

In survival analysis, the data are often not completely observed. For example, a single right censoring occurs commonly in the reponse time data. In this case, each lifetime X may be observed exactly or, alternatively, may be known only up to a certain accuracy. Thus, in the second case it is impossible to determine the exact value of the number of observations falling into a cell (Pearson statistic). Goodness-of-fit analysis is substantially complicated by the presence of censoring.

In the case of type II censoring (when censoring occurs at specified ordered failures), Mihalko and Moore used sample percentiles as cell boundaries to obtain Pearson type tests of fit that have limiting chi-square distributions for the composite null hypothesis. Habib and Thomas [8] showed that in the case of randomly censored values $Z_n = \sqrt{n}[\widehat{F}_n(t) - F(t, \widehat{\theta}_n)]$, there exists a limiting Gaussian process, where $\widehat{F}_n(t)$ is the product-limit estimator for $F(x, \theta)$ and $\widehat{\theta}_n$ is the maximum likelihood estimator. They applied a Chernoff–Lehmann result to modify the Pearson statistic which was shown to have a limiting chi-square null distribution. Nikulin and Solev showed in [14] (see also [7]) that, in the presence of doubly censored data, $U_n = \sqrt{n}[F_x^n(t) - F_x(t, \widehat{\theta})]$ converge weakly to a Gaussian process, where $F_x^n(t)$ is the Tsai and Growly estimator and the test estimator Y_n^2 has a limiting chi-square distribution with $k - 1$ degrees of freedom (k is the number of the cells), under the condition that an adaptive procedure has been used for grouping data to control the probability of error type I. In [17], Zhang used a chi-squared type statistic Y_n^2 to test the validity of the logistic regression model based on case-control data by adapting the goodness-of-fit test of Nikulin–Rao–Robson–Moore.

A general Pearson chi-squared goodness-of-fit test statistic for randomly censored data was considered by Kim in [10], where nonnegative-definite quadratic forms of cell frequencies obtained from the product-limit estimator allowed for random cells and general estimators of nuisance parameters. In [1], Akritas introduced chi-squared statistics for randomly censored data based on the number of uncensored observations in each cell. Several tests have been suggested for the case of a simple null hypothesis with randomly censored data. In this paper, we apply a chi-squared type test based on the number of uncensored observations in each cell for the generalized power Weibull family (see [2]). For the composite null hypothesis, the tests development is based on the weak convergence of the modified empirical process for which parameters are estimated. We note that the goodness-of-fit problem has also been studied by Voinov and Nikulin for the case of discrete data where the data is complete and/or right-censored. Unfortunately, the goodness-of-fit problem with right-censored discrete data has not been investigated extensively.

1.2. Maximum likelihood estimator and right-censored data

We are interested in testing the null hypothesis that F belongs to a given parametric family of distributions (composite null hypothesis). We consider a random censoring model which occurs frequently in industrial life-testing and medical follow-up studies.

In the random censorship model, we assume that responses $X_1, \dots, X_n$ are independent, nonnegative, random variables with a continuous distribution function F. Censoring variables $Y_1, \dots, Y_n$ are also nonnegative and are assumed to be a random sample drawn independently of the variables X_j from a population with a continuous distribution function $G \in \mathcal{G}$. We say that the variables X_j are censored on the right by the variables Y_j since we only may observe the values $Z_j = \min(X_j, Y_j)$ and $\delta_j = I[Z_j = X_j]$; the latter value indicates whether Z_j is an uncensored observation or not.

*St.Petersburg Department of the Steklov Mathematical Institute, St.Petersburg, Russia.

†University Victor Segalen, Bordeaux 2, France.

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We are interested in testing the null hypothesis that F belongs to a given parametric family of distributions

$$F \in \{F_\theta, \theta = (\theta_1, \dots, \theta_r)' \in \Theta\}.$$

We set $W_j = (Z_j, \delta_j)$. Thus, we consider a model in which our observations $(W_1, \dots, W_n)$ have distribution $P^n_{\theta,G}$ depending on the parameter of interest $\theta \in \Theta$ and the nuisance parameter $G \in \mathcal{G}$. Assume that for a test statistic $T_n = T_n(W_1, \dots, W_n)$ we reject the null hypothesis H_0 if $T_n > t$. In this case, the significant level $q_n(t)$ is

$$q_n(t) = \sup_{(\theta,G)\in\Theta\times\mathcal{G}} P^n_{\theta,G}\{T_n > t\}.$$

