Exact SDP Reformulations for Adjustable Robust Quadratic Optimization with Affine Decision Rules

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Abstract

In this paper, we deal with exact semidefinite programming (SDP) reformulations for a class of adjustable robust quadratic optimization problems with affine decision rules. By virtue of a special semidefinite representation of the non-negativity of separable non-convex quadratic functions on box uncertain sets, we establish an exact SDP reformulation for this adjustable robust quadratic optimization problem on spectrahedral uncertain sets. Note that the spectrahedral uncertain set contains commonly used uncertain sets, such as ellipsoids, polytopes, and boxes. As special cases, we also establish exact SDP reformulations for this adjustable robust quadratic optimization problems when the uncertain sets are ellipsoids, polytopes, and boxes, respectively. As applications, we establish the corresponding results for fractionally adjustable robust quadratic optimization problems.

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1 Introduction

As a special class of nonlinear optimization problems, quadratic optimization has received extensive attention by many researchers and used in a wide range of fields, such as game theory, signal processing, and portfolio optimization. In recent years, there have been various excellent works on the investigation of quadratic optimization problems from different perspectives, see [1, 2, 15, 16, 23, 33, 34, 36] and the references therein.

Recently, quadratic optimization problems with uncertain data have attracted extensive interest of many researchers due to their applications in different fields of mathematics, engineering, and economics. Robust optimization approach [4, 6, 17] is a powerful methodology for dealing with quadratic optimization problems with uncertain data. For example, by using the robust optimization methodology, second-order cone programming reformulation problems for convex quadratic optimization problems on different kinds of uncertain sets are obtained in [26]. Exact second-order cone programming relaxations are established in [19] for non-convex minimax separable quadratic optimization problems with multiple separable quadratic constraints. In [24], exact copositive optimization reformulations are obtained for robust quadratic optimization problems with uncertain parameters containing both continuous and integer components. In [13], a deterministic approach is given to examine robust optimality conditions and find robust efficient solutions of convex quadratic multiobjective optimization problems with uncertain data. By virtue of a new robust type characteristic cone constraint qualification, second-order conic programming dual of robust convex quadratic optimization problems on polytopic and norm uncertain sets is considered in [37].

Note that all of the above papers are concentrated on the investigation of static (single-stage) robust optimization models, containing only "here and now" decisions variables. This means that before obtaining complete information about uncertain parameters, we must now determine their values [9, 10, 21, 27–29, 31]. However, many dynamic decision models contain not only "here-and-now" (first-stage) decision variables, but also "wait-and-see" (second-stage) decision variables which are assigned numerical values after some of the uncertain parameters are known. Adjustable robust optimization, introduced in [3], is an important deterministic methodology to deal with optimization problems involving both "here-and-now" and "wait-and-see" decision variables. Although there are few papers in the literature devoted to adjustable robust quadratic optimization problems, see, for example, [7, 8, 11, 12, 32, 35], adjustable robust quadratic optimization problems have received far less attention than others. This is a main motivation for the investigation of adjustable robust *quadratic* optimization problems in this paper.

Motivated by the works reported in [7, 12], this paper will establish exact SDP reformulations for a class of adjustable robust quadratic optimization problems with affine decision rules on spectrahedral uncertain sets [25, 30]. More precisely, by using a special semidefinite representation of the non-negativity of separable non-convex quadratic functions on box uncertain sets, we show that this adjustable robust quadratic

optimization problem admits an exact SDP reformulation problem in the sense that they share the same optimal values and their optimal solutions are one-to-one correspondence. The result obtained provides us with a way to investigate adjustable robust quadratic optimization problems by considering the corresponding SDP reformulation problems. For ellipsoidal, polytopic, and box uncertain sets, we also establish exact SDP reformulations for adjustable robust quadratic optimization problems. We show that our results cover as special cases of some optimization problems considered in the recent literature [12, 32]. Furthermore, as an application, the proposed approach is applied to investigate exact SDP reformulations for fractionally adjustable robust quadratic optimization problems on spectrahedral uncertain sets.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and introduce an adjustable robust quadratic optimization problem. In Section 3, we establish exact SDP reformulations for adjustable robust quadratic optimization problems on spectrahedral uncertain sets. In Section 4, we consider exact SDP reformulations for fractionally adjustable robust quadratic optimization problems.

2 Preliminaries and Auxiliary Results

Unless otherwise specified, $\mathbb{R}^n$ signifies the n-dimensional Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y \rangle := x^\top y$, for all $x, y \in \mathbb{R}^n$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$. The space of all symmetric $n \times n$ matrices is denoted by $\mathbb{S}^n$. $M \in \mathbb{S}^n$ is said to be a positive semidefinite matrix, denoted by $M \succeq 0$, iff $x^\top M x \geq 0, \forall x \in \mathbb{R}^n$. Moreover, $M \in \mathbb{S}^n$ is said to be a positive definite matrix, denoted by $M \succ 0$, iff $x^\top M x > 0, \forall x \in \mathbb{R}^n \setminus \{0_n\}$. The symbol I_n stands for the $n \times n$ identity matrix, and the symbol $O_{n \times n} \in \mathbb{R}^{n \times n}$ stands for the $n \times n$ matrix of all zeros. The matrix $D := \mathrm{diag}(r_1, \ldots, r_n) \in \mathbb{R}^{n \times n}$ stands for the diagonal matrix with $r_i \in \mathbb{R}, i = 1, \ldots, n$.

In what follows, let $Q_0 \in \mathbb{S}^n$, $q_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}$, $Q_i := \mathrm{diag}(a_1^i, \ldots, a_n^i) \in \mathbb{R}^{n \times n}$, $q_i := (q_i^1, \ldots, q_i^n) \in \mathbb{R}^n$, $q_k^i := (q_k^{i1}, \ldots, q_k^{in}) \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$ and $\xi_k^i \in \mathbb{R}$, $k = 1, \ldots, p$, $i = 1, \ldots, l$. In this paper, we consider the following uncertain quadratic optimization problem

$$(\mathrm{UP}) \qquad \begin{cases} \inf\limits_{x \in \mathbb{R}^n} & f_0(x) \\ s.t. & f_i(x, u^i) \leq 0, \ i = 1, \ldots, l, \end{cases}$$

where $f_0(x) := x^\top Q_0 x + q_0^\top x + \xi_0$ and $f_i(x, u^i) := x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^p u_k^i \left((q_k^i)^\top x + \xi_k^i \right)$, $i = 1, \ldots, l$. $u^i := (u_1^i, \ldots, u_p^i) \in \mathbb{R}^p$, $i = 1, \ldots, l$, are uncertain parameters, belonging to the following spectrahedral uncertain sets [25, 30]

$$\mathcal{U}^i := \left\{ u^i := (u^i_1, \ldots, u^i_p) \in \mathbb{R}^p \;\middle|\; A^i + \sum_{k=1}^{p} u^i_k A^i_k \succeq 0 \right\}, \; i = 1, \ldots, l, \quad (1)$$

where A^i and A^i_k, $i = 1, \ldots, l$, are symmetric matrices. Note that the spectrahedral uncertain sets in (1) are closed convex sets covering the most commonly used uncertain sets often encountered in robust optimization problems, such as ellipsoids, polytopes, and boxes [14, 29].

