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SOME APPLICATIONS OF THE DUHAMEL PRODUCT

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The Duhamel product of functions f and g is defined by the formula

$$(f \circledast g)(x) = \frac{d}{dx} \int_0^x f(x-t)g(t)dt.$$

In the present paper, the Duhamel product is used in the study of spectral multiplicity for direct sums of operators and in the description of cyclic vectors of the restriction of the integration operator $f(x,y) \mapsto \int_0^x \int_0^y f(t,\tau)d\tau\, dt$ *in two variables to its invariant subspace consisting of functions that depend only on the product xy. Bibliography:* 13 *titles.*

Introduction

1. Let $\mathrm{Hol}\,(\mathbb{D})$ be the space of functions that are holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; we consider this space with the topology of compact convergence. In the space $\mathrm{Hol}\,(\mathbb{D})$, the Duhamel product is defined as the derivative of the Mikusinski convolution:

$$(f \circledast g)(z) \overset{\text{def}}{=} \frac{d}{dz} \int_0^z f(z-t)g(t)dt = \int_0^z f'(z-t)g(t)dt + f(0)g(z),$$

where the integrals are taken over the segment joining the points 0 and z (see [1]).

The Duhamel product is widely applied in various domains of calculus, for example, in the theory of differential equations with constant coefficients and in solution of some boundary-value problems of mathematical physics; the real-valued analog of this product plays an important role in the Mikusinski operator calculus (see [1–3]). In the work [1], Wigley applied the Duhamel product for the first time to describe closed ideals of the algebra $\mathrm{Hol}\,(\mathbb{D})$. In the author's works [4, 5], the product was applied in the description of cyclic vectors of the integration operator J, $(Jf)(x) = \int_0^x f(t)dt$, for some function spaces and in the proof of the unicellularity of the integration operator in the Banach space of functions holomorphic in $\mathbb{D}$. See [6–8] for other applications of the Duhamel product.

2. In Sec. 2 of this work, we apply the Duhamel product in the study of multiplicity of spectra for direct sums of operators. In Sec. 3, we apply an analog of the Duhamel product for functions of two variables to describe cyclic vectors of the restriction of the double integration operator,

$$f(x,y) \mapsto \int_0^x \int_0^y f(t,\tau)d\tau\, dt,$$

to an invariant subsapce of a special form.

Some of the results of this work have been obtained long ago; they are contained in the autor's thesis [9]. The author decided to publish these results in a journal paper, in particular, since new applications were found.

3. We use more or less standard notation. We denote by $L(X)$ the algebra of bounded linear operators in a Banach space X, $\mathrm{Lat}\,A$ denotes the lattice of invariant subspaces of an operator A, and $\mathrm{span}\,\{x_i\}$ is the closed linear hull of vectors x_i.

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Recall that a subspace $E \subset X$ is called a cyclic (generating) subspace for an operstor $A \in L(X)$ if span $\{A^n E : n \geq 0\} = X$. A vector $x \in X$ is called cyclic if

$$\text{span}\,\{A^n x \colon n \geq 0\} = X.$$

We denote by Cyc (A) the set of cyclic vectors of an operator A. The multiplicity of the spectrum of an operator A is

$$\mu(A) \stackrel{\text{def}}{=} \inf\bigl\{\dim E : \text{span}\,\{A^n E : n \geq 0\} = X\bigr\}$$

(or the symbol ∞). An operator $A \in L(X)$ is called cyclic if $\mu(A) = 1$; an operator A is called unicellular if the lattice Lat A is linearly ordered with respect to inclusions.

§1. Duhamel product and some of its properties

Let Hol $(\mathbb{D})$ be the space of functions that are holomorphic in the unit disk $\mathbb{D}$; we consider this space with the topology of uniform convergence on compact subsets of $\mathbb{D}$. For two functions $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$ and $g(z) = \sum_{n\geq 0} \widehat{g}(n) z^n \in \text{Hol}\,(\mathbb{D})$ (where $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$ is the nth Taylor coefficient of a function f), the Duhamel product is defined as follows (see [1]):

$$(f \circledast g)(z) \stackrel{\text{def}}{=} \frac{d}{dz} \int_0^z f(z-t) g(t)\, dt, \tag{1}$$

where the integral is taken over the segment joining the points 0 and z. It is easy to see that the Duhamel product satisfies all the axioms of multiplication, Hol $(\mathbb{D})$ is an algebra with respect to $\circledast$ as well, and the function $f(z) \equiv \mathbb{1}$ is the unit element of the algebra $(\text{Hol}\,(\mathbb{D}), \circledast)$.

Let B be the Borel transformation acting from Hol $(\mathbb{D})$ into the space $\mathbb{C}[[\mathbb{Z}]]$ of formal power series over the field $\mathbb{C}$ of complex numbers; this transformation is defined by the following formula:

$$B\biggl(\sum_{n\geq 0} \widehat{f}(n) z^n\biggr) \stackrel{\text{def}}{=} \sum_{n\geq 0} n! \widehat{f}(n) \mathbb{Z}^n.$$

The inverse Borel transformation B^{-1} acts by the following formula:

$$B^{-1}\biggl(\sum_{n\geq 0} a_n \mathbb{Z}^n\biggr) \stackrel{\text{def}}{=} \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

The following simple but useful statement holds.