We have to investigate the limit behavior of $q_n(t)$ as $n \to \infty$ under some conditions on the parametric set,

$$F \in \{F_\theta, \theta = (\theta_1, \dots, \theta_r)' \in \Theta\} \quad \text{and} \quad G \in \mathcal{G}.$$

The main conditions are as follows:

$$\inf_{G\in\mathcal{G}} G(x) > 0 \quad \text{for any } x > 0$$

and for any $\delta > 0$ there exist $x > 0$ such that

$$\sup_{\theta\in\Theta} (1 - F_\theta(x)) \le \delta.$$

We set

$$L_\theta(Z_i, \delta_i) = [f_\theta(Z_i)(1 - G)]^{\delta_i}[g(Z_i)(1 - F_\theta(Z_i)]^{(1-\delta_i)},$$

where G is the censoring distribution, g is its density, and f_θ is the density function of F_θ. The information matrix I_θ is

$$I_\theta = -E\left[\frac{\partial^2 \log L_\theta(Z_i, \delta_i)}{\partial\theta_l \partial\theta_m}\right].$$

1.3. Goodness-of-fit statistics for random censored data

Consider testing the hypothesis

$$H_0 : F \in \{F_\theta, \theta = (\theta_1, \dots, \theta_r)' \in \Theta\}.$$

In Sec. 2.2 below, we assume that we work with the generalized power Weibull family of distribution functions. Similarly to the usual chi-squared tests, we consider a partition of the sample space into k cells.

Taking into account the nature of randomly censored data, a test statistic for this hypothesis is constructed by partitioning two half-lines corresponding to $\delta = 0$ (censored) and $\delta = 1$ (uncensored), respectively. Let $A_j = [a_{j-1}, a_j)$, $j = 1, \dots, k$, where $0 = a_0 < a_1 < \dots < a_{k-1} < a_k = \infty$, be a partition of $[0, \infty)$ chosen to have enough observations in each interval. Set

$$\widehat{H}_1(z) = \frac{1}{n}\Sigma_{i=1}^n I(Z_i < z, \delta_i = 1),$$

$$N_{1j} = \Sigma_{i=1}^n I(Z_i \in A_j, \delta_i = 1) = n\int_{A_j} d\widehat{H}_1(z), \quad j = 1, \dots, k,$$

$$\widehat{p}_{1j} = \int_{A_j} (1 - \widehat{G})dF_{\widehat{\theta}_n}, \quad \text{and} \quad \widehat{p}_{0j} = \int_{A_j} (1 - F_{\widehat{\theta}_n})d\widehat{G},$$

where $\widehat{\theta}_n$ is the maximum likelihood estimator, $\widehat{G} = 1 - (1 - \widehat{H})/(1 - F_{\widehat{\theta}_n})$, and $\widehat{H}(z) = n^{-1}\Sigma_{i=1}^n I(Z_i < z)$ is the emprical distribution function.

The equalities $N_{1j} - n\widehat{p}_{1j} = -(N_{0j} - n\widehat{p}_{0j})$ hold, where $N_{0j} = \Sigma_{i=1}^n I(Z_i \in A_j, \delta_i = 0)$. Thus, to construct a goodness-of-fit statistic, it is enough to consider the vector

$$\widehat{V}_n = n^{-1/2}((N_{11} - n\widehat{p}_{11}), \dots, (N_{1k} - n\widehat{p}_{1k}))'. \tag{1}$$

Under some assumptions (see [1]), the vector $\widehat{V}_n$ defined in (1) has asymptotically a k-variate normal distribution,

$$\widehat{V}_n(\theta_0) \text{ is } AN(0_k, W(\theta_0)),$$

where

$$W(\theta_0, G) = \Sigma - BI_{\theta_0}^{-1}B', \qquad \Sigma = \Sigma(\theta_0) = \operatorname{diag}(p_{11}, \ldots, p_{1k}), \qquad p_{1j} = \int_{A_j} (1-G)dF_\theta,$$

$$B = B(\theta) = (b_{ji})_{k\times r}, \qquad b_{ji} = \int_{A_j} \overline{G}\varphi_\theta^{(i)} dF_\theta, \qquad \overline{G} = 1 - G, \qquad \varphi_\theta^{(i)} = \frac{\partial[\log(f_\theta/(1-F_\theta))]}{\partial\theta_i},$$

I_{θ_0} is the Fisher information matrix corresponding to the random censoring model, and θ_0 denotes the true underlying value of θ.

