Journal of Mathematical Sciences, Vol. 130, No. 4, 2005

NONUNIQUENESS OF A SOLUTION TO THE PROBLEM ON MOTION OF A RIGID BODY IN A VISCOUS INCOMPRESSIBLE FLUID

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This paper is devoted to the problem on motion of a rigid body in a viscous incompressible fluid. It is proved that there exist at least two weak solutions of this problem if collisions of the body with the boundary of the flow domain are allowed. These solutions have different behavior of the body after the collision. Namely, for the first solution, the body goes away from the boundary after the collision. In the second solution, the body and the boundary remain in contact. Bibliography 15 *titles.*

To Vsevolod Alekseevich Solonnikov
on the occasion of his jubilee

1. Introduction

The problem of motion of an absolutely rigid body in a viscous incompressible fluid (problem A below) was considered by many authors; this problem is studied relatively well (see [1–13]). At the same time, there exists a lot of open problems. In particular, the uniqueness of a weak solution has not been proved. Since problem A includes the Navier–Stokes equations, we cannot hope to get a fast answer to the latter problem in the case of three space variables. It might seem that in the two-dimensional case, the problem should have a unique solution. In this paper, we show that this is not always so.

We have to distinguish the following two cases. In the first case, we consider the problem without mutual collisions of the bodies or collisions of the bodies with the boundary (i.e., we consider the period before the first collision). In the second case, collisions are allowed. The first case is simpler; it was investigated in detail. In particular, it was proved in [13] that problem A with two space variables has a unique strong solution on any time interval in which there are no collisions of the body with the boundary. In this paper, we consider the second, more general, case. Our result is more negative than positive. Namely, we show that in the existing generalized setting of problem A, at least two weak solutions exist. At the same time, the problem does not have strong solutions on a time interval including collision moments (see [7, 11]).

We restrict our considerations to the two-dimensional case. In addition, we assume that the volume contains a single rigid body and the densities of the body and fluid coincide (and equal 1).

To define a generalized solution of problem A, we need some special function spaces. Let $\mathbf{u}$ be a vector field in $\mathbb{R}^2$. We denote by $D(\mathbf{u})$ the tensor with components

$$D_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

If $\mathbf{u}$ is a velocity, then $D(\mathbf{u})$ is the tensor of deformation velocities. This tensor has the following important property: if $\mathcal{S}$ is a connected domain, then $D(\mathbf{u}) = 0$ in $\mathcal{S}$ if and only if there exists a vector $\mathbf{a}$ and a skew-symmetric tensor $\mathcal{Q}$ such that

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathcal{Q}\mathbf{x}$$

for all $\mathbf{x} \in \mathcal{S}$. In this case, the velocity field $\mathbf{u}$ corresponds to a rigid body motion. One can find a proof of this fact in [14]. Thus, we may consider a rigid body as a medium for which the field of deformation velocities vanishes. Our setting of the problem is based on the mentioned property of the tensor $D(\mathbf{u})$.

Consider domains $\mathcal{S}$ and Ω in $\mathbb{R}^2$ such that $\mathcal{S} \subset \Omega$. We work with the following function spaces:

$$H(\Omega) = \{\mathbf{u} \in L_2(\Omega) \mid \ \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega\},$$
$$V(\Omega) = \{\mathbf{u} \in W_2^1(\Omega) \mid \ \operatorname{div} \mathbf{u} = 0 \ \text{ in } \ \Omega \quad \text{and} \ \ \mathbf{u} = 0 \quad \text{on} \quad \ \partial\Omega\},$$

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 306, 2003, pp. 199–209. Original article submitted October 20, 2003.

and

$$K(\mathcal{S},\Omega) = \{\mathbf{u} \in V(\Omega) \mid D(\mathbf{u}) = 0 \text{ in } \mathcal{S}\}.$$

We denote by $V'(\Omega)$ the dual space of $V(\Omega)$. We identify the space $H(\Omega)$ with its dual space. The space $K(\mathcal{S},\Omega)$ was introduced and studied in [2, 3, 6] (see also [7, 11]).