The problem (UP) with adjustable decision variables can be captured by

$$\text{(UTP)} \qquad \begin{cases} \inf\limits_{x \in \mathbb{R}^n, y(\cdot) \in \mathbb{R}^m} & f_0(x) + g_0(y(z)) \\ \text{s.t.} & f_i(x, u^i) + g_i(y(z)) \leq 0, \; i = 1, \ldots, l, \end{cases}$$

where $g_i(y(z)) := (y(z))^\top B_i y(z) + b_i^\top y(z) + t_i$, $i = 0, 1, \ldots, l$, with $B_i := \mathrm{diag}(\theta^i_1, \ldots, \theta^i_m) \in \mathbb{R}^{m \times m}$, $b_i := (b^1_i, \ldots, b^m_i) \in \mathbb{R}^m$ and $t_i \in \mathbb{R}$. Here, $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the first-stage (here-and-now) decision variable and $y(\cdot) \in \mathbb{R}^m$ is the second-stage (wait-and-see) decision variable, which is an adjustable decision variable that depends on the uncertain parameter $z := (z_1, \ldots, z_m) \in \mathbb{R}^m$ in a box uncertain set

$$\mathcal{U}_{box} := \prod_{j=1}^{m} [\beta_j, \gamma_j], \; \text{with } \beta_j, \gamma_j \in \mathbb{R} \text{ and } \beta_j \leq \gamma_j, \; j = 1, \ldots, m.$$

In what follows, we assume that $y(\cdot)$ is an affinely adjustable variable in the sense that it satisfies the affine decision rule [5] given by

$$y(z) := \rho + Wz, \qquad (2)$$

where $\rho := (\rho_1, \ldots, \rho_m) \in \mathbb{R}^m$ and $W := \mathrm{diag}(\omega_1, \ldots, \omega_m) \in \mathbb{R}^{m \times m}$ are non-adjustable variables.

For (UTP) with the affine decision rule (2), it is usually associated with the robust (worst-case) counterpart given below:

$$\text{(RTP)} \qquad \begin{cases} \inf\limits_{\substack{x \in \mathbb{R}^n, \rho \in \mathbb{R}^m, \\ W \in \mathbb{R}^{m \times m}}} & f_0(x) + \max\limits_{z \in \mathcal{U}_{box}} g_0(y(z)) \\ \text{s.t.} & f_i(x, u^i) + g_i(y(z)) \leq 0, \; \forall u^i \in \mathcal{U}^i, \; i = 1, \ldots, l, \\ & y(z) = \rho + Wz, \; \forall z \in \mathcal{U}_{box}. \end{cases}$$

To present the exact SDP reformulation of (RTP), we recall the following results, which play a key role in the sequel.

Lemma 2.1 [12] *Let* $\alpha \in \mathbb{R}$, $v := (v_1, \ldots, v_m) \in \mathbb{R}^m$ *and* $W := \mathrm{diag}(\omega_1, \ldots, \omega_m)$, *where* $\omega_j \in \mathbb{R}$, $j = 1, \ldots, m$. *Then, the following statements are equivalent:*

(i) *The following implication holds*

$$z := (z_1, \ldots, z_m) \in \prod_{j=1}^{m}[\beta_j, \gamma_j] \Rightarrow \alpha + \sum_{j=1}^{m} v_j z_j + \sum_{j=1}^{m} \omega_j z_j^2 \geq 0,$$

where $\beta_j, \gamma_j \in \mathbb{R}$ *with* $\beta_j \leq \gamma_j$, $j = 1, \ldots, m$.

(ii) *There exist* $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, m$, *such that* $\sum_{j=1}^{m} \alpha_j \leq \alpha$ *and*

$$\alpha_j h_j^1 + v_j h_j^2 + \omega_j h_j^3 \in \Sigma_4^2[z_j], \ j = 1, \ldots, m,$$

where $\Sigma_4^2[z_j]$ *denotes the set consisting of all the sum of squares polynomials with variable* z_j *and degree at most 4,* $h_j^1(z_j) := (1 + z_j^2)^2$, $h_j^2(z_j) := (\beta_j + \gamma_j z_j^2)(1 + z_j^2)$ *and* $h_j^3(z_j) := (\beta_j + \gamma_j z_j^2)^2$ *for* $z_j \in \mathbb{R}$.

(iii) *There exist* $\alpha_j \in \mathbb{R}$ *and* $X^j := \begin{pmatrix} X_{11}^j & X_{12}^j & X_{13}^j \\ X_{12}^j & X_{22}^j & X_{23}^j \\ X_{13}^j & X_{23}^j & X_{33}^j \end{pmatrix} \succeq 0$, $j = 1, \ldots, m$, *such that* $\sum_{j=1}^{m} \alpha_j \leq \alpha$ *and*

$$\begin{cases} X_{11}^j = \alpha_j + v_j \beta_j + \omega_j \beta_j^2, \\ X_{12}^j = X_{23}^j = 0, \\ 2X_{13}^j + X_{22}^j = 2\alpha_j + v_j(\beta_j + \gamma_j) + 2\omega_j \beta_j \gamma_j, \\ X_{33}^j = \alpha_j + v_j \gamma_j + \omega_j \gamma_j^2. \end{cases}$$

3 Exact SDP Reformulations for (RTP)

In this section, we establish an exact SDP reformulation for (RTP) in terms of a special semidefinite representation of the non-negativity of separable non-convex quadratic functions on box uncertain sets. For convenience, let $u := (u^1, \ldots, u^l) \in \mathbb{R}^{pl}$ with $u^i := (u_1^i, \ldots, u_p^i) \in \mathbb{R}^p$, $\delta := (\delta_1, \ldots, \delta_l) \in \mathbb{R}^{lm}$ with $\delta_i := (\delta_i^1, \ldots, \delta_i^m) \in \mathbb{R}^m$, $\sigma := (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m$, $X := (X^{11}, \ldots, X^{1m}, \ldots, X^{l1}, \ldots, X^{lm}) \in \mathbb{R}^{3 \times 3lm}$ with $X^{ij} \in \mathbb{S}^3$, $Y := (Y^1, \ldots, Y^m) \in \mathbb{R}^{3 \times 3m}$ with $Y^j \in \mathbb{S}^3$, $i = 1, \ldots, l$, $j = 1, \ldots, m$.

