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PARTIAL GEOMETRIC REGULARITY OF SOME OPTIMAL CONNECTED TRANSPORTATION NETWORKS

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We consider a continuous optimization model of a one-dimensional connected transportation network under the assumption that the cost of transportation with the use of network is negligible in comparison with the cost of transportation without it. We investigate the connections between this problem and its important special case, the minimization of the average distance functional. For the average distance minimization problem we formulate a number of conditions for the partial geometric regularity of a solution in $\mathbb{R}^n$ *with an arbitrary dimension* $n \geqslant 2$*. The corresponding results are applied to solutions to the general optimization problem. Bibliography: 26 titles. Illustrations: 1 Figure.*

§ 1. Introduction

We assume that the distribution of the population in some region (city) is determined by a nonnegative finite Borel measure φ^+ with compact support in $\mathbb{R}^n$. It is required to find an optimal transportation network (schemes of urban public transport and/or underground) which could be the most convenient for the population to reach service centers and working places provided that the distribution of working places and service centers is determined by a nonnegative finite Borel measure φ^- with compact support in $\mathbb{R}^n$. A given function A: $\mathbb{R}^+ \to \overline{\mathbb{R}}^+$ is interpreted as the effective cost of the movement of every citizen without using the transportation network (i.e., "on foot" or by their own transport"), so that the cost for covering the distance t is $A(t)$. In this paper, we consider a simplified model. The network is simulated by a closed connected set $\Sigma \subset \mathbb{R}^n$ and the cost for movement with the use of the transportation network is assumed to be zero. The corresponding optimization problem is formulated as follows.

Problem 1.1. Find a set (an optimal transportation network) $\Sigma = \Sigma_{\text{opt}} \subset \mathbb{R}^n$ minimizing the cost for movement of the population, provided that free travel over the network $\Sigma \mapsto MK(\varphi^+, \varphi^-, \Sigma)$ is allowed, among all closed connected sets $\Sigma \subset \mathbb{R}^n$ satisfying the length constraint $\mathscr{H}^1(\Sigma) \leqslant l$ ($l > 0$ is given).

The cost for movement $MK(\varphi^+, \varphi^-, \cdot)$ is strictly defined with the use of the Monge–Kantorovich optimal mass transport problem (cf. details in [**1–4**]). We suggest two equivalent formulas for computing the

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cost functional. The assumption of the zero cost for movement with the use of the transportation network has meaning if this cost is determined by the ticket price which is independent of the covering distance and/or tariff zone; moreover, this cost for every citizen is negligible comparing with the cost for movement without the use of the transportation network. This model is also applicable to the optimization of computer net (with the corresponding interpretation of the initial data).

In this paper, we show that the general model of the choice of an optimal transportation network is reduced to the minimization of the average distance functional. This problem was studied, in particular, in [**5**–**8**].

Problem 1.2. Let φ be a finite nonnegative Borel measure with compact support in $\mathbb{R}^n$. It is required to find a set $\Sigma = \Sigma_{\text{opt}} \subset \mathbb{R}^n$ minimizing the average distance functional F_φ defined by the formula

$$F_\varphi(\Sigma) := \int_{\mathbb{R}^n} A(\operatorname{dist}(x,\Sigma))\, d\varphi(x),$$

over all closed connected sets $\Sigma \subset \mathbb{R}^n$ satisfying the length constraint $\mathscr{H}^1(\Sigma) \leqslant l$ ($l > 0$ is given). Here, $A\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a given nondecreasing function.

It is easy to see from the definition of $MK(\varphi^+, \varphi^-, \cdot)$ that this problem is a special case of Problem 1.1 with $\varphi^- := 0$ (i.e., if the single goal of the population is to reach the transportation network with the minimal cost). If we replace the connectedness and the length constraint on Σ by the condition on the maximal number of points of Σ (i.e., we minimize the functional F_φ over discrete sets Σ such that $\#\Sigma \leqslant k$, where k is a given natural number), then this problem becomes the well-known optimal location problem (k-median problem). This problem can be interpreted as a problem of determining the optimal location of k service centers provided that the density of population φ is known (see more about such problems, for example, in [**9**–**11**]).

Optimization models like Problem 1.2 stem from various problems, not only in urban planning and economics, but also in image processing [**12**], mathematical statistics [**13**, **14**], the optimal irrigation problem [**5**, **15**], and in approximation of a solution to the classical travelling salesman problem (the so-called the "lazy travelling salesman problem") [**16**]). Solutions to Problem 1.2 will be referred to as *average distance minimizers*.

In this paper, we prove some results about the topological structure of the sets minimizing the average distance and establish a partial regularity of the minimizing sets in the case of an arbitrary dimension $n \geqslant 2$. For this purpose, we develop the technique from [**87**] to the case of two-dimensional spaces ($n = 2$) and $A(t) = t$ and also the technique from [**7**]. However, unlike [**8**, **7**], we will use it for obtaining estimates for the generalized mean curvature of the sets under consideration. Based on the obtained results, we prove some assertions concerning the geometric regularity of optimal transportation networks that form a solution to the general optimization problem. We emphasize that we mean the geometric regularity, i.e., solutions to the corresponding optimization problems are "good" from the geometric point of view (for example, they have finitely many endpoints and branching points, every branching point is, a "regular tripod", etc.), but this does not mean that these solutions are smooth curves.

§ 2. Monge–Kantorovich Problem with the Dirichlet Condition

Assume that the choice by the population of directions of daily transport movements is described by the Monge–Kantorovich optimal mass transport model. Since the cost for movement with the use of Σ is assumed to be zero, the total cost for daily transport movement to the service centers and/or working places is determined by a functional I_Σ on Borel measures in $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$I_\Sigma(\Gamma) := \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} A(d_\Sigma(x,y))\, d\Gamma(x,y),$$

where

$$d_\Sigma(x,y) := d(x,y) \wedge (\operatorname{dist}(x,\Sigma) + \operatorname{dist}(y,\Sigma)).$$

In other words, to move from a point x to the point y, each citizen chooses the most convenient route among the following two variants: 1) a route without the use of the transportation network, i.e., only by the own transport or "on foot" (in this case, it is necessary to cover the distance $d(x,y)$), 2) a route with maximal use of the transportation network (in this case, it is required only to pay for the covered distance $\operatorname{dist}(x,\Sigma) + \operatorname{dist}(y,\Sigma)$ without the use of the transportation network). For the sake of simplicity, we assume that the distance d is metrically equivalent to the Euclidean distance.

It is reasonable to regard the measure Γ as a "transportation plan" (i.e., intuitively, we can assume that $\Gamma(x,y)$ is the number of citizens moving from the point x to the point y). Then it must satisfy the condition

$$(\pi^\pm_\# \Gamma) \llcorner \mathbb{R}^n \setminus \Sigma = \varphi^\pm \llcorner \mathbb{R}^n \setminus \Sigma, \tag{2.1}$$

where $\pi^\pm\colon\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are projections on the first and second copies of $\mathbb{R}^n$ respectively; namely, $\pi^\pm(x^+,x^-) := x^\pm$. If $\varphi^\pm$ are concentrated outside Σ and $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$, then the condition (2.1) simply means that all the citizens reach their goals (i.e., service centers and/or working places). If $\varphi^+(\mathbb{R}^n) \neq \varphi^-(\mathbb{R}^n)$, then the goal of some citizens is only to reach Σ (or to reach the service centers and/or working places only from Σ): for example, in the limiting case $\varphi^- = 0$, in the model there are no service centers and working places, and a single goal of citizens is to reach Σ.

The generalized Monge–Kantorovich transport problem with the Dirichlet condition on Σ is to find a Borel measure Γ_{opt} on $\mathbb{R}^n \times \mathbb{R}^n$, called an *optimal transport plan*, minimizing the functional I_Σ over all Borel measures Γ, called *admissible transport plans*, on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the condition (2.1) (cf. [**17**]). We keep the notation $MK(\varphi^+,\varphi^-,\Sigma)$ for the Monge–Kantorovich problem with the Dirichlet condition on Σ, as well as for the corresponding minimal value of the functional I_Σ. We write $MK(\varphi^+,\varphi^-)$ if $\Sigma = \varnothing$. Finally, as a rule, we will not indicate the measure $\varphi^\pm$ in the notation and write simply $MK(\Sigma)$ instead of $MK(\varphi^+,\varphi^-,\Sigma)$.

It is easy to show that the Monge–Kantorovich problem has a solution under rather natural assumptions on the data. Namely, the following assertion holds.

Theorem 2.1. *Suppose that a function A is lower semicontinuous and a set $\Sigma \subset \mathbb{R}^n$ is compact. Then the functional I_Σ attains the minimum over the set of all admissible transport plans.*

Proof. Let $\{\Gamma_\nu\}$ be a minimizing sequence of admissible transport plans for I_Σ. By Lemma 2.2, for every $\nu \in \mathbb{N}$ there is a new admissible transport plan Γ'_ν such that $I_\Sigma(\Gamma_\nu) = I_\Sigma(\Gamma'_\nu)$ and all Γ'_ν have uniformly bounded complete mass. Since all admissible transport plans are supported in the compact set $(\operatorname{supp}\varphi^+ \cup \Sigma) \times (\operatorname{supp}\varphi^- \cup \Sigma)$, we can choose a subsequence of Γ'_ν (not relabeled) that $*$-weakly converges to some Borel measure Γ on $\mathbb{R}^n \times \mathbb{R}^n$. By Lemma 2.3 with $\Sigma_\nu := \Sigma$, we find that Γ satisfies the condition (2.1), i.e., it is an admissible transport plan However, the function $d_\Sigma\colon\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Since A is lower semi-continuous, the integrand of I_Σ is also lower semicontinous. Consequently, the functional I_Σ is lower semicontinous in the sense of the $*$-weak convergence of measures. Hence

$$I_\Sigma(\Gamma) \leqslant \liminf_\nu I_\Sigma(\Gamma'_\nu) = \liminf_\nu I_\Sigma(\Gamma_\nu) = \inf I_\Sigma$$

(since $\{\Gamma_\nu\}$ minimizes I_Σ). Thus, the functional I_Σ attains the minimum at Γ. □

The following simple lemmas were used in the previous proof.

Lemma 2.2. *For any admissible transport plan* Γ *there is another admissible transport plan* Γ' *such that*

$$\Gamma'(\mathbb{R}^n \times \mathbb{R}^n) \leqslant \varphi^+(\mathbb{R}^n) + \varphi^-(\mathbb{R}^n), \quad I_\Sigma(\Gamma') = I_\Sigma(\Gamma).$$

Proof. Setting $\Gamma' := \Gamma - \Gamma \llcorner \Sigma \times \Sigma$, we find that Γ' is an admissible transport plan. Furthermore, $I_\Sigma(\Gamma) = I_\Sigma(\Gamma')$ and $\Gamma'(\Sigma \times \Sigma) = 0$. Consequently,

$$\begin{aligned}\Gamma'(\mathbb{R}^n \times \mathbb{R}^n) &= \Gamma'((\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^n) + \Gamma'(\Sigma \times (\mathbb{R}^n \setminus \Sigma)) \leqslant \Gamma'((\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^n) + \Gamma'(\mathbb{R}^n \times (\mathbb{R}^n \setminus \Sigma)) \\ &= \Gamma((\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^n) + \Gamma(\mathbb{R}^n \times (\mathbb{R}^n \setminus \Sigma)) = \varphi^+(\mathbb{R}^n \setminus \Sigma) + \varphi^-(\mathbb{R}^n \setminus \Sigma) \\ &\leqslant \varphi^+(\mathbb{R}^n) + \varphi^-(\mathbb{R}^n).\end{aligned}$$

The proof is complete. □

Lemma 2.3. *Let* $\{\Gamma_\nu\}$ *be a sequence of admissible transport plans for* $MK(\varphi^+, \varphi^-, \Sigma_\nu)$*, and let the sets* Σ_ν *be compact; moreover,* $\Sigma_\nu \to \Sigma$ *in the sense of Hausdorff and* $\Gamma_\nu \rightharpoonup \Gamma$ *in the sense of the* $*$*-weak convergence of measures. Then* Γ *is an admissible transport plan for* $MK(\varphi^+, \varphi^-, \Sigma_\nu)$.

Proof. We have $\pi^\pm_\# \Gamma_\nu = \varphi^\pm$ over $\mathbb{R}^n \setminus \Sigma_\nu$. Since $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff, any function with compact support in $\mathbb{R}^n \setminus \Sigma$ is also compactly supported in $\mathbb{R}^n \setminus \Sigma_\nu$ for sufficiently large $\nu \in \mathbb{N}$. Consequently, for any $\psi \in C_0(\mathbb{R}^n \setminus \Sigma)$ we have

$$\int\limits_{\mathbb{R}^n} \psi \, d\pi^\pm_\# \Gamma = \lim_\nu \int\limits_{\mathbb{R}^n} \psi \, d\pi^\pm_\# \Gamma_\nu = \int\limits_{\mathbb{R}^n} \psi \, d\varphi^\pm.$$

In particular, $\pi^\pm_\# \Gamma(e_0) = \varphi^\pm(e_0)$ for any open set $e_0 \subset\subset \mathbb{R}^n \setminus \Sigma$. Thus, $\pi^\pm_\# \Gamma(\mathbb{R}^n \setminus \Sigma) = \varphi^\pm(\mathbb{R}^n \setminus \Sigma)$, and, consequently, $\pi^\pm_\# \Gamma = \varphi^\pm$ on $\mathbb{R}^n \setminus \Sigma$. □

We denote by $\Gamma_{\text{opt}}(\Sigma)$ the set of all optimal transport plans Γ_{opt} with a given Σ:

$$\Gamma_{\text{opt}}(\Sigma) := \text{Argmin}\,\{I_{B,\Sigma}(\Gamma) : \Gamma \text{ is an admissible transport plan}\}.$$

§ 3. Equivalent Formulation of the Transportation Problem

We consider the transportation problem from other point of view. Now we are interested in the question what transport routes are chosen by citizens for their daily transport movements. These routes will be described by Lipschitz paths in $\mathbb{R}^n$.

Two Lipschitz paths $\widehat{\theta}_1$, $\widehat{\theta}_2$: $[0,1] \to \mathbb{R}^n$ are said to be *equivalent* if there exists a continuous surjective strictly increasing function φ: $[0,1] \to [0,1]$, called "re-paremetrization" such that

$$\widehat{\theta}_1(t) = \widehat{\theta}_2(\varphi(t)) \text{ for all } t \in [0,1].$$

Denote by Θ the set of equivalence classes of Lipschitz paths. It is obvious that elements of Θ present directed rectifiable curves. We often identify elements of Θ (i.e., directed rectifiable curves) with their parametrizations (i.e., with the Lipschitz functions that parametrize the curves). We equip the set Θ with the metric

$$d_\Theta(\theta_1, \theta_2) := \inf\left\{ \max_{t \in [0,1]} |\widehat{\theta}_1(t) - \widehat{\theta}_2(t)| : \widehat{\theta}_i \text{ is the parametrization of } \theta_i, \ i = 1, 2, \right\}, \tag{3.1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. It is easy to see that the convergence $\theta_\nu \to \theta$ in the metric of Θ implies the Hausdorff convergence of the corresponding traces, but the converse assertion is false.

If $\{\theta_1, \theta_2\} \subset \Theta$ and $\theta_1(1) = \theta_2(0)$, then we define $\theta_1 \circ \theta_2 \in \Theta$ as a directed rectifiable curve admitting the parametrization

$$\theta_1 \circ \theta_2(t) := \begin{cases} \theta_1(2t), & t \in [0, 1/2], \\ \theta_2(2t-1), & t \in (1/2, 1]. \end{cases}$$

The choice by the population of routes is described by the measure $\eta \in \mathscr{M}^+(\Theta)$ which, roughly speaking, indicates how many citizens choose each special route. A measure $\eta \in \mathscr{M}^+(\Theta)$ is called an *admissible transport measure* if

$$(p_{0\#}\eta)\llcorner(\mathbb{R}^n \setminus \Sigma) = \varphi^+\llcorner(\mathbb{R}^n \setminus \Sigma), \quad (p_{1\#}\eta)\llcorner(\mathbb{R}^n \setminus \Sigma) = \varphi^-\llcorner(\mathbb{R}^n \setminus \Sigma), \tag{3.2}$$

where p_i: $\Theta \to \mathbb{R}^n$ is defined by the formula $p_i(\theta) := \theta(i)$, $i = 0, 1$. We introdcued the functional $\overline{C}_\Sigma$ in $\mathscr{M}^+(\Theta)$ as follows:

$$\overline{C}_\Sigma(\eta) := \int\limits_\Theta A(\mathscr{H}^1(\theta \setminus \Sigma))\, d\eta(\theta).$$

As is proved in Proposition 3.1, the functional $\overline{C}_\Sigma$ attains the minimum over the set of all admissible transport measures with respect to some measure η_{opt}. Denote by $E_{\text{opt}}(\Sigma)$ the set of all optimal transport measures η_{opt} for a given Σ:

$$E_{\text{opt}}(\Sigma) := \operatorname{Argmin}\{\overline{C}_\Sigma(\eta) : \eta \text{ is an admissible transport measure}\}.$$

We can see that the minimization of the functional $\overline{C}_\Sigma$ over the set of all admissible transport measures is nothing else as an equivalent formulation of the classical Monge–Kantorovich transport problem.