Lemma 1. *Let f, $g \in$ Hol $(\mathbb{D})$. The following equalities are valid for the product $\circledast$ in* Hol $(\mathbb{D})$*:*

(a) $f \circledast g = \sum_{n\geq 0} \sum_{k=0}^{n} \widehat{f}(k) \widehat{g}(n-k) k!\,(n-k)! \frac{1}{n!} z^n$;

(b) $f \circledast g = (Bf)(J)g = (Bg)(J)f$, *where J is the integration operator in* Hol $(\mathbb{D})$ *and* $(Bf)(J)g \stackrel{\text{def}}{=} \sum_{n\geq 0} n! \widehat{f}(n) (J^n g)(z)$;

(c) $f \circledast g = B^{-1}(Bf \cdot Bg)$.

Proof. Proofs of statements (a) and (c) can be found in [1]; let us prove item (b). First we show that the series $\sum_{n\geq 0} n!\, \widehat{f}(n)(J^n g)(z)$ defines a function in Hol $(\mathbb{D})$. For this purpose, we show that the series above converges uniformly on any compact subset of the disk $\mathbb{D}$.

Fix a number r, $0 < r < 1$. It is enough to show that the series converges absolutely and uniformly on the disk $|z| < r < 1$.

The formulas

$$(J^n g)(rz) = r^n J^n(g(rz)) \quad (n = 0, 1, 2, \dots)$$

and estimates

$$|(J^n g)(z)| = \left| \int_0^z \frac{(z-t)^{n-1}}{(n-1)!} g(t)dt \right| \le \frac{1}{(n-1)!} \sup_{|t|\le|z|} |g(t)|$$

imply that

$$\begin{gathered} \sum_{n\ge0} n!\, |\widehat{f}(n)|\, |(J^n g)(rz)| = \sum_{n\ge0} n!\, |\widehat{f}(n)| r^n |J^n(g_r(z))| \\ \le \sum_{n\ge0} n!\, |\widehat{f}(n)| \frac{1}{(n-1)!} r^n \sup_{|z|<r} |g_r(z)| = \sup_{|z|<r} |g_r(z)| \sum_{n\ge0} n|\widehat{f}(n)| r^n < +\infty \end{gathered}$$

(since it follows from the inclusion $f \in \operatorname{Hol}(\mathbb{D})$ that $\sum\limits_{n\ge0} n|\widehat{f}(n)|r^n < +\infty$). The desired convergence is established. Now we see that

$$\begin{aligned} (Bf)(J)g &= \sum_{n\ge0} n!\, \widehat{f}(n)(J^n g) = \sum_{n\ge0} n!\, \widehat{f}(n) \left(J^n \sum_{m\ge0} \widehat{g}(m) z^m \right) = \sum_{n\ge0} \sum_{m\ge0} \widehat{f}(n)\, \widehat{g}(m) n!\, (J^n z^m) \\ &= \sum_{n\ge0} \sum_{m\ge0} \widehat{f}(n)\, \widehat{g}(m) \frac{n!}{(m+1)\cdots(m+n)} z^{n+m} = \sum_{n\ge0} \sum_{m\ge0} \widehat{f}(n)\, \widehat{g}(m) \frac{n!\, m!}{(n+m)!} z^{n+m} = f \circledast g. \end{aligned}$$

By symmetry, the second equality in (b) is checked in a similar way. □

Finally, let as mention the following basic property of the Duhamel product; this property was established by Wigley in [1].

Lemma 2. *A function $f \in \operatorname{Hol}(\mathbb{D})$ is $\circledast$-invertible in $\operatorname{Hol}(\mathbb{D})$ if and only if $f(0) \neq 0$.*

One can find in [1] more information concerning the Duhamel product.

§2. Special criteria of adding for multiplicities of spectra

In this section, we apply the Duhamel product to calculate multiplicities of spectra for direct sums of the form $J \oplus A$, where J is the integration operator in the Wiener algebra of functions $f(z) = \sum\limits_{n\ge0} \widehat{f}(n) z^n$ that are holomorphic in $\mathbb{D}$ and satisfy the condition $\sum\limits_{n\ge0} |\widehat{f}(n)| < +\infty$ and A is an operator in a proper Banach space (Theorem 3). This result seems to be new even for the operator $J \oplus J$ in $W(\mathbb{D}) \oplus W(\mathbb{D})$, though a similar statemenr for $L^p[0,1] \oplus L^p[0,1]$ $(1 \le p < +\infty)$ was established in the Malamud work [10].