In the standard situation, the rank of W is equal to $k-1$. If $A = W^-$ is a generalized inverse of W, then under H_0, the statistic

$$Y_n^2 = \widehat{V}_n'(\widehat{\theta}_n)A(\widehat{\theta}_n, \widehat{G}_n)\widehat{V}_n(\widehat{\theta}_n)$$

has asymptotically a chi-squared distribution with $k-1$ degrees of freedom.

2. Weibull families

2.1. Introduction

The Weibull distribution, named after W. Weibull (1939), is commonly used for analyzing life time data. The Weibull family accommodates both increasing and decreasing failure rates. It general, this family is adequate for modeling monotone hazard rates, and large data are needed to discriminate it from different monotone hazard rate models such as the gamma and log-normal models. But the Weibull family does not allow for nonmonotone failure rates, which are common in survival analysis and reliability.

A possible approach to construction of flexible parametric models is to embed appropriate competing models into a larger model by adding a shape parameter. This embedding approach not only provides a broader range of hazard shapes, but also allows the methods of ordinary parametric inference to be used for discrimination and leads to an assessment of each competing model to a more comprehensive one.

Several models, such as Stacy's (1962) generalized gamma, Prentice's (1975) generalized F distribution, the two families introduced by Slymen and Lachenbruch (1984), have been introduced for modeling nonmonotone failure-rate data (see also Bagdonavicius and Nikulin [4]). A variety of methods for estimation and testing based on general principals such as methods of moments, least squares, and maximum likelihood, have been examined and discussed for these models. Often, such methods present difficulties, especially in the presence of censoring. To avoid the problem of model validity, the nonparametric approach, supported by the well-developed Kaplan–Meier product-limit estimator and related techniques, is often regarded as a preferable one. However, this alternative is often inefficient.

In this paper, we review two Weibull extensions, the generalized Weibull and exponentiated Weibull families, and present a new one: the power generalized Weibull family. These extensions of the Weibull family not only allow for a broader class of monotone hazard rates but also contain distributions with unimodal and bathtub hazard shapes. The *generalized Weibull family*, first suggested by Mudholkar et al. in [13] for constructing isotones, has the following distribution function $F(t)$ and quantil function t_p:

$$F(t) = 1 - (1 - \gamma(t/\sigma)^\nu)^{\frac{1}{\gamma}}$$

and

$$t_p = \begin{cases} \sigma[1-(1-p)^\gamma/\gamma]^{\frac{1}{\nu}} & \text{if } \gamma \neq 0, \\ \sigma[-\log(1-p)]^{\frac{1}{\nu}} & \text{if } \gamma = 0, \end{cases}$$

where $\nu, \sigma > 0$ and $-\infty < \gamma < \infty$. The generalized Weibull family turns into the Weibull distribution if $\gamma = 0$, exponential distribution if $\nu = 1$ and $\gamma = 0$, and the log-logistic distribution if $\gamma = -1$, which is often used as a model in survival studies. Moreover, common parametric distributions such as the lognormal and gamma distributions, are very well approximated by members of the family. What is more important, if $\lambda \leq 0$ and $\alpha \geq 0$, then the family coincides with Burr type XII distributions (see, for example, [15]). It is easy to verify that this family is closed under the proportional hazard relationship (see [13]).

The *exponentiated Weibull family* was originally proposed by Mudholdkar and Srivastava (1993) in the context of the bathtub shaped failure rate data for the reanalysis of the Bus-Motor-Failure Data. The exponentiated Weibull family has the distribution function

$$F(t) = [1 - \exp(-(t/\sigma)^{\alpha})]^{\frac{1}{\gamma}}, \quad \text{where} \quad \nu, \gamma, \sigma > 0,$$

and quantil function

$$t_p = \sigma\Big[-\log(1 - p^{\gamma})\Big]^{\frac{1}{\nu}}.$$

It is obvious that if $\gamma = 1$, then we get the Weibull family, and if $\alpha = 2$, then the family coincides with the Burr type X family of distributions. Distribution properties, extreme value, and extreme spacing distributions for members of the exponentiated Weibull family can be found in [13].