Proposition 3.1. *If A is a nondecreasing lower semicontinuous function and $\Sigma \subset \mathbb{R}^n$ is a closed connected set of finite length, then*

$$MK(\Sigma) = \min\left\{\overline{C}_\Sigma(\eta) : \eta \textit{ is an admissible transport measure}\right\}. \tag{3.3}$$

Moreover, if $\eta_{\text{opt}} \in E_{\text{opt}}(\Sigma)$, then $\Gamma := (p_0 \times p_1)_\# \eta_{\text{opt}} \in \Gamma_{\text{opt}}(\Sigma)$. Conversely, there exists a Borel mapping q_Σ: $\Omega \times \Omega \to \Theta$, where

$$\Omega := \operatorname{co}\left(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^- \cup \Sigma\right),$$

such that $\Gamma_{\text{opt}} \in \Gamma_{\text{opt}}(\Sigma)$ implies $\eta := q_{\Sigma\#}\Gamma_{\text{opt}} \in E_{\text{opt}}(\Sigma)$. Finally,

$$\operatorname{supp}\eta_{\text{opt}} \subset \widetilde{\Theta}(\Sigma) := \{\theta \in \Theta : d_\Sigma(\theta(0), \theta(1)) = \mathscr{H}^1(\theta \setminus \Sigma)\}$$

for all $\eta_{\text{opt}} \in E_{\text{opt}}(\Sigma)$.

Proof. If η is an admissible transport measure, then $\Gamma := (p_0 \times p_1)_\#\eta$ is an admissible transport plan. Recalling the definition of the distance d_Σ, we write

$$\begin{aligned} \overline{C}_\Sigma(\eta) &= \int\limits_\Theta A(\mathscr{H}^1(\theta \setminus \Sigma))\, d\eta \geqslant \int\limits_\Theta A(d_\Sigma((p_0 \times p_1)(\theta)))\, d\eta \\ &= \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} A(d_\Sigma(x,y))\, d\Gamma = I_\Sigma(\Gamma) \geqslant MK(\Sigma). \end{aligned} \tag{3.4}$$

We consider a multivalued mapping Q_Σ: $\Omega \times \Omega \multimap \widetilde{\Theta}(\Sigma)$ such that

$$Q_\Sigma(x,y) := \{\theta \in \Theta : \theta(0) = x, \theta(1) = y, d_\Sigma(x,y) = \mathscr{H}^1(\theta \setminus \Sigma)\}.$$

We note that the graph of Q_Σ defined by the formula

$$\begin{aligned} \operatorname{Graph} Q_\Sigma &:= \{(x,y,\theta) : (x,y) \in \Omega \times \Omega, \theta \in Q_\Sigma(x,y)\} \\ &= \{(x,y,\theta) \in \Omega \times \Omega \times \Theta : x = \theta(0), y = \theta(1)\} \cap (\Omega \times \Omega \times \widetilde{\Theta}(\Sigma)) \end{aligned}$$

is a closed set because $\widetilde{\Theta}(\Sigma)$ is closed (by Lemma 3.2); moreover, it is obvious that the set $Q_\Sigma(x,y)$ is not empty for any pair of points $(x,y)\in\Omega\times\Omega$ (since Σ is connected). Then by the measurable choice theorem (cf. [**18**, Theorem III.6] or [**19**, Theorem 5.2.1]) this mapping admits a Borel selector $q_\Sigma\colon \Omega\times\Omega\to\widetilde{\Theta}(\Sigma)$. If $\Gamma_{\rm opt}\in\Gamma_{\rm opt}(\Sigma)$ is an arbitrary optimal transport plan, then $\eta_{\rm opt}:=q_{\Sigma\#}\Gamma_{\rm opt}$ is an admissible transport measure; moreover,

$$\begin{aligned} MK(\Sigma) &= \int\limits_{\Omega\times\Omega} A(d_\Sigma(x,y))\,d\Gamma_{\rm opt} = \int\limits_{\Omega\times\Omega} A(\mathscr{H}^1(q_\Sigma(x,y)\setminus\Sigma))\,d\Gamma_{\rm opt} \\ &= \int\limits_{\Theta} A(\mathscr{H}^1(\theta\setminus\Sigma))\,d\eta_{\rm opt} = \overline{C}_\Sigma(\eta_{\rm opt}) \geqslant \inf\overline{C}_\Sigma, \end{aligned} \tag{3.5}$$

where the infimum of $\overline{C}_\Sigma$ is taken over the set of all admissible transport measures. Together with (3.4), this proves that $\eta_{\rm opt}$ is the optimal transport measure Taking for η the optimal transport measure, from (3.4) we find that $(p_0\times p_1)_\#\eta$ is the optimal transport plan. Finally, since all the inequalities in (3.4) become equalities if η is an optimal transport measure, we conclude that η is concentrated on $\widetilde{\Theta}(\Sigma)$ in the sense that $\eta(\Theta\setminus\widetilde{\Theta}(\Sigma))=0$. Since $\widetilde{\Theta}(\Sigma)$ is closed, we obtain the last assertion of the proposition. □

The following simple lemma was used in the proof.

Lemma 3.2. *If $\Sigma\subset X$ is a Borel set of finite length $\mathscr{H}^1(\Sigma)<\infty$, then the mapping $\theta\in\Theta\mapsto\mathscr{H}^1(\theta\setminus\Sigma)$ is lower semicontinuous. Moreover, the set $\widetilde{\Theta}(\Sigma)$ defined in Proposition* 3.1 *is closed.*

Proof. Let $\theta_\nu\to\theta$ in Θ as $\nu\to\infty$. For $\varepsilon>0$ we denote by $\overline{U}_\varepsilon$ a closed ε-neighborhood of set θ. Let $\mu:=\mathscr{H}^1\llcorner\Sigma$. Since $\theta_\nu\subset U_\varepsilon$ for all sufficiently large ν, we have $\mu(\theta_\nu)\leqslant\mu_\nu(\overline{U}_\varepsilon)$. Thus,

$$\limsup_\nu \mu(\theta_\nu)\leqslant\limsup_\nu \mu(\overline{U}_\varepsilon)=\mu(\overline{U}_\varepsilon).$$

Passing to the limit as $\varepsilon\to0^+$ in the last inequality and taking into account the convergence $\mu(\overline{U}_\varepsilon)\to\mu(\theta)$, we find

$$\limsup_\nu \mathscr{H}^1(\theta_\nu\cap\Sigma)\leqslant\mathscr{H}^1(\theta\cap\Sigma).$$

By [**20**, Theorem 4.4.7],

$$\liminf_\nu \mathscr{H}^1(\theta_\nu)\geqslant\mathscr{H}^1(\theta).$$

Hence

$$\begin{aligned} \liminf_\nu \mathscr{H}^1(\theta_\nu\setminus\Sigma) &= \liminf_\nu(\mathscr{H}^1(\theta_\nu)-\mathscr{H}^1(\theta_\nu\cap\Sigma)) \\ &\geqslant \liminf_\nu(\mathscr{H}^1(\theta_\nu)-\limsup_\nu\mathscr{H}^1(\theta_\nu\cap\Sigma)=\mathscr{H}^1(\theta)-\mathscr{H}^1(\theta\cap\Sigma)=\mathscr{H}^1(\theta\setminus\Sigma). \end{aligned}$$

Finally, to prove the closedness of $\widetilde{\Theta}(\Sigma)$, we note that if $\{\theta_\nu\}\subset\widetilde{\Theta}(\Sigma)$, $\theta_\nu\to\theta$ in Θ as $\nu\to\infty$, then $\theta_\nu(0)\to\theta(0)$ and $\theta_\nu(1)\to\theta(1)$ in $\mathbb{R}^n$ as $\nu\to\infty$. Consequently,

$$d_\Sigma(\theta_\nu(0),\theta_\nu(1))\to d_\Sigma(\theta(0),\theta(1)).$$

On the other hand,

$$\mathscr{H}^1(\theta\setminus\Sigma)\leqslant\liminf_\nu\mathscr{H}^1(\theta_\nu\setminus\Sigma)=\liminf_\nu d_\Sigma(\theta_\nu(0),\theta_\nu(1))=d_\Sigma(\theta(0),\theta(1)). \tag{3.6}$$

Since $\mathscr{H}^1(\theta\setminus\Sigma)\geqslant d_\Sigma(\theta(0),\theta(1))$, the inequality (3.6) becomes equality. Consequently, $\theta\in\widetilde{\Theta}(\Sigma)$. □

§ 4. Choice of Optimal Transportation Network

If Σ is given, $MK(\varphi^+,\varphi^-,\Sigma)$ indicates the minimal cost of movement from φ^+ to φ^- with the possibility of "free" use of Σ. In this case, Problem 1.1 has meaning. The following existence theorem hodls.

Theorem 4.1. *Let a function A be lower semicontinuous. Then Problem* 1.1 *has a solution.*

Proof. Let

$$m := \inf\left\{MK(\varphi^+,\varphi^-,\Sigma) : \Sigma \subset \mathbb{R}^n \text{ is closed and connected, } \mathscr{H}^1(\Sigma) \leqslant l\right\}.$$

Consider a sequence $\{\Sigma_\nu\}_{\nu=1}^\infty$ of closed connected subsets of $\mathbb{R}^n$ such that $\mathscr{H}^1(\Sigma_\nu) \leqslant l$ for all $\nu \in \mathbb{N}$ and

$$MK(\varphi^+,\varphi^-,\Sigma_\nu) \searrow m,$$

i.e., a minimizing sequence of the functional $MK(\varphi^+,\varphi^-,\cdot)$.

Lemma 2.2 asserts that for every $\nu \in \mathbb{N}$ there exists an optimal transport plan Γ_ν for the problem $MK(\varphi^+,\varphi^-,\Sigma_\nu)$ such that

$$\Gamma_\nu(\mathbb{R}^n \times \mathbb{R}^n) \leqslant \varphi^+(\mathbb{R}^n) + \varphi^-(\mathbb{R}^n).$$

Without loss of generality, we can assume that $\Sigma_\nu \subset \Omega$ for some compact set $\Omega \subset \mathbb{R}^n$. Otherwise, there is a sequence $\{x_\nu\}$, $x_\nu \in \Sigma_\nu$ such that $|x_\nu| \to \infty$ as $\nu \to \infty$. However, by the inequalities $\operatorname{diam} \Sigma_\nu \leqslant \mathscr{H}^1(\Sigma_\nu) \leqslant l$ for all $y_\nu \in \Sigma_\nu$, we have $|y_\nu| \to \infty$ as $\nu \to \infty$. Because of the compactness of supp $\varphi^\pm$, for any $x \in$ supp $\varphi^\pm$ we have $\operatorname{dist}(x,\Sigma_\nu) \to \infty$ as $\nu \to \infty$. If

$$\Gamma_\nu(\{(x,y) : \operatorname{dist}(x,\Sigma_\nu) + \operatorname{dist}(y,\Sigma_\nu) < d(x,y)\}) > 0$$

for some subsequence $\nu \to \infty$, then for this subsequence we have

$$MK(\varphi^+,\varphi^-,\Sigma_\nu) = I_{\Sigma_\nu}(\Gamma_\nu) \to +\infty,$$

which is impossible. Thus, for all sufficiently large $\nu \in \mathbb{N}$

$$\Gamma_\nu(\mathbb{R}^n \times \mathbb{R}^n) = \Gamma_\nu(\{(x,y) : d_{\Sigma_\nu}(x,y) = d(x,y)\}). \tag{4.1}$$

Since Γ_ν are concentrated on (supp $\varphi^+ \cup \Sigma_\nu) \times$ (supp $\varphi^- \cup \Sigma_\nu$) and supp $\varphi^\pm$ are compact sets, (4.1) means that Γ_ν are concdentrated on supp $\varphi^+ \times$ supp φ^- for sufficiently large ν. But such a situation is possible only if $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$. Furthermore, (4.1) implies the equalities $MK(\varphi^+,\varphi^-,\Sigma_\nu) = MK(\varphi^+,\varphi^-) = m$, which means that any admissible set Σ (even $\Sigma = \varnothing$) is a minimum point of the functional under consideration.

Thus, either $\Sigma_\nu \subset \Omega$ for some compact set $\Omega \subset \mathbb{R}^n$ or the existence of a solution is automatically guaranteed. Without loss of generality, we will assume that Ω is so large that it contains supp φ^+ and supp φ^- (recall that both sets are compact by assumption).

The space of closed subsets Ω equipped with the Hausdorff metric is a compact metric space in view of the Blashke theorem [**20**, Theorem 4.4.6]. Therefore, up to a subsequence (we preserve the same notation) we can assert that $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff; moreover, the set $\Sigma \subset \mathbb{R}^n$ is closed and connected. By the Golab theorem [**20**, Theorem 4.4.7], we have $\mathscr{H}^1(\Sigma) \leqslant l$. We note that the convergence of sets in the sense of Hausdorff implies that $d(x,\Sigma_\nu) \to d(x,\Sigma)$ for all $x \in \mathbb{R}^n$. Consequently,

$$d_{\Sigma_\nu}(x,y) \to d_\Sigma(x,y)$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ as $\nu \to \infty$. Furthermore, since all the functions d_{Σ_ν} are Lipschitz continuous with the same Lipschitz constant, we conclude that $d_{\Sigma_\nu} \to d_\Sigma$ uniformly on $\Omega \times \Omega$ as $\nu \to \infty$. Since A is lower semicontinuous, we find

$$A(d_\Sigma(x,y)) \leqslant \liminf_\nu A(d_{\Sigma_\nu}(x_\nu,y_\nu))$$

as $x_\nu \to x$ and $y_\nu \to y$ in Ω as $\nu \to \infty$. Consequently, introducing u_ν: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and u: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the formula

$$u_\nu(x,y) := A(d_{\Sigma_\nu}(x,y)), \quad u^-(x,y) := A(d_\Sigma(x,y)),$$

we can formulate the previous assertion as follows:

$$\Gamma^- \lim_\nu u_\nu \geqslant u^-. \tag{4.2}$$

By the choice of Γ_ν, the sequence $\{\Gamma_\nu(\mathbb{R}^n \times \mathbb{R}^n)\}_{\nu=1}^\infty$ is bounded and Γ_ν are concentrated in the compact set

$$(\operatorname{supp} \varphi^+ \cup \Sigma_\nu) \times (\operatorname{supp} \varphi^- \cup \Sigma_\nu) \subset \Omega \times \Omega.$$

Hence we can conclude that up to a subsequence (again, not relabeled) $\Gamma_\nu \rightharpoonup \Gamma$ $*$-weakly in the sense of measures, where Γ is a Borel measure on $\mathbb{R}^n \times \mathbb{R}^n$. By Lemma 2.3, Γ is an admissible transport plan for the problem $MK(\varphi^+, \varphi^-, \Sigma)$. Taking into account (4.2) and applying Lemma 4.2 with $X := \mathbb{R}^n \times \mathbb{R}^n$ and $\eta_\nu := \Gamma_\nu$, $\eta := \Gamma$, we find

$$\begin{aligned} I_\Sigma(\Gamma) := \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} u^-(x,y)\, d\Gamma(x,y) &\leqslant \liminf_\nu \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} u_\nu(x,y)\, d\Gamma_\nu(x,y) \\ &= \liminf_\nu I_{\Sigma_\nu}(\Gamma_\nu) = \liminf_\nu MK(\varphi^+, \varphi^-, \Sigma_\nu) = m. \end{aligned}$$

It remains to note that

$$MK(\varphi^+, \varphi^-, \Sigma) \leqslant I_\Sigma(\Gamma).$$

Consequently, $MK(\varphi^+, \varphi^-, \Sigma) = m$, i.e., Σ is a minimum point of the functional under consideration. $\square$

Lemma 4.2. *We assume that a sequence of nonnegative Borel functions $\{u_\nu\}$: $X \to \mathbb{R}$ defined on a local-compact or Σ-compactly metric space X, satisfies the inequality*

$$\Gamma^- \liminf_\nu u_\nu \geqslant u^-. \tag{4.3}$$

Then for any sequence of Borel measures $\eta_\nu \rightharpoonup \eta$ converging in the $*$-weak topology of measures we have

$$\int\limits_X u^- \, d\eta \leqslant \liminf_\nu \int\limits_X u_\nu \, d\eta_\nu.$$

Proof. It is easy to check that

$$\Gamma^- \liminf u_\nu = \sup_{\nu \in \mathbb{N}} \tau_\nu, \text{where}\, \tau_\nu := \big(\inf_{m \geqslant \nu} u_m\big)^-. \tag{4.4}$$

We fix $j \in \mathbb{N}$ and estimate

$$\begin{aligned} \liminf_{\nu\to\infty} \int\limits_X u_\nu \, d\eta_\nu &\geqslant \liminf_{\nu\to\infty} \int\limits_X \big(\inf_{m \geqslant j} u_m\big)\, d\eta_\nu \\ &\geqslant \liminf_{\nu\to\infty} \int\limits_X \big(\inf_{m \geqslant j} u_m\big)^- \, d\eta_\nu = \liminf_{\nu\to\infty} \int\limits_X \tau_j \, d\eta_\nu \geqslant \int\limits_X \tau_j \, d\eta. \end{aligned}$$

Since the last assertion is valid for any $j \in \mathbb{N}$, taking into account the Beppo Levi theorem and (4.4), we arrive at the required assertion. $\square$

We consider an important case of Problem 1.1 with $\varphi^- = 0$. It is obvious that $MK(\varphi^+, \varphi^-, \cdot) = F_{\varphi^+}(\cdot)$, where the functional $F_\varphi(\cdot)$ (φ is a finite nonnegative Borel measure), called the *average distance functional*, is defined by the formula

$$F_\varphi(\Sigma) := \int\limits_{\mathbb{R}^n} A(\operatorname{dist}(x, \Sigma))\, d\varphi(x).$$

Thus, Problem 1.1 is reduced to Problem 1.2 of minimizing the average distance functional with $\varphi := \varphi^+$.