Now, we propose a SDP reformulation for the problem (RTP) as follows:

(SDP)
$$
\begin{cases}
\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\delta\in\mathbb{R}^{lm},u\in\mathbb{R}^{pl},\rho\in\mathbb{R}^m,\\ X\in\mathbb{R}^{3\times3lm},W\in\mathbb{R}^{m\times m},\sigma\in\mathbb{R}^m,Y\in\mathbb{R}^{3\times3m}}} \quad x^\top Q_0 x + q_0^\top x + \xi_0 + \tau \\[2ex]
s.t. \quad \sum_{j=1}^m \delta_i^j \le -\left(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \right. \\[2ex]
\qquad \left. + \sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i \right), \ i = 1, \ldots, l, \\[2ex]
A^i + \sum_{k=1}^p u_k^i A_k^i \succeq 0, \ i = 1, \ldots, l, \\[2ex]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \ j = 1, \ldots, m, \\[2ex]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[1ex]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[1ex]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right)(\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[1ex]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\[2ex]
\sum_{j=1}^m \sigma_j \le \tau - \sum_{j=1}^m \theta_j^0 \rho_j^2 - \sum_{j=1}^m b_0^j \rho_j - t_0, \\[2ex]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \ j = 1, \ldots, m, \\[2ex]
Y_{11}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \beta_j - \theta_j^0 \omega_j^2 \beta_j^2, \\[1ex]
Y_{12}^j = Y_{23}^j = 0, \\[1ex]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right)(\beta_j + \gamma_j) - 2\theta_j^0 \omega_j^2 \beta_j \gamma_j, \\[1ex]
Y_{33}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \gamma_j - \theta_j^0 \omega_j^2 \gamma_j^2.
\end{cases}
$$

The following theorem describes an exact SDP reformulation for (RTP) in the sense that (RTP) and (SDP) share the same optimal values and their optimal solutions are one-to-one correspondence.

Theorem 3.1 *Consider the problem* (RTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (1) *and its reformulation problem* (SDP). *Then,*

$$\inf(\text{RTP}) = \inf(\text{SDP}).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (RTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3\times3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3\times3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (SDP).

Proof Obviously, (RTP) is equivalent to

$$
\text{(RTP}_0) \quad
\begin{cases}
\displaystyle \inf_{\substack{x \in \mathbb{R}^n, \tau \in \mathbb{R}, u \in \mathbb{R}^{pl}, \\ \rho \in \mathbb{R}^m, W \in \mathbb{R}^{m \times m}}} \quad x^\top Q_0 x + q_0^\top x + \xi_0 + \tau \\[1ex]
\quad s.t. \quad x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^{p} u_k^i \left((q_k^i)^\top x + \xi_k^i \right) + (y(z))^\top B_i y(z) \\
\qquad\qquad + b_i^\top y(z) + t_i \le 0, \ i = 1, \dots, l, \\
\quad A^i + \sum_{k=1}^{p} u_k^i A_k^i \succeq 0, \ i = 1, \dots, l, \\
\quad (y(z))^\top B_0 y(z) + b_0^\top y(z) + t_0 \le \tau, \\
\quad y(z) = \rho + W z, \ \forall z \in \mathcal{U}_{box}.
\end{cases}
$$

Note that τ is an auxiliary variable and adding it does not cause the optimal value to change. Therefore, it is sufficient to show that the feasible sets between (RTP$_0$) and (SDP) are equivalent.

Now, let (x, ρ, W, τ, u) be a feasible solution of (RTP$_0$). This means that

$$
x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^{p} u_k^i \left((q_k^i)^\top x + \xi_k^i \right)
$$
$$
+ (y(z))^\top B_i y(z) + b_i^\top y(z) + t_i \le 0, \ i = 1, \dots, l, \tag{3}
$$
$$
A^i + \sum_{k=1}^{p} u_k^i A_k^i \succeq 0, \ i = 1, \dots, l,
$$
$$
(y(z))^\top B_0 y(z) + b_0^\top y(z) + t_0 \le \tau, \tag{4}
$$

and

$$
y(z) = \rho + W z, \ \forall z \in \mathcal{U}_{box}. \tag{5}
$$

By (3) and (5), we obtain

$$
- \left(\sum_{s=1}^{n} a_s^i (x_s)^2 + \sum_{s=1}^{n} \left(q_i^s + \sum_{k=1}^{p} u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^{p} u_k^i \xi_k^i + \sum_{j=1}^{m} \theta_j^i (\rho_j)^2 + \sum_{j=1}^{m} b_i^j \rho_j + t_i \right)
$$
$$
- \left(\sum_{j=1}^{m} 2 \rho_j \theta_j^i \omega_j + \sum_{j=1}^{m} b_i^j \omega_j \right) z_j - \sum_{j=1}^{m} \theta_j^i (\omega_j)^2 z_j^2 \ge 0, \ \forall z \in \mathcal{U}_{box}, i = 1, \dots, l. \tag{6}
$$

From Lemma 2.1 and (6), there exist $\delta_i^j \in \mathbb{R}$ and $X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i = 1, \dots, l, \ j = 1, \dots, m,$ such that

$$\sum_{j=1}^{m} \delta_i^j \leq -\left(\sum_{s=1}^{n} a_s^i (x_s)^2 + \sum_{s=1}^{n} \left(q_i^s + \sum_{k=1}^{p} u_k^i q_k^{is} \right) x_s + \xi_i \right.$$
$$\left. + \sum_{k=1}^{p} u_k^i \xi_k^i + \sum_{j=1}^{m} \theta_j^i (\rho_j)^2 + \sum_{j=1}^{m} b_i^j \rho_j + t_i \right), \qquad (7)$$

and

$$\begin{cases} X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i (\omega_j)^2 \beta_j^2, \\ X_{12}^{ij} = X_{23}^{ij} = 0, \\ 2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^i (\omega_j)^2 \beta_j \gamma_j, \\ X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i (\omega_j)^2 \gamma_j^2. \end{cases} \qquad (8)$$

Moreover, (4) and (5) amount to

$$\tau - \sum_{j=1}^{m} \theta_j^0 (\rho_j)^2 - \sum_{j=1}^{m} b_0^j \rho_j - t_0 - \left(\sum_{j=1}^{m} 2\rho_j \theta_j^0 \omega_j + \sum_{j=1}^{m} b_0^j \omega_j \right) z_j$$
$$- \sum_{j=1}^{m} \theta_j^0 (\omega_j)^2 z_j^2 \geq 0, \ \forall z \in \mathcal{U}_{box}. \qquad (9)$$

