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EXACT SMALL BALL CONSTANTS FOR SOME GAUSSIAN PROCESSES UNDER THE L^2-NORM

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We find some logarithmic and exact small deviation asymptotics for the L^2-norms of certain Gaussian processes closely connected with a Wiener process. In particular, processes obtained by centering and integrating Brownian motion and Brownian bridge are examined. Bibliography: 28 titles.

1. Introduction

The problem of small deviations for norms of Gaussian processes has obtained much attention in recent years (see, e.g., the reviews [20] and [18]).

Let $X(t), 0 \le t \le 1$, be a Gaussian process with mean zero and covariance function $\sigma(s,t) = EX(t)X(s)$ for $s,t \in [0,1]$. Let

$$||X||_2 = \left(\int_0^1 X^2(t)dt\right)^{1/2}$$

and

$$Q(X;\varepsilon) = P(||X||_2 \le \varepsilon).$$

An interesting problem is to describe the behavior of $Q(X;\varepsilon)$ as $\varepsilon \to 0$. As an example, consider a Wiener process $W(t), 0 \le t \le 1$, and a Brownian bridge $B(t), 0 \le t \le 1$. The following small deviation asymptotics as $\varepsilon \to 0$ have been obtained long ago:

$$P(||W||_2 \le \varepsilon) \sim 4\pi^{-1/2}\varepsilon \exp(-(1/8)\varepsilon^{-2}) \tag{1}$$

and

$$P(||B||_2 \le \varepsilon) \sim 2\sqrt{2}\pi^{-1/2} \exp(-(1/8)\varepsilon^{-2}). \tag{2}$$

These asymptotics follow from known exact distributions of the L^2-norm for W and B. However, for other Gaussian processes, such formulas are seldom available.

Theoretically, the problem of small deviation asymptotics was solved by Sytaya [24], but in an implicit way. Therefore, the efforts of many scientists, beginning with the works [4], [12], and [27], were aimed at simplifications of the expression for $Q(X;\varepsilon)$ (see the references in [20] and [5]).

It readily follows from the Kac–Siegert formula that

$$\int_0^1 X^2(t)dt = \sum_{n=1}^{\infty} \lambda_n \xi_n^2, \tag{3}$$

where ξ_n, $n \ge 1$, are independent standard normal r.v.'s and $\lambda_n > 0, n \ge 1$, are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 \sigma(s,t) f(s) ds, \quad 0 \le t \le 1. \tag{4}$$

Thus, equivalently, we arrive at the problem on the asymptotic behavior of the value $P(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le \varepsilon^2)$ as $\varepsilon \to 0$. We may say that the latter problem is solved if the eigenvalues λ_n are found explicitly. However, these eigenvalues are known only for a limited number of examples (see [17], [5], and [20]).

The aim of the present work is to calculate the exact small deviation asymptotics for some Gaussian processes, which are of interest in statistics but have not been considered in the previous papers. The results were obtained in October of 2001 when the second author was visiting the Rome University "La Sapienza" and published in the preprint [1]. They were also presented at the POMI seminar of Prof. I. A. Ibragimov in the spring of 2002.

Some general exact small deviation results have been obtained later (see [6–8] and [21, 22]).

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2. Gaussian processes under consideration

In some statistical problems, it is reasonable to consider centered empirical processes and the corresponding limiting Gaussian processes. In particular, we are interested in the centered (by its mass) Wiener process

$$W^c(t) = W(t) - \int_0^1 W(u)du$$

and the centered Brownian bridge

$$B^c(t) = B(t) - \int_0^1 B(u)du.$$

The idea of such type of centering is very old and dates back at least to Watson [25], who used it for testing nonparametric hypotheses on the circle.

