Journal of Optimization Theory and Applications
https://doi.org/10.1007/s10957-024-02476-5

Metric Subregularity and $\varpi(\cdot)$-Normal Regularity Properties

Florent Nacry[1] · **Vo Anh Thuong Nguyen**[1] · **Juliette Venel**[2]

Received: 21 November 2023 / Accepted: 5 June 2024

Abstract
In this paper, we establish through an openness condition the metric subregularity of a multimapping with normal $\omega(\cdot)$-regularity of either the graph or values. Various preservation results for prox-regular and subsmooth sets are also provided.

Keywords Normal regularity · Prox-regularity · Subsmoothness · Metric regularity · Metric subregularity

Mathematics Subject Classification 49J52 · 49J53

1 Introduction

Let $M : X \rightrightarrows Y$ be a multimapping between two Banach spaces with a closed convex graph $\{(x, y) \in X \times Y : y \in M(x)\} =: \mathrm{gph}\, M$ and let $(\overline{x}, \overline{y}) \in \mathrm{gph}\, M$. In 1975-1976, C. Ursescu [32] and S.M. Robinson [27] independently established that the existence of a real $\gamma \geq 0$ such that

$$d\big(x, M^{-1}(y)\big) \leq \gamma d\big(y, M(x)\big) \quad \text{for all } (x, y) \text{ near } (\overline{x}, \overline{y}) \tag{1.1}$$

is equivalent to the inclusion $\overline{y} \in \mathrm{core}\, M(X)$, where

$$\mathrm{core}\, M(X) := \big\{y \in M(X) : \forall y' \in Y, \exists r > 0, y + ry' \in M(X)\big\} \supset \mathrm{int}\, M(X).$$

Dedicated to Prof. Boris Mordukhovich on the occasion of his 75th birthday

Communicated by René Henrion.

✉ Florent Nacry
florent.nacry@univ-perp.fr

1 Université Perpignan Via Domitia, LAMPS, 66000 Perpignan, France

2 Université Polytechnique Hauts-de-France, INSA Hauts-de-France, CERAMATHS, 59313 Valenciennes, France

Here and below, $M^{-1}: Y \rightrightarrows X$ denotes the inverse of the multimapping M defined by $M^{-1}(y) := \{x \in X : y \in M(x)\}$. In fact, the latter inclusion $\overline{y} \in \operatorname{core} M(X)$ guarantees the existence of a positive constant c such that

$$d\big(x, M^{-1}(y)\big) \le (c - \|y - \overline{y}\|)^{-1}(1 + \|x - \overline{x}\|)d\big(y, M(x)\big),$$

for all $x \in X$, $y \in B(\overline{y}, c)$.
When the above inequality (1.1) holds, one naturally says that the multimapping M is γ-metrically regular at $\overline{x}$ for $\overline{y}$. Metric regularity property has a long and deep story which goes back to the pioneers works by L.A. Lyusternik [20] and L.M. Graves [10] and has been developed since in numerous papers and books (see, e.g., [8, 14, 21, 30] and the references therein). Such a property is known to be equivalent either to some Lipschitz behavior of the multimapping M^{-1} or to an openness (with linear rate) type condition, namely the existence of positive constants $\alpha, \beta > 0$ such that

$$B[y, \alpha\beta t] \subset M(B[x, t\alpha]),$$

for all $t \in]0, 1]$ and all $(x, y) \in \operatorname{gph} M$ near $(\overline{x}, \overline{y})$. Besides the so-called Robinson-Ursescu theorem (which can be viewed as an extension of the famous Banach-Schauder open mapping theorem) metric regularity is strongly involved in subdifferential calculus, estimates of coderivatives and optimality conditions (see, e.g., [21, 24, 30] and the references therein).

Over the years, Robinson-Ursescu type theorems for multimappings with possibly nonconvex graph have been provided. A first natural way to go beyond convexity in such a context lies in the concept of paraconvexity. Recall that the (above) multimapping M is said to be (θ, C)-paraconvex [26] for a real $\theta > 0$ and a real $C \ge 0$ whenever for all $x_1, x_2 \in X$, for all $t \in [0, 1]$,

$$tM(x_1) + (1-t)M(x_2) \subset M(tx_1 + (1-t)x_2) + C \min(t, 1-t)\, \|x_1 - x_2\|^{\theta}\, \mathbb{B}_Y, \tag{1.2}$$

where $\mathbb{B}_Y$ denotes the closed unit ball of Y. It is worth pointing out that the latter class of multimappings contains the class of multimappings with convex graphs; in fact (as easily seen) the latter inclusion (1.2) with $C = 0$ characterizes the convexity of gph M. A. Jourani [15] proved in 1996 under the θ-paraconvexity of the multimapping M^{-1} with $\theta \ge 1$ that the inequality (1.1) is equivalent to the inclusion $\overline{y} \in \operatorname{int} M(X)$. In [18], H. Huang and R.X. Li established that if M^{-1} is paraconvex, then M is metrically regular at $\overline{x}$ for $\overline{y}$ whenever

$$B(\overline{y}, \beta) \subset M\big(B(\overline{x}, \alpha)\big) \tag{1.3}$$

for some reals $\alpha, \beta > 0$. In fact, under the latter inclusion, from [18, Theorem 2.2] we have the following estimate for any reals $\eta > 0$ and $\eta' > 0$ with $\eta + \eta' = \beta$,

$$d\big(x, M^{-1}(y)\big) \le \eta^{-1}\big(\alpha + C\eta^{\theta} + \|x - \overline{x}\|\big)d\big(y, M(x)\big)$$

for all $x \in X$ and for all $y \in B(\overline{y}, \eta')$, where $\theta, C > 0$ are such that M^{-1} is (θ, C)-paraconvex.

In 2012, X.Y. Zheng and K.F. Ng [36] showed that prox-regularity of sets [25] is also a suitable concept to develop nonconvex versions of Robinson-Ursescu theorem. More precisely, Zheng and Ng established in the Hilbert framework that if gph M is (r, δ)-prox-regular at $(\overline{x}, \overline{y})$, that is, for some reals $r, \delta > 0$

$$\big\langle (x^\star, y^\star), (u, v) - (x, y) \big\rangle \leq \frac{1}{2r} \|(u, v) - (x, y)\|^2,$$

for every $(u, v), (x, y) \in \text{gph}\, M \cap \big(B(\overline{x}, \delta) \times B(\overline{y}, \delta)\big) =: G_M(\delta, \overline{x}, \overline{y})$ and every $(x^\star, y^\star) \in N^C(\text{gph}\, M; (x, y)) \cap (\mathbb{B}_{X^\star} \times \mathbb{B}_{Y^\star}) =: N^C_M(\overline{x}, \overline{y})$, then M is metrically regular at $\overline{x}$ for $\overline{y}$ whenever the inclusion (1.3) holds for some reals $\alpha \in]0, \frac{\delta}{3}[$, $\beta \in]0, \delta[$ satisfying the inequality $\beta > \frac{4\alpha^2+\beta^2}{2r}$. Here and below, $N^C(S; x)$ denotes the Clarke normal cone of a set $S \subset X$ at $x \in S$. Two years later, X.Y. Zheng and Q.H. He provided in [37] a Robinson-Ursescu type theorem for multimappings with some variational behavior of order one, namely with a (σ, δ)-subsmooth [3] graph at $(\overline{x}, \overline{y})$, that is, for some positive constant σ

$$\big\langle (x^\star, y^\star), (u, v) - (x, y) \big\rangle \leq \sigma \|(u, v) - (x, y)\|,$$

for every $(u, v), (x, y) \in G_M(\delta, \overline{x}, \overline{y})$ and every $(x^\star, y^\star) \in N^C_M(\overline{x}, \overline{y})$.

As shown by [2], the above results of [36, 37] can be extended to the class of multimappings M with normally $\omega(\cdot)$-regular graph, that is, for some function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$

$$\big\langle (x^\star, y^\star), (u, v) - (x, y) \big\rangle \leq \omega(\|(u, v) - (x, y)\|)$$

for appropriate points u, v, x, y and unit normals $x^\star, y^\star$ (see Sect. 2 for the definition and more details). More precisely, it is established in [2] that a multimapping M with a normally $\omega(\cdot)$-regular graph satisfying the openness condition (1.3) for some reals $\alpha, \beta, \rho > 0$ such that

$$\beta > \frac{3\alpha}{\rho} + \big(1 + \frac{1}{\rho}\big)\omega\Big(\sqrt{4\alpha^2 + (\beta - \frac{\alpha}{\rho})^2}\Big), \tag{1.4}$$

is γ-metrically regular at $\overline{x}$ for $\overline{y}$ for some real $\gamma \leq \rho$. The authors of [2] derive from their study various preservation results for $\omega(\cdot)$-normally regular sets which complement previous works devoted to the stability of prox-regularity and subsmoothness properties [1, 6, 17, 33, 34] (see also the recent paper by G.E. Ivanov [11]).

In the present paper we first show that the metric subregularity property of the multimapping M (that is the inequality (1.1) with $y = \overline{y}$ (see Sect. 2)) is a suitable assumption to get the normal $\omega(\cdot)$-regularity of the inverse image $M^{-1}(\overline{y})$. We also establish the normal regularity of a generalized equation set, say $S := \{x \in X : f(x) \in M(x)\}$ with $f : X \to Y$ a (single-valued) mapping under a metric subregularity inequality, namely

$$d(x, S) \leq \gamma d\Big(\big(x, f(x)\big), \text{gph}\, M\Big) \quad x \text{ near } \overline{x}.$$

Of course, we require in both cases a normal $\omega(\cdot)$-regularity property on the involved multimapping M (either on the coderivative or on the graph). We then naturally replace (in the line of [2]) the latter metric subregularity assumption by the inclusion (1.3) with $\alpha, \beta > 0$ such that $\beta > \omega\big(\sqrt{\alpha^2+\beta^2}\big)$. At last but not least, we show that a Lipschitz (with respect to the Hausdorff-Pompeiu distance) multimapping with normal $\omega(\cdot)$-regularity values near $\overline{x}$ enjoys some metric subregularity property at $\overline{x}$ for $\overline{y}$.

The paper is organized as follows:

Section 2 is devoted to the necessary background in variational analysis. In Sect. 3, we provide general sufficient conditions ensuring the preservation of normal $\omega(\cdot)$-regularity for generalized equations. Sections 4 and 5 focus on normally regular versions of Robin-Ursescu theorem. Preservation results in the line of Sect. 3 are also provided.

2 Notation and Preliminaries

Our notation is quite standard. In the whole paper, all vector spaces are *real* vector spaces. The set of nonnegative real numbers is denoted $\mathbb{R}_+ := [0, +\infty[$.

Let $(X, \|\cdot\|)$ be a (real) normed space. We denote by $B[x, r]$ (resp. $B(x, r)$) the closed (resp. open) ball centered at $x \in X$ of radius $r > 0$. The boundary (resp. the interior) of a nonempty set $S \subset X$ is denoted by bdry S (resp. int S). For the unit balls in X (that is, centered at 0_X with radius 1) it will be convenient to set

$$\mathbb{B}_X := B[0_X, 1] \quad \text{and} \quad \mathbb{U}_X := B(0_X, 1).$$

We also set $\mathbb{S}_X := \{x \in X : \|x\| = 1\} = \mathbb{B}_X \backslash \mathbb{U}_X$. The (topological) dual $X^\star$ of X is endowed with its natural norm $\|\cdot\|_\star$ defined by

$$\|x^\star\|_\star := \sup_{x \in \mathbb{B}_X} \big\langle x^\star, x \big\rangle \quad \text{for all } x^\star \in X^\star,$$

where $\langle x^\star, x\rangle := x^\star(x)$. As usual, we define the distance function from the nonempty set S by setting

$$d(x, S) = d_S(x) := \inf_{s \in S} \|x - s\| \quad \text{for all } x \in X.$$

For every $x \in X$, the possibly empty set of all nearest points of x in S is defined by

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

The Hausdorff-Pompeiu excess of the set S over another nonempty subset $S' \subset X$ is defined by

$$\mathrm{exc}(S, S') := \sup_{x \in S} d(x, S') = \inf\{r > 0 : S \subset S' + r\mathbb{B}_X\}. \tag{2.1}$$

2.1 Normal Cones and Subdifferentials

Let X, Y be two normed spaces. The *Fréchet* (resp. *Mordukhovich limiting* (resp. *Clarke*)) *normal cone* [1] of a set $S \subset X$ at $x \in S$ is denoted by $N^F(S; x)$ (resp. $N^L(S; x)$ (resp. $N^C(S; x)$)). By convention, we put

$$N^F(S; x) = N^L(S; x) = N^C(S; x) := \emptyset \quad \text{for all } x \notin S. \tag{2.2}$$