Similarly, from Lemma 2.1 and (9), there exist $\sigma_j \in \mathbb{R}$ and $Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq$

$0, \ j = 1, \ldots, m$, such that

$$\sum_{j=1}^{m} \sigma_j \leq \tau - \sum_{j=1}^{m} \theta_j^0 (\rho_j)^2 - \sum_{j=1}^{m} b_0^j \rho_j - t_0, \qquad (10)$$

and

$$\begin{cases} Y_{11}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \beta_j - \theta_j^0 (\omega_j)^2 \beta_j^2, \\ Y_{12}^j = Y_{23}^j = 0, \\ 2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^0 (\omega_j)^2 \beta_j \gamma_j, \\ Y_{33}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \gamma_j - \theta_j^0 (\omega_j)^2 \gamma_j^2. \end{cases} \qquad (11)$$

Together with (7), (8), (10) and (11), it follows that the problems (RTP) and (SDP) are equivalent in the sense that

$$\inf (\text{RTP}) = \inf (\text{SDP}).$$

Moreover, (x, ρ, W) is a feasible solution of the problem (RTP) if and only if there exist $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ and $Y \in \mathbb{R}^{3 \times 3m}$ such that $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ is an optimal solution of the problem (SDP). The proof is complete.

Remark 3.1 Note that similar results for exact SDP reformulations of adjustable robust linear optimization problems have been investigated in [12, Theorem 3.1]. However, Theorem 3.1 extends these results from linear optimization models to quadratic optimization models.

The following example illustrates how to obtain robust optimal solutions and the corresponding optimal value of (RTP) on spectrahedral uncertain sets by Theorem 3.1.

Example 3.1 For problem (UTP). Let $n = p = m := 2$ and $l := 4$. The uncertain sets $\mathcal{U}^i$, $i = 1, \ldots, 4$, are defined by

$$\mathcal{U}^i := \left\{ u^i := (u_1^i, u_2^i) \in \mathbb{R}^2 \mid \frac{(u_1^i)^2}{2} + \frac{(u_2^i)^2}{3} \leq 1 \right\}, i = 1, \ldots, 4.$$

Obviously, by (1), we have

$$A^i = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1^i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } A_2^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, i = 1, \ldots, 4.$$

Let $f_0(x) := x_1 + x_2$, $g_0(y(z)) := 0$, $f_1(x, u^1) := x_1^2 - 3x_1 - x_2 + u_1^1$, $g_1(y(z)) := (0, -1)^\top y(z) - 3$, $f_2(x, u^2) := -2x_2$, $g_2(y(z)) := 2$, $f_3(x, u^3) := -3x_1$, $g_3(y(z)) := 3$, $f_4(x, u^4) := x_2^2 - x_1 - u_2^4$, and $g_4(y(z)) := (1, 0)^\top y(z)$. Let the affine decision rule $y(\cdot)$ be given by $y(z) := \rho + Wz$, where $\rho := (\rho_1, \rho_2) \in \mathbb{R}^2$, $W := \text{diag}(\omega_1, \omega_2) \in \mathbb{R}^{2 \times 2}$ and $z := (z_1, z_2) \in \mathcal{U}_{box} := [-1, 1] \times [0, 2]$. In this setting, (RTP) becomes

$$
\begin{cases}
\inf_{\substack{x \in \mathbb{R}^2, \rho \in \mathbb{R}^2, \\ W \in \mathbb{R}^{2 \times 2}}} x_1 + x_2 \\
s.t. \quad x_1^2 - 3x_1 - x_2 + u_1^1 - \rho_2 - \omega_2 z_2 - 3 \leq 0, \\
\quad\quad -2x_2 + 2 \leq 0, \\
\quad\quad -3x_1 + 3 \leq 0, \\
\quad\quad x_2^2 - x_1 - u_2^4 + \rho_1 + \omega_1 z_1 \leq 0, \\
\quad\quad \forall (z_1, z_2) \in \mathcal{U}_{box}, \forall (u_1^i, u_2^i) \in \mathcal{U}^i, i = 1, \ldots, 4.
\end{cases}
$$

It is easy to show that the problem (RTP) admits an optimal solution (x, ρ, W) with $x = (1, 1)$, $\rho = (-151, 151)$ and $W = O_{2 \times 2}$. Moreover, $\min (\text{RTP}) = 2$.

Now, we apply Theorem 3.1 to show that (x, ρ, W) is an optimal solution of (RTP). Clearly, the SDP reformulation of (RTP) becomes

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^2,\delta\in\mathbb{R}^8,u\in\mathbb{R}^8,\\ \rho\in\mathbb{R}^2,X\in\mathbb{R}^{3\times24},W\in\mathbb{R}^{2\times2}}} \quad x_1+x_2 \\[2em]
s.t. \quad \displaystyle\sum_{j=1}^{2}\delta_i^j \leq -\left(\sum_{s=1}^{2}a_s^i x_s^2 + \sum_{s=1}^{2}\left(q_i^s + \sum_{k=1}^{2}u_k^i q_k^{is}\right)x_s + \xi_i + \sum_{k=1}^{2}u_k^i \xi_k^i\right. \\[2em]
\qquad\quad \left. + \sum_{j=1}^{2}\theta_j^i \rho_j^2 + \sum_{j=1}^{2}b_i^j \rho_j + t_i\right),\ i=1,\ldots,4, \\[2em]
A^i + \displaystyle\sum_{k=1}^{2}u_k^i A_k^i \succeq 0,\ i=1,\ldots,4, \\[2em]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0,\ i=1,\ldots,4,\ j=1,2, \\[2em]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\beta_j - \theta_j^i\omega_j^2\beta_j^2, \\[1em]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[1em]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)(\beta_j+\gamma_j) - 2\theta_j^i\omega_j^2\beta_j\gamma_j, \\[1em]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\gamma_j - \theta_j^i\omega_j^2\gamma_j^2.
\end{cases}
$$

Using the Matlab toolbox CVX [18], we solve (SDP). The solver returns min (RTP) = 2 and an optimal solution of (SDP) as $(x, \rho, W, \delta, u, X)$, where $x = (1.0000, 1.0000)$, $\rho = (-151.0951, 151.4487)$, $W = \begin{pmatrix} 0.1148 & 0.0000 \\ 0.0000 & -0.1148 \end{pmatrix}$, $\delta = (50.1210, 50.0255,$ $4.9909 \times 10^{-10}, 1.4983 \times 10^{-9}, 2.2255 \times 10^{-9}, 2.2255 \times 10^{-9}, 49.9106, 50.1210)$, $u = (u^1, u^2, u^3, u^4)$ with $u^i = (-1.9376 \times 10^{-20}, -2.7868 \times 10^{-20})$, $i=1,\ldots,4$, and $X := (X^{11}, X^{12}, X^{21}, X^{22}, X^{31}, X^{32}, X^{41}, X^{42})$ with X^{11}
$= \begin{pmatrix} 50.1210 & 0.000 & 10.0608 \\ 0.0000 & 80.1204 & 0.0000 \\ 10.0608 & 0.0000 & 50.1210 \end{pmatrix}$, $X^{12} = \begin{pmatrix} 50.0255 & 0.000 & 9.9271 \\ 0.0000 & 79.9670 & 0.0000 \\ 9.9271 & 0.0000 & 49.7958 \end{pmatrix}$,