In the further study of solutions to Problems 1.1 and 1.2, we assume the following conditions on A:

(α_1) A is Lipschitz continuous on $[0,L]$, i.e., there exists a constant $\Lambda > 0$ such that

$$|A(x) - A(y)| \leqslant \Lambda |x - y|$$

if $\{x,y\} \subset [0,L]$, where $L := \operatorname{diam} \operatorname{supp} \varphi$ for Problem 1.2 and $L := \operatorname{diam} (\operatorname{supp} \varphi^+ \cup \operatorname{supp} \varphi^-)$ for Problem 1.1;

(α_2) for any $c > 0$ there is a number $\lambda = \lambda(c) > 0$ such that

$$|A(x) - A(y)| \geqslant \lambda |x - y|$$

for $\{x,y\} \subset [c, \operatorname{diam} \operatorname{supp} \varphi]$. This means that the function A is injective (i.e., strictly increasing since A is nondecreasing) on $[0, \operatorname{diam} \operatorname{supp} \varphi]$.

These conditions are satisfied, for example, by the function $A(t) := t^p$, $p \geqslant 1$. Hence the results concerning Problem 1.2 under these conditions will be applicable, in particular, to the average distance functionals

$$F_\varphi(\Sigma) := \int \operatorname{dist}^p(x, \Sigma)\, d\varphi(x), \quad p \geqslant 1.$$

§ 5. Topological Properties of the Average Distance Minimizers

In this section, we consider the main topological properties of average distance minimizers. For this purpose, we recall some notions.

Definition 5.1. Let Σ be a connected topological space. A point $x \in \Sigma$ is called a *noncut point* of Σ if $\Sigma \setminus \{x\}$ is connected. Otherwise, x is called a *cut point* of Σ.

We will use the notion of the order of space at a point and also the notions of branching points and endpoints.

Definition 5.2. Let Σ be a topological space. We say that the order of Σ at a point $x \in \Sigma$ does not exceed $\mathfrak{n}$ and write

$$\operatorname{ord}_x \Sigma \leqslant \mathfrak{n},$$

where $\mathfrak{n}$ is a cardinal, if for any $\varepsilon > 0$ there is an subset $U \subset \Sigma$ such that $x \in U$, $\operatorname{diam}(U) < \varepsilon$, and $\#\partial U \leqslant \mathfrak{n}$, where # denotes the cardinality.

We say that the order of Σ at a point $x \in \Sigma$ is equal to $\mathfrak{n}$ and write

$$\operatorname{ord}_x \Sigma = \mathfrak{n}$$

if $\mathfrak{n}$ is the least cardinal such that $\operatorname{ord}_x \Sigma \leqslant \mathfrak{n}$.

A point x is called a *branching point* of Σ if $\operatorname{ord}_x \Sigma = \mathfrak{n}$, $\mathfrak{n} \geqslant 3$. A point x is called an *endpoint* of Σ if $\operatorname{ord}_x \Sigma = 1$.

By [**21**, Theorem V.1], for $\operatorname{ord}_x \Sigma \leqslant 1$ the point x is a noncut point of Σ. In particular, each endpoint of Σ is a noncut point.

At a point $x \in \Sigma := \bigcup_{i=1}^{k} [x, a_i]$ in the union of k arcs $[x, a_1], \ldots, [x, a_k]$ in Σ intersecting only at the point x, i.e., $[x, a_i] \cap [x, a_j] = \{x\}$ for all $i, j = 1, \ldots, k$, the space Σ has order k. The converse assertion is also true (cf. [**21**, Sec. 51.II, Assertion 8]).

Proposition 5.3 (Menger n-Beinsatz)**.** *Let $x \in \Sigma$, where Σ is a locally connected continuum. Then*

$$\operatorname{ord}_x \Sigma \geqslant k, \quad k \in \mathbb{N},$$

implies the existence of at least k arcs $\{\Sigma_i\}_{i=1}^k$, $\Sigma_i \subset \Sigma$, $i = 1,\dots,k$, starting at the point x and satisfying the condition $\Sigma_i \cap \Sigma_j = \{x\}$ for all $i, j = 1,\dots,k$.

We recall the following well-known notion.

Definition 5.4. By the *Hausdorff dimension* $\dim \varphi$ of a Borel measure φ on $\mathbb{R}^n$ we mean

$$\dim \varphi := \sup\{k\colon\ \varphi \ll \mathscr{H}^k\},$$

where $\mathscr{H}^k$ is the k-dimensional Hausdorff measure.

For example, $\varphi \ll \mathscr{L}^n$ implies $\dim \varphi = n$.

Let Σ_{opt} be a solution to Problem 1.2. We will denote by $k\colon \mathbb{R}^n \to \Sigma_{\mathrm{opt}}$ a Borel projections on Σ_{opt} (i.e., a Borel mapping such that $d(x,k(x)) = \operatorname{dist}(x,\Sigma_{\mathrm{opt}})$ for all $x \in \mathbb{R}^n$) and $\psi := k_\# \varphi$.

The following assertion about topological properties of the average distance minimizers valid for $n \geqslant 2$, generalizes the result of [**6**] to the case $\varphi \ll \mathscr{L}^n$ and, in addition, some results of [**7**].

Theorem 5.5. *Let Σ_{opt} be a solution to Problem 1.2, where A satisfies conditions (α_1) and (α_2) and $\varphi(\Sigma_{\mathrm{opt}}) = 0$ (the last conditions is satisfied, for example, if $\dim \varphi > 1$ and, in particular, if $\varphi \ll \mathscr{L}^n$). Then*

(i) *Σ_{opt} does not contain simple closed curves (homeomorphic images of S^1) and, in particular, every noncut point of Σ_{opt} is an endpoint.*

In addition, we assume that there exists a point $y \in \Sigma_{\mathrm{opt}}$ such that $\psi(\{y\}) > 0$. Then there exists a constant $C > 0$ such that for any noncut point $x \in \Sigma_{\mathrm{opt}}$ we have $\psi(\{x\}) \geqslant C$. In this case,

(ii) *the number of noncut points (consequently, endpoints) of Σ_{opt} is finite,*
(iii) *there are finitely many branching points of Σ_{opt},*
(iv) *every branching point of Σ_{opt} is a triple point, i.e., $\operatorname{ord}_x \Sigma_{\mathrm{opt}} = 3$.*

Remark 5.6. As was shown in [**7**], for $n = 2$ assertion (i) is equivalent to the assertion that the set $\mathbb{R}^2 \setminus \Sigma_{\mathrm{opt}}$ is connected.

In the proof of this and following assertions, we will use the following notion. By a *centered Steiner net* of a finite set of points $N \subset \partial B_r(x)$ we mean any set $St(N,x,r)$ minimizing the length functional

$$\Sigma \mapsto \mathscr{H}^1(\Sigma)$$

over all closed connected sets $\Sigma \subset \overline{B}_r(x)$ such that $N \cup \{x\} \subset \Sigma$ (the existence of such a set is known). We note that such a set is not necessarily unique (for example we can take the set N of vertices of a square). It is easy to estimate the length of the centered Steiner net consisting of two points. Namely, if $N = \{a,b\} \subset \partial B_1(0)$ and the minimal arc in $\partial B_1(0)$ joining a with b has angle $\delta < 2\pi/3$, then

$$\mathscr{H}^1(St(N,0,1)) \leqslant 2\sin(\delta/2 + \pi/6). \tag{5.1}$$

Indeed, by definition,

$$\mathscr{H}^1(St(N,0,1)) \leqslant \mathscr{H}^1(\Sigma),$$

where Σ denotes the union of three segments joining a, b, and x at $2\pi/3$ angles, so that (5.1) is obtained by a direct calculation of $\mathscr{H}^1(\Sigma)$.

Now we are ready to prove Theorem 5.5.

Proof. The fact that there are no simple closed curves in $\Sigma_{\rm opt}$ was proved in [**7**]. By Proposition 5.3, if Σ is a locally connected continuum containing no simple closed curves (homeomorphic images of S^1), then every noncut point of Σ is an endpoint. Indeed, if $x \in \Sigma$ is a noncut point and $\mathrm{ord}_x\Sigma \geqslant 2$, then there exist at least two arcs $[x,a_1] \subset \Sigma$ and $[x,a_2] \subset \Sigma$ such that $[x,a_1] \cap [x,a_2] = \{x\}$. However, $\Sigma \setminus \{x\}$ is arcwise connected and, consequently, there is an arc $\Sigma \subset \Sigma \setminus \{x\}$ connecting a_1 and a_2. Then $[x,a_1] \circ \Sigma \circ [a_2,x] \subset \Sigma$ is a simple closed curve (hereinafter, the symbol $\circ$ denotes the composition of curves). Assertion (i) is proved.

Suppose that $\psi(\{y\}) > 0$ and $x \in \Sigma_{\rm opt}$ is an arbitrary noncut point of $\Sigma_{\rm opt}$, $x \neq y$. Such a point exists because, by [**21**, Sec. 47, Theorem IV.5], any continuum (a compact connected space) contains at least two noncut points. We begin by proving the inequality

$$\psi(\{x\}) \geqslant C\psi(\{y\}) \tag{5.2}$$

with some $C > 0$ independent of x. For this purpose, we consider the decreasing sequence $\{D_\nu\}_{\nu\in\mathbb{N}}$ of subsets of $\Sigma_{\rm opt}$ defined in Lemma 5.10 with $x \in D_\nu$. Assume that the diameters of D_ν are so small that $y \notin D_\nu$ for all $\nu \in \mathbb{N}$. Introduce the notation $\varepsilon_\nu := \mathrm{diam}\, D_\nu$ and note that $\mathscr{H}^1(D_\nu) \geqslant \varepsilon_\nu$. For the sake of brevity, we often omit the subscript ν and write simply, for example, ε instead of ε_ν.

Let $\Sigma^1_\varepsilon := \Sigma_{\rm opt} \setminus D_\nu$. Then

$$F_\varphi(\Sigma^1_\varepsilon) = \int\limits_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\Sigma^1_\varepsilon))\,d\varphi(x) \leqslant \int\limits_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\Sigma_{\rm opt}))\,d\varphi(x) + \Lambda\varepsilon\psi(D_\nu) = F_\varphi(\Sigma_{\rm opt}) + \Lambda\varepsilon\psi(D_\nu).$$

Taking into account that $y \notin D_\nu$ for all $\nu \in \mathbb{N}$ and setting $T'_y := k^{-1}(y) \setminus k^{-1}(D_1) = k^{-1}(y)$, we find $\varphi(T'_y) = \psi(\{y\})$. Using Lemma 5.7, we find a compact connected set $\Sigma^2_\varepsilon \supset \Sigma^1_\varepsilon$ such that

$$\mathscr{H}^1(\Sigma^2_\varepsilon) \leqslant \mathscr{H}^1(\Sigma^1_\varepsilon) + \varepsilon \leqslant \mathscr{H}^1(\Sigma_{\rm opt})$$

and for all sufficiently small $\varepsilon > 0$ we have the inequality

$$\begin{aligned} F_\varphi(\Sigma^2_\varepsilon) &= \int\limits_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\Sigma^2_\varepsilon))\,d\varphi(x) \\ &\leqslant \int\limits_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\Sigma^1_\varepsilon))\,d\varphi(x) - C\psi(\{y\})\varepsilon = F_\varphi(\Sigma^1_\varepsilon) - C\psi(\{y\})\varepsilon, \end{aligned}$$

where $C > 0$ is independent of ε, y, and x. Thus,

$$F_\varphi(\Sigma^2_\varepsilon) \leqslant F_\varphi(\Sigma_{\rm opt}) + \varepsilon\psi(D_\nu) - C\varepsilon\psi(\{y\}).$$

Since $\varepsilon \to 0$ and $\psi(D_\nu) \to \psi(\{x\})$ as $\nu \to \infty$, we see that the last inequality contradicts the optimality of $\Sigma_{\rm opt}$ for sufficiently large ν only if the inequality (5.2) fails.

Since x is chosen arbitrarily, for any noncut point $x \in \Sigma_{\rm opt}$ we have $\psi(\{x\}) > C$, where $C > 0$ is some constant independent of x. The fact that the set of noncut points of $\Sigma_{\rm opt}$ (assertion (ii)) is finite follows from the relation $\psi(\mathbb{R}^n) = \varphi(\mathbb{R}^n) < +\infty$, whereas the set of branching points of $\Sigma_{\rm opt}$ (assertion (iii)) is finite because of Lemma 5.11.

It remains to prove assertion (iv). Assume the contrary. Suppose that $\mathrm{ord}_x\Sigma_{\rm opt} > 3$ for some $x \in \Sigma_{\rm opt}$. By Proposition 5.3, there is at least four closed arcs $\Sigma_i \subset \Sigma_{\rm opt}$, $i = 1,\dots,4$, that start at the point x and do not meet at other points (i.e., $\Sigma_i \cap \Sigma_j = \{x\}$ for $i \neq j$). We choose a sufficiently small number $\varepsilon = \varepsilon_\nu := 1/\nu$ (where $\nu \in \mathbb{N}$) such that the closed ball $\overline{B}_\varepsilon(x)$ does not contain any other branching points and noncut points (this is possible because the number of such points is finite by assertions (ii) and (iii)) and

$$\Sigma_i \cap \partial B_\varepsilon(x) \neq \varnothing, \quad i = 1,\dots,4.$$

Let $a^i_\varepsilon \in \Sigma_i \cap \partial B_\varepsilon(x)$ denote the first point at which Σ_i tangents $\partial B_\varepsilon(x)$, i.e., for all $y \in (x,a^i_\varepsilon)$ we have $y \in B_\varepsilon(x)$, $i = 1,\dots,4$. By Lemma 5.9(ii), among these four points there are at least two points (without loss of generality we denote them by a_ε^1 and a_ε^2) such that the minimal arc joining them in $\partial B_\varepsilon(x)$ has angle $\delta < 2\pi/3$.

Introduce the notation

$$\Sigma_\varepsilon^1 := \bigl(\Sigma_{\rm opt} \setminus ((x,a_\varepsilon^1) \cup (x,a_\varepsilon^2))\bigr) \cup St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon).$$

The set Σ_ε^1 is connected since the removed arcs did not contain any branching point of the set $\Sigma_{\rm opt}$ by construction. It is obvious that this set is compact. We have

$$\mathscr{H}^1(\Sigma_\varepsilon^1) \leqslant \mathscr{H}^1(\Sigma_{\rm opt}) - \mathscr{H}^1((x,a_\varepsilon^1)) - \mathscr{H}^1((x,a_\varepsilon^2)) + \mathscr{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)).$$

Note that $\mathscr{H}^1((x,a_\varepsilon^i)) \geqslant \varepsilon$, $i = 1,2$, and

$$\mathscr{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)) = \varepsilon \mathscr{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1)).$$

By (5.1), we have

$$\mathscr{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1) \leqslant 2 - \beta,$$

where $\beta > 0$ depends only on δ. Combining the above estimates, we find

$$\mathscr{H}^1(\Sigma_\varepsilon^1) \leqslant \mathscr{H}^1(\Sigma_{\rm opt}) - \beta\varepsilon.$$

Furthermore, we have

$$\begin{aligned} F_\varphi(\Sigma_\varepsilon^1) &= \int_{\mathbb{R}^n} A(\operatorname{dist}(x, \Sigma_\varepsilon^1))\, d\varphi(x) \\ &\leqslant \int_{\mathbb{R}^n} A(\operatorname{dist}(x, \Sigma_{\rm opt}))\, d\varphi(x) + \Lambda\varepsilon\psi(B_\varepsilon(x) \setminus \{x\}) = F_\varphi(\Sigma_{\rm opt}) + \Lambda\varepsilon\psi(B_\varepsilon(x) \setminus \{x\}). \end{aligned} \tag{5.3}$$

Let z be a noncut point of $\Sigma_{\rm opt}$. Then, as was just proved, $\psi(\{z\}) > 0$. We choose $\nu' \in \mathbb{N}$ such that $z \notin B_{\varepsilon'}(x)$, where $\varepsilon' := \varepsilon_{\nu'}$. Setting $T_y' := k^{-1}(z) \setminus k^{-1}(B_{\varepsilon'}(x))$, we find $T_z' = k^{-1}(z)$. Consequently, $\varphi(T_z') = \psi(\{z\})$. Using Lemma 5.7, we obtain a compact connected set $\Sigma_\varepsilon^2 \supset \Sigma_\varepsilon^1$ such that

$$\mathscr{H}^1(\Sigma_\varepsilon^2) \leqslant \mathscr{H}^1(\Sigma_\varepsilon^1) + \varepsilon \leqslant \mathscr{H}^1(\Sigma_{\rm opt})$$

and

$$\begin{aligned} F_\varphi(\Sigma_\varepsilon^2) &= \int_{\mathbb{R}^n} A(\operatorname{dist}(\xi, \Sigma_\varepsilon^2))\, d\varphi(\xi) \\ &\leqslant \int_{\mathbb{R}^n} A(\operatorname{dist}(\xi, \Sigma_\varepsilon^1))\, d\varphi(\xi) - C\psi(\{z\})\varepsilon = F_\varphi(\Sigma_\varepsilon^1) - C\psi(\{z\})\varepsilon, \end{aligned} \tag{5.4}$$

where $C > 0$ is independent of ε. Since $\psi(B_\varepsilon(x) \setminus \{x\}) = o(1)$ as $\varepsilon \to 0$, from (5.3) and (5.4) it follows that for sufficiently small $\varepsilon > 0$

$$F_\varphi(\Sigma_\varepsilon^2) < F_\varphi(\Sigma_{\rm opt}).$$

This estimate, together with the inequality $\mathscr{H}^1(\Sigma_\varepsilon^2) \leqslant \mathscr{H}^1(\Sigma_{\rm opt})$, contradicts the optimality of $\Sigma_{\rm opt}$. $\square$

The following lemmas were used in the above proof.