The *coderivative* associated to a concept of normal cone $\mathcal{N}$ in $X \times Y$ of a multimapping $M : X \rightrightarrows Y$ at $(x, y) \in \operatorname{gph} M$ is defined for every $y^\star \in Y^\star$ by

$$D_{\mathcal{N}} M(x, y)(y^\star) := \left\{x^\star \in X^\star : (x^\star, -y^\star) \in \mathcal{N}\big(\operatorname{gph} M; (x, y)\big)\right\}.$$

The *Fréchet* (resp. *Mordukhovich limiting* (resp. *Clarke*)) *subdifferential* of an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ finite at $x \in X$ is defined by saying that $x^\star \in X^\star$ belongs to $\partial_F f(x)$ (resp. $\partial_L f(x)$ (resp. $\partial_C f(x)$)) when $(x^\star, -1)$ belongs to the corresponding normal cone of the epigraph of f at $(x, f(x))$. Through to (2.2), we easily see that

$$\partial_F f(x) = \partial_L f(x) = \partial_C f(x) = \emptyset \quad \text{if } f(x) = +\infty.$$

The above normal cones and subdifferentials do not depend on equivalent norms on X. In particular, *the subdifferential of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is always considered for an equivalent norm with respect to the initial one given on X.*

Given any subdifferential ∂ and its corresponding normal cone N, it is well known that

$$\partial \psi_S(x) = N(S; x) \quad \text{for all } x \in X,$$

where (as usual) ψ_S denotes the indicator of the subset S of X (in the sense of variational analysis) that is, for every $x \in X$,

$$\psi_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

We also recall that the subdifferential ∂ enjoys Fermat optimality condition, namely

$$0 \in \partial f(x) \quad \text{if } x \text{ is a local minimizer of } f.$$

If f is convex, it is known that the subdifferential ∂ coincides with the *Moreau-Rockafellar* subdifferential, that is,

$$\partial f(x) = \left\{x^\star \in X^\star : \langle x^\star, x' - x\rangle \le f(x') - f(x) \ \forall x' \in X\right\}.$$

[1] Also known as F (resp. L (resp. C))-normal cone.

For the particular case of the convex function $f := \|\cdot\|$, it is known (and not difficult to prove) that for all $x \in X$

$$\partial\|\cdot\|(x) = \begin{cases} \{x^\star \in X^\star : \|x^\star\|_\star = 1, \langle x^\star, x\rangle = \|x\|\} & \text{if } x \neq 0 \\ \mathbb{B}_{X^\star} & \text{otherwise} \end{cases}$$

Recall that the Fréchet normal cone N^F is linked to its subdifferential ∂_F through the equality

$$\partial_F d_S(x) = N^F(S; x) \cap \mathbb{B}_{X^\star} \quad \text{for all } x \in S. \tag{2.3}$$

A similar equality holds for Mordukhovich-limiting normal cone N^L and the subdifferential ∂_L while we only have

$$\partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}_{X^\star} \quad \text{for all } x \in S. \tag{2.4}$$

We end this section with fundamental results on subdifferentials. We start with the famous sum rule for the Clarke's subdifferential (see, e.g., [30, Theorem 2.98] (see also the references [5, 24])).

Theorem 2.1 (sum rule for C-subdifferential) *Let X be a normed space and let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be two functions which are finite at $\overline{x} \in X$. If $f_1(\overline{x}) < +\infty$ and f_2 is Lipschitz continuous near $\overline{x}$, then one has*

$$\partial_C(f_1 + f_2)(\overline{x}) \subset \partial_C f_1(\overline{x}) + \partial_C f_2(\overline{x}).$$

A quite similar result holds for the Mordukhovich limiting subdifferential (see the monographs [21, 30]).

Theorem 2.2 (sum rule for L-subdifferential) *Let X be an Asplund space and let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be two functions which are finite at $\overline{x} \in X$. If f_1 is Lipschitz continuous at $\overline{x} \in X$ and f_2 is lower semicontinuous near $\overline{x}$, then one has*

$$\partial_L(f_1 + f_2)(\overline{x}) \subset \partial_L f_1(\overline{x}) + \partial_L f_2(\overline{x}).$$

The next proposition (see, e.g., [30, Theorem 2.135] (see also the references [5, 24])) gives an estimate for the Clarke subdifferential $\partial_C(g \circ G)(\overline{x})$ for the composition of a locally Lipschitz function g with an inner strictly Hadamard differentiable vector-valued mapping G.

Proposition 2.1 *Let $G : X \to Y$ be a mapping between two normed spaces X and Y which is strictly Hadamard differentiable at a point $\overline{x} \in X$ and let $g : Y \to \mathbb{R}$ be a function Lipschitz continuous near $G(\overline{x})$. Then, the function $g \circ G$ is Lipschitz continuous near $\overline{x}$ and*

$$\partial_C(g \circ G)(\overline{x}) \subset DG(\overline{x})^\star\big(\partial_C g(G(\overline{x}))\big).$$

The following proposition provides a description of $N^C(\text{gph}\, M; \cdot)$ for a multimapping $M(\cdot)$ defined as the translation of a fixed set S, that is, $M(x) := f(x) + S$ for a given mapping f. For the proof, we refer the reader to [30, Proposition 2.129].

Proposition 2.2 *Let S be a nonempty set of a normed space Y and let $f : X \to Y$ be a mapping from a normed space X into Y. Let $M : X \rightrightarrows Y$ be the multimapping defined by*

$$M(x) := f(x) + S \quad \textit{for all } x \in S.$$

If f is strictly Hadamard differentiable at $\overline{x} \in X$, then for every $y \in M(\overline{x})$

$$N^C(\text{gph}\, M; (\overline{x}, y)) = \left\{(-Df(\overline{x})^\star(y^\star), y^\star) : y^\star \in N^C\big(S; y - f(\overline{x})\big)\right\}.$$

For more details on normal cones, coderivatives and subdifferentials, we refer the reader to the books [5, 21, 24, 28, 30] and the references therein.

2.2 Normally $\omega(\cdot)$-regular Sets

This section is devoted to the class of normally $\omega(\cdot)$-regular sets introduced in [2]. Let us start by giving the definition of such sets:

Definition 2.1 Let S be a subset of a normed space $(X, \|\cdot\|)$ and let $\omega : \mathbb{R}_+ \to \mathbb{R}$ be a function with $\omega(0) = 0$. Given a concept of normal cone $\mathcal{N}$ in X, one says that S is $\mathcal{N}$*-normally* $\omega(\cdot)$*-regular* relative to an open set $V \subset X$ (with respect to the norm $\|\cdot\|$) whenever

$$\left\langle x^\star, x' - x\right\rangle \le \|x^\star\|_\star \omega(\left\|x' - x\right\|),$$

for all $x, x' \in S \cap V$ and for all $x^\star \in \mathcal{N}(S; x)$. When V is the whole space X, we will just say that S is $\mathcal{N}$-normally $\omega(\cdot)$-regular. It will be also convenient to say that S is C-normally (resp. F-normally) $\omega(\cdot)$-regular whenever $\mathcal{N}$ is the normal cone N^C (resp. N^F).

The class of normally $\omega(\cdot)$-regular sets contains the class of (σ, δ)-subsmooth sets [37] which (roughly speaking) expresses a variational behavior of order one.

Definition 2.2 Let S be a subset of a normed space $(X, \|\cdot\|)$ and let $\overline{x} \in S$. One says that S is (σ, δ)*-subsmooth* at $\overline{x}$ for some reals $\sigma, \delta > 0$ provided that

$$\left\langle x^\star, x' - x\right\rangle \le \sigma \|x^\star\|_\star \left\|x' - x\right\|,$$

for all $x, x' \in S \cap B(\overline{x}, \delta)$ and for all $x^\star \in N^C(S; x)$.

Clearly, if a subset S of a normed space X is (σ, δ)-subsmooth at $\overline{x} \in S$, then it is C-normally $\omega(\cdot)$-regular relative to $B(\overline{x}, \delta)$ with $\omega(t) := \sigma t$ for every real $t \geq 0$. The above definition of (σ, δ)-subsmooth property is quite related to the original definition of subsmooth sets by D. Aussel, A. Daniilidis and L. Thibault [3] where the authors declare that the set S is subsmooth at $\overline{x}$ whenever for every $\varepsilon > 0$ we can find some $\delta > 0$ such that

$$\langle x^\star, x' - x \rangle \leq \varepsilon \|x^\star\|_\star \|x' - x\|.$$

for all $x, x' \in S \cap B(\overline{x}, \delta)$ and all $x^\star \in N^C(S; x)$. It is readily seen that the set S is subsmooth at $\overline{x} \in S$ if and only if for all $\sigma > 0$ there is a real $\delta_\sigma > 0$ such that S is (σ, δ_σ)-subsmooth at $\overline{x} \in S$. Given a real $\sigma > 0$, there are (σ, δ)-subsmooth sets S in $\mathbb{R}^2$ for every $\delta > 0$ which fails to be Fréchet-Clarke regular at $\overline{x} \in S$ (i.e., $N^F(S; \overline{x}) = N^C(S; \overline{x})$). Such sets are not hemi-subsmooth at $\overline{x}$ (hence one-sided subsmooth at $\overline{x}$/subsmooth at $\overline{x}$ (see, e.g., [31, Chapter 8])). An example of such a set S has been given by X.Y. Zheng and Q.H. He in their 2014 paper [37]: namely, for any real $\sigma > 0$ the set $S = \text{epi } f$ where

$$f(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ -\sigma x & \text{otherwise.} \end{cases}$$

is (σ, δ)-subsmooth at $\overline{x} := (0, 0)$ for every real $\delta > 0$ for the 1-norm $\|\cdot\|_1$ in $\mathbb{R}^2$ and fails to be Fréchet-Clarke regular at $(0, 0)$.

The class of normally $\omega(\cdot)$-regular sets also contains the class of uniform prox-regular sets in Hilbert spaces. Prox-regularity has been well recognized as a fundamental tool in variational analysis which allows to go beyond convexity property in many topics of modern analysis (see the survey [6] and the references therein). Taking the above definitions of $\omega(\cdot)$-regularity and (σ, δ)-subsmoothness into account, it will be convenient for us to use as definition of r-prox-regularity of sets ([25]) the following property.

Definition 2.3 A nonempty closed set S of a Hilbert space $\mathcal{H}$ is said to be *r-prox-regular* (or prox-regular with constant/thickness r) for some $r \in]0, +\infty]$ if

$$\langle v, x' - x \rangle \leq \frac{\|v\|}{2r} \left\|x' - x\right\|^2, \tag{2.5}$$

for all $x, x' \in S$, for all $v \in N^C(S; x)$ (or $N^F(S; x)$ (or $N^L(S; x)$).

An r-prox-regular closed set S of a Hilbert space $\mathcal{H}$ is clearly C-normally $\omega(\cdot)$-regular relative to the whole space $\mathcal{H}$ with $\omega(t) := \frac{1}{2r}t^2$ for every real $t \geq 0$. It is also known (and not difficult to check) that the (nonempty closed) set $S \subset \mathcal{H}$ is r-prox-regular if and only if (2.5) holds for all $x, x' \in S$ with $\|x' - x\| < 2r$ and for all $v \in N^F(S; x) \cap \mathbb{B}_{\mathcal{H}}$. The r-prox-regularity of the (nonempty closed) set S in $\mathcal{H}$ is often defined by means of the characterization property requiring that for any $x \in S$ and any nonzero $v \in N^F(S; x)$ with $\|v\| \leq 1$, one has $x \in \text{Proj}_S(x + tv)$ for every non-negative real $t \leq r$. The closed subset S in $\mathcal{H}$ is also known to be r-prox-regular if and only if $\text{Proj}_S(x)$ is a singleton for every $x \in U_r(S) := \{d_S < r\}$ and the induced (single-valued) mapping is continuous on $U_r(S)$.

Given a real $r > 0$, we can check (see, e.g., [1, Theorem 4.1]) that the epigraph of an r^{-1}-semiconvex continuous function $f : \mathcal{H} \to \mathbb{R}$ (that is, $f + \frac{1}{2r}\|\cdot\|^2$ is convex continuous) is r-prox-regular. Recall also (see [31, Proposition 15.35]) that a mapping $F : \mathcal{H} \to \mathcal{H}$ (resp. a function $F : \mathcal{H} \to \mathbb{R}$) which is differentiable with L-Lipschitz derivative has its graph (resp. epigraph) $1/L$-prox-regular.

The class of strongly convex sets also deserves to be mentioned:

Definition 2.4 A nonempty closed set C of a Hilbert space $\mathcal{H}$ is said to be *R-strongly convex* (or strongly convex with radius R) for some $R \in]0, +\infty[$ if

$$\left\langle v, x' - x\right\rangle \leq -\frac{\|v\|}{2R}\left\|x' - x\right\|^2,$$

for all $x, x' \in C$, for all $v \in N^F(C; x)$.