These families are suitable for modeling data that indicate nonmonotone hazard rates and can be also adopted for testing goodness-of-fit of the Weibull family as a submodel. An estimation procedure and chi-square goodness-of-test have been developed for them.

The *generalized power Weibull family* is another extension of the Weibull family. This family, which was presented at first by Bagdonavicius and Nikulin in [2], contains four shapes of the hazard function and is mostly used in the reliability and survival analysis domains. This family is often used for constructing accelerated failures times (AFT) models which describe dependence of the lifetime distribution on explanatory variables. Now we describe this family.

2.2. The generalized power Weibull family

The generalized power Weibull family is mostly conveniently specified in terms of its survival function,

$$S(t;\sigma,\nu,\gamma) = \exp\left\{1 - \left(1 + \left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\right\}, \quad (\sigma,\nu,\gamma > 0), \quad t > 0,$$

and its cumulative distribution function,

$$F(t;\sigma,\nu,\gamma) = 1 - S(t;\sigma,\nu,\gamma) = 1 - \exp\left\{1 - \left(1 + \left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\right\}, \qquad (\sigma,\nu,\gamma > 0), \quad t > 0.$$

The corresponding probability density function is

$$f(t) = \frac{\nu}{\gamma\sigma^{\nu}} t^{\nu-1}\left\{1 + \left(\frac{t}{\sigma}\right)^{\nu}\right\}^{(\frac{1}{\gamma}-1)} \exp\left\{1 - \left(1 + \left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\right\}.$$

The quantile function of the generalized power Weibull family is

$$t_p = \sigma\{(1 - \log(1-p))^{\gamma} - 1\}^{\frac{1}{\nu}}, \quad 0 < p < 1,$$

and its hazard function is

$$\alpha(t,\nu,\sigma,\gamma) = \frac{\nu}{\gamma\sigma^{\nu}} t^{\nu-1}\left\{1 + \left(\frac{t}{\sigma}\right)^{\nu}\right\}^{\frac{1}{\gamma}-1}.$$

Particular cases of the generalized power Weibull distribution are:

$\gamma = 1$: the family of Weibull distributions;

$\gamma = 1$ and $\nu = 1$: the family of exponential distributions.

2.3. Hazard function shapes

The generalized power Weibull family has very nice properties. Depending on the parameter values, the hazard rate can be constant, monotone (increasing or decreasing), ∩-shaped, and ∪-shaped.