Let $\mathcal{S}(t)$ be the domain in Ω occupied by the body at the time moment $t \in [0,T]$, $T < \infty$. Since the body is absolutely rigid and its form is not changed during motion, there exists a family of orientation preserving isometries $\mathcal{M}_{s,t} : \mathbb{R}^2 \to \mathbb{R}^2$ $\big(s,t \in [0,T]\big)$ such that $\mathcal{S}(t) = \mathcal{M}_{s,t}\mathcal{S}(s)$. We use the same notation for the sets $\mathcal{S}(t)$ and for their indicators. Thus,

$$\mathcal{S}(\mathbf{x},t) = \begin{cases} 1, & \mathbf{x} \in \mathcal{S}(t), \\ 0, & \mathbf{x} \in \Omega \setminus \mathcal{S}(t). \end{cases}$$

Finally, we denote by $L^q(0,T;K(\mathcal{S},\Omega))$ the space of functions $\mathbf{u} \in L^q(0,T;V(\Omega))$ such that $\mathbf{u}(\cdot,t) \in K(\mathcal{S}(t),\Omega)$ for almost all $t \in [0,T]$.

The following definition of a generalized solution to problem A was introduced in [2, 3].

Definition 1.1. *We say that a pair of functions $\{\mathbf{u},\mathcal{S}\}$ such that*

$$\mathbf{u} \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;K(\mathcal{S},\Omega))$$

and

$$\mathcal{S} \in C^{0,1/q}(0,T;L^q(\Omega)), \quad 1 \le q < \infty,$$

is a generalized solution to problem A if

(1) *$\mathcal{S}(\cdot,t)$ is the indicator of a domain $\mathcal{S}(t) \subset \Omega$;*
(2) *there exists a family of orientation preserving isometries $\mathcal{M}_{s,t} : \mathbb{R}^2 \to \mathbb{R}^2\big(s,t \in [0,T]\big)$ such that*

$$\mathcal{S}(t) = \mathcal{M}_{s,t}\mathcal{S}(s) \quad \text{for all} \quad s,t \in [0,T]; \tag{1.1}$$

(3) *the integral identities*

$$\int\limits_{\Omega_T} \Big(\mathbf{u}\cdot\mathbf{w}_t + \big(\mathbf{u}\otimes\mathbf{u} - D(\mathbf{u})\big) : D(\mathbf{w}) + \mathbf{g}\cdot\mathbf{w}\Big)\, d\mathbf{x}dt = -\int\limits_{\Omega} \mathbf{u}_0 \cdot \mathbf{w}_0 \, d\mathbf{x} \tag{1.2}$$

and

$$\int\limits_{\Omega_T} \mathcal{S}\left(\eta_t + \mathbf{u}\cdot\nabla\eta\right) d\mathbf{x}dt = -\int\limits_{\Omega} \mathcal{S}_0\eta_0 \, d\mathbf{x} \tag{1.3}$$

hold for any functions

$$\mathbf{w} \in W_2^1(\Omega_T) \cap L^2(0,T;K(\mathcal{S},\Omega))$$

and $\eta \in C^1(\Omega_T)$ such that $\mathbf{w}|_{t=T} = 0$ and $\eta|_{t=T} = 0$.

Here $\Omega_T = \Omega \times [0,T]$, $\mathbf{g}$ is a vector-function determining the exterior mass forces, and $\mathbf{u}_0$, $\mathbf{w}_0$, $\mathcal{S}_0$, and η_0 denote the values of the corresponding functions at $t = 0$.

The following existence result for problem A has been proved in [2, 3] (see also [7, 10]).