It is known (see, e.g., the survey [9]) that a strongly convex set is nothing but the intersection of a family of closed balls with common radius R (hence convex and bounded). Of course, a strongly convex set is $\omega(\cdot)$-normally regular with $\omega(t) := -\frac{t^2}{2R}$ for all $t \in [0, +\infty[$.

The class of $\mathcal{N}$-normally $\omega(\cdot)$-regular sets is also quite related to other previous concepts of nonsmooth sets: $C^{1,\varphi}$-regularity for functions and sets [16], super-regularity [19], Clarke regularity [7]. Before closing this subsection devoted to nonsmooth sets, let us give a result ensuring the normal $\omega(\cdot)$-regularity for the graph of a multimapping M given as a sum of a mapping and a set. For the proof, we refer the reader to [2, Theorem 4.4].

Proposition 2.3 *Let $f : X \to Y$ be a mapping between two normed spaces X and Y and let S be a subset of Y which is C-normally $\omega(\cdot)$-regular relative to the whole space Y for some nondecreasing function $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$. Assume that:*

(i) there exists a real $K \geq 0$ such that

$$\|f(x) - f(x')\| \leq K\|x - x'\| \quad \text{for all } x, x' \in X;$$

(ii) the mapping f is differentiable on X and there exists a real $L \geq 0$ such that

$$\left\|Df(x) - Df(x')\right\| \leq L\left\|x - x'\right\| \quad \text{for all } x, x' \in X.$$

Let $\|\cdot\|_{X\times Y}$ a norm on $X \times Y$ associated to the product topology (of the norm topologies of X and Y) and such that

$$\max(\|x - x'\|, \|y - y'\|) \leq \|(x, y) - (x', y')\|_{X\times Y} \quad \text{for all } x, x' \in X, \text{all } y, y' \in Y.$$

Then, the graph of the multimapping $M(\cdot) := f(\cdot) + S$ is C-normally $\rho(\cdot)$-regular where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\rho(t) := \omega\big(\max(1, K)t\big) + \frac{Lt^2}{2} \quad \text{for all } t \in \mathbb{R}_+.$$

2.3 Metric Subregularity

This section is devoted to the necessary background on metric subregularity theory needed in the paper. For more details on this topic, we refer to [14, 28, 30] and the references therein.

Let $M : X \rightrightarrows Y$ be a multimapping from a normed space X to another normed space Y and let $(\overline{x}, \overline{y}) \in \operatorname{gph} M$. One says that M is *metrically subregular* at $\overline{x}$ for $\overline{y}$ whenever there exist a real $\gamma \geq 0$ and a neighborhood U of $\overline{x}$ such that

$$d\big(x, M^{-1}(\overline{y})\big) \leq \gamma d\big(\overline{y}, M(x)\big) \quad \text{for all } x \in U. \tag{2.6}$$

The *modulus of metric subregularity* $\operatorname{subreg}[M](\overline{x} \mid \overline{y})$ of M at $\overline{x}$ for $\overline{y}$ is defined as the infimum of all $\gamma \in [0, +\infty[$ for which there is a neighborhood U of $\overline{x}$ such that the inequality (2.6) is fulfilled.

The following proposition is due to X.Y. Zheng and K.F. Ng ([35]). It provides important quantitative properties on a multimapping M which fails to fulfill the metric subregularity inequality (2.6) for some point x, that is,

$$\gamma d\big(\overline{y}, M(x)\big) < d\big(x, M^{-1}(\overline{y})\big).$$

Proposition 2.4 *Let $M : X \rightrightarrows Y$ be a multimapping with closed graph between two Banach spaces X and Y and let $\overline{y} \in Y$. Assume that there exist $x \in X$ and two reals $\gamma, r \in]0, +\infty[$ such that*

$$\gamma d\big(\overline{y}, M(x)\big) < r < d\big(x, M^{-1}(\overline{y})\big).$$

Then, for all reals $\eta, \varepsilon \in]0, +\infty[$, there exist $(u, v) \in \operatorname{gph} M$ satisfying:

(i) $\|u - x\| < r$ *and* $0 < \|v - \overline{y}\| < \min\{\frac{r}{\gamma}, d(\overline{y}, M(u)) + \varepsilon\}$;
(ii) $\|v - \overline{y}\| \leq \|b - \overline{y}\| + \frac{1}{\gamma}\big(\|a - u\| + \eta\|b - v\|\big)$ *for all* $(a, b) \in \operatorname{gph} M$;
(iii) $(0, 0) \in \{0\} \times \partial\|\cdot\|(v - \overline{y}) + \frac{1}{\gamma}\big(\mathbb{B}_{X^\star} \times \eta\mathbb{B}_{Y^\star}\big) + N^C\big(\operatorname{gph} M; (u, v)\big)$.

With the above result at hands, Zheng and Ng provide in [35] an important estimate for the modulus of subregularity:

Proposition 2.5 *Let $M : X \rightrightarrows Y$ be a multimapping with closed graph between two Banach spaces and let $(\overline{x}, \overline{y}) \in \operatorname{gph} M$. Then, one has*

$$\operatorname{subreg}[M](\overline{x}|\overline{y}) \leq \inf_{\varepsilon > 0} \sup \left\{ \sup_{b \in \mathbb{B}_{X^\star}} \|D_C M(x, y)^{-1}(b^\star)\| : \begin{cases} x \in B(\overline{x}, \varepsilon) \setminus M^{-1}(\overline{y}) \\ y \in M(x) \cap B(\overline{y}, \varepsilon) \end{cases} \right\}.$$

Remark 2.1 We point out that Proposition 2.4 (resp. Proposition 2.5) also holds for the Mordukhovich limiting normal cone $N^L\big(\operatorname{gph} M; (u, v)\big)$ (resp. the coderivative $D_L M(x, y)^{-1}(b^\star)$) in the context of Asplund spaces X and Y. ■

3 Preservation of Normal $\omega(\cdot)$-regularity of Sets Under Metric Subregularity

We first give sufficient conditions ensuring the normal $\omega(\cdot)$-regularity (so in particular the prox-regularity) for sets of the form

$$\{x \in X : \overline{y} \in M(x)\} =: M^{-1}(\overline{y}), \tag{3.1}$$

where $M : X \rightrightarrows Y$ is a multimapping and $\overline{y} \in Y$. It should be noted that a large number of sets can be rewritten as in (3.1). For instance, this is the case of constraint sets, namely sets of the form

$$\{f_1 \leq 0, \dots, f_m \leq 0, f_{m+1} = 0, \dots, f_{m+n} = 0\} \tag{3.2}$$

which is nothing but $M^{-1}(\overline{y})$ with $\overline{y} = 0$ and

$$M(x) := -\big(f_1(x), \dots, f_m(x), f_{m+1}(x), \dots, f_{m+n}(x)\big) +]-\infty, 0]^m \times \{0_{\mathbb{R}^n}\} \tag{3.3}$$

Another case which deserves to be stated lies in the intersection of finitely many sets. Indeed, for any sets $S_1, \dots, S_n \subset X$ we note that

$$\bigcap_{k=1}^{n} S_k = \{x \in X : 0 \in M(x)\} \quad \text{with } M(x) := -(x, \dots, x) + \prod_{k=1}^{n} S_k. \tag{3.4}$$

More generally, for a single-valued mapping $f : X \to Y$ and $B \subset Y$, the inverse image $f^{-1}(B)$ reduces to $M^{-1}(\overline{y})$ with $\overline{y} = 0$ and

$$M(x) := -f(x) + B. \tag{3.5}$$

Observe that in the above cases (3.3), (3.5) and (3.4), the involved multimapping $M(\cdot)$ is nothing but the translation of a fixed set. Such multimappings will be at the heart of Sect. 5.

Let us mention that the uniform prox-regularity of level and sublevel sets has been studied in [1, 2, 33, 34]. In [34], J.-P. Vial established the uniform prox-regularity (called weak convexity therein) of a sublevel set $S = \{f \leq 0\}$ in $\mathbb{R}^n$ of a weakly convex function f satisfying

$$\inf_{\zeta \in \partial f(x), x \in \mathrm{bdry}\, S} \|\zeta\| > 0.$$

We also mention the work [33] where the author gives sufficient conditions ensuring the prox-regularity of the set $S' = \{f_1 \leq 0, \dots, f_m \leq 0\}$ with $f_i \in C^2(\mathbb{R}^n)$ and for some positive constants α, β, M

$$\alpha \leq |\nabla f_i(x)| \leq \beta \quad \text{and} \quad |D^2 f_i(x)| \leq M.$$

In [1], the authors establish the prox-regularity of the set S' for possibly nonsmooth functions f_i defined on a Hilbert space $\mathcal{H}$ with their C-subdifferentials enjoying an hypomonotone property and under a generalized Slater's condition, namely the existence of a real $\delta > 0$ such that for every $x \in \text{bdry}\, S'$ there is a unit vector v_x such that for all $k = 1, \ldots, m$ and for all $\zeta \in \partial_C f_k(x)$,

$$\langle \zeta, v_x \rangle \leq -\delta.$$

The prox-regularity of the level $L := \{F = 0\}$ of a smooth function F is also developed in [1], under an openness condition, say for some real $\delta > 0$

$$\delta\mathbb{B} \subset DF(x)(\mathbb{B}) \quad \text{for all } x \in \text{bdry}\, L.$$

Regarding the prox-regularity of a constraint set with finitely many inequality and equality constraints (see (3.2)), let us say that is has been studied in [1] and [2] through two approaches, namely two different openness conditions either on the derivatives or on a perturbation of the involved graphs.

Sublevel and level sets can be seen as a particular case of inverse image. The prox-regularity of the inverse image $f^{-1}(B)$ has been first studied in the survey [6] under a theoretical condition

$$N^F(f^{-1}(B); x) \cap \mathbb{B} \subset Df(x)^\star(N(B; f(x) \cap \gamma\mathbb{B})$$

and also investigated in the papers [1] and [2]. Subsmoothness of inverse images have been studied in [17]. Besides the prox-regularity of inverse images, let us point out that the direct image case has been also examinated in [2, 6, 11].

Let us also mention that intersection of two uniformly prox-regular sets, say S_1 and S_2 may fail to be prox-regular, even in $\mathbb{R}^2$ (see, e.g., [6]). To the best of our knowledge, prox-regularity of the intersection $S_1 \cap S_2$ holds under anyone of the following conditions:

- the strong convexity of either S_1 or S_2;
- an openness condition on involved tangent cones, namely the existence of a real $s > 0$ such that for all $\overline{x} \in \text{bdry}(S_1 \cap S_2)$,

$$s\mathbb{B} \subset T(S_1; x_1) \cap \mathbb{B} - T(S_2; x_2) \cap \mathbb{B}, \quad x_i \in S_i, \text{ near } \overline{x},$$

- an openness condition on S_1, S_2, more precisely, the existence of $\alpha, \beta, s > 0$ such that for all $\overline{x} \in \text{bdry}(S_1 \cap S_2)$,

$$\beta\mathbb{B}_{\mathcal{H}^6} \subset -\Delta_{B[(\overline{x},\overline{x},\overline{x}),\alpha]^2} + \Delta_{\mathcal{H}^3} \times \mathcal{H} \times S_1 \times S_2, \tag{3.6}$$

where $\Delta_{E^m} := \{(x, \ldots, x) : x \in E\} \subset E^m$.

For a detailed overview on the preservation of nonsmooth sets under set operations, we refer the reader to the recent book by L. Thibault [31] (see the comments at the end of Chapters 8, 15 and 16).

We start our study of preservation of $\omega(\cdot)$-regularity in this section by giving an important estimate for the normal cone $N(M^{-1}(\overline{y});\overline{x})$ under a metric subregularity assumption on the multimapping M. Given a multimapping $M : X \rightrightarrows Y$ between two normed spaces X and Y, we define $\Delta_M : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\Delta_M(x, y) := d\big(y, M(x)\big) \quad \text{for all } (x, y) \in X \times Y.$$

It is known that a certain Lipschitz behavior of $\Delta_M(\cdot, y)$ is equivalent to the Aubin-Lipschitz property of M (see, e.g., [30, Proposition 7.7]). Subgradients of the function Δ_M have been studied in the literature by L. Thibault [29], B.S. Mordukhovich and N.M. Nam ([22, 23]) and M. Bounkhel [4, Chapter 4]. It is also known (see, e.g., [30, Proposition 4.162]) that

$$\partial_F \Delta_M(\overline{x}, \overline{y}) = N^F\big(\mathrm{gph}\, M; (\overline{x}, \overline{y})\big) \cap (X^\star \times \mathbb{B}_{Y^\star})$$

and

$$N^F\big(\mathrm{gph}\, M; (\overline{x}, \overline{y})\big) = \mathbb{R}_+ \partial_F \Delta_M(\overline{x}, \overline{y}).$$

Similar equalities hold true for the Mordukhovich-limiting normal cone and subdifferentials whenever X and Y are Banach spaces and gph M is closed.