Note that $(W(\mathbb{D}), \circledast)$ is a Banach algebra. The unique maximal ideal of this algebra consists of functions f from $W(\mathbb{D})$ such that $f(0) = 0$. Hence, the space of maximal ideals $\mathfrak{M}(W(\mathbb{D}), \circledast)$ of this algebra consists of a single homomorphism, namely, the evolution functional at the point 0 (i.e., $\mathfrak{M}(W(\mathbb{D}), \circledast) = \{0\}$). Hence, the Gelfand transformation is trivial. (More information concerning the algebra $(W(\mathbb{D}), \circledast)$ can be found, for example, in the author's work [7].)

Theorem 3. *Let X be a separable Banach space and let J, $(Jf)(z) = \int_0^z f(t)dt$, be the integration operator in the Wiener algebra $W(\mathbb{D})$. If an operator $\mathcal{K} \in L(X)$ satisfies the condition $\|\mathcal{K}^n\| \le c\frac{1}{n!}$, $n \ge 0$, for some $c > 0$, then*

$$\mu(J \oplus \mathcal{K}) = \mu(J) + \mu(\mathcal{K}) = 1 + \mu(\mathcal{K}).$$

Proof. For any operators $A_1 \in L(X_1)$ and $A_2 \in L(X_2)$, the estimates

$$\max\{\mu(A_1), \mu(A_2)\} \le \mu(A_1 \oplus A_2) \le \mu(A_1) + \mu(A_2)$$

hold. Hence, if $\mu(\mathcal{K}) = +\infty$, then the proof is trivial. Assume that $\mu(\mathcal{K}) = n < +\infty$. Assume, in addition, that $\mu(J \oplus \mathcal{K}) = \mu(\mathcal{K}) = n$. Let $\{f_i \oplus x_i\}_{i=1}^n$ be a cyclic set of vectors for the operator $J \oplus \mathcal{K}$. In this case, $\{f_i(z)\}_{i=1}^n \in \operatorname{Cyc}(J)$. Hence, there exists an index i_0, $1 \le i_0 \le n$, such that $f_{i_0}(0) \ne 0$. Without loss of generality, we assume that $i_0 = 1$, i.e., $f_1(0) \ne 0$. Under this condition, the element f_1 is invertible in $(W(\mathbb{D}), \circledast)$. Hence, there exists a function $F_1 \in W(\mathbb{D})$ such that $(F_1 \circledast f_1)(z) \equiv \mathbb{1}$. In this case, $F_1(0) \ne 0$. Consider the matrix

$$\mathcal{F}(z) = \begin{pmatrix} F_1 & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ -f_2 \circledast F_1 & \mathbb{1} & \mathbb{O} & & \mathbb{O} \\ -f_3 \circledast F_1 & \mathbb{O} & \mathbb{1} & \cdots & \mathbb{O} \\ \vdots & & & & \vdots \\ -f_n \circledast F_1 & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{1} \end{pmatrix}.$$

It follows from statement (b) of Lemma 1 that

$$(B\mathcal{F})(J)f \stackrel{\text{def}}{=} \begin{pmatrix} (BF_1)(J) & \mathbb{O} & \cdots & & \mathbb{O} \\ (B(-f_2 \circledast F_1))(J) & I & \cdots & & \mathbb{O} \\ (B(-f_3 \circledast F_1))(J) & \mathbb{O} & I & \cdots & \mathbb{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (B(-f_n \circledast F_1))(J) & \mathbb{O} & & \cdots & I \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix}$$

$$= \begin{pmatrix} (BF_1)(J)f_1 \\ (B(-f_2 \circledast F_1))(J)f_1 + f_2 \\ (B(-f_3 \circledast F_1))(J)f_1 + f_3 \\ \vdots \\ (B(-f_n \circledast F_1))(J)f_1 + f_n \end{pmatrix} = \begin{pmatrix} F_1 \circledast f_1 \\ (-f_2 \circledast F_1) \circledast f_1 + f_2 \\ (-f_3 \circledast F_1) \circledast f_1 + f_3 \\ \vdots \\ (-f_n \circledast F_1) \circledast f_1 + f_n \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix}.$$

Since $\circledast\text{-det}(B\mathcal{F})(0) = (BF_1)(0) \ne 0$, the operators $(B\mathcal{F})(J)$ and $(B\mathcal{F})(\mathcal{K})$ are invertible in the spaces $W^n(\mathbb{D}) \stackrel{\text{def}}{=} W(\mathbb{D}) \times \cdots \times W(\mathbb{D})$ and $X^n \stackrel{\text{def}}{=} X \times \cdots \times X$. The reasoning in the proof of Lemma 1.2.3 of the author's work [9] shows that the family

$$\left\{((B\mathcal{F})(J)f)_i \oplus \left((B\mathcal{F})(\mathcal{K})x\right)_i : i = 1, 2, \dots, n\right\}$$

is cyclic for the operator $J \oplus \mathcal{K}$. Thus, we obtain a new cyclic family of the form $\{\mathbb{1} \oplus \overline{x}_1, \mathbb{O} \oplus \overline{x}_2, \dots, \mathbb{O} \oplus \overline{x}_n\}$. Hence, there exists a family of polynomials $\{P_{m,i}\}_{i=1}^n$ such that

$$\lim_{m \to \infty} P_{m,1}(J)\mathbb{1} = \mathbb{O} \quad \text{in} \quad W(\mathbb{D})$$