Theorem 1.2. *Let Ω and $\mathcal{S}_0$ be bounded connected domains of class $C^{1,1}$ in $\mathbb{R}^2$ such that $\mathcal{S}_0 \subset \Omega$. Assume that $\mathbf{u}_0 \in H(\Omega)$. Then a generalized solution to problem A exists for any $T \in (0,\infty)$. In addition, the isometries $\mathcal{M}_{s,t}$ are Lipschitz continuous in s and t.*

The main result of this paper states that problem A is not uniquely solvable. In the following two sections, we prove the following statement.

Theorem 1.3. *There exist $\mathcal{S}_0 \subset \Omega$, $\mathbf{u}_0 \in H(\Omega)$, and $\mathbf{g} \in L^2(0,T;V'(\Omega))$ such that problem A has at least two different solutions.*

The idea of the proof is as follows. The domains Ω and $\mathcal{S}$ are disks. In Sec. 2, we construct a velocity field for which the body approaches the boundary $\partial\Omega$ of the domain Ω, touches this boundary at $t_* \in (0,T)$, and then moves away from the boundary. Proposition 2.4 states that there exists a field $\mathbf{g}$ of exterior mass forces for which the constructed velocity field is a solution to problem A. Further, in Sec. 3 we construct a second solution for the same function $\mathbf{g}$. This solution coincides with the first one until the moment t_* and is extended so that the body does not move away from the boundary. The possibility of existence of solutions of the second kind was mentioned in [12]. From the physical point of view, such solutions are not admissible since the adhesion of the body to the boundary of the fluid domain presupposes the existence of forces which are not present in the statement of the problem. The solution has to "decide" whether the body stays near the boundary or moves away from it. Thus, Definition 1.1 must contain additional relations which would forbid "incorrect" solutions.

2. Construction of the first solution

Denote by $B_\rho(\mathbf{b})$ the open disk of radius ρ centered at a point $\mathbf{b}$. Assume that $\Omega = B_R(\mathbf{0})$ and $\mathcal{S}(t) = B_r(\mathbf{a}(t))$, where $R > r > 0$, $\mathbf{0} = (0,0)$, and $\mathbf{a}(t) = (a(t), 0)$. Later we fix a function $a \in W^1_\infty(0,T)$ such that $\mathcal{S}(t) \subset \Omega$ for all t, i.e., $|a(t)| \le R - r$. Often, it is convenient for us to use the function $\sigma(t) = a(t)(R-r)^{-1}$ instead of $a(t)$. We also assume that there exists $t_* \in (0,T)$ such that the body touches the boundary at time t_*, i.e., $\operatorname{dist}(\mathcal{S}(t_*), \partial\Omega) = 0$. Note that $\sigma(t) \in [0,1)$ for $t \in [0,t_*)$ and $\sigma(t) \to 1$ as $t \to t_*$. Let us define a mapping $\mathbf{F}(\cdot,t)$ taking a point $\boldsymbol{\xi} \in \Omega$ to $\mathbf{x} \in \Omega$ by the following rule:

$$x_1 = F_1(\xi_1,\xi_2,t) = \xi_1 + \sigma(t)\Big(R - \sqrt{\xi_1^2 + \xi_2^2}\,\Big),$$
$$x_2 = F_2(\xi_1,\xi_2,t) = \xi_2.$$

It is easy to check that $\mathbf{F}(\Omega,t) = \Omega$ and

$$\mathcal{S}(t) = \mathbf{F}\big(B_r(\mathbf{0}),t\big). \tag{2.1}$$

In addition, if $\boldsymbol{\xi} \in \partial\Omega$, then $\mathbf{F}(\boldsymbol{\xi},t) = \boldsymbol{\xi}$. The inverse mapping $\mathbf{G}$ is as follows:

$$\xi_1 = G_1(x_1,x_2,t) = \frac{1}{1-\sigma^2}\Big(x_1 - \sigma R + \sigma\sqrt{x_2^2(1-\sigma^2) + (\sigma R - x_1)^2}\,\Big),$$
$$\xi_2 = G_2(x_1,x_2,t) = x_2,$$

where σ is $\sigma(t)$. Thus, $\mathbf{G}(\Omega,t) = \Omega$ and $\mathbf{G}\big(\mathcal{S}(t),t\big) = B_r(\mathbf{0})$. Since the mapping $\mathbf{G}$ is defined for $\sigma \in (-1,1)$ only, we first assume that $t \in [0,t_*)$.