Lemma 3.1 *Let $M : X \rightrightarrows Y$ be a multimapping between two normed spaces X and Y and let $(\overline{x}, \overline{y}) \in \mathrm{gph}\, M$. The following hold:*

(a) *One has*

$$\partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x}) \times \{0\} \subset \{0\} \times \mathbb{B}_{Y^\star} + \partial_C \Delta_M(\overline{x}, \overline{y})$$

and

$$\partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x}) \subset D_C M(\overline{x}, \overline{y})(\mathbb{B}_{Y^\star}).$$

(b) *Assume that X and Y are Asplund spaces. Then, one has*

$$\partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x}) \subset D_L M(\overline{x}, \overline{y})(\mathbb{B}_{Y^\star}).$$

If in addition Δ_M is lower semicontinuous near $(\overline{x}, \overline{y})$, then one has

$$\partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x}) \times \{0\} \subset \{0\} \times \mathbb{B}_{Y^\star} + \partial_L \Delta_M(\overline{x}, \overline{y}). \tag{3.7}$$

(c) *Assume that there exists a real $\gamma \geq 0$ and a real $\delta > 0$ such that*

$$d\big(x, M^{-1}(\overline{y})\big) \leq \gamma d\big(\overline{y}, M(x)\big) \quad \text{for all } x \in B(\overline{x}, \delta).$$

Then, one has

$$N^F\big(M^{-1}(\overline{y}); \overline{x}\big) \cap \mathbb{B}_{X^\star} = \partial_F d\big(\cdot; M^{-1}(\overline{y})\big)(\overline{x}) \subset \gamma \partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x}). \tag{3.8}$$

Proof (*a*) We denote $\|\cdot\|_1$ the 1-norm on $X \times Y$. Let $x^\star \in \partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x})$. Define $\varphi, \theta : X \times Y \to \mathbb{R}$ by setting for all $(x, y) \in X \times Y$

$$\varphi(x, y) := \Delta_M(x, y) + \|y - \overline{y}\| \quad \text{and} \quad \theta(x, y) := \psi_{\operatorname{gph} M}(x, y) + \|y - \overline{y}\|.$$

It is readily seen that

$$\varphi \leq \theta \quad \text{and} \quad \varphi(\overline{x}, \overline{y}) = \theta(\overline{x}, \overline{y}) = 0. \tag{3.9}$$

Fix any real $\varepsilon > 0$. Through the definition of F-subgradients, we can find some real $\eta > 0$ such that

$$\begin{aligned} \big\langle (x^\star, 0), (x', y') - (\overline{x}, \overline{y}) \big\rangle &\leq d_{M(x')}(\overline{y}) - d_{M(\overline{x})}(\overline{y}) + \varepsilon\|x' - \overline{x}\| \\ &\leq \Delta_M(x', y') + \|y' - \overline{y}\| + \varepsilon\|(x', y') - (\overline{x}, \overline{y})\|_1 \\ &\leq \varphi(x', y') - \varphi(\overline{x}, \overline{y}) + \varepsilon\|(x', y') - (\overline{x}, \overline{y})\|_1, \end{aligned} \tag{3.10}$$

for all $x' \in B(\overline{x}, \eta)$ and all $y' \in Y$. It follows that (see Theorem 2.1)

$$(x^\star, 0) \in \partial_F \varphi(\overline{x}, \overline{y}) \subset \partial_C \varphi(\overline{x}, \overline{y}) \subset \{0\} \times \mathbb{B}_{Y^\star} + \partial_C \Delta_M(\overline{x}, \overline{y}).$$

The first inclusion in (a) is then established. Regarding the second inclusion, it suffices to note that (thanks to (3.10) and (3.9))

$$\big\langle (x^\star, 0), (x', y') - (\overline{x}, \overline{y}) \big\rangle \leq \theta(x', y') - \theta(\overline{x}, \overline{y}) + \varepsilon\|(x', y') - (\overline{x}, \overline{y})\|_1,$$

for all $x' \in B(\overline{x}, \eta)$ and all $y' \in Y$ to get (as above)

$$(x^\star, 0) \in \partial_F \theta(\overline{x}, \overline{y}) \subset \partial_C \theta(\overline{x}, \overline{y}) \subset \{0\} \times \mathbb{B}_{Y^\star} + N^C\big(\operatorname{gph} M; (\overline{x}, \overline{y})\big).$$

(*b*) The proof is similar to (*a*) (see Theorem 2.2 for the inclusion (3.7)).
(*c*) Let $x^\star \in N^F\big(M^{-1}(\overline{y}); \overline{x}\big) \cap \mathbb{B}_{X^\star} = \partial_F d_{M^{-1}(\overline{y})}(\overline{x})$ (see (2.3)). Fix any real $\varepsilon > 0$. By definition of F-subgradients, there is a real $\eta > 0$ with $\eta < \delta$ such that

$$\big\langle x^\star, x' - \overline{x} \big\rangle \leq d_{M^{-1}(\overline{y})}(x') - d_{M^{-1}(\overline{y})}(\overline{x}) + \gamma\varepsilon\|\overline{x} - x'\|,$$

for every $x' \in B(\overline{x}, \eta)$. It follows from this

$$\Big\langle \gamma^{-1} x^\star, x' - \overline{x} \Big\rangle \leq d_{M(x')}(\overline{y}) - d_{M(\overline{x})}(\overline{y}) + \varepsilon\|\overline{x} - x'\| \quad \text{for all } x' \in B(\overline{x}, \eta)$$

and this translates the inclusion $x^\star \in \gamma\,\partial_F d\big(\overline{y}, M(\cdot)\big)(\overline{x})$. □

We easily derive from (c) of the above proposition the following result:

Proposition 3.1 *Let $M : X \rightrightarrows Y$ be a multimapping between two normed spaces and let $\overline{y} \in Y$ with $M^{-1}(\overline{y}) \neq \emptyset$. Assume that:*

(i) *There exists a real* $\gamma \geq 0$ *such that for each* $\overline{x} \in M^{-1}(\overline{y})$ *there is a real* $\delta > 0$ *satisfying*

$$d\big(x, M^{-1}(\overline{y})\big) \leq \gamma d\big(\overline{y}, M(x)\big) \quad \text{for all } x \in B(\overline{x}, \delta).$$

(ii) *There exists a function* $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\omega(0) = 0$ *and a real* $c \geq 0$ *such that for all* $x, x' \in M^{-1}(\overline{y})$ *and for all* $x^\star \in \partial_F d\big(\overline{y}, M(\cdot)\big)(x)$,

$$\big\langle x^\star, x' - x\big\rangle \leq \big(\|x^\star\| + c\big)\omega(\|x' - x\|).$$

Then, the set $M^{-1}(\overline{y})$ *is* F*-normally* $\rho(\cdot)$*-regular with* $\rho(\cdot) := (1 + \gamma c)\omega(\cdot)$.

Proof If suffices to observe that for any $x, x' \in M^{-1}(\overline{y})$ and any $x^\star \in N^F\big(M^{-1}(\overline{y}); x\big) \cap \mathbb{B}_{X^\star}$, we have (see Lemma 3.1) $\gamma^{-1}x^\star \in \partial_F d\big(\overline{y}, M(\cdot)\big)(x)$ which gives through (ii)

$$\big\langle x^\star, x' - x\big\rangle \leq \gamma(\gamma^{-1}\|x^\star\| + c)\omega(\|x' - x\|) \leq (1 + \gamma c)\omega(\|x' - x\|).$$

The proof is complete. □

Remark 3.1 For the above F-normal $\omega(\cdot)$-regularity of $M^{-1}(\overline{y})$, we claim that (ii) in Proposition 3.1 can be replaced by the assumption of C-normal $\omega(\cdot)$-regularity of gph M at $(\overline{x}, \overline{y}) \in$ gph M. We endow $X \times Y$ with the norm $\|\cdot\|_2$ whose dual norm $\|\cdot\|_\star$ is known (and easily seen) to satisfy

$$\|(x^\star, y^\star)\|_\star \leq \|x^\star\| + \|y^\star\| \quad \text{for all } (x^\star, y^\star) \in X^\star \times Y^\star. \tag{3.11}$$

Fix any $x, x' \in M^{-1}(\overline{y})$ and any $x^\star \in N^F\big(M^{-1}(\overline{y}); x\big) \cap \mathbb{B}_{X^\star}$. According to Lemma 3.1(c)-(a), we can find $y^\star \in \mathbb{B}_{Y^\star}$ such that

$$(x^\star, \gamma y^\star) \in N^C(\text{gph } M; (x, y))$$

and this allows us to use the C-normal $\omega(\cdot)$-regularity of gph M at $(\overline{x}, \overline{y}) \in$ gph M to get

$$\big\langle (x^\star, \gamma y^\star), (x', \overline{y}) - (x, \overline{y})\big\rangle \leq \|(x^\star, \gamma y^\star)\|_\star \omega(\|(x', \overline{y}) - (x, \overline{y})\|_2).$$

This and the inequality (3.11) easily give

$$\big\langle x^\star, x' - x\big\rangle \leq (\|x^\star\| + \gamma\|y^\star\|)\omega(\|(x' - x, 0)\|) \leq (1 + \gamma)\omega(\|x' - x\|)$$

which translates the F-normal $(1 + \gamma)\omega(\cdot)$-regularity of the set $M^{-1}(\overline{y})$ at the point $\overline{x}$.

We point out that such a remark obviously holds for the L-normal $\omega(\cdot)$-regularity of gph M at $(\overline{x}, \overline{y})$ whenever X and Y are Asplund spaces. ■

Remark 3.2 Putting together (c) and the second inclusion of (a) in Lemma 3.1, we easily see that the F-normal $\omega(\cdot)$-regularity property for the set $M^{-1}(\overline{y})$ at $\overline{x} \in M^{-1}(\overline{y})$ also holds true under the following inequality

$$\langle x^\star, x' - x \rangle \leq (\|x^\star\| + c)\omega(\|x' - x\|),$$

for all $x, x' \in M^{-1}(\overline{y})$ and all $x^\star \in D_C M(x, \overline{y})(\mathbb{B}_{Y^\star})$. ■

We pass now to the normal $\omega(\cdot)$-regularity for a (solution set of) generalized equation, say $S := \{x \in X : f(x) \in F(x)\}$ with f (resp. F) single-valued (resp. set-valued). Setting $M(x) := -f(x) + F(x)$ for every $x \in X$, we observe that $S = M^{-1}(0)$ so the above proposition gives sufficient conditions ensuring the desired normal $\omega(\cdot)$-regularity of the set S. For the above multimapping M, note that (i) and (ii) of Proposition 3.1 can be rewritten as:

(i'): There exists a real $\gamma \geq 0$ such that for each $\overline{x} \in S$ there is a real $\delta > 0$ satisfying

$$d(x, S) \leq \gamma d\big(f(x), F(x)\big) \quad \text{for all } x \in B(\overline{x}, \delta). \tag{3.12}$$

(ii'): There exists a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ and a real $c \geq 0$ such that for all $x, x' \in S$ and all $x^\star \in \partial_F d\big(f(\cdot), F(\cdot)\big)(x)$

$$\langle x^\star, x' - x \rangle \leq \big(\|x^\star\| + c\big)\omega(\|x' - x\|).$$

According to Remark 3.1, we know that (ii') above could be replaced by the C-normal $\omega(\cdot)$-regularity of gph M (and by the L-regularity in the context of Asplund spaces). As shown by the next result, such a regularity can only be required for gph F.

Proposition 3.2 *Let X, Y be normed spaces, $F : X \rightrightarrows Y$ be a multimapping and let $f : X \to Y$ be a mapping. Assume that $S := \{x \in X : f(x) \in F(x)\} \neq \emptyset$ along with:*

(i) There exists a real $\gamma \geq 0$ such that for each $\overline{x} \in S$, there exists a real $\delta > 0$ satisfying

$$d(x, S) \leq \gamma d\big((x, f(x)), \text{gph } F\big) \quad \textit{for all } x \in B(\overline{x}, \delta).$$

(ii) There exists a nondecreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that the set gph F *is C-normally $\omega(\cdot)$-regular with respect to the product norm $\|\cdot\|_2$ on $X \times Y$.*

(iii) The mapping f is K-Lipschitz continuous on X for some real $K \geq 0$ and differentiable on X with L-Lipschitz continuous derivative Df for some real $L \geq 0$.

Then, the set S is F-normally $\widehat{\omega}(\cdot)$-regular with $\widehat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\widehat{\omega}(t) := \gamma\Big(\omega(\kappa t) + \frac{L}{2}t^2\Big) \quad \textit{for all } t \in \mathbb{R}_+,$$

where $\kappa := (1 + K^2)^{1/2}$.