and

$$\lim_{m \to \infty} \sum_{i=1}^n P_{m,i}(\mathcal{K})\overline{x}_i = x \quad \text{in} \quad X,$$

where $x \in X$ is an arbitrary element. Applying formula (b) of Lemma 1, we deduce that

$$\lim_{m \to \infty} q_{m,1}(z) = 0 \quad \text{in} \quad W(\mathbb{D}),$$

where

$$q_{m,1}(z) \stackrel{\text{def}}{=} (B^{-1}P_{m,1})(z) = \sum_{k \ge 0} \frac{1}{k!}\widehat{P}_{m,1}(k)z^k$$

and B^{-1} is the inverse Borel transformation. Now the conditions of Theorem 3 imply that

$$\|P_{m,1}(\mathcal{K})\| = \left\| \sum_{k \ge 0} \widehat{P}_{m,1}(k)\mathcal{K}^k \right\| \le \sum_{k \ge 0} |\widehat{P}_{m,1}(k)| \, \|\mathcal{K}^k\|$$

$$= \sum_{k \ge 0} \frac{1}{k!}|\widehat{P}_{m,1}(k)|k! \, \|\mathcal{K}^k\| \le c \sum_{k \ge 0} |\widehat{q}_{m,1}(k)| = c\left\|q_{m,1}\right\|_{W(\mathbb{D})} \to 0, \quad m \to \infty.$$

In particular, it follows that $\lim_{m\to\infty} P_{m,1}(\mathcal{K})\overline{x}_1 = 0$, and, consequemtly, $\lim_{m\to\infty} \sum_{i=2}^{n} P_{m,i}(\mathcal{K})\overline{x}_i = x$. Since x is an arbitrary vector, we see that $\{\overline{x}_i\}_{i=2}^{n}$ is a cyclic family for the operator $\mathcal{K}$, and $\mu(\mathcal{K}) \le n-1$. We get a contradiction since $\mu(\mathcal{K}) = n$ by our assumption. $\square$

Remark 1. There exists a more direct construction of a cyclic family of the form $\{\mathbb{1}\oplus\overline{x}_1, \mathbb{O}\oplus\overline{x}_2, \dots, \mathbb{O}\oplus\overline{x}_n\}$. Indeed, assume that $f_k(0) \ne 0$ for $k = 1, 2, \dots, l$ and $f_k(0) = 0$ for $k = l+1, \dots, n$. Set $g_k = f_k$ and $y_k = x_k$ for $k = 1, \dots, l$ and $g_k = f_k - f_1$ and $y_k = x_k - x_1$ for $k = l+1, \dots, n$. In this case, the family $\{g_i \oplus y_i\}_{i=1}^{n}$ is cyclic. Since $g_k(0) \ne 0$ for $k = 1, 2, \dots, n$, there exist functions $F_k \in W(\mathbb{D})$ such that $F_k \circledast g_k = \mathbb{1}$, $k = 1, 2, \dots, n$. Set $\widetilde{x}_k = \sum_{m\ge 0} m!\,\widehat{F}_k(m)\mathcal{K}^m y_k$. In this case,

$$g_k \oplus y_k = \sum_{m\ge 0} m!\,\widehat{g}_k(m)(J\oplus\mathcal{K})^m(\mathbb{1}\oplus\widetilde{x}_k).$$

Hence,

$$g_k \oplus y_k \in \operatorname{span}\{(J\oplus\mathcal{K})^m(\mathbb{1}\oplus\widetilde{x}_i) : m \ge 0,\ i = 1, \dots, n\}$$

for $k = 1, \dots, n$, and the family $\{\mathbb{1}\oplus\widetilde{x}_i\}_{i=1}^{n}$ is cyclic.

Finally, we set $\overline{x}_1 = \widetilde{x}_1$ and $\overline{x}_k = \widetilde{x}_1 - \widetilde{x}_k$, $k = 2, \dots, n$, to get a cyclic family of the form $\{\mathbb{1}\oplus\overline{x}_1, \mathbb{O}\oplus\overline{x}_2, \dots, \mathbb{O}\oplus\overline{x}_n\}$.

Remark 2. The proof of Theorem 3 shows that the assumptions can be weakened as follows: it is enough to assume that for any $x \in X$ there exists a constant $C_x > 0$ such that $\|\mathcal{K}^n x\| \le C_x \frac{1}{n!}$ for $n \ge 0$.

In the next theorem, we give a criterion of additivity of multiplicities of spectra. Let us make the following remark. Let $\mathfrak{A}$ be a Banach algebra with unit e. To unify the terminology, we say that an element $x \in \mathfrak{A}$ is cyclic in $\mathfrak{A}$ if $\operatorname{clos}\{\mathfrak{A}x\} = \mathfrak{A}$, where $\mathfrak{A}x \stackrel{\text{def}}{=} \{ax : a \in \mathfrak{A}\}$. In this case, we write $x \in \operatorname{Cyc}\mathfrak{A}$. Obviously, an element x is cyclic in $\mathfrak{A}$ if only if this element is invertible. In particular, $e \in \operatorname{Cyc}\mathfrak{A}$. Thus, it is natural to assume that $\mu(\mathfrak{A}) = 1$.