$X^{21} = \begin{pmatrix} 0.4991 & 0.000 & -0.4993 \\ 0.0000 & 0.0004 & 0.0000 \\ -0.4993 & 0.0000 & 0.4991 \end{pmatrix} \times 10^{-9}$,

$X^{22} = \begin{pmatrix} 0.1498 & 0.000 & -0.0500 \\ 0.0000 & 0.3997 & 0.0000 \\ -0.0500 & 0.0000 & 0.1498 \end{pmatrix} \times 10^{-8}$,

$X^{31} = \begin{pmatrix} 0.2225 & 0.000 & -0.0757 \\ 0.0000 & 0.5965 & 0.0000 \\ -0.0757 & 0.0000 & 0.2225 \end{pmatrix} \times 10^{-8}$,

$X^{32} = \begin{pmatrix} 0.2225 & 0.000 & -0.0757 \\ 0.0000 & 0.5965 & 0.0000 \\ -0.0757 & 0.0000 & 0.2225 \end{pmatrix} \times 10^{-8}$,

$X^{41} = \begin{pmatrix} 50.0255 & 0.000 & 9.9271 \\ 0.0000 & 79.9670 & 0.0000 \\ 9.9271 & 0.0000 & 49.7958 \end{pmatrix}$, $X^{42} = \begin{pmatrix} 50.1210 & 0.000 & 10.0608 \\ 0.0000 & 80.1204 & 0.0000 \\ 10.0608 & 0.0000 & 50.1210 \end{pmatrix}$.

Thus, by Theorem 3.1, (x, ρ, W) is an optimal solution of (RTP).

Now, let us consider special cases of (RTP). In the special case when $B_0 = O_{m \times m}$, $b_0 = 0_m$ and $t_0 = 0$, (RTP) collapses to

(RTP$_1$)

$$
\begin{cases}
\displaystyle\inf_{\substack{x \in \mathbb{R}^n, \rho \in \mathbb{R}^m, \\ W \in \mathbb{R}^{m \times m}}} \quad x^\top Q_0 x + q_0^\top x + \xi_0 \\[2mm]
s.t. \quad x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^{p} u_k^i \left((q_k^i)^\top x + \xi_k^i \right) + (y(z))^\top B_i y(z) \\
\qquad + b_i^\top y(z) + t_i \le 0, \ \forall (u_1^i, \ldots, u_p^i) \in \mathcal{U}^i, \ i = 1, \ldots, l, \\
\qquad y(z) = \rho + W z, \ \forall z \in \mathcal{U}_{box},
\end{cases}
$$

and its SDP reformulation becomes

(SDP$_1$)

$$
\begin{cases}
\displaystyle\inf_{\substack{x \in \mathbb{R}^n, \delta \in \mathbb{R}^{lm}, u \in \mathbb{R}^{pl}, \rho \in \mathbb{R}^m, \\ X \in \mathbb{R}^{3 \times 3lm}, W \in \mathbb{R}^{m \times m}}} \quad x^\top Q_0 x + q_0^\top x + \xi_0 \\[3mm]
s.t. \quad \displaystyle\sum_{j=1}^{m} \delta_i^j \le -\left(\sum_{s=1}^{n} a_s^i x_s^2 + \sum_{s=1}^{n} \left(q_i^s + \sum_{k=1}^{p} u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^{p} u_k^i \xi_k^i \right. \\[3mm]
\qquad \left. + \sum_{j=1}^{m} \theta_j^i \rho_j^2 + \sum_{j=1}^{m} b_i^j \rho_j + t_i \right), \ i = 1, \ldots, l, \\[3mm]
A^i + \displaystyle\sum_{k=1}^{p} u_k^i A_k^i \succeq 0, \ i = 1, \ldots, l, \\[3mm]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \ j = 1, \ldots, m, \\[5mm]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[2mm]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[2mm]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[2mm]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2.
\end{cases}
$$

By virtue of Theorem 3.1, we can easily obtain the following result.

Corollary 3.1 *Consider the problem* (RTP$_1$) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (1) *and its reformulation problem* (SDP$_1$). *Then,*

$$
\inf(\text{RTP}_1) = \inf(\text{SDP}_1).
$$

Moreover, (x, ρ, W) *is an optimal solution of* (RTP$_1$) *if and only if there exist* $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$ *and* $X \in \mathbb{R}^{3 \times 3lm}$ *such that* $(x, \rho, W, \delta, u, X)$ *is an optimal solution of the problem* (SDP$_1$).

In the special case when (RTP) with $t_0 = 0$ and $Q_i = B_i = O_{m \times m}, i = 0, 1, \ldots, l$, (RTP) collapses to the following adjustable robust linear optimization

(RTP$_2$)

$$
\begin{cases}
\displaystyle\inf_{\substack{x \in \mathbb{R}^n, \rho \in \mathbb{R}^m, \\ W \in \mathbb{R}^{m \times m}}} \quad q_0^\top x + \xi_0 + \max_{z \in \mathcal{U}_{box}} b_0^\top y(z) \\[2mm]
s.t. \quad q_i^\top x + \xi_i + \sum_{k=1}^{p} u_k^i \left((q_k^i)^\top x + \xi_k^i \right) + b_i^\top y(z) \\
\qquad + t_i \le 0, \ \forall (u_1^i, \ldots, u_p^i) \in \mathcal{U}^i, \ i = 1, \ldots, l, \\
\qquad y(z) = \rho + W z, \ \forall z \in \mathcal{U}_{box},
\end{cases}
$$