Proof We denote $\|\cdot\|_2$ the 2-norm on $X \times Y$ and $\mathbb{B}_\star$ the unit ball of $(X \times Y)^\star$. First, let us define the function $\theta : X \times Y \to \mathbb{R}$ by setting

$$\theta(x, y) := d\big((x, y), \operatorname{gph} F\big) \quad \text{for all } (x, y) \in X \times Y.$$

We also need to consider the mapping $\varphi : X \to X \times Y$ defined by

$$\varphi(x) := \big(x, f(x)\big) \quad \text{for all } x \in X$$

which is obviously κ-Lipschitz continuous and differentiable at each point $x \in X$ with

$$D\varphi(x)(h) = \big(h, Df(x)(h)\big) \quad \text{for all } h \in X.$$

Let $x, x' \in S$ and $u^\star \in N^F(S; x) \cap \mathbb{B}_{X^\star}$. Putting together the equalities $d_S(x) = 0 = (\theta \circ \varphi)(x)$, the inequality given by (i) and the fact that the Fréchet subdifferential is always included in the Clarke's one, we get

$$u^\star \in \partial_F d_S(x) \subset \gamma \partial_F(\theta \circ \varphi)(x) \subset \gamma \partial_C(\theta \circ \varphi)(x). \tag{3.13}$$

Using Proposition 2.1 and the inclusion (2.4) we obtain

$$\partial_C(\theta \circ \varphi)(x) = D\varphi(x)^\star\big(\partial_C\theta(\varphi(x))\big) \subset D\varphi(x)^\star\Big(N^C\big(\operatorname{gph} F; (x, f(x))\big) \cap \mathbb{B}_\star\Big). \tag{3.14}$$

The inclusions (3.14) and (3.13) obviously give some $(x^\star, y^\star) \in N^C\big(\operatorname{gph} F; (x, f(x))\big) \cap \mathbb{B}_{X^\star \times Y^\star}$ such that $u^\star = \gamma D\varphi(x)^\star(x^\star, y^\star)$. We then have

$$\Big\langle \gamma^{-1} u^\star, h \Big\rangle = D\varphi(x)^\star(x^\star, y^\star)(h) = \big\langle (x^\star, y^\star), (h, Df(x)(h)) \big\rangle \quad \text{for all } h \in X.$$

Now, we observe that

$$\begin{aligned} \Big\langle \gamma^{-1} u^\star, x' - x \Big\rangle &= \big\langle (x^\star, y^\star), (x' - x, Df(x)(x' - x)) \big\rangle \\ &= \big\langle (x^\star, y^\star), (x', f(x')) - (x, f(x)) \big\rangle \\ &\quad + \big\langle y^\star, -f(x') + f(x) + Df(x)(x' - x) \big\rangle. \end{aligned}$$

Combining (ii), the inclusions $(x', f(x')), (x, f(x)) \in \operatorname{gph} F$ and $(x^\star, y^\star) \in \mathbb{B}_\star$ and (iii), we get

$$\begin{aligned} \big\langle (x^\star, y^\star), (x', f(x')) - (x, f(x)) \big\rangle &\le \omega\big(\|(x', f(x')) - (x, f(x))\|_2\big) \\ &\le \omega\big(\kappa \|x' - x\|\big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\big\langle y^\star, -f(x') + f(x) + Df(x)(x'-x)\big\rangle\\ &\quad = \left\langle y^\star, Df(x)(x'-x) - \int_0^1 Df(x+t(x'-x))(x'-x)dt\right\rangle,\end{aligned}$$

hence (noticing that $\|y^\star\| \leq 1$)

$$\begin{aligned}&\big\langle y^\star, -f(x') + f(x) + Df(x)(x'-x)\big\rangle\\ &\quad \leq \int_0^1 \|Df(x) - Df(x+t(x'-x))\|\|x'-x\|dt\\ &\quad \leq L\|x'-x\|^2 \int_0^1 tdt = \frac{L}{2}\|x'-x\|^2.\end{aligned}$$

We conclude that

$$\big\langle u^\star, x'-x\big\rangle \leq \gamma\omega\big(\kappa\|x'-x\|\big) + \frac{\gamma L}{2}\|x'-x\|^2$$

and this translates the fact that the set S is F-normally $\widehat{\omega}(\cdot)$-regular. □

Remark 3.3 In view of (3.14), it is clear that we can replace (ii) by (ii"):
(ii") There exists a nondecreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that

$$\big\langle (x^\star, y^\star), (x', f(x')) - (x, f(x))\big\rangle \leq \omega(\|(x', f(x')) - (x, f(x))\|),$$

for all $x, x' \in S$ and all $(x^\star, y^\star) \in \partial_C d(\cdot, \operatorname{gph} F)(x, f(x))$. ■

If F is constant in Proposition 3.2 (say $F \equiv B \subset Y$) we easily see that (i) (of Proposition 3.2) becomes

$$d(x, S) \leq \gamma d\big((x, f(x)), X \times B\big) \quad \text{for all } x \in B(\overline{x}, \delta).$$

This inequality obviously holds for the distance d on $X \times Y$ associated to the norm $\|\cdot\|_2$ whenever

$$d\big(x, f^{-1}(B)\big) \leq \gamma d\big(f(x), B\big) \quad \text{for all } x \in B(\overline{x}, \delta),$$

which is exactly the estimate in (3.12). Therefore, a direct application of Proposition 3.2 gives the normal $\widehat{\omega}(\cdot)$-regularity of $S = \{x \in X : f(x) \in B\} = f^{-1}(B)$. In fact, a direct and similar proof for the constant case $F \equiv B$ allows to slightly improve the above modulus $\widehat{\omega}$ of normal regularity:

Proposition 3.3 *Let X, Y be normed spaces, $B \subset Y$ and let $f : X \to Y$ be a mapping. Assume that $S := f^{-1}(B) \neq \emptyset$ along with:*

(i) There exists a real $\gamma \geq 0$ such that for each $\overline{x} \in S$, there exists a real $\delta > 0$ satisfying

$$d\big(x, f^{-1}(B)\big) \leq \gamma d\big(f(x), B\big) \ \ \textit{for all } x \in B(\overline{x}, \delta).$$

(ii) There exists a nondecreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that the set B is C-normally $\omega(\cdot)$-regular.

(iii) The mapping f is K-Lipschitz continuous on X for some real $K \geq 0$ and differentiable on X with L-Lipschitz continuous derivative Df for some real $L \geq 0$.

Then, the set S is F-normally $\widehat{\omega}(\cdot)$-regular with $\widehat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\widehat{\omega}(t) := \gamma\Big(\omega\big(Kt\big) + \frac{L}{2}t^2\Big) \ \ \textit{for all } t \in \mathbb{R}_+.$$

Proof Let $x, x' \in S$ and $u^\star \in N^F(S; x) \cap \mathbb{B}_{X^\star}$. According to Lemma 3.1, we have

$$u^\star \in \gamma \partial_F d\big(0, -f(\cdot) + B\big)(\overline{x}) = \gamma \partial_F d(f(\cdot), B)(\overline{x}).$$

Applying Proposition 2.1 and using the inclusion (2.4), we can write $u^\star = \gamma Df(x)^\star(y^\star) = \gamma\big(y^\star \circ Df(x)\big)$ for some $y^\star \in N^C\big(B; f(x)\big) \cap \mathbb{B}_{Y^\star}$. It then suffices to observe (as in the proof of Proposition 3.2) that

$$\big\langle y^\star, f(x') - f(x)\big\rangle \leq \omega\big(\|f(x') - f(x)\|\big) \leq \omega\big(K\|x' - x\|\big).$$

and

$$\begin{aligned}
&\big\langle y^\star, -f(x') + f(x) + Df(x)(x' - x)\big\rangle \\
&\quad \leq \int_0^1 \|Df(x) - Df(x + t(x' - x))\| \|x' - x\| dt \\
&\quad \leq L\|x' - x\|^2 \int_0^1 t dt = \frac{L}{2}\|x' - x\|^2
\end{aligned}$$

to get the desired F-normal $\widehat{\omega}(\cdot)$-regularity of S. □

4 Metric Subregularity for Multimappings with Normally $\omega(\cdot)$-regular Graph

The main aim of the present section is to replace the metric subregularity assumption in Proposition 3.1 and Proposition 3.2 by some openness type conditions. This leads to develop the following result ensuring the metric subregularity of a multimapping with normally $\omega(\cdot)$-regular graph. We follow for a large part the proof of [2, Theorem 2] (see the introduction for the precise statement). As usual, we set $1/\rho := 0$ whenever $\rho := +\infty$.

Theorem 4.1 *Let X, Y be Banach spaces, $M : X \rightrightarrows Y$ be a multimapping with closed graph, $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function with $\omega(0) = 0$. Let also $\overline{y} \in Y$ with $M^{-1}(\overline{y}) \neq \emptyset$. Assume that:*

(i) there exist $\alpha, \beta \in]0, +\infty[$ and $\rho \in]0, +\infty]$ such that

$$\beta > \frac{\alpha}{\rho} + \Big(1 + \frac{1}{\rho}\Big)\omega\Big(\big(\alpha^2 + \beta^2\big)^{\frac{1}{2}}\Big) \quad \textit{and}$$
$$B(\overline{y}, \beta) \subset M(B[\overline{x}, \alpha]) \quad \textit{for all } \overline{x} \in M^{-1}(\overline{y}).$$

(ii) the set gph M *is C-normally $\omega(\cdot)$-regular (in $X \times Y$ endowed with the 2-norm $\|\cdot\|_2$) relative to the open set $V := \bigcup_{x \in M^{-1}(\overline{y})} B\big((x, \overline{y}), \sqrt{\alpha^2 + \beta^2}\big)$;*

Then, there exists a real $\gamma \in [0, \rho[$ such that for every $\overline{x} \in M^{-1}(\overline{y})$, there exists a real $\delta > 0$ satisfying

$$d\big(x, M^{-1}(\overline{y})\big) \leq \gamma d\big(\overline{y}, M(x)\big) \quad \textit{for all } x \in B(\overline{x}, \delta).$$

Proof By contradiction, assume that for each $\gamma \in [0, \rho[$, there is $\overline{x} \in M^{-1}(\overline{y})$ such that for every real $\delta > 0$, there is $x \in B(\overline{x}, \delta)$ satisfying

$$d\big(x, M^{-1}(\overline{y})\big) > \gamma d(\overline{y}, M(x)).$$

Fix for a moment any $\rho' \in]0, \rho[$. Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of $]0, \rho'[$ with $\varepsilon_n \to 0$. Choose some integer $N \geq 1$ such that $\theta_n := \frac{1}{n(\rho' - \varepsilon_n)} < \frac{\beta}{2}$ and $\frac{4}{n^2} + \theta_n^2 < \alpha^2 + \beta^2$ for every integer $n \geq N$. Fix any integer $n \geq N$. There are $\overline{x}_n \in M^{-1}(\overline{y})$ and $x_n \in X$ with $\|x_n - \overline{x}_n\| < \frac{1}{n}$ such that

$$d\big(x_n, M^{-1}(\overline{y})\big) > (\rho' - \varepsilon_n) d\big(\overline{y}, M(x_n)\big).$$

According to Proposition 2.4, there is $(u_n, v_n) \in$ gph M such that

$$\|u_n - x_n\| < d\big(x_n, M^{-1}(\overline{y})\big) \quad \text{and} \quad 0 < \|v_n - \overline{y}\| < \frac{d\big(x_n, M^{-1}(\overline{y})\big)}{\rho' - \varepsilon_n} \tag{4.1}$$

along with

$$(0, 0) \in \{0\} \times \partial \|\cdot\|(v_n - \overline{y}) + \frac{1}{\rho' - \varepsilon_n}(\mathbb{B}_{X^\star} \times \varepsilon_n \mathbb{B}_{Y^\star}) + N^C\big(\operatorname{gph} M; (u_n, v_n)\big).$$

Since $v_n - \overline{y} \neq 0$, there is $z_n^\star \in \mathbb{S}_{Y^\star}$ such that $\langle z_n^\star, v_n - \overline{y}\rangle = \|v_n - \overline{y}\|$ and $(x_n^\star, y_n^\star) \in N^C\big(\operatorname{gph} M; (u_n, v_n)\big)$ satisfying

$$(x_n^\star, y_n^\star) \in (0, -z_n^\star) + \frac{1}{\rho' - \varepsilon_n}(\mathbb{B}_{X^\star} \times \varepsilon_n \mathbb{B}_{Y^\star}).$$

Therefore, we can write $y_n^\star = -z_n^\star + \frac{\varepsilon_n}{\rho'-\varepsilon_n} b_n^\star$ for some $b_n^\star \in \mathbb{B}_{Y^\star}$ along with

$$\|x_n^\star\| \leq \frac{1}{\rho' - \varepsilon_n} \quad \text{and} \quad \|y_n^\star\| \leq 1 + \frac{\varepsilon_n}{\rho' - \varepsilon_n}. \tag{4.2}$$