Theorem 4. *Let X be a separable Banach space. Let $\mathfrak{A}$ be a Banach algebra with respect to two multiplications denoted by $\cdot$ and $*$ and let $(\mathfrak{A}, *)$ be a commutative algebra. Let τ be a generator of the algebra $(\mathfrak{A}, *)$ with unit e. Finally, let $A \in L(X)$ be an operator with simple spectrum (i.e., $\mu(A) = 1$) and such that*

(1) *A admits a $(\mathfrak{A}, \cdot)$-calculus;*

(2) *there exists a constant $c > 0$ such that $\|P(\tau)(A)x\| \le c\|P(A)x\|$ for any polynomial P with respect to the multiplication $*$ and for any $x \in X$;*

(3) *if φ is an invertible element of the algebra $(\mathfrak{A}, *)$, then $\varphi(A) \ne \mathbb{O}$.*

Then $\mu(M_\tau \oplus A) = 2$, where M_τ is the operator of multiplication by τ.

Proof. Assume that $\mu(M_\tau \oplus A) = 1$. Let $f \oplus x$ be a cyclic vector for $M_\tau \oplus A$. In this case, there exist polynomials P_n such that

$$\lim_n P_n(M_\tau \oplus A)(f \oplus x) = e \oplus \mathbb{O}.$$

It follows that

$$\lim_n P_n(\tau)f = e \quad \text{and} \quad \lim_n P_n(A)x = \mathbb{O}. \tag{2}$$

Since $f \in \operatorname{Cyc}(\mathfrak{A}, *)$ and $P_n(\tau)f = P_n(\tau) * f$, the element f is invertible in $(\mathfrak{A}, *)$. Hence, $\lim_n P_n(\tau) = f^{-1*}$. By condition (1), the operator A admits a calsulus with respect to the multiplication; hence, $\lim_n P_n(\tau)(A) = f^{-1*}(A)$, and $\lim_n P_n(\tau)(A)x = f^{-1*}(A)x$. By condition (2), relations (2) imply that $\lim_n P_n(\tau)(A)x = \mathbb{O}$. Hence, $f^{-1*}(A)x = \mathbb{O}$. Finally, it follows from condition (3) that $f^{-1*}(A) \ne \mathbb{O}$. Since the operators $f^{-1*}(A)$ and A commute, we deduce that $x \notin \operatorname{Cyc}(A)$. This contradiction completes the proof. $\square$

Corollary 5. *Let $A \in L(X)$ be a cyclic noninvertible contraction and let J be the integration operator in the space $W(\mathbb{D})$. If*

$$\|q(A)x\| \le c\|P(A)x\| \tag{3}$$

for any $x \in \operatorname{Cyc}(A)$ and any polynomial P, where $q = B^{-1}P$, then $\mu(J \oplus A) = 2$.

Proof. First we note that the condition $0 \in \sigma(A)$ is necessary for the additivity of multiplicities of spectra. To prove the corollary, let us check the conditions of Theorem 4. Set $\mathfrak{A} = W(\mathbb{D})$. We take the Duhamel product $\circledast$ as $*$:

$$(f \circledast g)(z) = \frac{d}{dt}\int_0^z f(z-t)g(t)dt.$$

It was noted above that $W(\mathbb{D})$ is a Banach algebra with respect to $\circledast$. In fact, this statement is a corillary of the following reasoning: if f, $g \in W(\mathbb{D})$, then $\|J^n g\| = \left\|\frac{z^n}{n!} \circledast g\right\| \le \frac{1}{n!}\|g\|$, and it follows from item (b) of Lemma 1 that

$$\|f \circledast g\| = \|(Bf)(J)g\| = \left\|\sum_{n\ge 0} n!\,\widehat{f}(n)(J^n g)\right\| \le \|g\| \sum_{n\ge 0} |\widehat{f}(n)| = \|f\|\,\|g\|.$$

Obviously, condition (1) of Theorem 4 is fulfilled. Inequality (3) implies condition (2) of Theorem 4. Condition (3) follows from the theorem on mappings of a spectrum: $\varphi(0) \in \sigma(\varphi(A))$, and if a function φ is invertible in $(W(\mathbb{D}), \circledast)$, then $\varphi(0) \ne 0$; thus, $\varphi(A) \ne \mathbb{O}$. Since the integration operator J is the operator of multiplication by z in the sense of $\circledast$, it remains to apply Theorem 4. □

Corollary 6. *Let X be a Banach space of functions that are analytic in $\mathbb{D}$ and satisfy the following conditions:*

(1) *$\{z^n\}_{n\ge 0}$ is a complete system in X;*

(2) *the operator S of shift, $Sf = zf$, acts continuously in X;*

(3) *for any $z \in \mathbb{D}$, the functional $f \to f(z)$ from X into $\mathbb{C}$ is continuous;*

(4) *for any $\varepsilon > 0$ and any $x \in X$,*

$$\sum_{n\ge 0} \frac{1}{n!}\frac{1}{\varepsilon^n}\|z^n x\|_X < +\infty.$$

Let J be the integration operator in the space $l^1_A(\|S^n\|)$. Then $\mu(J \oplus S) = 2$.