Fig. 1. Mappings **F** and **G**.

Note that we do not treat (x_1,x_2,t) and (ξ_1,ξ_2,t) as the Euler or Lagrange coordinates. The advantage of coordinates $\boldsymbol{\xi}$ is that the domain $\Omega \setminus \mathcal{S}(t)$ is more symmetric; namely, Ω and $\mathcal{S}$ are concentric disks (see Fig. 1).

Relate polar coordinates (ρ,θ) to $\boldsymbol{\xi} = (\xi_1,\xi_2)$:

$$\xi_1 = \rho\cos\theta \quad \text{and} \quad \xi_2 = \rho\sin\theta.$$

In these coordinates, the Jacobians of the mappings $\mathbf{F}$ and $\mathbf{G}$ are given by the following simple expressions:

$$J_F(\boldsymbol{\xi},t) = \det\Big(\frac{\partial\mathbf{F}}{\partial\boldsymbol{\xi}}\Big) = 1 - \sigma(t)\cos\theta$$

and

$$J_G\big|_{\mathbf{x}=\mathbf{F}(\boldsymbol{\xi},t)} = \det\Big(\frac{\partial\mathbf{G}}{\partial\mathbf{x}}\Big)\Big|_{\mathbf{x}=\mathbf{F}(\boldsymbol{\xi},t)} = J_F^{-1} = \frac{1}{1-\sigma(t)\cos\theta}.$$

The mapping $\mathbf{F}$ determines the dynamics of the geometry of the fluid motion. We assume that the body moves without rotation from the center of the domain Ω to its boundary. Thus, there exists a family $\mathcal{M}_{s,t}$

of isometries satisfying relations (1.1). These isometries are shifts of the domain $\mathcal{S}$ along the x_1 axis. Let us construct a velocity field $\mathbf{u}$ corresponding to the situation considered above. This field must satisfy the following relations:

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad D(\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{S}.$$

Since in the two-dimensional case to a velocity field there corresponds a unique flow function, we first construct a flow function ψ, and then find the velocity field $\mathbf{u}$.

Consider a smooth function $\lambda : [0, \infty) \to [0, 1]$ such that $\lambda(\rho) = 1$ for $\rho < r$ and $\lambda'(r) = \lambda(R) = \lambda'(R) = 0$. For example, one may take

$$\lambda(\rho) = (R - r)^{-3}(\rho - R)^2(2\rho - 3r + R).$$

In particular, if $R = 3$ and $r = 1$, then $\lambda(\rho) = \frac{1}{4}\rho(\rho - 3)^2$. Define the flow function as follows:

$$\psi(\mathbf{x}, t) = \dot{a}(t)\, x_2\, \varphi(\mathbf{G}(\mathbf{x}, t)),$$

where $\varphi(\boldsymbol{\xi}) = \lambda\big(\sqrt{\xi_1^2 + \xi_2^2}\,\big)$ and $\dot{a} = da/dt$. Level lines of this function are shown schematically in Fig. 2.