Another operation, which has received great attention in the recent years, is the integration of Gaussian processes (see, e.g., [2], [15], and [16]). Let

$$\overline{W}(t) = \int_0^t W(u)du \quad \text{and} \quad \overline{B}(t) = \int_0^t B(u)du, \quad 0 \le t \le 1,$$

be the integrated Wiener process and Brownian bridge, respectively. In addition to purely probabilistic studies of these processes, we note that $\overline{B}(t)$ has appeared in [10] in the context of goodness-of-fit testing. Similarly, $\overline{W}(t)$ may be used for samples of the Poisson size if the empirical process is replaced by the Kac process [13] and W is the limiting process.

We can also combine the operations of centering and integration. In this context, Henze and Nikitin [11] (see also [16]) considered two different processes: the centered integrated Brownian bridge

$$B^0(t) = \overline{B}(t) - \int_0^1 \overline{B}(u)du$$

and the integrated centered Brownian bridge

$$B^*(t) = \int_0^t (B(s) - \int_0^1 B(u)du)ds = \overline{B}(t) - t\overline{B}(1).$$

The latter process may be regarded as the bridge of the integrated Brownian bridge $\overline{B}(t)$. These studies were also motivated by the construction of new Watson type goodness-of-fit tests (see [11]).

Quite smilarly, we can consider the centered integrated Wiener process

$$W^0(t) = \overline{W}(t) - \int_0^1 \overline{W}(u)du$$

and the integrated centered Wiener process

$$W^*(t) = \int_0^t (W(s) - \int_0^1 W(u)du)ds = \overline{W}(t) - t\overline{W}(1),$$

which, apparently, have not been considered previously. Note that the operations of centering and integrating do not commute, so that B^0 differs from B^*, as well as W^0 differs from W^*.

Chen and Li [2] considered the m-fold integrated Wiener process

$$W_m(t) = \frac{1}{m!}\int_0^t (t-s)^m dW(s), \quad m \ge 0.$$

They showed that

$$\lim_{\varepsilon \to 0} \varepsilon^{2/2m+1} \log P(||W_m||_2 \le \varepsilon) = -D_m$$

as $\varepsilon \to 0$, where

$$D_m = \frac{1}{2}(2m+1)\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}.$$

For $m = 0$ (the Wiener process), the right-hand side is equal to $-(1/8)$ according to (1); for $m = 1$ (the process $\overline{W}(t)$), the right-hand side is equal to $-(3/8)$. The latter result was obtained for the first time in [15].

3. Centered Wiener process and centered Brownian bridge

The process $B^c(t) = B(t) - \int_0^1 B(u)du$ was introduced by Watson [25]. He showed that the covariance function of this process is

$$\sigma_{B^c}(s,t) = s \wedge t - st + \frac{1}{2}(s^2 + t^2 - s - t) + \frac{1}{12},\ 0 \le s,t \le 1,$$

and that the spectrum of the corresponding integral operator consists of the eigenvalues $\lambda_n = (2n\pi)^{-2}, n \ge 1$, of multiplicity 2. This makes the computation of small deviations more difficult since the r.v.'s ξ_n^2 in (3) are to be replaced by the r.v.'s χ_2^2 . Fortunately, it was proved long ago (see, e.g., [23, p. 148]) that the following equality in distribution holds:

$$||B^c||_2 = \pi^{-1} \sup_{0 \le t \le 1} |B(t)|.$$

The random variable $\sup_{0\le t\le 1} |B(t)|$ has the well-known Kolmogorov distribution function, for which there exist closed-form expressions applicable both for small and large values of the argument (see, e.g., [19, §18]). Hence,

$$P(||B^c||_2 \le \varepsilon) = P(\sup_{0 \le t \le 1} |B(t)| \le \varepsilon\pi) \sim \sqrt{\frac{2}{\pi}}\varepsilon^{-1} \exp\left(-\frac{1}{8}\varepsilon^{-2}\right)$$

as $\varepsilon \to 0$. To study the small deviations of the process W^c, we note that

$$\sigma_{W^c}(s,t) = s \wedge t + \frac{1}{2}(s^2 + t^2) - s - t + \frac{1}{3}.$$