Proof. It is enough to check that

$$\|q(S)x\| \le c\|p(S)x\|$$

for any polynomial p and any $x \in X$ such that $x(0) \ne 0$, where $q(z) \stackrel{\text{def}}{=} \sum_{n\ge 0} \frac{\widehat{p}(n)}{n!} z^n$. Take $x \in X$ such that $x(0) \ne 0$. Fix an arbitrary ε, $0 < \varepsilon < 1$. Fix a number $0 < \delta < \varepsilon$ such that $\inf_{|z|\le\delta} |x(z)| \stackrel{\text{def}}{=} \alpha > 0$. By the closed graph theorem, the embedding of the space X into $C(|z| \le \delta)$ is continuous. Hence,

$$\|p(z)x(z)\|_X \ge c \max_{|z|\le\delta} |p(z)x(z)| \ge c\alpha \max_{|z|\le\delta} |p(z)| \ge \delta^n c\alpha |\widehat{p}(n)|.$$

Now we deduce the desired estimate as follows:

$$\|q(S)x\|_X = \left\|\sum_{n\ge 0} \frac{\widehat{p}(n)}{n!} z^n x\right\|_X \le \sum_{n\ge 0} \frac{\widehat{p}(n)}{n!}\|z^n x\|_X \le \frac{1}{c\alpha}\|p(z)x\|_X \sum_{n\ge 0} \frac{1}{n!\delta^n}\|z^n x\|_X = c_1\|p(S)x\|_X.$$

It remains to note that $l^1_A(\|S^n\|)$ is a Banach algebra with respect to the Duhamel product $\circledast$ as well. Indeed,

$$\|f \circledast g\|_{l^1_A(\|S^n\|)} = \left\|\sum_{n\ge 0}\sum_{k=0}^n \widehat{f}(k)\widehat{g}(n-k)\frac{k!(n-k)!}{n!} z^n\right\|_{l^1_A(\|S^n\|)} = \sum_{n\ge 0}\left|\sum_{k=0}^n \widehat{f}(k)\widehat{g}(n-k)\frac{k!(n-k)!}{n!}\right|\|S^n\|$$

$$\le \sum_{n\ge 0}\left(\sum_{k=0}^n |\widehat{f}(k)|\,|\widehat{g}(n-k)|\right)\|S^n\| = \|f_+ \cdot g_+\|_{l^1_A(\|S^n\|)} \le c\|f_+\|_{l^1_A(\|S^n\|)}\|g_+\|_{l^1_A(\|S^n\|)} = c\|f\|_{l^1_A(\|S^n\|)}\|g\|_{l^1_A(\|S^n\|)},$$

where $f_+(z) \stackrel{\text{def}}{=} \sum_{n\ge 0} |\widehat{f}(n)| z^n$ and $g_+(z) \stackrel{\text{def}}{=} \sum_{n\ge 0} |\widehat{g}(n)| z^n$. The proof is complete. □

§3 Operator of double integration

In this section, we consider an analog of the Duhamel product for functions of two variables. We apply this analog to describe cyclic vectors of the restriction of the operator of double integration to an invariant subspace of a special form.

Denote by $C^{(n)}[0,1]\times[0,1]$, where $n\geq 2$, the Banach space of function having continuous partial derivatives of order not exceeding n on $[0,1]\times[0,1]$. We introduce a norm in $C^{(n)}[0,1]\times[0,1]$ by the following formula:

$$\|u\|_{C^{(n)}[0,1]\times[0,1]}=\sum_{|\alpha|\leq n}\|D^{\alpha}u\|_{C[0,1]\times[0,1]},$$

or, in more detail,

$$\|u\|_{C^{(n)}[0,1]\times[0,1]}=\sum_{0\leq|\alpha|\leq n}\max_{x\in[0,1]\times[0,1]}|D^{\alpha}u(x)|.$$

In the space $C^{(n)}[0,1]\times[0,1]$, we consider the Volterra integration operator in two variables,

$$(Wf)(x,y)\stackrel{\text{def}}{=}\int_0^x\int_0^y f(t,\tau)d\tau\,dt.$$

Denote by E_{xy} the subspace of the space $C^{(n)}[0,1]\times[0,1]$ consisting of functions that depend on the product xy. It is easy to see that $E_{xy}=\operatorname{span}\{(xy)^k:k\geq 0\}$ and $E_{xy}\in\operatorname{Lat}W$. Set $W_{xy}=W|E_{xy}$, i.e., $(W_{xy}f)(xy)\stackrel{\text{def}}{=}\int_0^x\int_0^y f(t\tau)d\tau\,dt$. In the space $C^{(n)}[0,1]\times[0,1]$, we define a convolution (analog of the Duhamel product for functions of two variables) as follows:

$$(f\circledast g)(x,y)\stackrel{\text{def}}{=}\frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y f(x-t,y-\tau)g(t,\tau)d\tau\,dt. \tag{4}$$