We also note that

$$d\big(x_n, M^{-1}(\overline{y})\big) \leq \|x_n - \overline{x}_n\| \leq \frac{1}{n}$$

which obviously gives (see (4.1))

$$\|v_n - \overline{y}\| < \frac{d\big(x_n, M^{-1}(\overline{y})\big)}{\rho' - \varepsilon_n} \leq \frac{1}{n(\rho' - \varepsilon_n)} = \theta_n.$$

and (see again (4.1))

$$\|u_n - \overline{x}_n\| \leq \|u_n - x_n\| + \|x_n - \overline{x}_n\| < d\big(x_n, M^{-1}(\overline{y})\big) + \|x_n - \overline{x}_n\| \leq \frac{2}{n}. \tag{4.3}$$

Now, set $\zeta_n := (\beta - \theta_n)\frac{v_n - \overline{y}}{\|v_n - \overline{y}\|}$ and observe that (keeping in mind the choice of N)

$$\|v_n - \zeta_n - \overline{y}\| = |\|v_n - \overline{y}\| - \beta + \theta_n| = \beta - \|v_n - \overline{y}\| - \theta_n < \beta,$$

that is, $v_n - \zeta_n \in B(\overline{y}, \beta)$. By assumption, we can find some $w_n \in B[\overline{x}_n, \alpha]$ such that $v_n - \zeta_n \in M(w_n)$. Through (4.3), we see that

$$\|w_n - u_n\| \leq \|w_n - \overline{x}_n\| + \|\overline{x}_n - u_n\| < \alpha + \frac{2}{n}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\|(w_n, v_n - \zeta_n) - (\overline{x}_n, \overline{y})\|_2^2 = \|w_n - \overline{x}_n\|^2 + \|v_n - \zeta_n - \overline{y}\|^2 \\ < \alpha^2 + \beta^2\end{aligned}$$

and

$$\|(u_n, v_n) - (\overline{x}_n, \overline{y})\|_2^2 = \|u_n - \overline{x}_n\|^2 + \|v_n - \overline{y}\|^2 \leq \frac{4}{n^2} + \theta_n^2 < \alpha^2 + \beta^2.$$

Thus, we have the inclusions

$$(w_n, v_n - \zeta_n), (u_n, v_n) \in V.$$

Denote $\|\cdot\|_\star$ the dual norm of the product norm $\|\cdot\|_2$ on $X \times Y$. Putting together the fact that gph M is C-normally $\omega(\cdot)$-regular, (4.2), (4.4) and the definition of ζ_n, we get

$$\begin{aligned}\left\langle (x_n^\star, y_n^\star), (w_n, v_n - \zeta_n) - (u_n, v_n)\right\rangle &\le \|(x_n^\star, y_n^\star)\|_\star \omega\big(\|(w_n - u_n, -\zeta_n)\|\big)\\ &\le \big(\|x_n^\star\| + \|y_n^\star\|\big)\omega\Big(\big(\|w_n - u_n\|^2 + \|\zeta_n\|^2\big)^{\frac{1}{2}}\Big)\\ &\le \big(1 + \frac{1+\varepsilon_n}{\rho' - \varepsilon_n}\big)\omega\Big(\big((\alpha + \frac{2}{n})^2 + (\beta - \theta_n)^2\big)^{\frac{1}{2}}\Big).\end{aligned}$$

By (4.4) and (4.2), we also have

$$\left\langle x_n^\star, w_n - u_n\right\rangle \ge -\|x_n^\star\|\|w_n - u_n\| \ge -\frac{1}{\rho' - \varepsilon_n}(\alpha + \frac{2}{n}).$$

Combining the definition of ζ_n with the equality $\langle z_n^\star, v_n - \overline{y}\rangle = \|v_n - \overline{y}\|$, we obtain

$$\left\langle y_n^\star, \zeta_n\right\rangle = \left\langle -z_n^\star + \frac{\varepsilon_n}{\rho' - \varepsilon_n} b_n^\star, \zeta_n\right\rangle \le \theta_n - \beta + \frac{\varepsilon_n}{\rho' - \varepsilon_n}(\beta - \theta_n).$$

Putting what precedes together, we arrive to (having in mind that $\omega(\cdot)$ is upper semi-continuous)

$$\begin{aligned}-\frac{1}{\rho' - \varepsilon_n}(\alpha + \frac{2}{n}) \le \theta_n - \beta &+ \frac{\varepsilon_n}{\rho' - \varepsilon_n}(\beta - \theta_n)\\ &+ \big(1 + \frac{1+\varepsilon_n}{\rho' - \varepsilon_n}\big)\omega\Big(\big((\alpha + \frac{2}{n})^2 + (\beta - \theta_n)^2\big)^{\frac{1}{2}}\Big).\end{aligned}$$

Letting $n \to \infty$ and $\rho' \uparrow \rho$ obviously yields

$$\beta \le \frac{\alpha}{\rho} + \big(1 + \frac{1}{\rho}\big)\omega\Big(\big(\alpha^2 + \beta^2\big)^{\frac{1}{2}}\Big)$$

which is the desired contradiction. The proof is then complete. □

Remark 4.1 If $\rho = +\infty$, the inequality in (ii) of Theorem 4.1 is reduced to

$$\beta > \omega\Big(\big(\alpha^2 + \beta^2\big)^{\frac{1}{2}}\Big). \tag{4.5}$$

Assume there is a real $r > 0$ such that $\omega(t) := \frac{t^2}{2r}$ for every $t \ge 0$ (which is the case if X and Y are Hilbert spaces and gph M is an r-prox-regular set of $X \times Y$). We then observe that the latter inequality (4.5) can be written as

$$\beta > \frac{\alpha^2 + \beta^2}{2r}$$

which is obviously satisfied for every $\alpha, \beta > 0$ with $\alpha < r$ and $|\beta - r| < \sqrt{r^2 - \alpha^2}$. ■

We derive from the latter theorem sufficient conditions ensuring the normal $\omega(\cdot)$-regularity of an inverse image $M^{-1}(\overline{y})$. Doing so, we complement Proposition 3.1.

Proposition 4.1 *Let X, Y be two Banach spaces, $M : X \rightrightarrows Y$ be a multimapping whose graph is closed and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function with $\omega(0) = 0$. Let $\overline{y} \in Y$ with $S := M^{-1}(\overline{y}) \neq \emptyset$. Assume that (i) and (ii) of Theorem 4.1 hold with $\rho < +\infty$. Then, the set S is F-normally $\theta(\cdot)$-regular with $\theta = (1 + \rho)\omega$.*

Proof Let $x, x' \in S$ and $x^\star \in N^F(S; x) \cap \mathbb{B}_{X^\star}$. Combining Theorem 4.1 and Lemma 3.1(a)-(c) we can find $b^\star \in \mathbb{B}_{Y^\star}$ such that $x^\star \in \rho D_C M(x, \overline{y})(b^\star)$, that is,

$$(\rho^{-1}x^\star, -b^\star) \in N^C\big(\mathrm{gph}\, M; (x, \overline{y})\big).$$

Let $\|\cdot\|_2$ be the product 2-norm of the norms of X and Y and let $\|\cdot\|_\star$ its associated dual norm. We can then write

$$\begin{aligned}\big\langle x^\star, x' - x\big\rangle = \big\langle (x^\star, -\rho b^\star), (x', \overline{y}) - (x, \overline{y})\big\rangle &\leq \|(x^\star, -\rho b^\star)\|_\star \omega\big(\|(x', \overline{y}) - (x, \overline{y})\|_2\big)\\ &\leq \big(\|x^\star\| + \rho\|b^\star\|\big)\omega\big(\|x' - x\|\big)\\ &\leq (1 + \rho)\omega\big(\|x' - x\|\big)\end{aligned}$$

which translates the desired F-normal $\theta(\cdot)$-regularity of the set S. □

Remark 4.2 If $\rho = +\infty$ in the latter proposition, then we can conclude with similar arguments that there exists a real $\lambda > 0$ such that the set S is F-normally $\theta(\cdot)$-regular with $\theta(\cdot) := \lambda\omega(\cdot)$. ■

Remark 4.3 It is readily seen that we can develop Theorem 4.1 and Proposition 4.1 under the L-normal regularity of $\mathrm{gph}\, M$ and the Asplund property of the Banach spaces X and Y. ■

We now focus on the normal $\omega(\cdot)$-regularity of the solution set of a generalized equation, say $f(x) \in F(x)$ for a single-valued mapping f and a multimapping F. We point out that the normal $\omega(\cdot)$-regularity for a set of the form $\{x \in X : 0 \in F_1(x) + F_2(x)\}$ with F_1, F_2 two multimappings has been established in [2] under an openness condition in the product space $(X \times Y)^2$, namely

$$\beta\mathbb{U}_{(X\times Y)^2} \subset -\{((x, y), (x, y)) : (x, y) \in (\overline{x}, \overline{y}) + \alpha\mathbb{B}_{X\times Y}\} + \mathrm{gph}\, F_1 \times \mathrm{gph}\,(-F_2),$$

for two constants $\alpha, \beta > 0$ satisfying (1.4) for some real $\rho > 0$.

Proposition 4.2 *Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a multimapping with closed graph and let $f : X \to Y$ be a mapping. Assume that $S := \{x \in X : f(x) \in F(x)\} \neq \emptyset$ along with:*

(i) The mapping f is K-Lipschitz continuous on X for some real $K \geq 0$ and differentiable on X with L-Lipschitz continuous derivative Df for some real $L \geq 0$.

(ii) *there exists an upper semicontinuous nondecreasing function* $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\omega(0) = 0$, *two reals* $\alpha, \beta \in]0, +\infty[$ *and an extended real* $\rho \in]0, +\infty]$ *satisfying with* $\kappa := (1 + K^2)^{\frac{1}{2}}$

$$\beta > \frac{\alpha}{\rho} + \Big(1 + \frac{1}{\rho}\Big)\Big(\omega\big(\kappa(\alpha^2 + \beta^2)^{\frac{1}{2}}\big) + \frac{L}{2}(\alpha^2 + \beta^2)\Big)$$

such that the set gph F *is* C*-normally* $\omega(\cdot)$*-regular and*

$$\beta\mathbb{U}_{(X\times Y)} \subset -\mathrm{gph}\, f \cap \big(B[\overline{x}, \alpha] \times Y\big) + \mathrm{gph}\, F \quad \text{for all } \overline{x} \in S. \tag{4.6}$$

Then, there exists a real $\gamma \in [0, \rho[$ *such that for each* $\overline{x} \in S$, *there exists a real* $\delta > 0$ *satisfying*

$$d(x, S) \le \gamma d\big((x, f(x)), \mathrm{gph}\, F\big) \quad \text{for all } x \in B(\overline{x}, \delta). \tag{4.7}$$

Further, the set S *is* F*-normally* $\widehat{\omega}(\cdot)$*-regular with* $\widehat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$\widehat{\omega}(t) := \gamma\Big(\omega\big(\kappa t\big) + \frac{L}{2}t^2\Big) \quad \text{for all } t \in \mathbb{R}_+.$$

Proof First, we define the multimapping $M : X \rightrightarrows X \times Y$ by setting

$$M(x) := -(x, f(x)) + \mathrm{gph}\, F \quad \text{for all } x \in X.$$

It is readily seen that the multimapping M has its graph closed. It is also evident that $M^{-1}(0, 0) = S$ and

$$d\big((0, 0), M(x)\big) = d\big((x, f(x)), \mathrm{gph}\, F\big) \quad \text{for all } x \in X.$$

On the other hand, the mapping $\varphi : X \to X \times Y$ defined by

$$\varphi(x) := (x, f(x)) \quad \text{for all } x \in X$$

is obviously κ-Lipschitz continuous on X endowed with the 2-norm $\|\cdot\|_2$ and differentiable at each point $x \in X$ with

$$D\varphi(x)(h) := \big(h, Df(x)(h)\big) \quad \text{for all } h \in X.$$

Further, we observe that the derivative $D\varphi$ is L-Lipschitz continuous on X since we have for every $x, x' \in X$

$$\begin{aligned}\sup_{h\in\mathbb{B}_X} \|D\varphi(x)(h) - D\varphi(x')(h)\|_2 &= \sup_{h\in\mathbb{B}_X} \big\|\big(0, (Df(x) - Df(x'))(h)\big)\big\|_2 \\ &\le \|Df(x) - Df(x')\| \le L\|x - x'\|.\end{aligned}$$

According to Proposition 2.3, we know that M has a C-normally $\omega_0(\cdot)$ regular graph (with respect to the norm $\|\cdot\|_2$) where $\omega_0(t) := \omega(\kappa t) + \frac{L}{2}t^2$ for all $t > 0$. The inequality (4.7) then follows from Theorem 4.1. It remains to apply Proposition 3.2 to conclude the proof. $\square$

5 Metric Subregularity for Multimappings with Normally ω-regular Values

Theorem 4.1 requires the normal $\omega(\cdot)$-regularity for the graph gph M of the involved multimapping M. Unfortunately, there are numerous and various multimappings which fail to enjoy such a property for a given function $\omega(\cdot)$. This can be easily seen with the subdifferential of a nonsmooth function, for instance

$$\mathrm{gph}\big(\partial|\cdot|\big) =]-\infty, 0[\times\{-1\} \cup \{0\} \times [-1, 1] \cup]0, +\infty[\times\{1\},$$

which is obviously non-prox-regular (even not subsmooth) in $\mathbb{R}^2$.