and its SDP reformulation becomes

$$
(\text{SDP}_2) \quad
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\delta\in\mathbb{R}^{lm},u\in\mathbb{R}^{pl},\rho\in\mathbb{R}^m, \\ X\in\mathbb{R}^{3\times 3lm},W\in\mathbb{R}^{m\times m},\sigma\in\mathbb{R}^m,Y\in\mathbb{R}^{3\times 3m}}} \quad q_0^\top x + \xi_0 + \tau \\[2em]
s.t. \quad \displaystyle\sum_{j=1}^m \delta_i^j \leq -\left(\sum_{s=1}^n\left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is}\right)x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i\right. \\[1.5em]
\qquad\qquad \left. + \displaystyle\sum_{j=1}^m b_i^j \rho_j + t_i\right), \ i=1,\ldots,l, \\[1.5em]
A^i + \displaystyle\sum_{k=1}^p u_k^i A_k^i \succeq 0, \ i=1,\ldots,l, \\[1.5em]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i=1,\ldots,l, \ j=1,\ldots,m, \\[2em]
X_{11}^{ij} = \delta_i^j - b_i^j \omega_j \beta_j, \\[0.5em]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[0.5em]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - b_i^j \omega_j\left(\beta_j + \gamma_j\right), \\[0.5em]
X_{33}^{ij} = \delta_i^j - b_i^j \omega_j \gamma_j, \\[0.5em]
\displaystyle\sum_{j=1}^m \sigma_j \leq \tau - \sum_{j=1}^m b_0^j \rho_j, \\[1.5em]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \ j=1,\ldots,m, \\[2em]
Y_{11}^j = \sigma_j - b_0^j \omega_j \beta_j, \\[0.5em]
Y_{12}^j = Y_{23}^j = 0, \\[0.5em]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - b_0^j \omega_j\left(\beta_j + \gamma_j\right), \\[0.5em]
Y_{33}^j = \sigma_j - b_0^j \omega_j \gamma_j.
\end{cases}
$$

Corollary 3.2 *Consider the problem* (RTP$_2$) *with* $\mathcal{U}^i$, $i=1,\ldots,l$, *given by* (1) *and its reformulation problem* (SDP$_2$). *Then,*

$$\inf(\text{RTP}_2) = \inf(\text{SDP}_2).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (RTP$_2$) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3\times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3\times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (SDP$_2$).

Remark 3.2 Corollary 3.2 improves upon the results obtained in [32, Theorem 2.3], where no adjustable variables are involved in the objective function of the adjustable robust linear optimization problem.

At the end of this section, we examine the category of uncertain sets such that (RTP) enjoys the SDP reformulation. We first consider the case when $\mathcal{U}^i$, $i=1,\ldots,l$, are

ellipsoidal uncertain sets, that is, $\mathcal{U}^i$ is given by

$$\mathcal{U}^i := \left\{ u^i := (u_1^i, \ldots, u_p^i) \in \mathbb{R}^p \ \middle| \ (u^i)^\top E^i u^i \leq 1 \right\}, \ i = 1, \ldots, l. \tag{12}$$

Here, $E^i \in \mathbb{S}^p$ and $E^i \succ 0$, $i = 1, \ldots, l$.

In this case, its SDP reformulation becomes

$$
\begin{cases}
\displaystyle \inf_{\substack{x\in\mathbb{R}^n, \tau\in\mathbb{R}, \delta\in\mathbb{R}^{lm}, u\in\mathbb{R}^{pl}, \rho\in\mathbb{R}^m, \\ X\in\mathbb{R}^{3\times3lm}, W\in\mathbb{R}^{m\times m}, \sigma\in\mathbb{R}^m, Y\in\mathbb{R}^{3\times3m}}} \quad x^\top Q_0 x + q_0^\top x + \xi_0 + \tau \\[2ex]
s.t. \quad \displaystyle\sum_{j=1}^{m} \delta_i^j \leq -\left(\sum_{s=1}^{n} a_s^i x_s^2 + \sum_{s=1}^{n} \left(q_i^s + \sum_{k=1}^{p} u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^{p} u_k^i \xi_k^i \right. \\[2ex]
\qquad \left. + \displaystyle\sum_{j=1}^{m} \theta_j^i \rho_j^2 + \sum_{j=1}^{m} b_i^j \rho_j + t_i \right), \ i = 1, \ldots, l, \\[2ex]
\begin{pmatrix} (E^i)^{-1} & u^i \\ (u^i)^\top & 1 \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \\[2ex]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \ j = 1, \ldots, m, \\[3ex]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[1ex]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[1ex]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[1ex]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\[1ex]
\displaystyle\sum_{j=1}^{m} \sigma_j \leq \tau - \sum_{j=1}^{m} \theta_j^0 \rho_j^2 - \sum_{j=1}^{m} b_0^j \rho_j - t_0, \\[2ex]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \ j = 1, \ldots, m, \\[3ex]
Y_{11}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \beta_j - \theta_j^0 \omega_j^2 \beta_j^2, \\[1ex]
Y_{12}^j = Y_{23}^j = 0, \\[1ex]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^0 \omega_j^2 \beta_j \gamma_j, \\[1ex]
Y_{33}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right) \gamma_j - \theta_j^0 \omega_j^2 \gamma_j^2.
\end{cases}
$$

Corollary 3.3 *Consider the problem* (RTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (12) *and its reformulation problem* (SDP). *Then,*

$$\inf(\text{RTP}) = \inf(\text{SDP}).$$

Moreover, (x, ρ, W) is an optimal solution of the problem (RTP) *if and only if there exist $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ and $Y \in \mathbb{R}^{3 \times 3m}$ such that $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ is an optimal solution of the problem* (SDP).

Proof Following similar arguments to those used in [14, Corollary 2.1], we can show that

$$A^i + \sum_{k=1}^{p} u_k^i A_k^i \succeq 0 \iff \begin{pmatrix} (E^i)^{-1} & u^i \\ (u^i)^\top & 1 \end{pmatrix} \succeq 0 \iff 1 - (u^i)^\top E^i u^i \geq 0, \ i = 1, \dots, l. \tag{13}$$

Then, the desired result is obtained.

Remark 3.3 Note that related results for SDP reformulations of static robust quadratic optimization problems on ellipsoidal uncertain sets have been investigated in [26, Lemma 2]. However, Corollary 3.3 extends these results from static to the adjustable setting.

We now consider the case where $\mathcal{U}^i$, $i = 1, \dots, l$, are cross-polytopes defined by

$$\mathcal{U}^i := \left\{ u^i := (u_1^i, \dots, u_p^i) \in \mathbb{R}^p \ \middle| \ \sum_{k=1}^{p} |u_k^i| \leq \lambda_i \right\}, \ i = 1, \dots, l, \tag{14}$$

where $\lambda_i > 0$, $i = 1, \dots, l$.