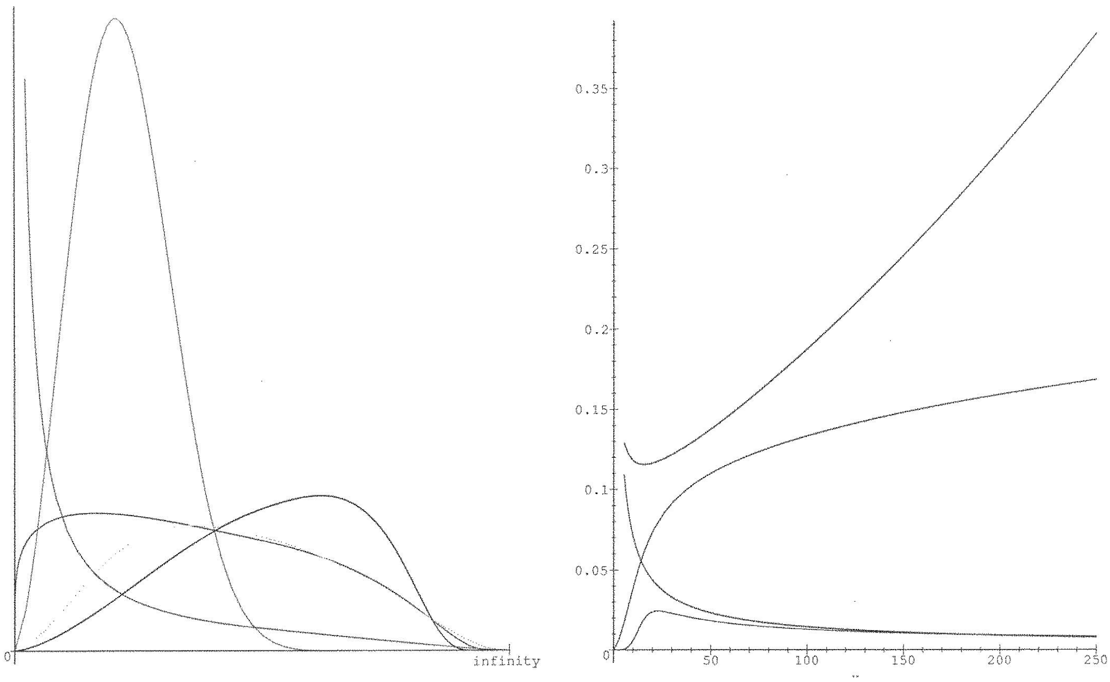

FIG. 1. Generalized power Weibull pdf (left) and its typical hazard shapes.

TABLE 1. SURVIVAL TIMES (IN DAYS) FOR PATIENTS AT ARM A OF THE HEAD-AND-NECK CANCER TRIAL.

Arm A : $7, 34, 42, 63, 64, 74^*, 83, 84, 91, 108, 112, 129, 133, 133, 139,$ $140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185^*, 218, 225, 241,$ $248, 273, 277, 279^*, 297, 319^*, 405, 417, 420, 440, 523, 523^*, 583, 594,$ $1101, 1116^*, 1146, 1226^*, 1349^*, 1412^*, 1417$

The sign * indicates that the observations lost to follow up.

The parameter space can be divided into several regions, over which the hazard function becomes IFR, DFR, bathtub failure rate, and unimodal failure rate, respectively.

For the generalized power Weibull family, the hazard function $\alpha(t, \theta)$ is

(a) monotone increasing if either $\nu > 1$ and $\nu > \gamma$ or $\nu = 1$ and $\gamma < 1$;

(b) monotone decreasing if either $0 < \nu < 1$ and $\nu < \gamma$ or $0 < \nu < 1$ and $\nu = \gamma$;

(c) $\cap$-shaped if $\gamma > \nu > 1$;

(d) $\cup$-shaped if $0 < \gamma < \nu < 1$.

2.4. Reanalysis of the Arm A data for the Head-and-Neck cancer study

The survival times Z (in days) for patients of Arm A of the Head-and-Neck cancer trial were first considered by Efron in [6] for 51 head-and-neck cancer patients.

Efron discretized the data into 47 intervals, each of one month length (1 month=30.438 days), and used the standard logistic regression techniques to estimate the hazard rate. He showed that the estimated hazard rate for the Head-and-Neck cancer study is unimodal.

It is noteworthy that the Weibull model, having only increasing and decreasing hazard shapes, is inadequate for the Head-and-Neck cancer data. Therefore, we considered extensions of the Weibull family which contain not only unimodal hazard rate, but are also computationally convenient for censored data.

Mudholkar and Srivastava analyzed in [13] these data by using the exponentiated Weibull family. They found that in terms of the chi-squared goodness-of-fit, the exponentiated Weibull distribution provided an acceptable fit to data. Here we propose the generalized power Weibull family (see [2]), which is mostly convenient specified in terms of its survival function,

$$S(t;\sigma,\nu,\gamma) = \exp\left\{1 - \left(1 + \left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\right\}, \quad (\sigma,\nu,\gamma > 0), \quad t > 0.$$

We fit this model to data and test the goodness-of-fit for this model under the assumption that

$$0 < a \le \sigma \le b < \infty, \quad 0 < c \le \gamma \le d < \infty, \quad \text{and} \quad \nu \le f < \infty$$

for some positive a, b, c, d, and f.

After transforming the data into month intervals, the maximum likelihood estimators of parameters of the generalized power Weibull fit, obtained by using a computational program which was prepared for estimating the maximum likelihood estimators of censored data for the Weibull famillies, are as follows:

$$\widehat{\nu} = 2.1887, \quad \widehat{\sigma} = 2.5458, \quad \text{and} \quad \widehat{\gamma} = 4.9950. \tag{2}$$

For the generalized power Weibull fit, $\widehat{\gamma} > \widehat{\nu} > 1$. Therefore, in agreement with Efron's analysis, the hazard function is unimodal (see [2]). This result is confirmed by Fig. 2.