Lemma 5.7. *Let $\Sigma_\nu \subset \mathbb{R}^n$ be a sequence of compact connected sets such that $\Sigma := \cap_\nu \Sigma_\nu \neq \varnothing$, and let φ be a positive Borel measure on $\mathbb{R}^n$ such that $\varphi(\Sigma) = 0$. Assume that for some $x \in \Sigma$ there exists a Borel set $T_x' \subset \mathbb{R}^n$ such that $\varphi(T_x') > 0$ and there is a number $\nu_0 \in \mathbb{N}$ such that for any point $y \in T_x'$*

$$0 < \operatorname{dist}(y, \Sigma) = d(y,x) < \operatorname{dist}(y, \Sigma_\nu \setminus \Sigma),$$

provided that $\nu \geqslant \nu_0$. Then, if condition (α_2) is satisfied, for any $\nu \in \mathbb{N}$ and a sufficiently small $\varepsilon > 0$ there is a compact connected set $\Sigma_{\nu,\varepsilon} \supset \Sigma_\nu$ such that $\mathscr{H}^1(\Sigma_{\nu,\varepsilon}) \leqslant \mathscr{H}^1(\Sigma_\nu) + \varepsilon$ and

$$F_\varphi(\Sigma_{\nu,\varepsilon}) \leqslant F_\varphi(\Sigma_\nu) - C\varphi(T_x')\varepsilon,$$

where the constant $C > 0$ is independent of ν and ε.

Proof. Using Lemma 5.8, for every $\delta > 0$ we can construct a compact connected set Γ'_δ on the surface $\partial I^n_\delta(x)$ of the n-dimensional cube $I^n_\delta(x)$ with side of length 4δ and center x such that

$$\mathscr{H}^1(\Gamma'_\delta) \leqslant C\delta, \quad \operatorname{dist}(y, \Gamma'_\delta) \leqslant \delta$$

for all $y \in \partial I^n_\delta(x)$, where $C > 0$ depends only on n. Denote by Γ_δ the union of sets Γ'_δ with the shortest line segment joining the latter with the point x. It is obvious that the set Γ_δ is compact and connected; moreover, $\mathscr{H}^1(\Gamma_\delta) \leqslant C'\delta$, where $C' > 0$ depends only on n.

We set $\varepsilon := C'\delta$ and

$$\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup \Gamma_\delta.$$

The set $\Sigma_{\nu,\varepsilon}$ is compact and connected; moreover, $\mathscr{H}^1(\Sigma_{\nu,\varepsilon}) \leqslant \mathscr{H}^1(\Sigma_\nu) + \varepsilon$. We set

$$\Sigma^c := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Sigma) \leqslant c\}$$

and note that there is $c > 0$ such that $\varphi(T'_x \setminus \Sigma^c) \geqslant \varphi(T'_x)/2$ (since $\varphi(T'_x \cap \Sigma^c) \to \varphi(T'_x \cap \Sigma) = 0$ as $c \downarrow 0$ by the assumption $\varphi(\Sigma) = 0$). We fix this number $c > 0$ and set $\varphi_c := \varphi \llcorner \mathbb{R}^n \setminus \Sigma^c$ and $\varphi'_c := \varphi \llcorner \Sigma^c$.

If $2\sqrt{n}\delta < c$ (i.e., $\Gamma_\delta \subset \partial I^n_\delta(x) \subset \Sigma^c$), then for any $y \in T'_x \setminus \Sigma^c$ and $\nu \geqslant \nu_0$ we have $\operatorname{dist}(y, \Sigma_\nu) = d(y, x)$; moreover,

$$\operatorname{dist}(y, \Sigma_{\nu,\varepsilon}) \leqslant \operatorname{dist}(y, \Gamma_\delta) \leqslant d(y,x) - 2\delta + \delta.$$

Consequently,

$$\operatorname{dist}(y, \Sigma_{\nu,\varepsilon}) \leqslant \operatorname{dist}(y, \Sigma_\nu) - C\varepsilon$$

for some constant $C > 0$ independent of ε and ν. Using (α_2), we find

$$\begin{aligned} \int_{\mathbb{R}^n} A(\operatorname{dist}(z, \Sigma_{\nu,\varepsilon}))\, d\varphi_c(z) - \int_{\mathbb{R}^n} A(\operatorname{dist}(z, \Sigma_\nu))\, d\varphi_c(z) &\leqslant -\lambda C \varepsilon \varphi_c(T'_x) \\ &\leqslant -C\varphi\left(T'_x\right)\varepsilon \end{aligned}$$

(where $\lambda = \lambda(c/2)$ and $C > 0$ can be different from line to line, but always independent of ν and ε). On the other hand, we have

$$\int_{\mathbb{R}^n} A(\operatorname{dist}(z, \Sigma_{\nu,\varepsilon}))\, d\varphi'_c(z) \leqslant \int_{\mathbb{R}^n} A(\operatorname{dist}(z, \Sigma_\nu)) d\varphi'_c(z).$$

Adding the above inequalities and taking into account that $\varphi = \varphi_c + \varphi'_c$, we obtain at the required assertion. □

Lemma 5.8. *For any $j \in \mathbb{N}$ there exists a compact connected set $\Gamma^j \subset I^n := [0,1]^n$ satisfying the conditions $\mathscr{H}^1(\Gamma^j) \leqslant n(j+1)^{n-1}$ and $\operatorname{dist}(y, \Gamma^j) \leqslant \sqrt{n}/2j$ for all $y \in I^n$.*

Proof. Let Γ^j be the set of all $(x_1, \ldots, x_n) \in I^n$ such that $jx_i \in \mathbb{N}$ for all $i = 1, \ldots, n$ except, perhaps, one, i.e., Γ^j is a uniform one-dimensional grid with step $1/j$ in I^n. The required properties of Γ^j can be verified directly. □

Lemma 5.9. *The following assertions hold.*

(i) *Among three vectors in $\mathbb{R}^n$ that do not belong to the same two-dimensional plane, there are two vectors such that the angle between them is strictly less than $2\pi/3$.*

(ii) *Among four vectors in $\mathbb{R}^n$ there are two vectors such that the angle between them is strictly less than $2\pi/3$.*

Proof. To prove (i), we note that we can assume $n = 3$ without loss of generality (since any three vectors belong to a three-dimensional subspace of $\mathbb{R}^n$). Let a, b, and c be arbitrary unit vectors in $\mathbb{R}^3$. We choose the coordinate axes in such a way that $a = (0,0,1)$ and $b = (b_1,0,b_3)$. Assume that the angle between a and b, as well as between a and c, is not less than $2\pi/3$ (otherwise, there is nothing to prove). If $c_2 = 0$, then all three vectors belong to the same two-dimensional plane $\{x_2 = 0\}$. Otherwise (i.e., $c_2 \neq 0$), we set $c' := (c_1,0,c_3)$. Since the angle between a and c is at least $2\pi/3$, we have $c_3 \leqslant -1/2$. We can assume that $c_1^2 + c_3^2 \neq 0$; otherwise, c is perpendicular to a, which contradicts the assumption. Thus, for the angle α' between a and c' we find

$$\cos\alpha' = \frac{c_3}{\sqrt{c_1^2 + c_3^2}} < c_3 \leqslant -\frac{1}{2}$$

since $\sqrt{c_1^2 + c_3^2} < 1$. Hence $\alpha' > 2\pi/3$. Thus, the angle δ' between b and c' is strictly less than $2\pi/3$ (since a, b, and c' belong to the same two-dimensional plane $\{x_2 = 0\}$), which implies $\cos\delta' > -1/2$. On the other hand,

$$\cos\delta' = \frac{b_1c_1 + b_3c_3}{\sqrt{c_1^2 + c_3^2}} = \frac{\cos\delta}{\sqrt{c_1^2 + c_3^2}}.$$

We assume that $\cos\delta < 0$ (otherwise, $\delta \leqslant \pi/2$ and there is nothing to prove). Then $\cos\delta > \cos\delta'$. Consequently, $\cos\delta > -1/2$. Hence $\delta < 2\pi/3$, which completes the proof of assertion (i).

To prove (ii), we can assume without loss of generality that $n = 4$ (since any four vectors in $\mathbb{R}^n$ belong to a four-dimensional subspace). We consider an arbitrary triple of vectors among the given four vectors in $\mathbb{R}^4$. By assertion (i), this triple either contains two vectors such that the angle between them is strictly less than $2\pi/3$ (and then there is nothing to prove) or lies inside a two-dimensional plane . Using assertion (i) again, in the last case, we find that either the angle between the fourth vector and some of the vectors of the triple is strictly less than $2\pi/3$ or all the four vectors belong to the same two-dimensional plane. Consequently, among them there is a pair of vectors forming an angle strictly less than $2\pi/3$. □

The following two topological lemmas are taken from [**22**] and [**6**] respectively.

Lemma 5.10. *Let Σ be a locally connected metric continuum containing more than one point, and let $x \in \Sigma$ be a noncut point of Σ. Then there exists a sequence of open sets $D_\nu \subset \Sigma$ satisfying the following conditions*:

(i) *$x \in D_\nu$ for all sufficiently large ν,*
(ii) *$\Sigma \setminus D_\nu$ is connected for every ν,*
(iii) $\operatorname{diam} D_\nu \searrow 0$ *as $\nu \to \infty$,*
(iv) *D_ν is connected for every ν.*

Lemma 5.11. *Let Σ be a local connected continuum containing no simple closed curves* (*homeomorphic images of S^1*). *If the set of endpoints of Σ is finite, then the set of branching points of Σ is also finite. Furthermore, in this case, for every branching point $x \in \Sigma$ the order* $\operatorname{ord}_x\Sigma$ *is finite.*

§ 6. Characterization of Branching Points via the Mean Curvature

Now, we can more exactly characterize branching points of the average distance minimizer Σ_{opt} (i.e., solutions to Problem 1.2). According to the results of numerical simulation in [**5**], we expect the following result: for at least a sufficiently good measure φ (i.e., if its density is summable with a suitable exponent relative to the Lebesgue measure) all the branching points of Σ_{opt} are *regular tripods*, i.e., points where exactly three smooth branches meet at $120°$ angles. However, even if we do not possess results concerning the strong regularity of $\Sigma_{\rm opt}$ up to branching points, we can prove a weaker result. For this purpose, we recall the notion of the generalized average curvature in [**23**]. By the *generalized mean curvature* H_Σ of a countably $(\mathscr{H}^k,k)$-rectifiable set $\Sigma\subset\mathbb{R}^n$ (or measure $\mathscr{H}^k\llcorner\Sigma$ in the terminology of [**23**]) we mean the vector-valued distribution

$$\langle X, H_\Sigma\rangle := \int\limits_\Sigma \operatorname{div}^\Sigma X\, d\mathscr{H}^k$$

for all $X\in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$. The mean curvature H_Σ is a Radon charge (a signed measure) if

$$|H_\Sigma|(D) := \sup\left\{\int\limits_{\Sigma\cap D} \operatorname{div}^\Sigma X\, d\mathscr{H}^k : X\in C_0^\infty(D;\mathbb{R}^n), ||X||_\infty = 1\right\}$$

is finite for any open set $D\subset\mathbb{R}^n$. Denote by H the generalized mean curvature of the average distance minimizer $\Sigma_{\rm opt}$. The following assertion holds.

Proposition 6.1. *Under the assumptions of Theorem 5.5, the generalized mean curvature H is a Radon charge*; *moreover, $H\ll\psi$.*

Proof. By Theorem 5.5, the set of noncut points of $\Sigma_{\rm opt}$ is finite. Therefore, it suffices to prove the existence of a constant $C>0$ such that

$$|H|(D)\leqslant C\psi(D) \tag{6.1}$$

for any open set $D\subset\mathbb{R}^n$ such that its closure $\overline{D}$ does not contain any noncut point of $\Sigma_{\rm opt}$. To prove this assertion, we assume that $|H|(D)\neq 0$ (otherwise, there is nothing to prove). For $m\in\mathbb{N}$ we consider a vector field $X_m\in C_0^\infty(D;\mathbb{R}^2)$ such that $||X_m||_\infty = 1$ and

$$\left|\int\limits_{\Sigma_{\rm opt}\cap D} \operatorname{div}^{\Sigma_{\rm opt}} X_\nu\, d\mathscr{H}^1 - |H|(D)\right| \leqslant 1/m.$$

For every $\varepsilon\in\mathbb{R}$ and $m\in\mathbb{N}$ we introduce a diffeomorphism Φ_ε^m by the formula

$$\Phi_\varepsilon^m(x) := x-\varepsilon X_m(x).$$

Let $\Sigma_\varepsilon^m := \Phi_m^\varepsilon(\Sigma_{\rm opt})$. By [**24**, Theorem 7.31],

$$\frac{d}{d\varepsilon}\mathscr{H}^1(\Sigma_\varepsilon^m) = \int\limits_{\Sigma_{\rm opt}\cap D} \operatorname{div}^{\Sigma_{\rm opt}} X_m\, d\mathscr{H}^1.$$

Consequently,

$$\delta := \mathscr{H}^1(\Sigma_{\rm opt}) - \mathscr{H}^1(\Sigma_\varepsilon^m) = \varepsilon \int\limits_{\Sigma_{\rm opt}\cap D} \operatorname{div}^{\Sigma_{\rm opt}} X_m\, d\mathscr{H}^1 + o(\varepsilon).$$

We assume that m is sufficiently large and set $\varepsilon=\varepsilon_\nu := 1/\nu$ (we will omit the subscript ν for brevity) for $\nu\in\mathbb{N}$, so that the set $\{\Sigma_\varepsilon^m\}$ is countable. In this case,

$$\delta\geqslant\varepsilon(|H|(D)-1/m)+o(\varepsilon)>0 \tag{6.2}$$

for small $\varepsilon>0$. Then

$$\begin{aligned} F_\varphi(\Sigma_\varepsilon^m) &= \int\limits_{\mathbb{R}^n} A(\operatorname{dist}(x,\Sigma_\varepsilon^m))\, d\varphi(x) \\ &\leqslant \int\limits_{\mathbb{R}^n} A(\operatorname{dist}(x,\Sigma_{\rm opt}))\, d\varphi(x) + \Lambda\varepsilon\psi(D) = F_\varphi(\Sigma_{\rm opt}) + \Lambda\varepsilon\psi(D). \end{aligned} \tag{6.3}$$

Let $y \in \Sigma_{\rm opt}$ be an arbitrary endpoint. If $\varepsilon > 0$ is sufficiently small, then, by the condition on D, we have $y \notin D_\varepsilon$, where

$$D_\varepsilon := \{x \in \mathbb{R}^n : \mathrm{dist}\,(x,D) \leqslant \varepsilon\}.$$

Consequently, denoting $T'_y := k^{-1}(y) \setminus k^{-1}(D_\varepsilon) = k^{-1}(y)$, we get $\varphi(T'_y) = \psi(\{y\})$. Using Lemma 5.7, we obtain a compact connected set $\widetilde{\Sigma}^m_\varepsilon \supset \Sigma^m_\varepsilon$ satisfying the condition

$$\mathscr{H}^1(\widetilde{\Sigma}^m_\varepsilon) \leqslant \mathscr{H}^1(\Sigma^m_\varepsilon) + \delta = \mathscr{H}^1(\Sigma_{\rm opt});$$

moreover, for a sufficiently small $\varepsilon > 0$ the following estimate holds:

$$\begin{aligned} F_\varphi(\widetilde{\Sigma}^m_\varepsilon) &= \int_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\widetilde{\Sigma}^m_\varepsilon))\,d\varphi(x) \\ &\leqslant \int_{\mathbb{R}^n} A(\mathrm{dist}\,(x,\Sigma^m_\varepsilon))\,d\varphi(x) - C\psi(\{y\})\delta = F_\varphi(\Sigma^m_\varepsilon) - C\psi(\{y\})\delta \end{aligned} \tag{6.4}$$

with some constant $C > 0$ independent of ε and m.

The inequalities (6.3) and (6.4) yield

$$F_\varphi(\widetilde{\Sigma}^m_\varepsilon) \leqslant F_\varphi(\Sigma_{\rm opt}) + \Lambda\varepsilon\psi(D) - C\psi(\{y\})\delta.$$

To avoid a contradiction with the optimality of $\Sigma_{\rm opt}$, we must show that the condition $C\psi(\{y\})\delta \leqslant \varepsilon\psi(D)$ is satisfied. By (6.2), this condition implies the inequality

$$C(\varepsilon(|H|(D) - 1/m) + o(\varepsilon))\psi(\{y\}) \leqslant \varepsilon\psi(D).$$

Dividing both sides of the last inequality by ε and passing to the limit as $\nu \to \infty$ (in other words, as $\varepsilon \to 0^+$) and, after that, as $m \to +\infty$, we arrive at (6.1) (perhaps, with some other value of C). □

Using the above assertion, we can characterize branching points of the average distance minimizer $\Sigma_{\rm opt}$ as follows.

Theorem 6.2. *Under the assumptions of Theorem* 5.5, *every branching point $x \in \Sigma_{\rm opt}$ is a triple point* (*i.e.,* $\Sigma_{\rm opt}$ *has order three*); *moreover, the Euclidean dimension of the set $k^{-1}(x)$ is at most $n-2$, i.e., $k^{-1}(x)$ is contained in some $(n-2)$-dimensional hyperplane. In particular, if* $\dim\varphi > n-2$ (*for example, if $\varphi \ll \mathscr{L}^n$*), *then $H(\{x\}) = 0$.*

Proof. Let $x \in \Sigma_{\rm opt}$ be a branching point. By Theorem 5.5(iv), it is a triple point. We assume that the Euclidean dimension of the set $k^{-1}(x)$ is equal to $m \leqslant n$, i.e., $k^{-1}(x)$ is contained in some m-dimensional hyperplane $\Pi_m \subset \mathbb{R}^n$ and is not contained in any $(m-1)$-dimensional hyperplane. We prove that $m \leqslant n-2$.