In this case, the SDP reformulation of (RTP) becomes

$$\left\{ \begin{array}{l} \inf\limits_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\delta\in\mathbb{R}^{lm},u\in\mathbb{R}^{pl},\rho\in\mathbb{R}^m,\\ X\in\mathbb{R}^{3\times 3lm},W\in\mathbb{R}^{m\times m},\sigma\in\mathbb{R}^m,Y\in\mathbb{R}^{3\times 3m}}} x^\top Q_0 x + q_0^\top x + \xi_0 + \tau \\[2em] s.t. \quad \sum_{j=1}^m \delta_i^j \le -\left(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \right. \\[1.5em] \qquad\qquad \left. + \sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i \right), \ i=1,\dots,l, \\[1.5em] \qquad \lambda_i + \sum_{k=1}^p u_k^i \ge 0, \ \lambda_i - \sum_{k=1}^p u_k^i \ge 0, \ i=1,\dots,l, \\[1.5em] \qquad X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i=1,\dots,l, \ j=1,\dots,m, \\[2em] \qquad X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right)\beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[1em] \qquad X_{12}^{ij} = X_{23}^{ij} = 0, \\[1em] \qquad 2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right)(\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[1em] \qquad X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right)\gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\[1em] \qquad \sum_{j=1}^m \sigma_j \le \tau - \sum_{j=1}^m \theta_j^0 \rho_j^2 - \sum_{j=1}^m b_0^j \rho_j - t_0, \\[1.5em] \qquad Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \ j=1,\dots,m, \\[2em] \qquad Y_{11}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right)\beta_j - \theta_j^0 \omega_j^2 \beta_j^2, \\[1em] \qquad Y_{12}^j = Y_{23}^j = 0, \\[1em] \qquad 2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right)(\beta_j + \gamma_j) - 2\theta_j^0 \omega_j^2 \beta_j \gamma_j, \\[1em] \qquad Y_{33}^j = \sigma_j - \left(2\rho_j \theta_j^0 \omega_j + b_0^j \omega_j \right)\gamma_j - \theta_j^0 \omega_j^2 \gamma_j^2. \end{array} \right.$$

Corollary 3.4 *Consider the problem* (RTP) *with* $\mathcal{U}^i$, $i=1,\dots,l$, *given by* (14) *and its reformulation problem* (SDP). *Then,*

$$\inf(\text{RTP}) = \inf(\text{SDP}).$$

Moreover, (x,ρ,W) *is an optimal solution of the problem* (RTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3\times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3\times 3m}$ *such that* $(x,\rho,W,\tau,\delta,u,X,\sigma,Y)$ *is an optimal solution of the problem* (SDP).

Proof Following similar arguments to those used in [14, Corollary 2.2], we can show that

$$A^i + \sum_{k=1}^p u_k^i A_k^i \succeq 0 \iff \lambda_i + \sum_{k=1}^p u_k^i \ge 0 \text{ and } \lambda_i - \sum_{k=1}^p u_k^i \ge 0$$

$$\iff \sum_{k=1}^p |u_k^i| \le \lambda_i, \ i=1,\dots,l. \tag{15}$$

Then, the desired result is obtained.

Remark 3.4 Note that related results for SDP reformulations of robust optimization problems on polytopic uncertain sets have been investigated in [26, Lemma 1] and [7, Theorem 1]. However, Corollary 3.4 extends the results obtained in [26, Lemma 1] from static to the adjustable setting. Furthermore, Corollary 3.4 also extends the results obtained in [7, Theorem 1] from linear optimization models to quadratic optimization models.

Let us now consider the case when $\mathcal{U}^i$, $i = 1, \ldots, l$, are boxes defined by

$$\mathcal{U}^i := \left\{ u^i := (u^i_1, \ldots, u^i_p) \in \mathbb{R}^p \ \middle| \ |u^i_k| \le \lambda_i, \ k = 1, \ldots, p \right\}, \ i = 1, \ldots, l, \ (16)$$

where $\lambda_i > 0$, $i = 1, \ldots, l$.

In this setting, the SDP reformulation of (RTP) becomes

$$
\begin{cases}
\inf_{\substack{x \in \mathbb{R}^n, \tau \in \mathbb{R}, \delta \in \mathbb{R}^{lm}, u \in \mathbb{R}^{pl}, \rho \in \mathbb{R}^m, \\ X \in \mathbb{R}^{3 \times 3lm}, W \in \mathbb{R}^{m \times m}, \sigma \in \mathbb{R}^m, Y \in \mathbb{R}^{3 \times 3m}}} x^\top Q_0 x + q_0^\top x + \xi_0 + \tau \\[2mm]
s.t. \quad \sum_{j=1}^m \delta^j_i \le - \left(\sum_{s=1}^n a^i_s x^2_s + \sum_{s=1}^n \left(q^s_i + \sum_{k=1}^p u^i_k q^{is}_k \right) x_s + \xi_i + \sum_{k=1}^p u^i_k \xi^i_k \right. \\[2mm]
\left. + \sum_{j=1}^m \theta^i_j \rho^2_j + \sum_{j=1}^m b^j_i \rho_j + t_i \right), \ i = 1, \ldots, l, \\[2mm]
\lambda_i + u^i_k \ge 0, \ \lambda_i - u^i_k \ge 0, \ k = 1, \ldots, p, \ i = 1, \ldots, l, \\[2mm]
X^{ij} := \begin{pmatrix} X^{ij}_{11} & X^{ij}_{12} & X^{ij}_{13} \\ X^{ij}_{12} & X^{ij}_{22} & X^{ij}_{23} \\ X^{ij}_{13} & X^{ij}_{23} & X^{ij}_{33} \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \ j = 1, \ldots, m, \\[2mm]
X^{ij}_{11} = \delta^j_i - \left(2\rho_j \theta^i_j \omega_j + b^j_i \omega_j \right) \beta_j - \theta^i_j \omega^2_j \beta^2_j, \\[1mm]
X^{ij}_{12} = X^{ij}_{23} = 0, \\[1mm]
2X^{ij}_{13} + X^{ij}_{22} = 2\delta^j_i - \left(2\rho_j \theta^i_j \omega_j + b^j_i \omega_j \right) (\beta_j + \gamma_j) - 2\theta^i_j \omega^2_j \beta_j \gamma_j, \\[1mm]
X^{ij}_{33} = \delta^j_i - \left(2\rho_j \theta^i_j \omega_j + b^j_i \omega_j \right) \gamma_j - \theta^i_j \omega^2_j \gamma^2_j, \\[2mm]
\sum_{j=1}^m \sigma_j \le \tau - \sum_{j=1}^m \theta^0_j \rho^2_j - \sum_{j=1}^m b^j_0 \rho_j - t_0, \\[2mm]
Y^j := \begin{pmatrix} Y^j_{11} & Y^j_{12} & Y^j_{13} \\ Y^j_{12} & Y^j_{22} & Y^j_{23} \\ Y^j_{13} & Y^j_{23} & Y^j_{33} \end{pmatrix} \succeq 0, \ j = 1, \ldots, m, \\[2mm]
Y^j_{11} = \sigma_j - \left(2\rho_j \theta^0_j \omega_j + b^j_0 \omega_j \right) \beta_j - \theta^0_j \omega^2_j \beta^2_j, \\[1mm]
Y^j_{12} = Y^j_{23} = 0, \\[1mm]
2Y^j_{13} + Y^j_{22} = 2\sigma_j - \left(2\rho_j \theta^0_j \omega_j + b^j_0 \omega_j \right) (\beta_j + \gamma_j) - 2\theta^0_j \omega^2_j \beta_j \gamma_j, \\[1mm]
Y^j_{33} = \sigma_j - \left(2\rho_j \theta^0_j \omega_j + b^j_0 \omega_j \right) \gamma_j - \theta^0_j \omega^2_j \gamma^2_j.
\end{cases}
$$