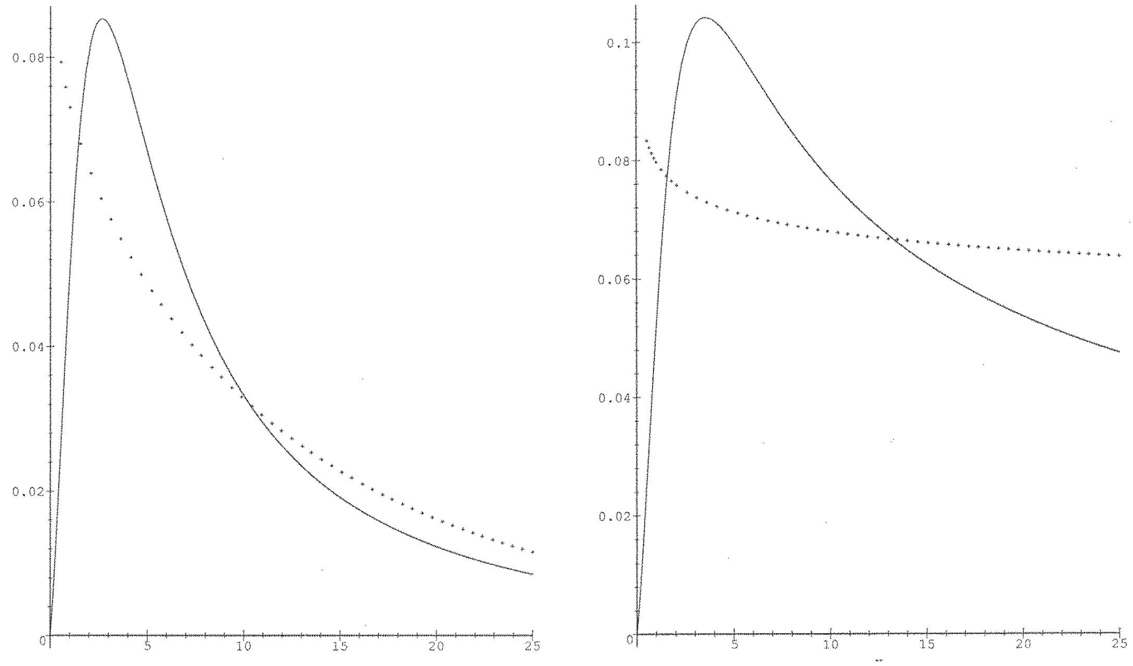

FIG. 2. The right-hand figure represents the fitted density function, and the left-hand figure represents the hazard function of the Arm A data for the Weibull (dotted line) and the generalized power Weibull (solid line) families.

We consider the composite null hypothesis and employ the maximum likelihood based on ungrouped observations to estimate the unknown parameters. Consider testing the hypothesis that the underlying distribution belongs to the generalized power Weibull family. Set

$$H_0 : F = 1 - \exp\{1 - (1 + \lambda t^{\nu})^{\alpha}\},$$

where

$$\lambda = 1/\sigma^{\nu} \quad \text{and} \quad \alpha = 1/\gamma.$$

We present the survival data Z_i in Table 1. Entries of the matrix B are given by

$$\widehat{b}_{j1} = \int_{A_j} \overline{\widehat{G}}\varphi_\theta^{(1)} dF_\theta|_{\widehat{\theta}} = (1-\widehat{H})\widehat{\alpha}z^{\widehat{\nu}}(1+\widehat{\lambda}z^{\widehat{\nu}})^{\widehat{\alpha}-1}|_{a_{j-1}}^{a_j} + n^{-1}\Sigma_{i=1}^n \widehat{\alpha}Z_i^{\widehat{\nu}}(1+\widehat{\lambda}Z_i^{\widehat{\nu}})^{\widehat{\alpha}-1} I_{[a_{j-1}<Z_i<a_j]}, \quad (3)$$