Differentiating the integral, we transform formula (4) to the following form:

$$\begin{aligned}(f\circledast g)(x,y)=&\int_0^x\int_0^y\frac{\partial^2}{\partial x\partial y}f(x-t,y-\tau)g(t,\tau)d\tau\,dt+\int_0^x\frac{\partial}{\partial x}f(x-t,0)g(t,y)dt\\&+\int_0^y\frac{\partial}{\partial y}f(0,y-\tau)g(x,\tau)d\tau+f(0,0)g(x,y).\end{aligned}$$

The formula above implies that if $f,\ g\in E_{xy}$, then

$$\begin{aligned}(f\circledast g)(xy)&=\int_0^x\int_0^y\frac{\partial^2}{\partial x\partial y}f((x-t)(y-\tau))g(t\tau)d\tau\,dt+f(0)g(xy)\\&=\int_0^x\int_0^y\left[f'((x-t)(y-\tau))+(x-t)(y-\tau)f''((x-t)(y-\tau))\right]g(t\tau)d\tau\,dt+f(0)g(xy).\end{aligned} \tag{5}$$

In particular, if $f,\ g\in E_{xy}$, then, introducing the variables $u=\frac{t}{x}$ and $v=\frac{\tau}{y}$, we deduce from formula (5) that the function $f\circledast g$ depends on the product xy only.

Set

$$X^{(s)}\stackrel{\text{def}}{=}\left\{f\in E_{xy}:\left.\frac{\partial^{i+j}f}{\partial x^i\partial y^j}\right|_{xy=0}=0,\ i+j=0,1,\dots,s\right\}\ (s=0,1,\dots,n)$$

and

$$X_{\lambda}\stackrel{\text{def}}{=}\left\{f\in E_{xy}:f(xy)=0,\ 0\leq xy<\lambda\right\}\ (0<\lambda<1).$$

Obviously, $X^{(s)}$ and X_λ are invariant subspaces of the operator W_{xy}, and

$$\{0\} \subset X_\mu \subset X_\lambda \subset X^{(n)} \subset X^{(n-1)} \subset \cdots \subset X^{(0)} \subset C^{(n)}[0,1] \times [0,1] \quad (\mu > \lambda).$$

Calculating the derivatives $\frac{\partial^{i+j}}{\partial x^i \partial y^j}$, we see that

$$\left.\frac{\partial^{i+j} f(xy)}{\partial x^i \partial y^j}\right|_{xy=0} = 0 \Longleftrightarrow \left. f^{(\max(i,j))}(xy)\right|_{xy=0} = 0.$$

Hence, the subspaces $X^{(s)}$ have the following form:

$$X^{(s)} = \{f \in E_{xy} : f(0) = f'(0) = \cdots = f^{(s)}(0) = 0\} \quad (s = 0, 1, \ldots, n).$$

In the following statement, we apply the Duhamel product to describe cyclic vectors of the restricted operator W_{xy}.

Proposition 7. *Let $f \in E_{xy}$. The inclusion $f \in \operatorname{Cyc}(W_{xy})$ holds if and only if $f(0) \neq 0$.*

Proof. Let g be an arbitrary function from E_{xy}. Equality (5) implies that $1\!\!1 \circledast g(xy) = g(xy)$,

$$W_{xy}^k g(xy) = \frac{(xy)^k}{(k!)^2} \circledast g(xy) \quad (k \geq 1),$$

and the operator $\mathcal{D}_f$ defined by $(\mathcal{D}_f g)(xy) \stackrel{\text{def}}{=} (f \circledast g)(xy)$ is a bounded operator in E_{xy} (in addition, E_{xy} is a Banach algebra with respect to $\circledast$).

It follows from these statements that

$$E_f \stackrel{\text{def}}{=} \operatorname{span}\left\{W_{xy}^k f(xy) : k \geq 0\right\} = \operatorname{span}\left\{\frac{(xy)^k}{(k!)^2} \circledast f(xy) : k \geq 0\right\}$$

$$= \operatorname{span}\left\{\mathcal{D}_f\left(\frac{(xy)^k}{(k!)^2}\right) : k \geq 0\right\} = \operatorname{clos}\mathcal{D}_f \operatorname{span}\{(xy)^k : k \geq 0\} = \operatorname{clos}\mathcal{D}_f E_{xy}.$$