Solving the integral equation (4) with this kernel by differentiation, we readily get the following boundary-value problem:

$$\lambda f''(t) = -(f(t) - \int_0^1 f(u)du),$$
$$f'(0) = f'(1) = 0.$$

The set of solutions of the latter problem consists of the eigenvalues $\lambda_n = (n^2\pi^2)^{-1}, n \ge 1$, with the eigenfunctions $f_n(t) = C\cos n\pi t, n \ge 1$. Thus, the spectrum coincides with that of the Brownian bridge B. Hence, we get a different proof of the well-known equality in distribution (see [3]):

$$||B||_2 = ||W^c||_2.$$

It follows from (2) that

$$P(||W^c||_2 \le \varepsilon) \sim 2\sqrt{2}\pi^{-1/2}\exp(-(1/8)\varepsilon^{-2})$$

as $\varepsilon \to 0$. We see that the centering by integral does not change the exponential term for small deviation probabilities of the Wiener process and Brownian bridge but it changes the factor at the exponential term.

4. Integrated Brownian bridge: exact small deviations

The covariance function of the integrated Brownian bridge $\overline{B}$ is

$$\sigma_{\overline{B}}(s,t) = \frac{1}{2}st(s \wedge t) - \frac{1}{6}(s \wedge t)^3 - \frac{1}{4}s^2t^2, \quad 0 \le s,t \le 1.$$

The spectrum of the corresponding integral equation has been found in [10]. The derivation of this spectrum is based on the transcendental equation

$$\tan x + \tanh x = 0. \tag{5}$$

Denoting the solutions of Eq. (5) by $k_1 < k_2 < \dots$, the eigenvalues are $\lambda_n = (k_n)^{-4}, n \ge 1$.

It was noticed in [10] that, since $\tanh x \sim 1$ for large x , the solutions k_j of Eq. (5), arranged in the ascending order of magnitude, satisfy the approximate relation $k_j \sim (j - \frac{1}{4})\pi$ for large j. Therefore,

$$\lambda_j \sim \sigma_j := \left(j - \frac{1}{4}\right)^{-4} \pi^{-4} \tag{6}$$

for large values of j. It follows from Theorem 2 of [17] or from Theorem 6.2 of [20] that

$$P(||\overline{B}||_2 \le \varepsilon) = P(\sum_{j=1}^{\infty} \lambda_n \xi_n^2 \le \varepsilon^2) \sim \Pi_{n=1}^{\infty} (\sigma_n/\lambda_n)^{1/2} P(\sum_{j=1}^{\infty} \sigma_n \xi_n^2 \le \varepsilon^2) \tag{7}$$

as $\varepsilon \to 0$ provided that the condition $\sum_{n=1}^{\infty} |1 - \lambda_n/\sigma_n| < +\infty$ is fulfilled. Hence, we only have to verify that

$$\sum_{j=1}^{\infty} |1 - (j - \frac{1}{4})^4 \pi^4 k_j^{-4}| < +\infty. \tag{8}$$

Clearly, $k_j \to \infty$ as $j \to \infty$ and $k_j = (j - 1/4)\pi + \delta_j$ for large j, where $\delta_j \to 0$ as $j \to \infty$. Substituting these relations into (5), we get the equality

$$\frac{\tan \delta_j - 1}{\tan \delta_j + 1} + \frac{1 - \exp(\pi 2 - 2j\pi - 2\delta_j)}{1 + \exp(\pi/2 - 2j\pi - 2\delta_j)} = 0,$$

which is equivalent to the following equation:

$$\tan \delta_j = \exp(\pi/2 - 2j\pi - 2\delta_j).$$

Obviously, $\delta_j = O(\exp(-2j\pi))$ as $j \to \infty$. It follows that

$$k_j = -\pi/4 + j\pi + O(\exp(-2j\pi)), \quad j \to \infty; \tag{9}$$

therefore, series (8) is convergent. Hence, (7) is proved.