By Lemma 6.3,

$$C \cap B_{r_0}(x) \subset k^{-1}(x)$$

for some convex m-dimensional cone (i.e., it cannot be reduced to a set of dimension less than m) $C \subset \Pi_m$ with vertex x and sufficiently small $r_0 > 0$. We prove that for all $y \in C$

$$\limsup_{r\to 0} \sup_{z\in\Sigma_{\rm opt}\cap\partial B_r(x)} \frac{\langle y-x, z-x\rangle}{|y-x|\cdot|z-x|} \leqslant 0. \tag{6.5}$$

Indeed, in the opposite case, for some $c > 0$ there is an infinitesimal sequence $\{r_\nu\}$, $r_\nu \to 0$ as $\nu \to \infty$ (the subscript ν will be omitted) such that for some $y_r \in C$ and $z_r \in \Sigma_{\rm opt} \cap \partial B_r(x)$ we have

$$\frac{\langle y-x, z-x\rangle}{|y-x|\cdot|z-x|} \geqslant c,$$

i.e., the angle between the vectors $y_r - x$ and $z_r - x$ does not exceed $\arccos c < \pi/2$. In the two-dimensional plane formed by the vectors $y_r - x$ and $z_r - x$, we consider the line l_r passing through z_r perpendicularly to $z_r - x$ and set $x_r := l_r \cap l'_r$, where l'_r is the line passing through x and y_r. If $r/c < r_0$, then $x_r \in B_{r_0}(x)$. Consequently, $x_r \in k^{-1}(x)$, which implies $\operatorname{dist}(x_r, \Sigma_{\text{opt}}) = d(x_r, x)$. On the other hand,

$$\operatorname{dist}(x_r, \Sigma_{\text{opt}}) \leqslant d(x_r, z_r) < d(x_r, x).$$

We arrive at a contradiction. Thus, (6.5) is proved (cf. Fig. 1).

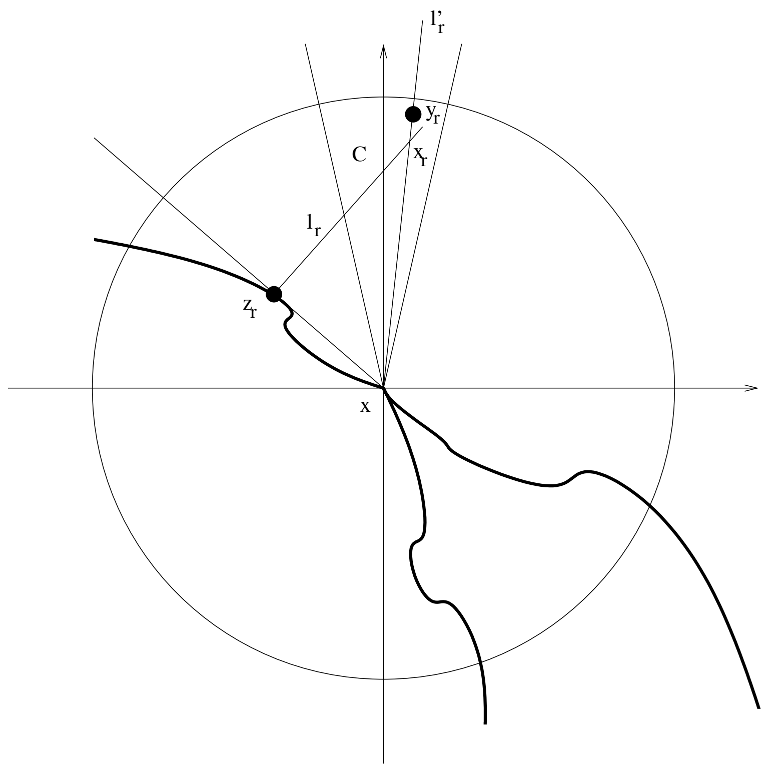

FIGURE 1. Proof of the inequality (6.5)

We argue in the same way as in the proof of Theorem 5.5(iv). Since x is a triple point, from Proposition 5.3 it follows that there are three closed arcs $\Sigma_i \subset \Sigma_{\text{opt}}$, $i = 1, \dots, 3$, that start at the point x and are pairwise disjoint outside x, i.e., $\Sigma_i \cap \Sigma_j = \{x\}$ for $i \neq j$. We assume that $\varepsilon > 0$ is so small that the closed ball $\overline{B}_\varepsilon(x)$ does not contain any other branching point of Σ_{opt} (it is possible because the set of branching points is finite in view of Theorem 5.5(iii)) and

$$\Sigma_i \cap \partial B_\varepsilon(x) \neq \varnothing, \quad i = 1, \dots, 3.$$

Let $a^i_\varepsilon \in \Sigma_i \cap \partial B_\varepsilon(x)$ denote the first point at which the arc Σ_i tangents $\partial B_\varepsilon(x)$, i.e., a point such that $y \in B_\varepsilon(x)$, $i = 1, \dots, 3$, for all $y \in (x, a^i_\varepsilon)$.

We chose a sequence $\varepsilon_\nu \to 0$ as $\nu \to \infty$ such that

$$\frac{a^i_{\varepsilon_\nu} - x}{\varepsilon_\nu} \to a^i \in \partial B_1(0), \quad i = 1, \dots, 3,$$

as $\nu \to \infty$. We will omit the subscript ν and write ε instead of ε_ν. Let us prove that the angle between any two vectors among a^i, $i = 1, \dots, 3$, is equal to $2\pi/3$. Indeed, in the opposite case, there exists a number $\delta < 2\pi/3$ such that for all sufficiently small ε there is a couple of points a^i_ε, $i = 1, \dots, 3$ (say, a^1_ε and a^2_ε), such that the minimal joining them arc in $\partial B_\varepsilon(x)$ has angle at most δ. We set

$$\Sigma^1_\varepsilon := (\Sigma_{\text{opt}} \setminus ((x, a^1_\varepsilon) \cup (x, a^2_\varepsilon))) \cup St(\{a^1_\varepsilon, a^2_\varepsilon\}, x, \varepsilon).$$

We note that the set Σ_ε^1 is connected since the "removed" arcs did not contain any branching point of Σ_{opt} by assumption. We also note that this set is closed. The estimate

$$\mathscr{H}^1(\Sigma_\varepsilon^1) \leqslant \mathscr{H}^1(\Sigma_{\text{opt}}) - \mathscr{H}^1((x,a_\varepsilon^1)) - \mathscr{H}^1((x,a_\varepsilon^2)) + \mathscr{H}^1(St(\{a_\varepsilon^1,a_\varepsilon^2\},x,\varepsilon))$$

together with the inequalities $\mathscr{H}^1((x,a_\varepsilon^i)) \geqslant \varepsilon$, $i=1,2$, and the relation

$$\mathscr{H}^1(St(\{a_\varepsilon^1,a_\varepsilon^2\},x,\varepsilon)) = \varepsilon\mathscr{H}^1(St(\{(a_\varepsilon^1-x)/\varepsilon,(a_\varepsilon^2-x)/\varepsilon\},0,1)) \geqslant \varepsilon(2-\beta)$$

where $\beta>0$ depends only on δ (the last estimate holds by (5.1)), implies

$$\mathscr{H}^1(\Sigma_\varepsilon^1) \leqslant \mathscr{H}^1(\Sigma_{\text{opt}}) - \beta\varepsilon.$$

Assume that a point $y \in \Sigma_{\text{opt}}$ satisfies the condition $\psi(\{y\})>0$. Taking into account that $y \notin B_\varepsilon(x)$ for sufficiently small $\varepsilon>0$ and setting $T_y' := k^{-1}(y)\setminus k^{-1}(B_\varepsilon(y)) = k^{-1}(y)$, we find $\varphi(T_y') = \psi(\{y\})$. Using Lemma 5.7, we find a closed connected set $\Sigma_\varepsilon^2 \supset \Sigma_\varepsilon^1$, such that

$$\mathscr{H}^1(\Sigma_\varepsilon^2) \leqslant \mathscr{H}^1(\Sigma_\varepsilon^1) + \beta\varepsilon = \mathscr{H}^1(\Sigma_{\text{opt}});$$

moreover, for any sufficiently small $\varepsilon>0$ the following estimate holds:

$$\begin{aligned} F_\varphi(\Sigma_\varepsilon^2) &= \int_{\mathbb{R}^n} A(\operatorname{dist}(z,\Sigma_\varepsilon^2))\,d\varphi(z) \\ &\leqslant \int_{\mathbb{R}^n} A(\operatorname{dist}(z,\Sigma_\varepsilon^1))\,d\varphi(z) - C\psi(\{x\})\beta\varepsilon = F_\varphi(\Sigma_\varepsilon^1) - C\varepsilon \end{aligned} \tag{6.6}$$

with some constant $C>0$ independent of ε. However,

$$\begin{aligned} F_\varphi(\Sigma_\varepsilon^1) &= \int_{\mathbb{R}^n} A(\operatorname{dist}(z,\Sigma_\varepsilon^1))\,d\varphi(z) \\ &\leqslant \int_{\mathbb{R}^n} A(\operatorname{dist}(z,\Sigma_{\text{opt}}))\,d\varphi(z) + \varepsilon\psi(B_\varepsilon(x)\setminus\{x\}) = F_\varphi(\Sigma_{\text{opt}}) + \varepsilon\psi(B_\varepsilon(x)\setminus\{x\}), \end{aligned}$$

which, together with (6.6), yields

$$F_\varphi(\Sigma_\varepsilon^2) \leqslant F_\varphi(\Sigma_{\text{opt}}) + \varepsilon\psi(B_\varepsilon(x)\setminus\{x\}) - C\varepsilon.$$

Since $\psi(B_\varepsilon(x)\setminus\{x\}) = o(1)$ as $\varepsilon\to 0^+$, the last estimate implies

$$F_\varphi(\Sigma_\varepsilon^2) < F_\varphi(\Sigma_{\text{opt}})$$

for all sufficiently small ε, which contradicts the optimality of Σ_{opt}. Thereby we complete the proof of the assertion that the angles between each pair of vectors a^i, $i=1,\dots,3$, are equal to $2\pi/3$.

By Lemma 5.9(i), the vectors a^i, $i=1,\dots,3$, belong to the same two-dimensional hyperplane. We choose the Cartesian coordinate system with the origin at x in such a way that the vectors a^i, $i=1,\dots,3$, belong to the hyperplane $\{x_3=x_4=\ldots=x_n=0\}$; moreover,

$$a^1=(1,0,0,\dots,0),\, a^2=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},0,\dots,0\right),\, a^3=\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,\dots,0\right).$$

For any $y \in C$, $y \neq x$, from (6.5) we find

$$y_1 = \langle y, a^1 \rangle = \lim_{\varepsilon \to 0^+} \frac{\langle y, a_\varepsilon^1 \rangle}{\varepsilon} \leqslant 0,$$
$$-\frac{y_1}{2} - \frac{y_2\sqrt{3}}{2} = \langle y, a^2 \rangle = \lim_{\varepsilon \to 0^+} \frac{\langle y, a_\varepsilon^2 \rangle}{\varepsilon} \leqslant 0,$$
$$-\frac{y_1}{2} + \frac{y_2\sqrt{3}}{2} = \langle y, a^3 \rangle = \lim_{\varepsilon \to 0^+} \frac{\langle y, a_\varepsilon^3 \rangle}{\varepsilon} \leqslant 0,$$

which implies $y_1 = y_2 = 0$. Hence the Euclidean dimension of C does not exceed $n-2$.

The assertion about the generalized curvature follows from Proposition 6.1 since $\psi(\{x\}) = 0$ by condition. □

The following lemma was used in the proof of Theorem 6.2.

Lemma 6.3. *Suppose that $\Sigma \subset \mathbb{R}^n$ is a closed set and $K\colon \mathbb{R}^n -\!\!\circ\, \Sigma$ is a multivalued projection on Σ. If $x \in K(\{y,z\})$, where $x \in \Sigma$, and $d(y,x) = d(z,x)$, then $x \in K(\overline{\mathrm{co}}\,\{x,y,z\})$.*

Proof. Since $x \in K(x')$ implies $x \in K(tx' + (1-t))$ for any $t \in [0,1]$, it suffices to prove that $x \in K(\overline{\mathrm{co}}\,\{y,z\})$ or, in other words, $x \in K(ty + (1-t)z)$ for any $t \in [0,1]$. We assume that $x \neq y$ (otherwise, the assertion is trivial). Suppose that the origin is at the point x and the coordinate axes are located in such a way that $y = (a,b,0,\dots,0)$, $z = (-a,b,0,\dots,0)$, $a > 0$, $b \geqslant 0$. By the assumptions of the lemma, $\Sigma \cap B_r(y) = \Sigma \cap B_r(z) = \varnothing$, where $r := \sqrt{a^2+b^2} = d(y,x) = d(z,x)$. In other words, for any point $x' = (x'_1, \dots, x'_n) \in \Sigma$

$$(x'_1 - a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 \geqslant a^2 + b^2,$$
$$(x'_1 + a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 \geqslant a^2 + b^2.$$

Then for any $x' \in \Sigma$ and $t \in [0,1]$

$$\begin{aligned} d^2(ty + (1-t)z, x') &= (x'_1 - (2t-1)a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 \\ &= (x'_1 + a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 + (2ta)^2 - 4ta(x'_1 + a) \\ &\geqslant a^2 + b^2 + (2ta)^2 - 4ta(x'_1 + a) = (2ta - a)^2 + b^2 - 4tax'_1 \\ &= ((2t-1)a)^2 + b^2 - 4tax'_1 = d^2(ty + (1-t)z, x) - 4tax'_1 \\ &\geqslant d^2(ty + (1-t)z, x), \end{aligned}$$

if $x'_1 \leqslant 0$. Similarly,

$$\begin{aligned} d^2(ty + (1-t)z, x') &= (x'_1 - (2t-1)a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 \\ &= (x'_1 - a)^2 + (x'_2 - b)^2 + \sum_{i=3}^n x_i'^2 + ((2t-2)a)^2 - (4t-4)a(x'_1 - a) \\ &\geqslant a^2 + b^2 + ((2t-2)a)^2 - (4t-4)a(x'_1 - a) \\ &= ((2t-1)a)^2 + b^2 - (4t-4)ax'_1 \\ &= d^2(ty + (1-t)z, x) - (4t-4)ax'_1 \geqslant d^2(ty + (1-t)z, x), \end{aligned}$$

if $x_1' > 0$ (since $t \leqslant 1$), which completes the proof. □

We note that Theorem 6.2 implies that any branching point of the average distance minimizer $\Sigma_{\rm opt}$ is a regular tripod (i.e., a point where exactly three smooth branches meet at $2\pi/3$ angles) only if $\Sigma_{\rm opt}$ is sufficiently regular (i.e., sufficiently smooth up to branching points).

Now we study the regularity of branching points of the average distance minimizer $\Sigma_{\rm opt}$. For this purpose, for an m-dimensional subspace $\Pi_m \subset \mathbb{R}^n$ and a compact set $\Omega \subset \mathbb{R}^n$ we introduce the notation

$$\rho(\Pi_m, x) := \inf\{r : k^{-1}(x) \cap \Omega \subset \overline{B}_r^m(x) \times \Pi_m^\perp\},$$
$$\rho(\Pi_m, \overline{B}_r(x)) := \sup\{\rho(\Pi_m, y) : y \in \overline{B}_r(x)\},$$

where $\Pi_m^\perp$ is the orthogonal complement in $\mathbb{R}^n$ of the subspace Π_m and $\overline{B}_r^m(x) \subset \Pi_m$ is the m-dimensional closed ball of radius r and center x.

Lemma 6.4. *Let a point $x \in \Sigma_{\rm opt}$, where $\Sigma_{\rm opt}$ is the average distance minimizer, be such that the Euclidean dimension of the set $k^{-1}(x)$ does not exceed $n-2$ (in particular, this condition is satisfied if x is a branching point of $\Sigma_{\rm opt}$). Under the assumptions of Theorem* 5.5*, there exists a two-dimensional subspace $\Pi_2 \subset \mathbb{R}^n$, $x \in \Pi_2$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\Pi_2, \overline{B}_r(z)) < \varepsilon$ if $\overline{B}_r(z) \subset B_\delta(x)$.*

Proof. Let S be an arbitrary $(m-2)$-dimensional hyperplane containing $k^{-1}(x)$ (if x is a branching point of $\Sigma_{\rm opt}$, then such a hyperplane exists in view of Theorem 6.2). We set $\Pi_2 := S^\perp$ (for $n = 2$ we set $\Pi_2 := \mathbb{R}^2$). It suffices to prove that the convergence $x_\nu \to x$ as $\nu \to \infty$ implies $\rho(\Pi_2, x_\nu) \to 0$, but, in the opposite case, $\rho(\Pi_2, x_\nu) \geqslant r_0$ for some number $r_0 > 0$ and subsequence x_ν (we preserve the same notation). Consequently, there is a sequence of points $z_\nu \notin B^m_{r_0/2}(x_\nu) \times \Pi_2^\perp$, $z_\nu \in \Omega$ such that the projection of each point z_ν to $\Sigma_{\rm opt}$ contains x_ν. Passing to a converging subsequence (in the same notation) $z_\nu \to z \in \Omega$ as $\nu \to \infty$, we find that the projection of z to $\Sigma_{\rm opt}$ contains x. However, $z \notin B^m_{r_0/2}(x) \times \Pi_2^\perp$, i.e., $\rho(\Pi_2, x) \geqslant r_0/2$, while $\rho(\Pi_2, x) = 0$ by construction. We obtain a contradiction. The proof is complete. □

We need the following assertion about the Ahlfors regularity of the average distance minimizers. It was proved in [**6**] in the two-dimensional case $n = 2$ and in [**7**] for an arbitrary dimension $n \geqslant 2$. This regularity condition is rather weak. However, as was shown in [**25**], for one-dimensional closed connected sets (it is the class in which we look for average distance minimizers) it provides some nice analytic properties. In particular, this condition guarantees a kind of "quantitative rectifiability" which is somewhat stronger than the classical rectifiability used in geometric measure theory.