Hence, $f \in \operatorname{Cyc}(W_{xy}) \Leftrightarrow \operatorname{clos}\mathcal{D}_f E_{xy} = E_{xy}$. To prove the proposition, we show that $\operatorname{clos}\mathcal{D}_f E_{xy} = E_{xy} \Leftrightarrow f(0) \neq 0$. Indeed, if $\operatorname{clos}\mathcal{D}_f E_{xy} = E_{xy}$, i.e., if $E_f = E_{xy}$, then $f(0) \neq 0$. Conversely, assume that $f(0) \neq 0$. We claim that in this case the operator $\mathcal{D}_f$ is invertible in E_{xy}. This claim implies the desired statement. Indeed, it follows from formula (5) that $\mathcal{D}_f = (\mathcal{K}_{\frac{\partial^2 f}{\partial x \partial y}} + f(0)I)$, where I is the identity operator in E_{xy} and

$$\mathcal{K}_{\frac{\partial^2 f}{\partial x \partial y}} g(xy) = \int_0^x \int_0^y \big[f'((x-t)(y-\tau)) + (x-t)(y-\tau)f''((x-t)(y-\tau))\big] g(t\tau) d\tau\, dt.$$

It is easy to show that $\mathcal{K}_{\frac{\partial^2 f}{\partial x \partial y}}$ is a compact operator in E_{xy}. On the other hand, by the Titchmarsh theorem on convolution for functions of several variables (see [11]), the condition $f(0) \neq 0$ implies the equality $\ker \mathcal{D}_f = \{0\}$. Hence, the operator $\mathcal{D}_f$ is invertible in E_{xy} by the Fredholm theorem. Thus, $E_f = E_{xy}$, i.e., $f \in \operatorname{Cyc}(W_{xy})$. □

Remark 3. Changing variables, it is posiible to show that

$$(W_{xy} f)(xy) = \int_0^x \int_0^y f(t\tau) d\tau\, dt = \int_0^{xy} \log\frac{xy}{v} f(v) dv. \tag{6}$$

Hence, Proposition 7 describes, in fact, cyclic vectors of the operator

$$(\mathcal{K}_{\log x} f)(x) \stackrel{\text{def}}{=} \int_0^x \log\frac{x}{y} f(y) dy$$

in the space $C^{(n)}[0,1]$. Indeed, set $u = t\tau$. Since $0 \le \tau \le y$, $0 \le u \le ty \overset{\text{def}}{=} v$ and $0 \le v \le xy$. Hence,

$$
\begin{aligned}
(W_{xy}f)(xy) &= \int_0^x \int_0^y f(t\tau)d\tau\, dt = \int_0^x \left(\int_0^{ty} f(u)\frac{du}{t} \right) dt = \int_0^x \left(\int_0^{ty} f(u)du \right) d\log t \\
&= \left(\log t \int_0^{ty} f(u)du \right)\Bigg|_0^x - \int_0^x \left(\log t\, d \int_0^{ty} f(u)du \right) = \log x \int_0^{xy} f(u)du - \int_0^x \log t f(ty)y\, dt \\
&= \log x \int_0^{xy} f(u)du - \int_0^{xy} \log \frac{v}{y} f(v)dv = \int_0^{xy} \log \frac{xy}{v} f(v)dv.
\end{aligned}
$$

This proves formula (6).

In conclusion, let us mention that the study of invariant subspaces of the operator W of double integration in the space $L^2[0,1] \times [0,1]$ was originated by the known Donoghue's work [12] and that the operator W is not unicellular. We also mention the Atzmon amd Manos' work [13], where the equality $\mu(W) = +\infty$ was proved. In the same work, a description of all reducing subspaces of the operator $L^2[0,1] \times [0,1]$ was given. The problem of complete description of $\operatorname{Lat} W$ is still open.

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REFERENCES

1. N. M. Wigley, "The Duhamel product of analytic functions," *Duke Math. J.*, **41**, 211–217 (1974).
2. J. Mikusinski, *Operator Calculus* [Russian translation], Moscow (1956).
3. M. A. Lavrent'ev and B. V. Shabat, *Methods of Functions of a Complex Variable* [in Russian], Moscow (1973).
4. M. T. Karaev, "Application of a convolution in the proof of unicellularity," *Zap. Nauchn. Semin. LOMI*, **135**, 66–68 (1984).
5. M. T. Karaev, "Application of the Duhamel product in description of invariant subspaces," *Trudy IMM AN Azerb.*, 137–146 (1995).
6. I. Dimovski, *Convolution Calculus*, Sofia (1982).
7. M. T. Karaev, "Application of the Duhamel product in problems of the basic property," *Izv. AN Azerb. SSR*, ser. fiz.-tekhn.-mat. nauk, No. 3–6, 145–150 (1990).
8. M. K. Fahe and N. I. Nagnibida, *Problem of Equivalence for Ordinary Linear Differential Operators* [in Russian], Novosibirsk (1987).
9. M. T. Karaev, "Addition of multiplicities of spectra and invariant subspaces," Ph.D. Thesis, Baku (1991).
10. M. M. Malamud, "Reproducing subspaces of Volterra operators," *Dokl. RAN*, **37**, 454–458 (1996).
11. J. Mikusinski, "Convolution of functions of several variables," *Stidua Math.*, **20**, 253–259 (1961).
12. W. F. Donoghue, "The lattice of invariant subspaces of a completely continuous quasinilpotent transformation," *Pacific J. Math.*, **7**, 1031–1035 (1957).
13. A. Atzmon and H. Manos, "The integration operator in two variables," *Proc. Amer. Math. Soc.*, **119**, 513–523 (1993).