Our aim in the present section is to provide some metric subregularity properties for multimappings normally $\omega(\cdot)$-regular-valued. Doing so, we need to develop an appropriate version of Proposition 2.4 where the normal cone $N^C\big(\mathrm{gph}\, M; (u, v)\big)$ is replaced by $N\big(M(x); v\big)$. Our arguments follow those in the work [36].

Proposition 5.1 *Let $M : X \rightrightarrows Y$ be a multimapping from a normed space X into a Banach space Y and let $\overline{y} \in Y$. Assume that there exists $x \in X$ with $M(x)$ nonempty and closed and two reals $\gamma, r \in]0, +\infty[$ such that*

$$\gamma d\big(\overline{y}, M(x)\big) < r < d\big(x, M^{-1}(\overline{y})\big). \tag{5.1}$$

Then, for all reals $\eta, \varepsilon \in]0, +\infty[$, there exist $v \in M(x)$ satisfying:

(i) $0 < \|v - \overline{y}\| < \min\{\frac{r}{\gamma}, d(\overline{y}, M(x)) + \varepsilon\}$;

(ii) $\|v - \overline{y}\| \le \|y - \overline{y}\| + \frac{\eta}{\gamma}\|y - v\|$ *for all* $y \in M(x)$;

(iii) $0 \in \partial\|\cdot\|(v - \overline{y}) + \frac{\eta}{\gamma}\mathbb{B}_{Y^\star} + N^C\big(M(x); v\big)$.

If in addition Y is an Asplund space, then (iii) can be replaced by

(iii′) $0 \in \partial\|\cdot\|(v - \overline{y}) + \frac{\eta}{\gamma}\mathbb{B}_{Y^\star} + N^L\big(M(x); v\big)$.

Proof Let $\eta, \varepsilon \in]0, +\infty[$. Choose some real $\gamma' > \gamma$ and some real $r' < r$ such that

$$\gamma' d\big(\overline{y}, M(x)\big) < r' < d\big(x, M^{-1}(\overline{y})\big).$$

According to the first inequality, we can find some $y_0 \in M(x)$ such that $\|y_0 - \overline{y}\| < \frac{r'}{\gamma'}$. Thanks to the second inequality in (5.1), we obviously have

$$B[x, r] \cap M^{-1}(\overline{y}) = \emptyset. \tag{5.2}$$

Let us set $\theta(\cdot) := \|\cdot - \overline{y}\| + \psi_{M(x)}(\cdot)$ which is lower semicontinuous and proper (keeping in mind that $M(x)$ is nonempty and closed). Note that

$$\theta(y_0) = \|y_0 - \overline{y}\| < \frac{r'}{\gamma'} \leq \inf_{y \in Y} \theta(y) + \frac{r'}{\gamma'}.$$

Fix any real $\eta' \in]0, \min\{\gamma, \eta\}[$ with $\frac{2\eta'\|y_0 - \overline{y}\|}{\gamma' - \eta'} < \varepsilon$. Observe that (Y, N) is a Banach space with $N(\cdot) := \eta'\|\cdot\|$. We are in a position to apply Ekeland variational principle. Doing so, we get some $v \in Y$ such that

$$\begin{aligned} N(y_0 - v) &\leq r' < r, \\ \theta(v) \leq \theta(y_0) &= \|y_0 - \overline{y}\| < \frac{r'}{\gamma'} < \frac{r}{\gamma} \end{aligned} \tag{5.3}$$

and

$$\theta(v) \leq \theta(y) + \frac{1}{\gamma'} N(y - v) \quad \text{for all } y \in Y. \tag{5.4}$$

From (5.3), we see that $v \in M(x)$ (so $v \neq \overline{y}$ by (5.2)) and $\|v - \overline{y}\| \leq \|y_0 - \overline{y}\|$. Then, by (5.4) we have

$$\theta(v) = \|v - \overline{y}\| \leq \|y - \overline{y}\| + \frac{\eta'}{\gamma'}\|y - v\| \quad \text{for all } y \in M(x). \tag{5.5}$$

Now, let us define $f : Y \to \mathbb{R}$ by setting

$$f(y) := \|y - \overline{y}\| + \frac{\eta'}{\gamma'}\|y - v\| \quad \text{for all } y \in Y.$$

Through (5.5), we see that v is a global minimizer of $f + \psi_{M(x)}$ and this implies that

$$0 \in \partial_C f(v) \subset \partial\|\cdot\|(v - \overline{y}) + \frac{\eta'}{\gamma'}\mathbb{B}_{Y^\star} + N^C\big(M(x); v\big) \tag{5.6}$$

along with

$$\begin{aligned} f(v) = \|v - \overline{y}\| &\leq \|y - \overline{y}\| + \frac{\eta'}{\gamma'}\|y - v\| \\ &\leq (1 + \frac{\eta'}{\gamma'})\|y - \overline{y}\| + \frac{\eta'}{\gamma'}\|\overline{y} - v\| \end{aligned}$$

for every $y \in M(x)$. Hence, we have

$$\|v - \overline{y}\| \leq \frac{1 + \eta'/\gamma'}{1 - \eta'/\gamma'}\|y - \overline{y}\| = \frac{\gamma' + \eta'}{\gamma' - \eta'}\|y - \overline{y}\| \quad \text{for all } y \in M(x).$$

We deduce from this

$$\|v-\overline{y}\|_Y \le \frac{\gamma'+\eta'}{\gamma'-\eta'} d\big(\overline{y}, M(x)\big) = d\big(\overline{y}, M(x)\big) + \frac{2\eta'}{\gamma'-\eta'} d\big(\overline{y}, M(x)\big). \quad (5.7)$$

Coming back to (5.3) and using the definition of η' yield

$$d\big(\overline{y}, M(x)\big) \le \|v-\overline{y}\| \le \|\overline{y}-y_0\| < \frac{(\gamma'-\eta')\varepsilon}{2\eta'}. \quad (5.8)$$

It remains to put together (5.7) and (5.8) to obtain

$$\|v-\overline{y}\| \le d\big(\overline{y}, M(x)\big) + \varepsilon.$$

The proof of $(i)-(ii)-(iii)$ is complete.

Regarding (iii'), if Y is an Asplund space we can write (see (5.6))

$$0 \in \partial_L f(v) \subset \partial\|\cdot\|(v-\overline{y}) + \frac{\eta'}{\gamma'}\mathbb{B}_{Y^\star} + N^L\big(M(x); v\big).$$

This finishes the proof. □

Remark 5.1 Assume that $M(\cdot) = -f(\cdot) + S$ for some mapping $f : X \to Y$ and some set $S \subset Y$. Given any $v \in Y$, we obviously have $N^C(M(x); v) = N^C(S; v + f(x))$, so (iii) in Proposition 5.1 can be rewritten as

$$0 \in \partial\|\cdot\|(v-\overline{y}) + \frac{\eta}{\gamma}\mathbb{B}_{Y^\star} + N^C(S; v + f(x))$$

without any assumption on the mapping f. ■

We are now in a position to establish the following result which complements Theorem 4.1.

Theorem 5.1 *Let $M : X \rightrightarrows Y$ be a multimapping from a normed space X into a Banach (resp. Asplund) space Y and let $\overline{y} \in Y$. Assume that $M(\cdot)$ is closed and C-normally (resp. L-normally) $\omega(\cdot)$-regular valued for some upper semicontinuous nondecreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$. Assume also that there exist $\alpha, \beta, \kappa \in]0, +\infty[$ with $\beta > \kappa\alpha + \omega(\beta + \kappa\alpha)$ such that:*

(i) *for all $\overline{x} \in M^{-1}(\overline{y})$, there exists a real $\eta > 0$ satisfying*

$$\operatorname{exc}\big(M(x), M(x')\big) \le \kappa\|x'-x\| \quad \text{for all } x \in B[\overline{x}, \alpha], \text{ all } x' \in B[\overline{x}, \eta];$$

(ii) *for all $\overline{x} \in M^{-1}(\overline{y})$, one has $B(\overline{y}, \beta) \subset M(B[\overline{x}, \alpha])$.*

Then, there exists a real $\gamma \ge 0$ such that for every $\overline{x} \in M^{-1}(\overline{y})$, there exists a real $\delta > 0$ satisfying

$$d\big(x, M^{-1}(\overline{y})\big) \le \gamma d\big(\overline{y}, M(x)\big) \quad \text{for all } x \in B(\overline{x}, \delta).$$

Proof We only deal with the C-normal $\omega(\cdot)$-regularity for the values of $M(\cdot)$ in the Banach space Y. We proceed as in the proof of Theorem 4.1. By contradiction, assume that for each $\gamma \geq 0$, there is $\overline{x} \in M^{-1}(\overline{y})$ such that for every real $\delta > 0$, there is $x \in B(\overline{x}, \delta)$ satisfying

$$d\big(x, M^{-1}(\overline{y})\big) > \gamma d\big(\overline{y}, M(x)\big). \tag{5.9}$$

Fix any real $\gamma > 0$ and any real $\kappa' > \kappa$. According to assumption (i), for each $\overline{x} \in M^{-1}(\overline{y})$ we can find a real $\eta_{\overline{x}} > 0$ (see (2.1)) such that

$$M(x) \subset M(x') + \kappa' \|x - x'\| \mathbb{B}_Y \quad \text{for all } x \in B[\overline{x}, \alpha], \text{ all } x' \in B[\overline{x}, \eta_{\overline{x}}]. \tag{5.10}$$

Pick any sequence $(\varepsilon_n)_{n\geq 1}$ in $]0, \gamma[$ with $\varepsilon_n \to 0$. Choose some integer $N \geq 1$ such that $\theta_n := \frac{1}{n(\gamma-\varepsilon_n)} < \frac{\beta}{2}$ for every integer $n \geq N$. Fix for a moment any integer $n \geq N$. Thanks to (5.9), there are $\overline{x}_n \in M^{-1}(\overline{y})$ and $x_n \in X$ with $\|x_n - \overline{x}_n\| < \min\{\frac{1}{n}, \eta_{\overline{x}_n}\}$ such that

$$d\big(x_n, M^{-1}(\overline{y})\big) > (\gamma - \varepsilon_n) d\big(\overline{y}, M(x_n)\big).$$

According to Proposition 5.1, there is $v_n \in M(x_n)$ such that

$$0 < \|v_n - \overline{y}\| < \frac{d\big(x_n, M^{-1}(\overline{y})\big)}{\gamma - \varepsilon_n}$$

along with

$$0 \in \partial \|\cdot\|(v_n - \overline{y}) + \frac{\varepsilon_n}{\gamma - \varepsilon_n} \mathbb{B}_{Y^\star} + N^C\big(M(x_n); v_n\big).$$

Since $v_n - \overline{y} \neq 0$, the latter inclusion gives $y_n^\star \in N^C\big(M(x_n); v_n\big)$, $z_n^\star \in \mathbb{S}_{Y^\star}$ with $\big\langle z_n^\star, v_n - \overline{y}\big\rangle = \|v_n - \overline{y}\|$ and $b_n^\star \in \mathbb{B}_{Y^\star}$ such that

$$y_n^\star = -z_n^\star + \frac{\varepsilon_n}{\gamma - \varepsilon_n} b_n^\star.$$

Setting $\zeta_n := (\beta - \theta_n)\frac{v_n - \overline{y}}{\|v_n - \overline{y}\|}$ and noticing that $v_n - \zeta_n \in B(\overline{y}, \beta)$, we can find some $w_n \in B[\overline{x}_n, \alpha]$ such that $v_n - \zeta_n \in M(w_n)$. Thanks to (5.10), there is $\xi_n \in M(x_n)$ and some $b_n \in \mathbb{B}_Y$ such that

$$v_n - \zeta_n = \xi_n + \kappa' \|x_n - w_n\| b_n.$$

Using the fact that $M(x_n)$ is C-normally $\omega(\cdot)$-regular, we obtain

$$\begin{aligned}\langle y_n^\star, -\zeta_n\rangle &= \langle y_n^\star, \xi_n - v_n + \kappa'\|x_n - w_n\| b_n\rangle \\ &= \langle y_n^\star, \xi_n - v_n\rangle + \kappa'\|x_n - w_n\| \langle y_n^\star, b_n\rangle \\ &\le \|y_n^\star\| \Big(\omega(\|\xi_n - v_n\|) + \kappa'\|x_n - w_n\|\Big) \\ &\le \Big(1 + \frac{\varepsilon_n}{\gamma - \varepsilon_n}\Big)\Big(\omega(\|\xi_n - v_n\|) + \kappa'\|x_n - w_n\|\Big).\end{aligned}$$