Now we use the technique developed in [5] for asymptotic evaluation of the probability on the right-hand side of (7). We adopt the same notation as in [5]. Let Φ be the distribution function of the standard normal law. Denote, for $t, u \ge 0$,

$$\begin{aligned} &\phi(t) = (t - 1/4)^{-4}, \quad f(t) = (1 + 2t)^{-1/2}, F(t) = 2\Phi(\sqrt{t}) - 1, \\ &I_0(u) = \int_1^{\infty} \log f(u\phi(t))dt, \quad I_1(u) = \int_1^{\infty} u\phi(t)(\log f)'(u\phi(t))dt, \\ &I_2(u) = \int_1^{\infty} (u\phi(t))^2 (\log f)''(u\phi(t))dt, \end{aligned}$$

and

$$C_\phi = \frac{1}{2} \sum_{j=1}^{\infty} \int_0^1 \log \frac{\phi(j)\phi(j+1)}{\phi^2(t+j)} dt.$$

The following Theorem 1 is a concretization of Corollary 3.2 of [5]. Note that, under our choice of ϕ, f, and F, all the regularity conditions of [5] are satisfied.

Theorem 1.

$$P\left(\sum_{j=1}^{\infty} \phi(j)\xi_j^2 \le r\right) \sim \sqrt{\frac{\Gamma(3/2)F((u\phi(1))^{-1})}{2\pi I_2(u)}} \exp(I_0(u) - C_\phi/2 + ur), \tag{10}$$

where $u = u(r)$ is any function such that

$$\lim_{r \to 0} \frac{I_1(u) + ur}{\sqrt{I_2(u)}} = 0. \tag{11}$$

We begin with the asymptotic analysis of $I_s(u), s = 0, 1, 2,$ as $u \to \infty$. Changing variables and integrating by parts in the definition of $I_0(u)$, we see that

$$\begin{aligned} I_0(u) &= -\frac{1}{2} \int_{3/4}^{\infty} \log(1 + \frac{2u}{t^4})dt = \frac{3}{8} \log(1 + \frac{512u}{81}) - 4u \int_{3/4}^{\infty} \frac{dt}{2u + t^4} \\ &= \frac{3}{8} \log(1 + \frac{512u}{81}) - 4u \int_0^{\infty} \frac{dt}{2u + t^4} + 4u \int_0^{3/4} \frac{dt}{2u + t^4} = \frac{3}{8} \log(1 + \frac{512u}{81}) - J_1(u) + J_2(u). \end{aligned}$$

From [9, formula 3.241] we deduce that

$$J_1(u) = (u/2)^{1/4}\pi.$$

By the Lebesgue Dominated Convergence Theorem,

$$J_2(u) = 2\int_0^{3/4} \frac{dt}{1+\frac{t^4}{2u}} \to 3/2$$

as $u \to \infty$. Our considerations imply that

$$I_0(u) \sim \frac{3}{8}\log\left(\frac{512u}{81}\right) - (u/2)^{1/4}\pi + 3/2 \tag{12}$$

as $u \to \infty$. Note that

$$(\log f(t))' = -\frac{1}{1+2t} \quad \text{and} \quad (\log f(t))'' = \frac{2}{(1+2t)^2}. \tag{13}$$

Taking into account formula (13) and repeating the reasoning applied in the analysis of $I_0(u)$, we see that

$$I_1(u) = -u\int_{3/4}^{\infty} \frac{dt}{2u+t^4} = -u\int_0^{\infty} \frac{dt}{2u+t^4} + u\int_0^{3/4} \frac{dt}{2u+t^4} \sim -2^{-9/4}\pi u^{1/4}$$

and

$$I_2(u) = 2u^2\int_{3/4}^{\infty} \frac{dt}{(2u+t^4)^2} = 2u^2\int_0^{\infty} \frac{dt}{(2u+t^4)^2} - 2u^2\int_0^{3/4} \frac{dt}{(2u+t^4)^2} \sim 3\pi 2^{-17/4}u^{1/4}.$$

If we take u so that $u = \pi^{4/3}2^{-3}r^{-4/3}$, then $u^{1/4} = \pi^{1/3}2^{-3/4}r^{-1/3}$, and $ur = -I_1(u) + O(1)$; hence, u satisfies condition (11).