Fig. 2. Level lines of the function ψ for various values of σ

The corresponding velocity field $\mathbf{u}(\mathbf{x}, t) = \nabla_x^{\perp}\psi(\mathbf{x}, t)$ has the following components:

$$\begin{aligned} u_1(\mathbf{x}, t) &= \frac{\partial\psi}{\partial x_2} = \dot{a}(t)\, \varphi\big(\mathbf{G}(\mathbf{x}, t)\big) + \dot{a}(t)\, x_2\, \mathbf{G}_{x_2} \cdot \nabla_\xi \varphi(\boldsymbol{\xi})|_{\boldsymbol{\xi}=\mathbf{G}(\mathbf{x},t)}, \\ u_2(\mathbf{x}, t) &= -\frac{\partial\psi}{\partial x_1} = -\dot{a}(t)\, x_2\, \mathbf{G}_{x_1} \cdot \nabla_\xi \varphi(\boldsymbol{\xi})|_{\boldsymbol{\xi}=\mathbf{G}(\mathbf{x},t)}. \end{aligned} \tag{2.3}$$

Note that $\mathbf{u}$ satisfies conditions (2.2). Let us study some properties of the function $\mathbf{u}$.

Lemma 2.1. *The function* $\mathbf{u}$ *belongs to the space* $L^\infty(0, t_*;\, L^2(\Omega))$ *if and only if* $\dot{\sigma} \in L^\infty(0, t_*)$.

We can find a closed-form expression of the norm of the function $\mathbf{u}$ in the space $L^2(\Omega)$:

$$\|\mathbf{u}\|^2_{L^2(\Omega)} = \|u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)} = \mu_1(R, r)\, \nu_1(\sigma)\, |\dot{\sigma}|^2, \tag{2.4}$$

where $\mu_1(R, r) = \int_r^R \big(\lambda'(\rho)\big)^2 \rho^3\, d\rho < \infty$ and

$$\nu_1(\sigma) = \int_0^{2\pi} \frac{\sin^2\theta}{1 - \sigma\cos\theta}\, d\theta = 2\pi\, \frac{1 - \sqrt{1 - \sigma^2}}{\sigma^2} = 2\pi\, \frac{1}{1 + \sqrt{1 - \sigma^2}}.$$

Since $\nu_1(\sigma) \in [\pi, 2\pi]$ for any σ, Lemma 2.1 is proved.

Lemma 2.2. *If* $(1 - \sigma)^{-3/2}|\dot{\sigma}|^2 \in L^1(0, t_*)$, *then the function* $\mathbf{u}$ *belongs to the space* $L^2(0, t_*; V(\Omega))$.

To prove this statement, it is enough to estimate $\Delta\psi$ in the space $L^2(0, t_*;\, L^2(\Omega))$. After some simple but bulky computations, we obtain the estimate

$$\|\Delta\psi\|^2_{L^2(\Omega)} \le \mu_2(R, r)\nu_2(\sigma)\, |\dot{\sigma}|^2,$$

where $\nu_2(\sigma) = (1 - \sigma)^{-3/2}$ and μ_2 is a positive constant depending on R and r only. The estimate above proves Lemma 2.2.

Lemma 2.3. *If* $\ddot{\sigma} \in L^2(0, t_*)$, *then* $\mathbf{u}_t \in L^2(0, t_*; V'(\Omega))$.

There exists a constant $\mu_3(R, r)$ depending on R and r only and such that

$$\|\psi_t\|_{L^2(\Omega)} \le \mu_3(R, r)\big(|\ddot{\sigma}| + |\dot{\sigma}|^2\big).$$

This estimate proves Lemma 2.3 since if the function $\ddot{\sigma}$ belongs to the space $L^2(0, t_*)$, then the function $\dot{\sigma}$ is bounded.

Consider the following function σ :

$$\sigma(t) = 1 - (t - t_*)^4 T^{-4}. \tag{2.5}$$

It is easy to see that $|\sigma(t)| < 1$ for $t \in [0, t_*) \cup (t_*, T]$ and $\sigma(t_*) = 1$. In addition, the function σ satisfies the conditions of Lemmas 2.1, 2.2, and 2.3. Hence, the function $\mathbf{u}$ defined by (2.3) belongs to the space $L^\infty(0, t_*; H(\Omega)) \cap L^2(0, t_*; V(\Omega))$ and the time derivative of this function belongs to the space $L^2(0, t_*; V'(\Omega))$. Hence, the function $\mathbf{u}$ is weakly continuous in t in $L^2(\Omega)$. What is more, $\mathbf{u}$ is strongly continuous in t in $L^2(\Omega)$ since equality (2.4) implies that $\|\mathbf{u}(\cdot, t)\|_{L^2(\Omega)}$ is continuous in t.