Theorem 6.5. *Suppose that $\Sigma_{\rm opt}$ is a solution to Problem* 1.2*, a function A satisfies condition (α_1), (α_2), and $\varphi \in L^p(\mathbb{R}^n)$, where $p = n/(n-1)$ for $n \geqslant 3$ and $p = 4/3$ for $n = 2$. Then $\Sigma_{\rm opt}$ is Ahlfors regular, i.e., there are two constants $c > 0$ and $C > 0$ such that for any positive number $\rho <$ diam $\Sigma_{\rm opt}$ and point $x \in \Sigma_{\rm opt}$*

$$c\rho \leqslant \mathscr{H}^1(\Sigma_{\rm opt} \cap B_\rho(x)) \leqslant C\rho \tag{6.7}$$

We also need the following estimates generalizing analogous estimates from [**8**] in the two-dimensional case $n = 2$ and $A(t) = t$.

Lemma 6.6. *Suppose that $\Sigma_{\rm opt}$ is an average distance minimizer and $x \in \Sigma_{\rm opt}$, $y \in \Sigma_{\rm opt}$ are such that there is an arc $\Sigma \subset \Sigma_{\rm opt}$ with the starting point x and endpoint y that contains no branching points of $\Sigma_{\rm opt}$, perhaps, except for the points x and y themselves. Then, under the assumptions of Theorem* 5.5*, for any arc Σ_0 with the starting point x and endpoint y we have*

$$\mathscr{H}^1(\Sigma) - \mathscr{H}^1(\Sigma_0) \leqslant C\psi(\Sigma \setminus \{x,y\}) \max_{x \in \Sigma} \operatorname{dist}(x, \Sigma_0) \leqslant C\psi(\Sigma \setminus \{x,y\}) d_H(\Sigma, \Sigma_0), \tag{6.8}$$

where $d_H(\Sigma,\Sigma_0)$ is the Hausdorff distance between the arcs Σ and Σ_0 and the constant $C>0$ depends only on the data of the problem. Furthermore, if $\Sigma\subset\Sigma_{\rm opt}\cap\overline{B}_r(x)$, where $r:=|x-y|$, then

$$d_H(\Sigma,[x,y])\leqslant Cr\psi(\Sigma\setminus\{x,y\}), \tag{6.9}$$

where $[x,y]$ is a line segment.

Proof. Let $\mathscr{H}^1(\Sigma)>\mathscr{H}^1(\Sigma_0)$ (otherwise, the inequality (6.8) is trivial). We set $\Sigma_1:=(\Sigma_{\rm opt}\setminus\Sigma)\cup\Sigma_0$. Then

$$\begin{aligned}
F_\varphi(\Sigma_1)&=\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma_1))\,d\varphi(z)\\
&\leqslant\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma_{\rm opt}))\,d\varphi(z)+\int\limits_{k^{-1}(\Sigma\setminus\{x,y\})}(A(\operatorname{dist}(z,\Sigma_0))-A(\operatorname{dist}(z,\Sigma)))\,d\varphi(z)\\
&\leqslant\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma_{\rm opt}))\,d\varphi(z)+\Lambda\psi(\Sigma\setminus\{x,y\})\max_{x\in\Sigma}\operatorname{dist}(x,\Sigma_0)\\
&=F_\varphi(\Sigma_{\rm opt})+\Lambda\psi(\Sigma\setminus\{x,y\})\max_{x\in\Sigma}\operatorname{dist}(x,\Sigma_0).
\end{aligned}$$

Using Lemma 5.7, we find a compact connected set $\Sigma_2\supset\Sigma_1$ such that

$$\mathscr{H}^1(\Sigma_2)\leqslant\mathscr{H}^1(\Sigma_1)+\mathscr{H}^1(\Sigma)-\mathscr{H}^1(\Sigma_0)\leqslant\mathscr{H}^1(\Sigma_{\rm opt})$$

and

$$F_\varphi(\Sigma_2)\leqslant F_\varphi(\Sigma_1)-C(\mathscr{H}^1(\Sigma)-\mathscr{H}^1(\Sigma_0))$$

with some $C>0$ independent of Σ, y, and x. Thus,

$$F_\varphi(\Sigma_2)\leqslant F_\varphi(\Sigma_{\rm opt})+\Lambda\psi(\Sigma\setminus\{x,y\})\max_{x\in\Sigma}\operatorname{dist}(x,\Sigma_0)-C(\mathscr{H}^1(\Sigma)-\mathscr{H}^1(\Sigma_0)).$$

Hence the last inequality contradicts the optimality of $\Sigma_{\rm opt}$ only if the first inequality in (6.8) fails. The second inequality in (6.8) obviously follows from the first one since $\max_{x\in\Sigma}\operatorname{dist}(x,\Sigma_0)\leqslant d_H(\Sigma,\Sigma_0)$.

To prove (6.9), we apply (6.8) to $\Sigma_0:=[x,y]$. We have

$$\mathscr{H}^1(\Sigma)-|x-y|\leqslant Cd_H(\Sigma,[x,y])\psi(\Sigma\setminus\{x,y\}). \tag{6.10}$$

Note that

$$\mathscr{H}^1(\Sigma)^2\geqslant 4d_H^2(\Sigma,[x,y])+|x-y|^2. \tag{6.11}$$

Indeed, let $z\in\Sigma$ satisfy the condition $\operatorname{dist}(z,[x,y])=d_H(\Sigma,[x,y])$. Then

$$\mathscr{H}^1(\Sigma)\geqslant|x-z|+|z-y|,$$

but

$$\begin{aligned}
|x-z|+|z-y|&\geqslant\min\{|x-p|+|p-y|:p\in\mathbb{R}^n,\operatorname{dist}(p,[x,y])=d_H(\Sigma,[x,y])\}\\
&=\left(4d_H^2(\Sigma,[x,y])+|x-y|^2\right)^{1/2}.
\end{aligned}$$

The inequalities (6.11), (6.10) and the estimate $|x-y|\leqslant\mathscr{H}^1(\Sigma)$ yield

$$4d_H^2(\Sigma,[x,y])\leqslant(\mathscr{H}^1(\Sigma)+|x-y|)(\mathscr{H}^1(\Sigma)-|x-y|)\leqslant 2\mathscr{H}^1(\Sigma)Cd_H(\Sigma,[x,y])\psi(\Sigma\setminus\{x,y\}).$$

Consequently,

$$d_H(\Sigma,[x,y])\leqslant C\mathscr{H}^1(\Sigma)\psi(\Sigma\setminus\{x,y\}).$$

Taking into account that $\mathscr{H}^1(\Sigma)\leqslant\mathscr{H}^1(\Sigma_{\rm opt}\cap B_r(x))\leqslant Cr$ in view of Theorem 6.5, we arrive at (6.9). □

The following auxiliary assertion establishes conditions of a certain geometric regularity of the average distance minimizer in a neighborhood of each point (namely, it asserts that in a sufficiently small neighborhood of each point this set is a collection of "not too oscillating" curves).

Proposition 6.7. *Under the assumptions of Theorem 5.5, for any point $x \in \Sigma_{\rm opt}$ there is a number $r_0 > 0$ such that for any arc $\theta \subset \Sigma_{\rm opt}$ with the starting point x*

$$\#\theta \cap \partial B_r(x) = 1$$

for any $r \in (0, r_0]$.

Proof. Denote by x_r^1 and x_r^2 the first and last tangent points of the sphere $\partial B_r(x)$ by the arc θ. Then for curvalinear segments of the arc $[x, x_r^2] \subset \theta$ we have $\operatorname{diam}[x, x_r^2] \to 0$ as $r \to 0^+$. Indeed, in the opposite case, $\operatorname{diam}[x, x_{r_\nu}^2] \geqslant c$ for some $c > 0$ and $r_\nu \to 0^+$. Consequently, $\Sigma_{\rm opt}$ must contain a simple closed curve, which contradicts Theorem 5.5(i). As a consequence, we find that $\psi([x, x_r^2]) \to \psi(x)$ as $r \to 0^+$. Consequently, $\psi([x, x_r^2] \setminus \{x, x_r^2\}) \to 0$ as $r \to 0^+$.

If $r > 0$ is sufficiently small, then $[0, x_r^2]$ contains no branching points and endpoints of $\Sigma_{\rm opt}$, perhaps, except for the point x itself. Using the estimate (6.8) in Lemma 6.6 with $\Sigma := [x, x_r^2]$ and Σ_0, the result of a rotation of $[x, x_r^1]$ after whcih the points x_r^1 and x_r^2 coincide, we find

$$\mathscr{H}^1([x_r^1, x_r^2]) \leqslant C\psi([x, x_r^2] \setminus \{x, x_r^2\}) \max_{z \in [x, x_r^2]} \operatorname{dist}(z, \Sigma_0). \tag{6.12}$$

Since

$$\max_{z \in [x, x_r^1]} \operatorname{dist}(z, \Sigma_0) \leqslant |x_r^2 - x_r^1| \leqslant \mathscr{H}^1([x_r^1, x_r^2]),$$

and

$$\max_{z \in [x_r^1, x_r^2]} \operatorname{dist}(z, \Sigma_0) \leqslant \max_{z \in [x_r^1, x_r^2]} |z - x_r^2| \leqslant \operatorname{diam}[x_r^1, x_r^2] \leqslant \mathscr{H}^1([x_r^1, x_r^2]),$$

from (6.12) it follows that

$$\mathscr{H}^1([x_r^1, x_r^2]) \leqslant C\mathscr{H}^1([x_r^1, x_r^2])\psi([x, x_r^2] \setminus \{x, x_r^2\}).$$

The last inequality is valid for sufficiently small $r > 0$ (such that $C\psi([x, x_r^2] \setminus \{x, x_r^2\}) < 1$) only if

$$\mathscr{H}^1([x_r^1, x_r^2]) = 0,$$

i.e., $x_r^1 = x_r^2$ or, in other words, $\#\theta \cap \partial B_r(x) = 1$. □

Lemma 6.8. *Suppose that $\Sigma_{\rm opt}$ is an average distance minimizer and $x \in \Sigma_{\rm opt}$ is neither a branching point nor an endpoint. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary closed convex set containing* supp φ *and $\Sigma_{\rm opt}$, and let $\Pi_2 \subset \mathbb{R}^n$ be an arbitrary two-dimensional plane. Assume that $\varphi \in L^\infty(\mathbb{R}^n)$. Then, under the assumptions of Theorem 5.5, there is a number $r' > 0$ such that for any $r \in (0, r')$*

$$\psi(B_r(x)) \leqslant C(r + \psi(B_{2r}(x))), \tag{6.13}$$

where the constant $C > 0$ depends on the data of the problem and, in addition, on $R := \operatorname{diam}\Omega$ and $\rho := \rho(\Pi_2, \overline{B}_{r_0}(x))$; moreover, $C \to 0$ as $\rho \to 0^+$. In particular, there is a constant $\rho' > 0$ depending only on the data of the problem and R such that $\rho < \rho'$ and $r < r'$ imply $C < 1/2$ and

$$\psi(B_r(x)) \leqslant \frac{Cr}{1 - 2C} + \psi(B_{r'}(x)) \left(\frac{2r}{r'}\right)^{\log_2 \frac{1}{C}}. \tag{6.14}$$

Proof. The relation (6.13) is obtained by a direct generalization of Lemma 2.10 in **[8]** to the case of an arbitrary dimension $n \geqslant 2$ with the help of (6.9). Namely, let $r_0 > 0$ be such that $\Sigma_{\rm opt} \cap \overline{B}_{r_0}(x)$ contains neither branching points nor endpoints of $\Sigma_{\rm opt}$; moreover, $\#\Sigma_{\rm opt} \cap B_r(x) = 2$ for any $r \leqslant r_0$ (such a number r_0 exists because the set of branching points and endpoints of $\Sigma_{\rm opt}$ is finite and by Proposition 6.7). Let

$r \leqslant r_0/2$. Since $\text{ord}\,_x\Sigma_{\text{opt}} = 2$, there are arcs $\Sigma_i \subset \Sigma_{\text{opt}}$, $i = 1,2$, starting at x. Denote by x_1 and x_2 the point where these arcs intersect $\partial B_{2r}(x)$. Since

$$d_H(\Sigma_{\text{opt}} \cap B_{2r}(x), [x,x_1] \cap [x,x_2]) \leqslant \max_{i=1,2} d_H(\Sigma_i \cap B_{2r}(x), [x,x_i]),$$

where $[x,x_i]$, $i = 1,2$, is the line segment with the starting point x and endpoint x_i, from (6.9) we find

$$d := d_H(\Sigma_{\text{opt}} \cap B_{2r}(x), [x,x_1] \cap [x,x_2]) \leqslant Cr\psi(B_{2r}(x)) \tag{6.15}$$

with some positive constant C. We denote by K the union of two cylinders with the axes of symmetry $[x,x_i]$, $i = 1,2$, and the radius of base d. Then

$$\Sigma_{\text{opt}} \cap B_{2r}(x) \subset K.$$

We set $K' := \text{co}\,(K \cap B_r(x))$ and note that

$$k^{-1}(B_r(x)) \cap \Omega \subset \widetilde{K} := \left\{ z \in \Omega \cap (\overline{B}^2_\rho(x) \times \mathbb{R}^{n-2}) : \text{dist}\,(x, K') \leqslant \min_{i=1,2} |z - x_i| \right\},$$

which implies

$$\psi(B_r(x)) \leqslant \varphi(\widetilde{K}). \tag{6.16}$$

However,

$$\widetilde{K} \leqslant T_1 \cup T_2 \cup E \cup C_1 \cup C_2,$$

where T_i, $i = 1,2$, are the intersections of the set $\Omega' := \Omega \cap (\overline{B}^2_\rho(x) \times \mathbb{R}^{n-2})$ and the cylinders with axes $[x,x_i]$, E is the intersection of the set Ω' and the sector that is formed by the planes passing through x perpendicularly to $[x,x_i]$ and is bounded by the boundaries of cylinders forming T_i, $i = 1,2$. Finally C_i, $i = 1,2$, are the intersection of Ω' with the sets bounded by the external (relative to x) boundaries of the cylinders forming T_i, and surfaces of revolution about $[x,x_i]$, such that the distance to any point x_i is equal to the distance to the $(n-1)$-dimensional circles S_i formed by the intersection of $\partial B_r(x)$ and the cylinders with radius d and axes $[x,x_i]$ (it is easy to see that these surfaces are the surface of corresponding cone of rotation with the axes $[x,x_i]$). A direct computation shows that $\mathscr{L}^n(T_i) \leqslant Cr$ and since the "aperture" of the sets C_1, C_2, and E is estimated by d/r, we have

$$\mathscr{L}^n(C_i) \leqslant Cd/r, \quad \mathscr{L}^n(E) \leqslant Cd/r$$

for sufficiently small r, where the constant $C > 0$ satisfies the assumptions of the lemma. Thus,

$$\mathscr{L}^n(\widetilde{K}) \leqslant C(r + 2d/r).$$

Taking into account (6.15), we find

$$\mathscr{L}^n(\widetilde{K}) \leqslant C(r + \psi(B_{2r}(x)).$$

Since $\varphi \in L^\infty(\Omega)$, we have

$$\varphi(\widetilde{K}) \leqslant C(r + \psi(B_{2r}(x)),$$

which implies (6.13) in view of (6.16).

Let us prove (6.14). If $0 < r < r'$, then there is a natural number $j \in \mathbb{N}$ such that $r'/2 \leqslant 2^j r < r$. Successively applying (6.13) j times, we find

$$\psi(B_r(x)) \leqslant Cr \sum_{i=0}^{j-1} (2C)^i + C^j \psi(B_{r'}(x)) \leqslant \frac{Cr}{1-2C} + C^j \psi(B_{r'}(x)). \tag{6.17}$$

Taking into account that $h > \log_2 r'/2r$, $C < 1/2$, we obtain the inequality

$$C^j \leqslant \left(\frac{2r}{r'}\right)^{\log_2 \frac{1}{C}}$$

which, together with (6.17), leads to (6.14). □

Now we can establish the regularity of branching points of the average distance minimizer $\Sigma_{\rm opt}$.