To apply formula (10), it is necessary to compute the constant C_ϕ; in our case,

$$\begin{aligned} C_\phi &= \frac{1}{2}\sum_{j=1}^{\infty}\int_0^1 \log\frac{(t+j-1/4)^8}{(j-1/4)^4(j+3/4)^4}dt \\ &= 4\sum_{j=1}^{\infty}[\int_0^1 \log(t+j-1/4)dt - \frac{1}{2}\log((j-1/4)(j+3/4))] \\ &= 2\sum_{j=1}^{\infty}[(2j+1/2)\log\frac{j+3/4}{j-1/4} - 2]. \end{aligned}$$

To simplify the last sum, we need some formulas from the theory of the gamma-function (see, e.g., [26, Ch. 12]). Consider the integral

$$I(z) = \int_0^{\infty} e^{-tz}\left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right\}t^{-1}dt \tag{14}$$

which is defined for any complex z with positive real part. We recall the Binet's integral representation for the logarithm of the gamma-function which is valid for any complex z with positive real part (see [26, §12.31]):

$$\log\Gamma(z+1) = (z+1/2)\log z - z + 1 + I(z) - I(1). \tag{15}$$

Setting $z = j + \frac{3}{4}$ and $z = j - \frac{1}{4}$ in (14) and taking the difference of the obtained identities, we see that

$$I(j-1/4) - I(j+3/4) = (j+1/4)\log\frac{j+3/4}{j-1/4} - 1.$$

Hence,

$$C_\phi = 2\sum_{j=1}^{\infty}[(2j+1/2)\log\frac{j+3/4}{j-1/4} - 2] = 4\sum_{j=1}^{\infty}[I(j-1/4) - I(j+3/4)] = 4I(3/4).$$

Applying Binet's formula once more, we see that

$$\log\Gamma(3/4) = \frac{1}{4}\log(3/4) + 3/4 + I(3/4) - I(1). \tag{16}$$

It is well known that $I(1) = 1 - \frac{1}{2}\log(2\pi)$ (see [26, Section 12.31]). We deduce from (16) that

$$I(3/4) = \log\Gamma(3/4) - \frac{1}{4}\log(3/4) + 3/4 - \frac{1}{2}\log(2\pi).$$

Finally, we conclude that

$$C_\phi = 4I(3/4) = 4\log\Gamma(3/4) - \log(3/4) + 3 - 2\log(2\pi)$$

and

$$\exp(-C_\phi/2) = 3^{1/2}\pi\Gamma^{-2}(3/4)\exp(-3/2).$$

Combining these partial results, we see that

$$F(1/u\phi(1)) = 2\Phi((\frac{3}{4})^2u^{-1/2}) - 1 \sim \sqrt{\frac{2}{\pi}}(\frac{3}{4})^2u^{-1/2}, \; \sqrt{2\pi I_2(u)} \sim 3^{1/2}2^{-13/8}\pi u^{1/8}$$

and

$$\exp(I_0(u) + ur) \sim (512u/81)^{3/8}\exp(3/2 - 3\pi 2^{-9/4}u^{1/4})$$

as $u \to \infty$. Formula (14) and the relation $u = \frac{1}{8}(\pi/r)^{4/3}$ imply that

$$P\biggl(\sum_{j=1}^{\infty}\sigma_j\xi_j^2 \le r\biggr) \sim \frac{2^{11/4}}{(\sqrt{3})\Gamma^2(3/4)}\exp(-(3/8)r^{-1/3})$$

as $r \to 0$. To get the final result, we must take into account the constant

$$C_{\sigma\lambda} = \Pi_{n=1}^{\infty}(\sigma_n/\lambda_n)^{1/2},$$

which we have to calculate numerically. It was shown above that the numbers λ_j and σ_j are very close; hence, the infinite product converges very fast. Using the ten largest eigenvalues λ_j found in [10], we get the value $C_{\sigma\lambda} \approx 1.0075...$ by simple calculations. Hence, formula (7) implies the following exact asymptotic:

$$P(||\overline{B}||_2 \le \varepsilon) \sim 1.0075...\frac{2^{11/4}}{(\sqrt{3})\Gamma^2(3/4)}\exp(-(3/8)\varepsilon^{-2/3}).$$

Note that the factor at the exponent does not depend on ε. More refined arguments of [22] and [6] prove that the constant on the right-hand side is, in fact, equal to $8/\sqrt{3\pi}$.