Extend the function $\mathbf{u}$ to $t = t_*$ by continuity. For $t \in (t_*, T]$, we define this function by equality (2.3), where the function σ is given by (2.5). The resulting function $\mathbf{u}$ has the following properties:

$$\mathbf{u} \in L^\infty(0, T; H(\Omega)) \cap L^2(0, T; V(\Omega)) \quad \text{and} \quad \mathbf{u}_t \in L^2(0, T; V'(\Omega)). \tag{2.6}$$

In addition,

$$\mathbf{u} \in L^2(0, T; K(\mathcal{S}, \Omega)).$$

Mow we can prove the following statement.

Proposition 2.4. *Let functions σ, $\mathbf{u}$, and $\mathcal{S}$ be given by equalities* (2.5), (2.3) *and* (2.1)*, respectively. Then there exists a function $\mathbf{g} \in L^2(0, T; V'(\Omega))$ such that the pair $\{\mathbf{u}, \mathcal{S}\}$ is a generalized solution to problem A.*

Proof. First we note that identity (1.3) holds by construction. It follows from inclusions (2.6) that we may write identity (1.2) as follows:

$$\int_{\Omega_T} \Big(\mathbf{u}_t \cdot \mathbf{w} - \big(\mathbf{u} \otimes \mathbf{u} - D(\mathbf{u})\big) : D(\mathbf{w})\Big)\, d\mathbf{x}dt = \int_{\Omega_T} \mathbf{g} \cdot \mathbf{w}\, d\mathbf{x}dt. \tag{2.7}$$

We have to find a function $\mathbf{g}$ such that equality (2.7) holds for any function $\mathbf{w} \in L^2(0, T; K(\mathcal{S}, \Omega))$.

By (2.6), the left-hand side of (2.7) is a continuous linear functional in $L^2(0, T; V(\Omega))$ applied to a function $\mathbf{w}$. Hence, there exists a function $\mathbf{g} \in L^2(0, T; V'(\Omega))$ such that identity (2.7) holds for any function $\mathbf{w} \in L^2(0, T; V(\Omega))$; hence, if holds for any $\mathbf{w} \in L^2(0, T; K(\mathcal{S}, \Omega))$. Proposition 2.4 is proved. □

3. Construction of the second solution

In the previous section, we have constructed a solution of problem A for which the vector $\mathbf{g}$ of mass forces is given by Proposition 2.4. Now we prove the existence of a second solution with the same function $\mathbf{g}$. Denote by $\mathbf{v}$ the velocity field for the new solution and denote by $\mathcal{R}$ the domain occupied with the rigid body. Set

$$\mathbf{v}(\cdot, t) = \mathbf{u}(\cdot, t) \quad \text{and} \quad \mathcal{R}(t) = \mathcal{S}(t), \quad t \in [0, t_*]. \tag{3.1}$$

Recall that the body touches the boundary at t_*, i.e., $\operatorname{dist}(\mathcal{S}(t_*), \partial\Omega) = 0$. Define the functions $\mathbf{v}$ and $\mathcal{R}$ for $t \in (t_*, T]$ as follows:

$$\begin{aligned} &\mathcal{R}(t) = \mathcal{R}_* = \mathcal{S}(t_*) \qquad \text{for} \quad t \in (t_*, T], \\ &\mathbf{v}(\mathbf{x}, t) = 0 \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}(t), \quad t \in (t_*, T]. \end{aligned} \tag{3.2}$$