$$\widehat{b}_{j2} = \int_{A_j} \overline{\widehat{G}}\varphi_\theta^{(2)} dF_\theta|_{\widehat{\theta}} = (1-\widehat{H})\log(1+\widehat{\lambda}z^{\widehat{\nu}})(1+\widehat{\lambda}z^{\widehat{\nu}})^{\widehat{\alpha}}|_{a_{j-1}}^{a_j} + n^{-1}\Sigma_{i=1}^n (1+\widehat{\lambda}Z_i^{\widehat{\nu}})^{\widehat{\alpha}} \log(1+\widehat{\lambda}Z_i^{\widehat{\nu}}) I_{[a_{j-1}<Z_i<a_j]}, \quad (4)$$

and

$$\widehat{b}_{j3} = \int_{A_j} \overline{\widehat{G}}\varphi_\theta^{(3)} dF_\theta|_{\widehat{\theta}} = (1-\widehat{H})\log(1+\widehat{\lambda}z^{\widehat{\nu}})(1+\widehat{\lambda}z^{\widehat{\nu}})^{\widehat{\alpha}}|_{a_{j-1}}^{a_j} + n^{-1}\Sigma_{i=1}^n (1+\widehat{\lambda}Z_i^{\widehat{\nu}})^{\widehat{\alpha}} \log(1+\widehat{\lambda}Z_i^{\widehat{\nu}}) I_{[a_{j-1}<Z_i<a_j]}, \quad (5)$$

where $\widehat{H} = n^{-1}\Sigma_{i=1}^n I(Z_i < z)$ is the empirical distribution function,

$$\varphi_\theta^{(1)} = \frac{\partial(\log(f_\theta/\overline{F}_\theta))}{\partial\lambda}, \quad \varphi_\theta^{(2)} = \frac{\partial(\log(f_\theta/\overline{F}_\theta))}{\partial\alpha}, \quad \text{and} \quad \varphi_\theta^{(3)} = \frac{\partial(\log(f_\theta/\overline{F}_\theta))}{\partial\nu},$$

where $\overline{F}_\theta = 1 - F_\theta$ and G has to be replaced by $\widehat{G}$ in (3)–(5). These quantities may be estimated by substituting the maximum likelihood estimators. Note that

$$\widehat{p}_{1j}(\theta) = (1-\widehat{H})(1+\widehat{\lambda}z^{\widehat{\nu}})^{\widehat{\alpha}}\Big|_{a_{j-1}}^{a_j} + n^{-1}\Sigma_{i=1}^n(1+\widehat{\lambda}Z_i^{\widehat{\nu}})^{\widehat{\alpha}} I(a_{j-1} < Z_i < a_j).$$

We considered three cells with $a_1 = 200$ and $a_2 = 600$ for the Head-and-Neck cancer data and applied the programs DQDAG, DLINRG, DMRRR, and DBLINF of IMSL (1999). For the Head-and-Neck cancer data, $Y_n^2 = \widehat{V}_n' A \widehat{V}_n = 3.2401$, which provides an acceptable fit to the data, where the statistic Y_n^2 has (asymptotically) a chi-squared distribution with $k = 2$ degrees of freedom.

Table 2. Reanalysis of Arm A of the Head-and-Neck cancer trial using the Weibull and the generalized power Weibull families.

class interval	observed frequency (S_j)	N_j	Weibull (E_j)	power generalized Weibull (E_j)
$0-1$	1	51	4.378366	1.48630
$1-2$	2	50	3.88426	3.59361
$2-3$	5	48	3.59399	4.29222
$3-4$	2	42	3.07046	3.87714
$4-6$	15	72	5.13863	6.29304
$6-8$	3	49	3.41218	3.81026
$8-11$	4	56	3.81915	3.80906
$11-14$	3	45	3.00793	2.64315
$14-18$	2	45	2.95668	2.31189
$18-24$	2	46	2.96665	2.03375
$24-31$	0	49	3.09877	1.84193
$31-38$	2	47	2.92581	1.54790
$38-47$	1	28	1.96539	0.9407
$\sum R_i^2$		21.3318	12.6865	
p-value		.021	.175	
χ_2^2			3.2401	

Remark. The appropriateness of the model is also tested by using the *signed deviance residuals*. For this purpose, the data are regrouped into 13 classes ($j = 1, \dots, 13$), see Table 2. The original data was discretized into 47 intervals, each of one month length ($i = 1, \dots, 47$). We set

$$R_j = \sqrt(2)\,\mathrm{sign}(S_j - E_j)\left[S_j \log(S_j/E_j) + (N_j - S_j)\log\frac{(N_j - S_j)}{(N_j - E_j)}\right]^{1/2},$$

where

$$N_j = \sum_* n_i, \quad S_j = \sum_* S_i, \quad \text{and} \quad E_j = \sum_* n_i \widehat{h}_i$$

for $j = 1, \dots, 13$. $\sum_*$ denotes the sum over the jth time period,

n_i is the number of patients at risk at the beginning of month i,

s_i is the number of patients dying,

h_i is the discrete hazard rate for month i.