5. Exact small deviation asymptotics for the integrated Wiener process

The same technique as in the previous section enables us to get the exact small deviation asymptotics of $Q(\overline{W};\varepsilon)$ as $\varepsilon \to 0$. The calculations are similar to those for the integrated Brownian bridge, and we omit some details. The covariance of the integrated Wiener process is

$$\sigma_{\overline{W}}(s,t) = \frac{1}{2}st(s\wedge t) - \frac{1}{6}(s\wedge t)^3, \quad 0 \le s,t \le 1.$$

The spectrum of the corresponding integral operator can be found from the following boundary-value problem:

$$\lambda f^{(IV)}(t) = f(t),$$
$$f(0) = f'(0) = f''(1) = f'''(1) = 0.$$

The solution of this problem can be found in [14]. The spectrum consists of the eigenvalues $\lambda_n = m_n^{-4}, n \geq 1$, where $m_1 < m_2 < ...$ are the solutions of the auxiliary transcendental equation

$$\cos m \cosh m + 1 = 0, \tag{17}$$

while the eigenfunctions are $f_n(t) = (\cosh m_n + \cos m_n)(\sinh m_n t - \sin m_n t) - (\sinh m_n + \sin m_n)(\cosh m_n t - \cos m_n t), n \geq 1$.

As in the previous section, it can be proved that

$$\lambda_j \sim \tau_j := (\pi(j - 1/2))^{-4}$$

with an exponential error for large j. Thus, we can find the asymptotics of $P(\sum_{j=1}^{\infty} \xi_j^2 (j - 1/2)^{-4} \leq r)$ taking into account the constant $C_{\tau\lambda} = \prod_{n=1}^{\infty} (\tau_n/\lambda_n)^{1/2}$.

Asymptotic computations of the functions $I_s(u), s = 0, 1, 2$, are very similar to those of the previous section. While the asymptotics of $I_1(u)$ and $I_2(u)$ (therefore, the choice of $u = u(r)$) coincide with those of the integrated Brownian bridge, the limit expression for $I_0(u)$ in this case is

$$I_0(u) \sim \frac{1}{4} \log(32u) - (u/2)^{1/4}\pi + 1.$$

Some differences also appear when we compute the constant C_ϕ. Repeating the reasoning of the previous section, we see that

$$\begin{aligned} C_\phi &= \frac{1}{2} \sum_{j=1}^{\infty} \int_0^1 \log \frac{(t + j - 1/2)^8}{(j - 1/2)^4 (j + 1/2)^4} dt = 4 \sum_{j=1}^{\infty} (j \log \frac{j + 1/2}{j - 1/2} - 1) \\ &= 4 \sum_{j=1}^{\infty} [I(j - 1/2) - I(j + 1/2)] = 4I(1/2). \end{aligned}$$

The integral $I(1/2) = (1 + \log 1/2)/2$ was calculated in [26, Section 12.31]. Thus, $C_\phi = 2 - 2\log 2$ and $\exp(-C_\phi/2) = 2\exp(-1)$.

To evaluate the constant $C_{\tau\lambda} = \prod_{n=1}^{\infty} (\tau_n/\lambda_n)^{1/2}$, we have to find numerically the first few roots $m_1, m_2, ...$ of Eq. (17) and to calculate the values $\lambda_j = (m_j)^{-4}, j \geq 1$. After easy calculations, we get the approximate value $C_{\tau\lambda} = 1.4142...$. Collecting the results above, we obtain the exact asymptotics

$$P(||\overline{W}||_2 \leq \varepsilon) \sim 1.4142... \cdot (8/\sqrt{3\pi})\varepsilon^{1/3} \exp(-(3/8)\varepsilon^{-2/3}).$$

Note that the methods of [22], [6], and [7] show that, in fact, $C_{\tau\lambda} = \sqrt{2}$.