In the domain $\Omega \setminus \mathcal{R}_*$, the function $\mathbf{v}$ is a solution of the Navier–Stokes equations with the right-hand side $\mathbf{g}$ and zero boundary values. Since $\mathbf{u}(\mathbf{x}, t_*) = 0$ for any $\mathbf{x} \in \Omega$, the integral identity

$$\int_{t_*}^{T} \int_{\Omega \setminus \mathcal{R}_*} \Big(\mathbf{v}_t \cdot \mathbf{w} - \big(\mathbf{v} \otimes \mathbf{v} - D(\mathbf{v})\big) : D(\mathbf{w}) - \mathbf{g} \cdot \mathbf{w}\Big)\, d\mathbf{x}dt = 0 \tag{3.3}$$

holds for any function $\mathbf{w} \in L^2(t_*, T; V(\Omega \setminus \mathcal{R}_*))$. The general theory of the Navier–Stokes equations (see [15]) implies the existence of a unique function $\mathbf{v} \in L^\infty(t_*, T; H(\Omega \setminus \mathcal{R}_*)) \cap L^2(t_*, T; V(\Omega \setminus \mathcal{R}_*))$ such that $\mathbf{v}_t \in L^2(t_*, T; V'(\Omega \setminus \mathcal{R}_*))$ and identity (3.3) holds. The domain $\Omega \setminus \mathcal{R}_*$ is not smooth, but this fact creates no complications since we consider the problem with zero boundary values.

Thus, the function $\mathbf{v}$ is defined everywhere in Ω_T and the function $\mathcal{R}$ is defined for all $t \in [0, T]$.

Proposition 3.1. *The functions* $\mathbf{v}$ *and* $\mathcal{R}$ *give us a generalized solution of problem A.*

Proof. Since $\mathbf{v} = 0$ in $\mathcal{R}(t)$ for $t \in [t_*, T]$, the inclusion $\mathbf{v} \in L^2(0, T; K(\mathcal{R}, \Omega))$ holds. Thus, we have to check identities (1.2) and (1.3) with $\mathbf{u}$ and $\mathcal{S}$ replaced by $\mathbf{v}$ and $\mathcal{R}$, respectively. By Proposition 2.4 and relations (3.1), it is enough to integrate in these identities over the time interval from t_* to T. Identity (1.3) follows from relation (3.2). We write identity (1.2) as follows:

$$\int_{t_*}^{T} \int_{\Omega} \Big(\mathbf{v}_t \cdot \mathbf{w} - \big(\mathbf{v} \otimes \mathbf{v} - D(\mathbf{v}) \big) : D(\mathbf{w}) - \mathbf{g} \cdot \mathbf{w} \Big) \, d\mathbf{x} dt = 0. \tag{3.4}$$

Let us check that equality (3.4) holds for any $\mathbf{w} \in L^2(t_*, T;\, K(\mathcal{R}, \Omega))$.

Note that equalities (3.4) and (3.3) differ by the integration domains and test functions only. We need the following property of functions from the space $K(\mathcal{R}, \Omega)$ (see [2, 3, 7, 11]).

Lemma 3.2. *Let* $\Omega, \mathcal{R} \subset \mathbb{R}^2$ *be connected domains of class* $C^{1,\alpha}$*, where* $\alpha \in [\frac{2}{3}, 1]$*, such that* $\mathcal{R} \subset \Omega$ *and* $\operatorname{dist}(\mathcal{R}, \partial\Omega) = 0$*. If* $\mathbf{w} \in K(\mathcal{R}, \Omega)$*, then* $\mathbf{w}(\mathbf{x}) = 0$ *for* $\mathbf{x} \in \mathcal{R}$*.*

Identity (3.4) is an immediate corollary of Lemma 3.2 and identity (3.3).

Proposition 3.1 is proved. □

Obviously, the constructed solution $\{\mathbf{v}, \mathcal{R}\}$ to problem A differs from the solution $\{\mathbf{u}, \mathcal{S}\}$ described in Sec. 2. Thus, Theorem 1.3 is proved.

Translated by S. Yu. Pilyugin.

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