On the other hand, we have

$$\|x_n - w_n\| \le \|x_n - \overline{x}_n\| + \|\overline{x}_n - w_n\| < \frac{1}{n} + \alpha$$

and this entails

$$\|\xi_n - v_n\| = \| - \zeta_n - \kappa'\|x_n - w_n\| b_n\| \le \ \beta - \theta_n + \kappa'(\frac{1}{n} + \alpha).$$

Combining the definition of ζ_n with the equality $\langle z_n^\star, v_n - \overline{y}\rangle = \|v_n - \overline{y}\|$, we obtain

$$\langle y_n^\star, \zeta_n\rangle = \Big\langle -z_n^\star + \frac{\varepsilon_n}{\gamma - \varepsilon_n} b_n^\star, \zeta_n\Big\rangle \le \theta_n - \beta + \frac{\varepsilon_n}{\gamma - \varepsilon_n}(\beta - \theta_n).$$

Putting what precedes together, we arrive to

$$\beta - \theta_n - \frac{\varepsilon_n}{\gamma - \varepsilon_n}(\beta - \theta_n) \le \big(1 + \frac{\varepsilon_n}{\gamma - \varepsilon_n}\big)\Big(\omega\big(\beta - \theta_n + \kappa'(\frac{1}{n} + \alpha)\big) + \kappa'(\frac{1}{n} + \alpha)\Big).$$

Keeping in mind that the function $\omega(\cdot)$ is upper semicontinuous and letting $n \to \infty$ and $\kappa' \downarrow \kappa$ give

$$\beta \le \kappa\alpha + \omega(\beta + \kappa\alpha)$$

which is the desired contradiction. The proof is then complete. □

The Lipschitz behavior with respect to the Hausdorff-Pompeiu excess in (ii) of Theorem 5.1 obviously holds for the Lipschitz translation of a fixed set, say $M(x) = -f(x) + B$ for some Lipschitz mapping f.

Corollary 5.1 *Let $f : X \to Y$ be a κ-Lipschitz continuous mapping between a normed space X and a Banach (resp. Asplund) space Y with $\kappa \ge 0$. Let also B be a closed C-normally (resp. L-normally) $\omega(\cdot)$-regular set for some upper semicontinuous non-decreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$. Assume that there exist two reals $\alpha, \beta > 0$ such that*

$$\beta > \kappa\alpha + \omega(\beta + \kappa\alpha) \quad \textit{and} \quad \beta\mathbb{U}_Y \subset -f(B[\overline{x}, \alpha]) + B \quad \textit{for all}\ \overline{x} \in f^{-1}(B).$$

Then, there exists a real $\gamma \geq 0$ such that for every $\overline{x} \in f^{-1}(B)$, there exists a real $\delta > 0$ satisfying

$$d\big(x, f^{-1}(B)\big) \leq \gamma d\big(f(x), B\big) \quad \textit{for all } x \in B(\overline{x}, \delta).$$

Proof It suffices to apply Theorem 5.1 with the multimapping $M(\cdot) := -f(\cdot) + B$ and the point $\overline{y} := 0$. □

Remark 5.2 We keep notation and assumptions of the latter corollary. If in addition f is strictly Hadamard differentiable, then we can combine Proposition 2.5 and Proposition 2.2 to obtain

$$\gamma \leq \inf_{\varepsilon > 0} \sup_{y^\star \in \Omega_\varepsilon} \|y^\star\|_{Y^\star},$$

where for each real $\varepsilon > 0$, we denote Ω_ε the set of $y^\star \in Y^\star$ for which there are $x \in B(\overline{x}, \varepsilon)$ with $\overline{y} + f(x) \notin S$ and $y \in B(\overline{y}, \varepsilon)$ with $y + f(x) \in S$ such that $y^\star \in N(S; f(x) + y)$ and $\|Df(x)^\star(y^\star)\| \leq 1$. ■

Coming back to Proposition 4.2 with $F \equiv B$, we see that the openness condition (4.6) can be written as

$$\beta \mathbb{U}_{(X \times Y)} \subset -\{(x, f(x)) : x \in B[\overline{x}, \alpha]\} + X \times B.$$

Using Corollary 5.1 allows us to drop the whole space X in the latter formula. More precisely:

Proposition 5.2 *Let X be a normed space, Y be a Banach (resp. an Asplund) space, $B \subset Y$ and let $f : X \to Y$ be a mapping. Assume that $S := f^{-1}(B) \neq \emptyset$ along with:*

(i) There exists a nondecreasing upper semicontinuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that the set B is C-normally (resp. L-normally) $\omega(\cdot)$-regular.
(ii) The mapping f is K-Lipschitz continuous on X for some real $K \geq 0$ and differentiable on X with L-Lipschitz continuous derivative Df for some real $L \geq 0$.
(iii) There exist two reals $\alpha, \beta > 0$ such that

$$\beta > K\alpha + \omega(\beta + K\alpha) \quad \textit{and} \quad \beta \mathbb{U}_Y \subset -f(B[\overline{x}, \alpha]) + B \quad \textit{for all } \overline{x} \in f^{-1}(B).$$

Then, there exists a real $\gamma \geq 0$ such that the set S is F-normally $\widehat{\omega}(\cdot)$-regular with $\widehat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\widehat{\omega}(t) := \gamma\Big(\omega\big(Kt\big) + \frac{L}{2}t^2\Big) \quad \textit{for all } t \in \mathbb{R}_+.$$

Proof It directly follows from Proposition 3.3 and Corollary 5.1. □

Proposition 5.2 allows to get sufficient conditions ensuring the normal regularity of an intersection set, say $S_1 \cap S_2$ with $S_1 \times S_2$ normally $\omega(\cdot)$-regular. Indeed, with

$f(x) := (x, x)$ (which is obviously Lipschitz continuous with Lipschitz derivative) we see that

$$f^{-1}(S_1 \times S_2) = S_1 \cap S_2.$$

Hence, we see through Proposition 5.2 that the latter set $S_1 \cap S_2$ is normally regular under an openness condition of the form

$$\beta \mathbb{U}_{X^2} \subset -\{(x, x) : x \in B[\overline{x}, \alpha]\} + S_1 \times S_2 \quad \text{for all} \in \overline{x} \in S_1 \cap S_2.$$

This complements (3.6) which comes from [2, Proposition 7]. A similar remark holds for the normal regularity of the constraint set (3.2) denoted S. In such a case, a suitable openness condition is given by

$$\beta \mathbb{U}_{\mathbb{R}^{m+n}} \subset -F(B[\overline{x}, \alpha]) +]-\infty, 0]^m \times \{0_{\mathbb{R}^n}\} \quad \text{for all } \overline{x} \in S,$$

where

$$F(x) := \big(f_1(x), \ldots, f_m(x), f_{m+1}(x), \ldots, f_{m+n}(x)\big).$$

Acknowledgements The authors are indebted to Professor Lionel Thibault and an anonymous referee for many valuable comments on the paper.

Data Availibility Not applicable.

References

1. Adly, S., Nacry, F., Thibault, L.: Preservation of prox-regularity of sets with applications to constrained optimization. SIAM J. Optim. **26**, 448–473 (2016)
2. Adly, S., Nacry, F., Thibault, L.: Prox-regularity approach to generalized equations and image projection. ESAIM Control Optim. Calc. Var. **24**, 677–708 (2018)
3. Aussel, D., Daniilidis, A., Thibault, L.: Subsmooth sets: functional characterizations and related concepts. Trans. Amer. Math. Soc. **357**, 1275–1301 (2005)
4. Bounkhel, M.: Regularity Concepts in Nonsmooth Analysis. Theory and Applications., Springer Optim. Appl., 59 Springer, New York, (2012)
5. Clarke, F.H.: Optimization and Nonsmooth Analysis, Classics Appl. Math., 5 Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1990)
6. Colombo, G., Thibault, L.: Prox-regular sets and applications, Handbook of nonconvex analysis and applications, Int. Press, Somerville, MA (2010) 99–182
7. Daniilidis, A., Luke, D.R., Tam, M.: Characterizations of super-regularity and its variants, Splitting algorithms, modern operator theory, and applications, 137–152. Springer, Cham (2019)
8. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings. Springer Ser. Oper. Res. Financ. Eng. Springer, New York (2014)
9. Goncharov, V.V., Ivanov, G.E.: Strong and weak convexity of closed sets in a Hilbert space, Operations research, engineering, and cyber security, 259-297, Springer Optim. Appl., 113, Springer, Cham, (2017)
10. Graves, L.M.: Some mapping theorems. Duke Math. J. **17**, 111–114 (1950)
11. Ivanov, G.E.: Nonlinear images of sets. I: Strong and weak convexity, J. Convex Anal. 27 (2020), 363-382
12. Ioffe, A.D.: Metric regularity - a survey Part 1. Theory, J. Aust. Math. Soc. 101 (2016), 188-243
13. Ioffe, A.D.: Metric regularity - a survey Part 2. Applications, J. Aust. Math. Soc. 101 (2016), 376-414

14. Ioffe, A.D.: Variational Analysis of Regular Mappings. Theory and Applications. Springer Monographs in Mathematics. Springer, Cham (2017)
15. Jourani, A.: Open mapping theorem and inversion theorem for γ-paraconvex multivalued mappings and applications. Studia Mathematica **117**, 123–136 (1996)
16. Jourani, A., Thibault, L., Zagrodny, D.: $C^{1,\omega(\cdot)}$-regularity and Lipschitz-like properties of subdifferential, Proc. Lond. Math. Soc. (3) 105(2012), 189-223
17. Jourani, A., Vilches, E.: Positively α-far sets and existence results for generalized perturbed sweeping processes. J. Convex Anal. **23**, 775–821 (2016)
18. Huang, H., Li, R.X.: Global error bounds for γ-paraconvex multifunctions. Set-Valued Var. Anal. **19**, 487–504 (2011)
19. Lewis, A.S., Luke, D.R., Malick, J.: Local linear convergence of alternating and averaged projections. Found. Comput. Math. **9**, 485–513 (2009)
20. Lyusternik, L.A.: On conditional extrema of functionals. Mat. Sb. **41**, 390–401 (1934)
21. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I: basic theory, Grundlehren der Mathematischen Wissenschaften, vol. 330. Springer-Verlag, Berlin (2006)
22. Mordukhovich, B.S., Nam, N.M.: Subgradient of distance functions with applications to Lipschitzian stability. Math. Program. **104**, 635–668 (2005)
23. Nam, N.M., Mordukhovich, B.S.: Subgradients of distance functions at out-of-set points. Taiwanese J. Math. **10**, 299–326 (2006)
24. Penot, J.-P.: Calculus without Derivatives, Graduate Texts in Mathematics, 266. Springer, New York (2013)
25. Poliquin, R.A., Rockafellar, R.T., Thibault, L.: Local differentiability of distance functions. Trans. Amer. Math. Soc. **352**, 5231–5249 (2000)
26. Rolewicz, S.: On paraconvex multifunctions. Operations Res. Verfahren **31**, 539–546 (1979)
27. Robinson, S.M.: Regularity and stability for convex multivalued functions. Math. Oper. Res. **1**, 130–143 (1976)
28. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis, Grundlehren der Mathematischen Wissenschaften, vol. 317. Springer, New York (1998)
29. Thibault, L.: Sweeping process with regular and nonregular sets. J. Diff. Eq. **193**, 1–26 (2003)
30. Thibault, L.: Unilateral Variational Analysis in Banach Spaces. Part I, General Theory, World Scientific (2023)
31. Thibault, L.: Unilateral Variational Analysis in Banach Spaces. Part II, Special Classes of Functions and Sets, World Scientific (2023)
32. Ursescu, C.: Multifunctions with closed convex graphs. Czechoslovak Math. J. **25**, 438–411 (1975)
33. Venel, J.: A numerical scheme for a class of sweeping processes. Numer. Math. **118**, 367–400 (2011)
34. Vial, J.-P.: Strong and weak convexity of sets and functions. Math. Oper. Res. **8**, 231–259 (1983)
35. Zheng, X.Y., Ng, K.F.: Metric subregularity and calmness for nonconvex generalized equations in Banach spaces. SIAM J. Optim. **20**, 2119–2136 (2010)
36. Zheng, X.Y., Ng, K.F.: Metric subregularity for proximal generalized equations in Hilbert spaces. Nonlinear Anal. **75**, 1686–1699 (2012)
37. Zheng, X.Y., He, Q.H.: Characterization for metric regularity for σ-subsmooth multifunctions. Nonlinear Anal. **100**, 111–121 (2014)