Theorem 6.9. *We assume that* $\varphi \in L^\infty(\mathbb{R}^n)$. *Under the assumption of Theorem* 5.5, *the following assertions hold*:

(i) *if* $x \in \Sigma_{\rm opt}$ *is not a branching point, then the Euclidean dimension of the set* $k^{-1}(x)$ *is at least* $n-1$,

(ii) *for every branching point* $x \in \Sigma_{\rm opt}$ *there is a number* $\delta > 0$ *such that the set* $\Sigma_{\rm opt} \cap B_\delta(x)$ *consists of exactly three* $C^{1,1}$ *arcs starting at x.*

Proof. Since the set of branching points is finite (cf. Theorem 5.5(iii)), there is $r_0 > 0$ such that $\Sigma_{\rm opt} \cap \overline{B}_{r_0}(x)$ does not contain branching points of the set $\Sigma_{\rm opt}$, perhaps, except for the point x itself. Moreover, by Proposition 6.7, one can choose r_0 so small that $\#\Sigma_{\rm opt} \cap \overline{B}_r(x) = k := {\rm ord}_x \Sigma_{\rm opt}$ for all $r \in (0, r_0]$. Then if $r > 0$ is sufficiently small, then $\Sigma_{\rm opt} \cap \overline{B}_r(x)$ is an arc provided that $x \in \Sigma_{\rm opt}$ is not a branching point and is the union of three $C^{1,1}$ arcs provided that $x \in \Sigma_{\rm opt}$ We assume that the Euclidean dimension of the set $k^{-1}(x)$ does not exceed $n-2$. We show that the corresponding arcs forming $\Sigma_{\rm opt} \cap \overline{B}_r(x)$ are of class $C^{1,1}$ (one if $k \leqslant 2$ or three if $k = 3$) for sufficiently small $r > 0$ i (thereby we prove assertion (ii)). Let $\Omega \subset \mathbb{R}^n$ be an arbitrary closed convex set containing supp φ and $\Sigma_{\rm opt}$. By Lemma 6.4, there exists a two-dimensional subspace $\Pi_2 \subset \mathbb{R}^n$, $x \in \Pi_2$, and $\delta \in (0, r_0]$ such that $\rho(\Pi_2, \overline{B}_r(z)) < \rho'$ (the constant ρ' is defined in Lemma 6.8) if $\overline{B}_r(z) \subset B_\delta(x)$. Thus if $r < r'$ (the constant r' is defined in Lemma 6.8) and $\overline{B}_r(z) \subset B_\delta(x)$, then (6.14) holds. Consequently, $\mathscr{H}^1$-a.a in $\Sigma_{\rm opt} \cap B_\delta(x)$ there exists the limit

$$\lim_{r \to 0^+} \frac{\psi(B_r(z))}{2r} \leqslant \frac{C}{1-2C} < +\infty.$$

In other words, $\psi \in L^\infty(B_\delta(x))$. Therefore, by Proposition 6.1, we have $|H| \in L^\infty(B_\delta(x))$. Consequently, each of the arcs forming $\Sigma_{\rm opt} \cap \overline{B}_r(x)$ (one if $k \leqslant 2$ or three if $k = 3$) is a $C^{1,1}$ curve by Lemma 6.10.

It remains to prove (i). This assertion is wrong only if $k = 2$ since for $k = 1$ (x is an endpoint) $\psi(\{x\}) > 0$ in view of Theorem 5.5. Hence the Euclidean dimension of the set $k^{-1}(x)$ is equal to n. As we just proved, $\Sigma := \Sigma_{\rm opt} \cap \overline{B}_r(x)$ for a sufficiently small $r > 0$ is and of class $C^{1,1}$. Consequently, by Lemma 6.11, there exists an $(n-1)$-dimensional hyperplane $\Pi_{n-1} \subset \mathbb{R}^n$, $x \in \Pi_{n-1}$ and a number $\rho > 0$ such that all the point of the relatively open $(n-1)$-dimensional ball $\mathscr{B}_\rho^{n-1} \subset \Pi_{n-1}(x)$ with center x and radius ρ have a single projection on Σ at the point x. If (i) fails, there is a sequence of points $\{z_\nu\} \in \Pi_{n-1}$ such that $z_\nu \to x$ as $\nu \to \infty$ and each point z_ν has a projection on $\Sigma_{\rm opt}$ at the point $y_\nu \notin \Sigma$; moreover, it has a projection at the point x. Then $y_\nu \to x$ as $\nu \to \infty$. Consequently, $y_\nu \in \Sigma$ for sufficiently large $\nu \in \mathbb{N}$. We obtain a contradiction. The proof is complete. □

The following lemmas were used in the proof of Theorem 6.9.

Lemma 6.10. *Let* $\Sigma \subset \mathbb{R}^n$ *be a Lipschitz arc with the starting point a and endpoint b. Assume that its generalized curvature satisfies the condition* $H \in L^\infty(\Sigma \setminus \{a, b\})$. *Then* Σ *belongs to the class* $C^{1,1}$.

Proof. We assume that Σ is an arc parametrized by length, i.e., Σ: $I := [0, \mathscr{H}^1(\Sigma)] \to \mathbb{R}^n$, $|\Sigma'(t)| = 1$ for a.a. $t \in I$. Then

$$\frac{d}{d\varepsilon} \int_I |\Sigma'(t) + \varepsilon(\varphi \circ \Sigma)'(t)|\, dt \Big|_{\varepsilon=0} = \int_I \frac{\Sigma'(t)}{|\Sigma'(t)|} \nabla \cdot \varphi(\Sigma(t)) \Sigma'(t)\, dt = \int_I H(\Sigma(t)) \varphi(\Sigma(t)) |\Sigma'(t)|\, dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Taking into account that $|\Sigma'(t)| = 1$ for a.a. $t \in I$, we find

$$\frac{d}{dt} \Sigma' \in L^\infty(I),$$

where the derivative is understood in the weak sense. In other words, $\Sigma \in C^{1,1}$. □

Lemma 6.11. *Let $\Sigma \subset \mathbb{R}^n$ be an arc of class $C^{1,1}$. Then for any point $x \in \Sigma$ there is an $(n-1)$-dimensional hyperplane $\Pi_{n-1}(x) \subset \mathbb{R}^n$, $x \in \Pi_{n-1}(x)$, and $\rho > 0$ such that $\mathscr{B}_\rho^{n-1} \subset k^{-1}(x)$, where $\mathscr{B}_\rho^{n-1} \subset \Pi_{n-1}(x)$ is a relative open $(n-1)$-dimensional ball in Π_{n-1} with center x and radius ρ.*

Proof. We assume that Σ is parametrized by length, i.e., Σ: $I := [0, \mathscr{H}^1(\Sigma)] \to \mathbb{R}^n$, $|\Sigma'(t)| = 1$ for all $t \in I$. The unit tangent vector $\tau(x) := \Sigma'(t)$ to Σ and the normal $(n-1)$-dimensional hyperplane $\Pi_{n-1}(x) \subset \mathbb{R}^n$ are defined at any point $x \in \Sigma$, $x = \Sigma(t)$.

We set

$$N(x)v := v - (v, \tau(x))\tau(x)$$

for any $v \in \mathbb{R}^n$. Since Σ belongs to the class $C^{1,1}$ and $(\tau(x), N(x)\cdot v) = 0$ for any $v \in \mathbb{R}^{n-1}$, for any $\varepsilon > 0$ there is $r > 0$ such that

$$\begin{aligned} &|(\Sigma'(t), N(\Sigma(s))v)| \leqslant \varepsilon |v|, \\ &|\Sigma(t) - \Sigma(s) - \Sigma'(t)(t-s)| \leqslant \varepsilon |t-s| \end{aligned} \tag{6.18}$$

for any $v \in \mathbb{R}^n$ and for any $s \in I$, $|s-t| \leqslant r$.

We define f: $I \times \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$f(t,v) := \Sigma(t) + N(\Sigma(t))v.$$

It is obvious that $f(t,\cdot) \in \Pi_{n-1}(\Sigma(t))$ for any $t \in I$. Let $x := \Sigma(t)$, $y := \Sigma(s)$. Since

$$\begin{aligned} |f(t,u) - f(s,v)| &= |x - y + N(x)(u-v) - (N(y)v - N(x)v)| \\ &\geqslant |x - y + N(x)(u-v)| - |(N(y)v - N(x)v)| \end{aligned}$$

and $|(N(y)v - N(x)v)| \leqslant L|y-x|\cdot|v|$ for some the constants $L > 0$ by the regularity of Σ, we have

$$|f(t,u) - f(s,v)| \geqslant |x - y + N(x)(u-v)| - L|y-x|\cdot|v|. \tag{6.19}$$

However, by (6.18), we have

$$\begin{aligned} |x-y+N(x)(u-v)| &= (|x-y|^2 + |u-v|^2 + (x-y, N(x)(u-v)))^{1/2} \\ &\geqslant (|x-y|^2 + |u-v|^2 - |(\Sigma'(t)(t-s), N(x)(u-v))| - \varepsilon|t-s|\cdot|u-v|)^{1/2} \\ &= (|x-y|^2 + |u-v|^2 - 2\varepsilon|t-s|\cdot|u-v|)^{1/2}. \end{aligned} \tag{6.20}$$

We take into account that

$$\begin{aligned} |t-s| - |\Sigma(t) - \Sigma(s)| &= |\Sigma'(t)(t-s)| - |\Sigma(t) - \Sigma(s)| \\ &\leqslant |\Sigma(t) - \Sigma(s) - \Sigma'(t)(t-s)| \leqslant \varepsilon|t-s|, \end{aligned}$$

which means

$$|t-s| \leqslant \frac{|x-y|}{1-\varepsilon}$$

and, consequently,

$$2\varepsilon|t-s|\cdot|u-v| \leqslant \varepsilon|t-s|^2 + \varepsilon|u-v|^2 \leqslant \frac{\varepsilon}{(1-\varepsilon)^2}\cdot|x-y|^2 + \varepsilon|u-v|^2.$$

Substituting the last inequality in (6.20) and setting $\varepsilon := 1/3$, we find

$$|x - y + N(x)\cdot(u-v)| \geqslant |x-y|/4 + 2|u-v|/3 \geqslant |x-y|/4. \tag{6.21}$$

Then (6.19) and (6.21) imply

$$|f(t,u) - f(s,v)| \geqslant |\Sigma(t) - \Sigma(s)|(1/4 - L|v|) \tag{6.22}$$

if $|t-s| \leqslant r$.

If the assertion is not true, then there is a sequence $\{z_k\} \in \Pi_{n-1}$ such that $z_k \to x$ as $k \to \infty$ and each point z_k has a projection on Σ at the point $y_k := \Sigma(t_k) \neq x$; moreover, at the point x. Then $\Sigma(t_k) \to x$.

Since Σ is assumed to be an arc, i.e., an injective curve, we have $t_k \to t$ as $k \to \infty$, where $x = \Sigma(t)$. Since $z_k = f(t_k, z_k - y_k) = f(t, z_k - x)$ and $z_k - y_k \to 0$, $z_k - x \to 0$ as $k \to \infty$, for sufficiently large $k \in \mathbb{N}$ from (6.22) we find

$$0 = |f(t, z_k - x) - f(t_k, z_k - y_k)| \geqslant |\Sigma(t) - \Sigma(t_k)|(1/4 - L|z_k - y_k|).$$

Consequently, $x = \Sigma(t) = \Sigma(t_k)$, which contradicts the assumption. □

§ 7. Regularity of the Average Distance Minimizer in the Case $n = 2$

As was shown in [**22**], the assumptions of Theorem 5.5 are satisfied for $n = 2, A(t) = t$, and $\varphi \in L^p(\mathbb{R}^n)$ if $p > 4/3$. Namely, the following assertion was proved in [**22**].

Lemma 7.1. *Suppose that $\Sigma_{\text{opt}} \subset \mathbb{R}^2$ is a solution to Problem* 1.2 *with $n = 2$, $A(t) = t$, and $\varphi \in L^p(\mathbb{R}^2)$, where $p > 4/3$. Then there exists a point $x \in \Sigma_{\text{opt}}$ such that $\psi(\{x\}) > 0$.*

From Lemma 7.1, Theorem 5.5, Proposition 6.7, and Theorem 6.9 we obtain the following assertion.

Corollary 7.2. *Suppose that $\Sigma_{\text{opt}} \subset \mathbb{R}^2$ is a solution to Problem* 1.2 *with $n = 2$, $A(t) = t$, and $\varphi \in L^p(\mathbb{R}^2)$, where $p > 4/3$. Then the following assertions hold*:

(i) *the number of noncut points* (*consequently, endpoints*) *of Σ_{opt} is finite,*

(ii) *the number of branching points of Σ_{opt} is finite,*

(iii) *every branching point $x \in \Sigma_{\text{opt}}$ is a triple point, i.e., at this point, Σ_{opt} has order three*; *moreover, the generalized curvature of Σ_{opt} is a vector-valued Radon measure and $H(\{x\}) = 0$,*

(iv) *for any point $x \in \Sigma_{\text{opt}}$ there is $r_0 > 0$ such that for any arc $\theta \subset \Sigma_{\text{opt}}$ starting ar x*

$$\#\theta \cap \partial B_r(x) = 1 \quad \textit{for any } r \in (0, r_0],$$

(v) *if $\varphi \in L^\infty(\mathbb{R}^2)$, then for every branching point $x \in \Sigma_{\text{opt}}$ there is $\delta > 0$ such that the set $\Sigma_{\text{opt}} \cap B_\delta(x)$ consists of exactly three $C^{1,1}$-arcs starting at x,*

(vi) *if $\varphi \in L^\infty(\mathbb{R}^2)$ and $x \in \Sigma_{\text{opt}}$ is not a branching point, then the Euclidean dimension of the set $k^{-1}(x)$ is at least* 1.

We note that assertion (v) of Corollary 7.2 is contained in [**8**], whereas assertion (iv) is also contained in the same paper, but in a weaker form (only in the case $\varphi \in L^\infty(\mathbb{R}^2)$).

§ 8. Reduction of the General Problem to the Minimization of the Average Distance Functional

In this section, using the technique developed in [**26**], we reduce the general problem 1.1 about the optimization of the transportation network to Problem 1.2 of minimizing the average distance functional.

Introduce the notation

$$\mathscr{R} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \operatorname{dist}(x,\Sigma) + \operatorname{dist}(y,\Sigma) > d(x,y)\}$$

and note that $d_\Sigma(x,y) = d(x,y)$ on $\mathscr{R}$. We set

$$\varphi_r^\pm := \pi_\#^\pm(\Gamma \llcorner \mathscr{R}), \qquad \varphi_s^\pm := \pi_\#^\pm(\Gamma \llcorner (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathscr{R}) \llcorner (\mathbb{R}^n \setminus \Sigma),$$

where Γ is the optimal transport plan for the problem $MK(\varphi^+, \varphi^-, \Sigma)$. It is obvious that $\mathscr{R} \subset (\mathbb{R}^n \setminus \Sigma) \times (\mathbb{R}^n \setminus \Sigma)$, Consequently, $\varphi_r^\pm$ are concentrated outside Σ. Since the measures $\varphi_s^\pm$ are also concentrated outside

Σ by construction, we have

$$\varphi^{\pm}\llcorner(\mathbb{R}^n\setminus\Sigma)=\varphi_r^{\pm}+\varphi_s^{\pm}.$$

Let

$$\varphi_s:=\varphi_s^{+}+\varphi_s^{-}.$$

We note that $\varphi^{\pm}$ and φ_s depend on Σ. If we need to indicate this dependence, we write $\varphi_\Sigma^{\pm}$ and $\varphi_{s,\Sigma}$ respectively.

Lemma 8.1. *Suppose that there exist Borel measures $\widetilde{\varphi}_r^{\pm}$ and $\widetilde{\varphi}_s^{\pm}$ on $\mathbb{R}^n\setminus\Sigma$ such that $\widetilde{\varphi}_r^{+}(\mathbb{R}^n)=\widetilde{\varphi}_r^{-}(\mathbb{R}^n)$ and*

$$\varphi^{\pm}\llcorner(\mathbb{R}^n\setminus\Sigma)=\widetilde{\varphi}_r^{\pm}+\widetilde{\varphi}_s^{\pm}.$$

Then

$$MK(\varphi^+,\varphi^-,\Sigma)\leqslant MK(\widetilde{\varphi}_r^{+},\widetilde{\varphi}_r^{-})+\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma))\,d\widetilde{\varphi}_s^{+}(z)+\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma))\,d\widetilde{\varphi}_s^{-}(z).$$

If $\widetilde{\varphi}_r^{\pm}=\varphi_r^{\pm}$, $\widetilde{\varphi}_s^{\pm}=\varphi_s^{\pm}$, and the function A is superadditive (i.e., $A(u)+A(v)\leqslant A(u+v)$ for all $\{u,v\}\subset\mathbb{R}^+$), then the above inequality becomes equality.

Proof. Let Γ be some optimal transport plan for the problem $MK(\varphi^+,\varphi^-,\Sigma)$. Denote by $\widetilde{\Gamma}_r$ the optimal transport plan for the problem $MK(\widetilde{\varphi}_r^{+},\widetilde{\varphi}_r^{-})$. Let $K\colon\mathbb{R}^n\multimap\Sigma$ denote the multivalued projection on Σ defined by the formula

$$K(x):=\{y\in\Sigma : d(x,y)=\operatorname{dist}(x,\Sigma)\},$$

and let $k\colon\mathbb{R}^n\to\Sigma$ be an arbitrary Borel measurable selector of this multivalued mapping. Finally, we set

$$\widetilde{\Gamma}^{+}(e):=\widetilde{\varphi}_s^{+}(\{x : (x,k(x))\in e\}),\qquad \widetilde{\Gamma}^{-}(e):=\widetilde{\varphi}_s^{-}(\{y : (k(y),y)\in e\})$$

for all Borel sets $e\subset\mathbb{R}^n\times\mathbb{R}^n$. It is obvious that the measures $\widetilde{\Gamma}^{\pm}$ are optimal transport plans for the problem $MK(\widetilde{\varphi}_s^{\pm},\widetilde{\psi}^{\mp})$, where $\widetilde{\psi}^{\mp}:=\pi_{\#}^{\mp}\widetilde{\Gamma}^{\pm}$. Since Γ is optimal and the measure $\widetilde{\Gamma}_r+\widetilde{\Gamma}^{+}+\widetilde{\Gamma}^{-}$ satisfies the condition (2.1), we have

$$\begin{aligned}MK(\varphi^+,\varphi^-,\Sigma)&=\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\Gamma(x,y)\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d(\widetilde{\Gamma}_r+\widetilde{\Gamma}^{+}+\widetilde{\Gamma}^{-})(x,y)\\&=\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\widetilde{\Gamma}_r(x,y)+\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\widetilde{\Gamma}^{+}(x,y)+\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\widetilde{\Gamma}^{-}(x,y).\end{aligned}$$

However,

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\widetilde{\Gamma}_r(x,y)\leqslant\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d(x,y))\,d\widetilde{\Gamma}_r(x,y)=MK(\widetilde{\varphi}_r^{+},\widetilde{\varphi}_r^{-})$$

and

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}A(d_\Sigma(x,y))\,d\widetilde{\Gamma}^{\pm}(x,y)=\int\limits_{\mathbb{R}^n}A(\operatorname{dist}(z,\Sigma))\,d\widetilde{\varphi}_s^{\pm}(z),$$

which completes the proof of the first assertion.