6. Centered integrated Brownian bridge and integrated centered Brownian bridge

Now we consider the processes

$$B^0(t) = \overline{B}(t) - \int_0^1 \overline{B}(u) du$$

and

$$B^*(t) = \int_0^t (B(s) - \int_0^1 B(u) du) ds = \overline{B}(t) - t\overline{B}(1)$$

introduced above. It follows from the results of [10] that for the first process with mean zero and covariance function

$$\sigma_{B^0}(s, t) = \frac{st \cdot s \wedge t}{2} - \frac{(s \wedge t)^3}{6} - \frac{s^2 t^2}{4} - \frac{s^2 + t^2}{6} - \frac{s^4 + t^4}{24} + \frac{s^3 + t^3}{6} + \frac{1}{45},$$

$0 \leq s, t \leq 1$, the spectrum has the form $\lambda_n = (\pi n)^{-4}, n \geq 1$.

We can apply the result of [19, §18, Example 2] (see also [17, Example 1]) to obtain the exact small deviation asymptotics

$$P(||B^0||_2 \leq \varepsilon) \sim 2^{5/2} 3^{-1/2} \pi^{-1/2} \varepsilon^{-1/3} \exp(-\frac{3}{8}\varepsilon^{-2/3}), \quad \varepsilon \to 0.$$

In the case of the process B^* whose covariance function is

$$\sigma_{B^*}(s,t) = \frac{st \cdot s\wedge t}{2} - \frac{(s\wedge t)^3}{6} - \frac{s^2t^2}{4} - \frac{st^2}{4} + \frac{st^3}{6} - \frac{ts^2}{4} + \frac{ts^3}{6} + \frac{st}{12}, \qquad 0 \le s,t \le 1,$$

the spectrum has a more complicated structure. In [10], it is shown that the spectrum contains the following two series of eigenvalues: $\lambda_n = (2\pi n)^{-4}$ and $\mu_n = (2k_n)^{-4}$, $n \ge 1$, where the k_n are solutions of Eq. (5).

By the Kac–Siegert formula,

$$||B^*||_2^2 = \sum_{n=1}^{\infty} \xi_n^2/16\pi^4 n^4 + \sum_{m=1}^{\infty} \eta_m^2/16k_m^4 := V_1 + V_2, \tag{18}$$

where $\{\eta_m\}$ is a sequence of independent standard normal variables that is independent of $\{\xi_n\}$. Applying again the result of Li [17] and formula (6), we can replace k_m by $\pi(m-1/4)$ in (18).

We can obtain the exact small deviation asymptotics of the sum (18) of two independent random variables of the same nature using the following theorem which was kindly communicated to us by Prof. M. Lifshits.

Theorem 2. *Let V_1, $V_2 > 0$ be two independent random variables with known behavior of small deviations. Namely, assume that*

$$P(V_1 \le r) \sim c_1 r^{a_1} \exp(-b_1 r^{-d})$$

and

$$P(V_2 \le r) \sim c_2 r^{a_2} \exp(-b_2 r^{-d})$$

as $r \to 0$. Then the following small deviation asymptotic is valid for their sum:

$$P(V_1 + V_2 \le r) \sim K r^{a_1+a_2-d/2} \exp(-S^{d+1} r^{-d}),$$

where

$$S = b_1^{1/(d+1)} + b_2^{1/(d+1)} \quad \text{and} \quad K = c_1 c_2 \sqrt{\frac{2\pi d}{d+1}} S^{d/2-1/2-a_1-a_2} b_1^{(2a_1+1)/2(d+1)} b_2^{(2a_2+1)/2(d+1)}.$$

The proof is elementary but rather laborious, and we omit it.