The estimate of the hazard function of ith interval is obtained by integrating the estimated hazard function over the given interval.

If the model is correct (in the sense that it contains the true hazard function), then the values R_j should be approximate standard normal deviates and the sum of squares $\sum R_i^2$ should be an approximate chi-squared distribution with 10 degrees of freedom.

The bottom of the table shows a significantly large value of $\sum R_j^2$ for the Weibull model, but the generalized power Weibull has an acceptable significance level of .175. The difference in $\sum R_j^2 = 21.3319 - 12.6865 = 8.6453$ is also significant compared with the distribution; this indicates a genuinely improved fit when we pass from the Weibull to the generalized power Weibull family.

REFERENCES

1. M. G. Akritas, "Pearson-type goodness-of-fit tests: the univariate case," *J. Amer. Statist. Assoc.*, **83**, 222–230 (1988).
2. V. Bagdonavicius and M. Nikulin, *Accelerated Life Models*, Chapman and Hall/CRC, Boca Raton (2002).
3. V. Bagdonavicius and M. Nikulin, "Generalized proportional hazard model based on modified partial likelihood," *Lifetime Data Analysis*, **5**, 138–147 (1999).
4. V. Bagdonavicius and M. Nikulin, *Semiparametric Models in Accelerated Life Testing*, Queen's Papers in Pure and Appl. Math., Vol. 98, Kingston Queen's Univ., Ontario (1995).
5. H. Chernoff and E. L. Leman, "The use of maximum likelihood estimates in chi-square tests for goodness-of-fit," *Ann. Statist.*, **2**, 579–586 (1954).
6. B. Efron, "Logistic regression, survival analysis, and the Kaplan–Meier curve," *J. Amer. Statist. Assoc.*, **83**, 414–425 (1988).
7. P. Greenwood and M. Nikulin, *A Guide to Chi-Squared Testing*, John Wiley and Sons, New York (1996).
8. M. Habib and D. Thomas, "Chi-square goodness-of-fit tests for randomly censored data," *Ann. Statist.*, **14**, 759–765 (1986).
9. J. D. Kalbfleisch and R. L. Prentice, *The Statistical Analysis of Failure Time Data*, John Wiley and Sons, New York (1980).
10. J. H. Kim, "Chi-square goodness-of-fit tests for randomly censored data," *Ann. Statist.*, **21**, 1621–1639 (1993).
11. J. Lawless, *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons, New York (2002).
12. D. P. Mihalko and D. S. Moore, "Chi-square tests of fit for type II censored data," *Ann. Statist.*, **8**, 625–644 (1980).
13. G. Mudholkar, D. Srivastava, and G. D. Kollia, "A generalization of the Weibull distribution with application," *J. Amer. Statist. Assoc.* **91**, 1575–1583 (1996).
14. M. Nikulin and V. Solev, "Chi-squared goodness-of-fit test for doubly censored data with applications in survival analysis and reliability, statistical, and probabilistic models in reliability," in: *Statistical and Probabilistic Models in Reliability*, Birkhäuser, Boston (1999), pp. 100–110.
15. V. Voinov and M. Nikulin, *Unbiased Estimators and Their Applications.* Vol. 1: *Univariate Case*, Kluwer Acad. Publ., Dordrecht (1993).
16. V. Voinov and M. Nikulin, *Unbiased Estimators and Their Applications.* Vol. 2: *Multivariate Case*, Kluwer Acad. Publ., Dordrecht (1996).

17. B. Zhang, "A chi-squared goodness-of-fit test for logistic regression models based on case-control data," *Biometrika*, **86**, 531–539 (1999).
18. A. W. Van der Vaart, *Asymptotic Statistics*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge Univ. Press (1998).