To prove the second assertion, we note that

$$MK(\varphi^+,\varphi^-,\Sigma)=I_\Sigma^0(\Gamma)+I_\Sigma'(\Gamma),$$

where

$$I^0_\Sigma(\Gamma) := \int\limits_{\mathscr{R}} A(d_\Sigma(x,y))\, d\Gamma(x,y),$$

$$I'_\Sigma(\Gamma) := \int\limits_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathscr{R}} A(d_\Sigma(x,y))\, d\Gamma(x,y).$$

Since $d_\Sigma = d$ on $\mathscr{R}$, we have

$$I^0_\Sigma(\Gamma) = MK(\varphi_r^+, \varphi_r^-).$$

Because of the superadditivity of A we have

$$\begin{aligned} I'_\Sigma(\Gamma) &\geqslant \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} A(d(x,K(x)))\, d\Gamma_{\mathscr{R}^c}(x,y) + \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} A(d(y,K(y)))\, d\Gamma_{\mathscr{R}^c}(x,y) \\ &= \int\limits_{\mathbb{R}^n} A(d(x,K(x)))\, d(\varphi_s^+ + \widetilde{\psi}^+)(x) + \int\limits_{\mathbb{R}^n} A(d(y,K(y)))\, d(\varphi_s^- + \widetilde{\psi}^-)(y) \\ &= \int\limits_{\mathbb{R}^n\setminus\Sigma} A(\operatorname{dist}(x,\Sigma))\, d\varphi_s^+(x) + \int\limits_{\mathbb{R}^n\setminus\Sigma} A(\operatorname{dist}(y,\Sigma))\, d\varphi_s^-(y), \end{aligned}$$

where $\Gamma_{\mathscr{R}^c} := \Gamma\llcorner(\mathbb{R}^n\times\mathbb{R}^n\setminus\mathscr{R})$ and $\widetilde{\psi}^\pm$ are Borel measures concentrated in Σ. □

The following simple assertion connects the minimization problem $MK(\varphi^+, \varphi^-, \cdot)$ with the corresponding minimization problem for the average distance functional.

Proposition 8.2. *Suppose that the set $\Sigma_{\rm opt}$ minimizes $MK(\varphi^+, \varphi^-, \cdot)$ over all admissible sets. If A is superadditive, then $\Sigma_{\rm opt}$ is a minimizer of the average distance functional*

$$F_\varphi(\Sigma) := \int\limits_{\mathbb{R}^n} A(\operatorname{dist}(x,\Sigma))\, d\varphi(x),$$

where $\varphi := \varphi_{s,\Sigma_{\rm opt}}$, over all admissible sets Σ. Moreover, if $\varphi^+ \neq \varphi^-$ and A is strictly increasing, then $\varphi \neq 0$.

Remark 8.3. It is easy to see that a monotone nondecreasing superadditive function $A\colon \mathbb{R}^+ \to \overline{\mathbb{R}}^+$ is strictly increasing if and only if it is not a constant (in other words, there is $l > 0$ such that $A(l) \neq A(0)$).

Proof. If $\Sigma_{\rm opt}$ is not a minimum point of F_φ, i.e., there exists an admissible set Σ' such that $F_\varphi(\Sigma') < F_\varphi(\Sigma_{\rm opt})$, then we obtain a contradiction with the optimality of $\Sigma_{\rm opt}$ for the functional $MK(\varphi^+, \varphi^-, \cdot)$ by the following chain of relations:

$$MK(\varphi^+, \varphi^-, \Sigma_{\rm opt}) = MK(\varphi_r^+, \varphi_r^-) + F_\varphi(\Sigma_{\rm opt}) > MK(\varphi_r^+, \varphi_r^-) + F_\varphi(\Sigma') \geqslant MK(\varphi^+, \varphi^-, \Sigma'),$$

where the last inequality holds in view of Lemma 8.1.

We show that $\varphi \neq 0$ if $\varphi^+ \neq \varphi^-$. For this purpose, we note that $\varphi = 0$ implies $\varphi_r^\pm = \varphi^\pm$, which means

$$MK(\varphi^+, \varphi^-, \Sigma_{\rm opt}) = MK(\varphi^+, \varphi^-)$$

by Lemma 8.1. Hence the empty set is optimal for Problem 1.1 (note that it is possible only if $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$), which contradicts Lemma 8.4. □

Lemma 8.4. *If $\varphi^+ \neq \varphi^-$ and A is strictly increasing, then there is $\Sigma \subset \Omega$ such that $\mathscr{H}^1(\Sigma) \leqslant l$ and $MK(\Sigma) < MK(\varnothing)$. In particular, if $\Sigma_{\rm opt}$ is a solution to Problem 1.1, then it is nonempty and is different from a point.*

Proof. Let η_{opt} be an optimal transport measure. To prove the required assertion, we assume the contrary. Suppose that the optimal set Σ_{opt} is nonempty. Since $\varphi^+ \neq \varphi^-$, we have $\eta_{\text{opt}} \neq 0$. Consequently, there exists a number $L > 0$ such that the set

$$\widehat{\Theta} := \{\theta \in \operatorname{supp} \eta_{\text{opt}} \colon \mathscr{H}^1(\theta) > L\}$$

satisfies the condition $\eta_{\text{opt}}(\widehat{\Theta}) > 0$. However, $\widetilde{\Theta}(\varnothing)$ consists of segments and $\operatorname{supp} \eta_{\text{opt}} \subset \widetilde{\Theta}(\varnothing)$ in view of Proposition 3.1. Therefore, $\mathscr{H}^1(\theta) \leqslant \operatorname{diam} \Omega$ for any $\theta \in \operatorname{supp} \eta_{\text{opt}}$. Hence $\operatorname{supp} \eta_{\text{opt}}$ is compact in Θ. Thus, $\widehat{\Theta}$ can be covered by finitely many balls of arbitrarily small radius and centers at $\widehat{\Theta}$. In particular, denoting

$$\varepsilon := \min\left\{\frac{l}{8}, \frac{L}{8}\right\},$$

we find that there exists a ball $B_\varepsilon([x_0,y_0]) \subset \Theta$, where $[x_0,y_0] \subset \Omega$ is a segment of length $L_0 := |x_0 - y_0| \geqslant L$ such that $\eta_{\text{opt}}(B_\varepsilon([x_0,y_0])) > 0$.

Let $\delta := \min\{l, L_0\}$, so that $\delta - 4\varepsilon > 0$. Consider the segment $\Sigma := [\overline{x},\overline{y}]$ centered in $[x_0,y_0]$ and having length δ, i.e.,

$$\overline{x} := x_0 + \frac{L_0 - \delta}{2L_0}(y_0 - x_0), \qquad \overline{y} := y_0 + \frac{L_0 - \delta}{2L_0}(x_0 - y_0).$$

For every $\theta \in B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta}$ (which is a segment) we introduce a new path $f(\theta)$ as the broken line $[\theta(0),\overline{x}] \circ [\overline{x},\overline{y}] \circ [\overline{y},\theta(1)]$. For any $\theta \in \Theta \setminus (B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta})$ we set $f(\theta) := \theta$. It is obvious that the mapping $f\colon \Theta \to \Theta$ is a Borel one. For $\theta \in B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta}$, using the triangle inequality, we find

$$|\theta(0) - \overline{x}| \leqslant \frac{L_0 - \delta}{2} + \varepsilon, \qquad |\overline{y} - \theta(1)| \leqslant \frac{L_0 - \delta}{2} + \varepsilon, \quad L_0 \leqslant \mathscr{H}^1(\theta) + 2\varepsilon.$$

Hence

$$\begin{aligned} A(\mathscr{H}^1(f(\theta) \setminus \Sigma)) &\leqslant A(|\theta(0) - \overline{x}| + |\overline{y} - \theta(1)|) \leqslant A\left(2\left(\frac{L_0 - \delta}{2} + \varepsilon\right)\right) \\ &= A(L_0 + 2\varepsilon - \delta) \leqslant A(\mathscr{H}^1(\theta) + 4\varepsilon - \delta) < A(\mathscr{H}^1(\theta)) = A(\mathscr{H}^1(\theta \setminus \varnothing)) \end{aligned}$$

for any $\theta \in B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta}$.

Let $\eta := f_\# \eta_{\text{opt}}$. Then

$$\begin{aligned} MK(\Sigma) &\leqslant \int_\Theta A(\mathscr{H}^1(\theta \setminus \Sigma))\, d\eta = \int_\Theta A(\mathscr{H}^1(f(\theta) \setminus \Sigma))\, d\eta_{\text{opt}} \\ &= \int_{B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta}} A(\mathscr{H}^1(f(\theta) \setminus \Sigma))\, d\eta_{\text{opt}} + \int_{\Theta \setminus (B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta})} A(\mathscr{H}^1(f(\theta) \setminus \Sigma))\, d\eta_{\text{opt}} \\ &< \int_{B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta}} (B,\theta) A(\mathscr{H}^1(\theta \setminus \varnothing))\, d\eta_{\text{opt}} + \int_{\Theta \setminus (B_\varepsilon([x_0,y_0]) \cap \widehat{\Theta})} A(\mathscr{H}^1(\theta \setminus \varnothing))\, d\eta_{\text{opt}} \\ &= MK(\varnothing). \end{aligned}$$

Taking into account that $\mathscr{H}^1(\Sigma) \leqslant l$ by construction, we conclude that $MK(\Sigma) < MK(\varnothing)$, which contradicts the optimality of the empty set. $\square$

We summarize the most interesting qualitative properties of solutions to Problem 1.1 in the following theorem which immediately follows from Proposition 8.2 and some results on partial regularity of the average distance minimizers.

Theorem 8.5. *Suppose that* $\varphi^+ \neq \varphi^-$ *and* $\dim \varphi^\pm > 1$ (*for example,* $\varphi^\pm \ll \mathscr{L}^n$). *Let* $\Sigma_{\mathrm{opt}} \subset \mathbb{R}^n$ *is a solution to Problem* 1.1*, where A is superadditive and satisfies conditions* (α_1) *and* (α_2). *Then* Σ_{opt} *possesses the following properties.*

(i) Σ_{opt} *does not contain simple closed curves* (*homeomorphic images of* S^1). *In particular for* $n = 2$ *the set* $\mathbb{R}^2 \setminus \Sigma_{\mathrm{opt}}$ *is connected.*

If, in addition, $\varphi^\pm \in L^p(\mathbb{R}^n)$, *where* $p = 4/3$ *for* $n = 2$ *and* $p = n/(n-1)$ *in the general case, then*

(ii) Σ_{opt} *is Ahlfors regular.*

If $n = 2$, $A(t) = t$, *and* $\varphi^\pm \in L^p(\mathbb{R}^n)$, *where* $p > 4/3$, *then the following additional properties hold*:

(iii) $\operatorname{ord}_x \Sigma_{\mathrm{opt}} \leqslant 3$ *for all* $x \in \Sigma_{\mathrm{opt}}$*; moreover, the set of endpoints and branching points* (*thus, the branching points are triple points*) *is finite.*

(iv) *The generalized mean curvature H of* Σ_{opt} *is a Radon measure and satisfies the condition* $H(\{x\}) = 0$ *for every branching point* $x \in \Sigma_{\mathrm{opt}}$. *This property can be regarded as a* "*weak form*" *of the assertion that every endpoint is a* "*regular tripod*"*, i.e., a triple point, where exactly three smooth branches meet at* 120° *angles.*

(v) *For any point* $x \in \Sigma_{\mathrm{opt}}$ *there is a number* $r_0 > 0$ *such that for any arc* $\theta \subset \Sigma_{\mathrm{opt}}$ *starting at x we have*

$$\#\theta \cap \partial B_r(x) = 1$$

for any $r \in (0, r_0]$.

(vi) *If* $\varphi^\pm \in L^\infty(\mathbb{R}^2)$*, then for every branching point* $x \in \Sigma_{\mathrm{opt}}$ *there is* $\delta > 0$ *such that the set* $\Sigma_{\mathrm{opt}} \cap B_\delta(x)$ *consists of exactly three* $C^{1,1}$*-arcs starting at x.*

(vii) *If* $\varphi^\pm \in L^\infty(\mathbb{R}^2)$ *and* $x \in \Sigma_{\mathrm{opt}}$ *is not a branching point, then the Euclidean dimension of the set* $k^{-1}(x)$ *is at least* 1.

References

1. V. N. Sudakov, "Geometric problems of infinite-dimensional probability distributions" [in Russian], *Tr. Mat. Institute AN SSSR* **141**, 1976.
2. L. Ambrosio, "Lecture Notes on Optimal Transport Problems", In: *Mathematical Aspects of Evolving Interfaces, Lect. Notes Math.* **1812**, Springer-Verlag, 2003, pp. 1–52.
3. L. C. Evans, "Partial differential equations and Monge-Kantorovich mass transfer", In: *Current Developments in Mathematics*, Cambridge MA, 1997, pp. 65–126.
4. S. T. Rachev and L. Rüschendorf, *Mass Transportation Problems. Vol. I Theory, Vol. II Applications*, Berlin, Springer-Verlag, 1998.
5. G. Buttazzo, E. Oudet, and E. Stepanov, "Optimal transportation problems with free Dirichlet regions," In: *Progress in Nonlinear Diff. Equations and Their Applications* **51**, Birkhäuser 2002, pp. 41–65.
6. G. Buttazzo and E. Stepanov, "Minimization problems for average distance functionals", In: *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi* (D. Pallara, ed.), *Quad. Mat. Sec. Univ. Napoli* **14** (2004), 47–83.
7. E. Paolini and E. O. Stepanov, Qualitative properties of maximum distance minimizers and average distance minimizers in R^n" [in Russian], *Probl. Mat. Analiz.* **28** (2004), 105–122; English transl.: *J. Math. Sci. New York* **122** (2003), no. 3, 3290–3309.

8. F. Santambrogio and P. Tilli, "Blow-up of optimal sets in the irrigation problem", *J. Geom. Anal.* **15** (2005), no. 2, 343–362.
9. A. Suzuki and Z. Drezner, "The *p*-center location," *Location Science* **4** (1996), no. 1-2, 69–82.
10. A. Suzuki and A. Okabe, "Using Voronoi diagrams," In: *Facility Location: a Survey of Applications and Methods*, Springer Verlag, 1995, pp. 103–118.
11. F. Morgan and R. Bolton, "Hexagonal economic regions solve the location problem," *Am. Math. Month* **109** (2001), 165–172.
12. T. DeRose, T. Duchamp, H. Hoppe, J. McDonald, W. Stützle, "Fitting of surfaces to scattered data", In: *Proc. Curves and Surfaces in Computer Vision and Graphics 3*, SPIE Proceedings, **1830**, 1992, pp. 212–220.
13. T. Duchamp, W. Stützle, "Geometric properties of principal curves in the plane", *Robust Statistics, Data Analysis, and Computer Intensive Methods: in Honor of Peter Huber's 60th Birthday* (H. Rieder, ed.), *Lect. Notes Stat.* **109**, New York, Springer-Verlag, 1996.
14. T. Duchamp and W. Stützle, "Extremal properties of principal curves in the plane", *Ann. Statistics* **109** (1996), no. 4, 1511–1520.
15. M. Bernot, V. Caselles, and J.-M. Morel, *Traffic Plans*, Preprint, ENS Cachan, 2005.
16. P. Pollack, "Lazy Travelling Salesman Problem", Master's thesis, The Technion–Israel Institute of Technology, Haifa, 2002.
17. G. Buttazzo and G. Bouchitté, "Characterization of optimal shapes and masses through Monge-Kantorovich equation", *J. European Math. Soc.* **3** (2001), 139–168.
18. C. Castaing and M. Valadier, "Convex analysis and measurable multifunctions", *Lect. Notes Math.* **580**, Berlin, Springer-Verlag, 1977.
19. S. M. Srivastava, *A Course on Borel Sets*, *Graduate Texts in Mathematics*, **180**, New York, Springer-Verlag, 1998.
20. L. Ambrosio and P. Tilli, *Topics on Analysis in Metric Spaces*, Oxford, Oxford Univ. Press, , 2004.
21. C. Kuratowski, *Topologie. Vol. 1*, Warszawa, Państwowe Wydawnictwo Naukowe, 1958.
22. G. Buttazzo and E. Stepanov, "Optimal transportation networks as free Dirichlet regions in the Monge-Kantorovich problem," *Ann. Scuola Norm. Sup. Pisa Cl. Sci. II* (2003), no.4, 631–678.
23. G. Bouchitté, G. Buttazzo, and I. Fragalà, "Mean curvature of a measure and related variational problems", *Ann. Scuola Norm. Sup. Cl. Sci.* **25** (1997), no. 4, 179–196.
24. L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford, Oxford Univ. Press, 2000.
25. G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Providence, Am. Math. Soc., 1993.
26. G. Buttazzo and E. Stepanov, "Transport density in Monge-Kantorovich problems with Dirichlet conditions", *Discrete and Continuous Dynamical Systems-A* **12** (2005), no. 4, 607–628.