Let us apply Theorem 2 to sum (18). In our case,

$$c_1 = 2^{11/6}3^{-1/2}\pi^{-1/2},\ a_1 = -1/6, b_1 = 3\cdot 2^{-13/3},\ d = 1/3;$$
$$c_2 = 1.0075...2^{11/4}3^{-1/2}\Gamma^{-2}(3/4),\ a_2 = 0, b_2 = 3\cdot 2^{-13/3}.$$

Hence, after some computations, we get the exact asymptotic

$$P(V_1 + V_2 \le r) \sim 1.0075...\cdot 3^{-1/2}2^{7/4}\Gamma^{-2}(3/4) r^{-1/3} \exp(-(3/8)r^{-1/3})$$

or, equivalently,

$$P(||B^*||_2 \le \varepsilon) \sim 1.0075...\cdot 3^{-1/2}2^{7/4}\Gamma^{-2}(3/4)\varepsilon^{-2/3} \exp(-(3/8)\varepsilon^{-2/3}).$$

Using the sharp value of the constant given in the end of Sec. 5, we see that the constant on the right-hand side is, in fact, equal to $4/\sqrt{3\pi}$.

7. Centered integrated Wiener process and integrated centered Wiener process

Now we consider the process

$$W^0(t) = \overline{W}(t) - \int_0^1 \overline{W}(u)du$$

introduced above. Its covariance function is

$$\sigma_{W^0}(s,t) = \frac{1}{2}(s\wedge t)\cdot st - \frac{1}{6}(s\wedge t)^3 - \frac{s^2+t^2}{4} + \frac{s^3+t^3}{6} - \frac{s^4+t^4}{24} + \frac{1}{20}, \qquad 0 \le s,t \le 1.$$

Differentiating the integral equation (4), we get the boundary-value problem:

$$\lambda f^{(IV)}(t) = f(t) - \int_0^1 f(u)du,$$
$$f'(0) = f''(1) = f'''(0) = f'''(1) = 0.$$

If $p(t) = f(t) - \int_0^1 f(s)ds$, then p satisfies a similar boundary-value problem:

$$\lambda p^{(IV)}(t) = p(t),$$
$$p'(0) = p''(1) = p'''(0) = p'''(1) = 0.$$

This problem has the following solutions (see case (1,3,2,3) in Eq. (4.3) of [14]): $p_n(t) = C(\cos k_n \cosh nt + \cosh k_n \cos nt)$ and $\lambda_n = k_n^{-4}, n \geq 1$, where, as above, the k_n are solutions of (5). It is easy to prove that $f_n(t) = p_n(t), n = 1, 2...$ (see [11]). Hence,

$$||W^0||_2 = ||\overline{W}||_2$$

in distribution, and the exact small deviation asymptotic of $||W^0||_2$ has the same form as in Sec. 6.

In the case of the integrated centered Wiener process

$$W^*(t) = \int_0^t (W(s) - \int_0^1 W(u)du)ds = \overline{W}(t) - t\overline{W}(1),$$

the covariance function has the form

$$\sigma_{W^*}(s,t) = \frac{1}{2}(s \wedge t)st - \frac{1}{6}(s \wedge t)^3 + \frac{s^3t + st^3}{6} - \frac{s^2t + st^2}{2} + \frac{st}{3}, \qquad 0 \leq s, t \leq 1.$$

The integral equation can be reduced to the boundary-value problem:

$$\lambda f^{(IV)}(t) = f(t),$$
$$f(0) = f(1) = f''(0) = f''(1) = 0.$$

The solutions of this problem (according to [14]) are $\lambda_n = (n\pi)^{-4}, n \geq 1$, and $f_n(t) = C \sin n\pi t, n \geq 0$. We have already met such a spectrum in our paper; thus,

$$||W^*||_2 = ||B^0||_2$$

in distribution, and we can apply the result obtained above. Hence,

$$P(||W^*||_2 \leq \varepsilon) \sim 2^{5/2}3^{-1/2}\pi^{-1/2}\varepsilon^{-1/3}\exp(-\frac{3}{8}\varepsilon^{-2/3}),\ \varepsilon \to 0.$$

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