Corollary 3.5 *Consider the problem* (RTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* ([16](#)) *and its reformulation problem* (SDP). *Then,*

$$\inf(\text{RTP}) = \inf(\text{SDP}).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (RTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3 \times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (SDP).

Proof Using similar arguments to those used in [14, Corollary 2.2], we can show that

$$A^i + \sum_{k=1}^{p} u_k^i A_k^i \succeq 0 \iff \lambda_i + u_k^i \geq 0 \text{ and } \lambda_i - u_k^i \geq 0 \iff |u_k^i| \leq \lambda_i, \ i = 1, \ldots, l.$$
(17)

Then, we obtain the desired result.

4 Applications

In this section, we apply the results obtained in the previous sections to fractionally adjustable robust optimization problems.

In what follows, suppose that g_0, f_i, g_i, $y(\cdot)$, $\mathcal{U}^i$ and $\mathcal{U}_{box}$ are considered as before. Let $Q_0, \widetilde{Q}_0 \in \mathbb{S}^n$, $q_0, \widetilde{q}_0 \in \mathbb{R}^n$, and $\xi_0, \widetilde{\xi}_0 \in \mathbb{R}$. We consider the following fractionally adjustable robust quadratic optimization

$$(\text{FTP}) \quad \begin{cases} \inf\limits_{\substack{x \in \mathbb{R}^n, \rho \in \mathbb{R}^m, \\ W \in \mathbb{R}^{m \times m}}} \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \dfrac{\max\limits_{z \in \mathcal{U}_{box}} g_0(y(z))}{\min\limits_{z \in \mathcal{U}_{box}} \widetilde{g}_0(y(z))} \\ \text{s.t.} \quad f_i(x, u^i) + g_i(y(z)) \leq 0, \ \forall u^i \in \mathcal{U}^i, \ i = 1, \ldots, l, \\ \qquad y(z) = \rho + Wz, \ \forall z \in \mathcal{U}_{box}, \end{cases}$$

where $\widetilde{g}_0(y(z)) := (y(z))^\top \widetilde{B}_0 y(z) + \widetilde{b}_0^\top y(z) + \widetilde{t}_0$ with $\widetilde{B}_0 := \text{diag}(\widetilde{\theta}_1^0, \ldots, \widetilde{\theta}_m^0) \in \mathbb{R}^{m \times m}$, $\widetilde{b}_0 := (\widetilde{b}_0^1, \ldots, \widetilde{b}_0^m) \in \mathbb{R}^m$ and $\widetilde{t}_0 \in \mathbb{R}$. Moreover, we assume that $x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0 > 0$ and $\widetilde{g}_0(y(z)) > 0$ are in the feasible set of (FTP).

The SDP reformulation of (FTP) is given as follows:

(FSDP)

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\delta\in\mathbb{R}^{lm},u\in\mathbb{R}^{pl},\rho\in\mathbb{R}^m,\\ X\in\mathbb{R}^{3\times3lm},W\in\mathbb{R}^{m\times m},\sigma\in\mathbb{R}^m,Y\in\mathbb{R}^{3\times3m}}} \quad \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[2em]
s.t. \quad \displaystyle\sum_{j=1}^{m}\delta_i^j \le -\left(\sum_{s=1}^{n}a_s^i x_s^2 + \sum_{s=1}^{n}\left(q_i^s + \sum_{k=1}^{p}u_k^i q_k^{is}\right)x_s + \xi_i + \sum_{k=1}^{p}u_k^i \xi_k^i\right.\\[1.5em]
\qquad\quad \left. + \displaystyle\sum_{j=1}^{m}\theta_j^i \rho_j^2 + \sum_{j=1}^{m}b_i^j \rho_j + t_i\right),\; i=1,\dots,l,\\[1.5em]
A^i + \displaystyle\sum_{k=1}^{p}u_k^i A_k^i \succeq 0,\; i=1,\dots,l,\\[1.5em]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij}\\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij}\\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij}\end{pmatrix}\succeq 0,\; i=1,\dots,l,\; j=1,\dots,m,\\[2em]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\beta_j - \theta_j^i\omega_j^2\beta_j^2,\\[0.8em]
X_{12}^{ij} = X_{23}^{ij} = 0,\\[0.8em]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)(\beta_j+\gamma_j) - 2\theta_j^i\omega_j^2\beta_j\gamma_j,\\[0.8em]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\gamma_j - \theta_j^i\omega_j^2\gamma_j^2,\\[1em]
\displaystyle\sum_{j=1}^{m}\sigma_j \le \tau - \sum_{j=1}^{m}\left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\rho_j^2 - \sum_{j=1}^{m}\left(b_0^j - \tau\widetilde{b}_0^j\right)\rho_j - \left(t_0 - \tau\widetilde{t}_0\right),\\[1.5em]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j\\ Y_{12}^j & Y_{22}^j & Y_{23}^j\\ Y_{13}^j & Y_{23}^j & Y_{33}^j\end{pmatrix}\succeq 0,\; j=1,\dots,m,\\[2em]
Y_{11}^j = \sigma_j - \left(2\left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\rho_j\omega_j + \left(b_0^j - \tau\widetilde{b}_0^j\right)\omega_j\right)\beta_j - \left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\omega_j^2\beta_j^2,\\[1em]
Y_{12}^j = Y_{23}^j = 0,\\[0.8em]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2\left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\rho_j\omega_j + \left(b_0^j - \tau\widetilde{b}_0^j\right)\omega_j\right)(\beta_j+\gamma_j)\\[0.8em]
\qquad\qquad\quad - 2\left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\omega_j^2\beta_j\gamma_j,\\[0.8em]
Y_{33}^j = \sigma_j - \left(2\left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\rho_j\omega_j + \left(b_0^j - \tau\widetilde{b}_0^j\right)\omega_j\right)\gamma_j - \left(\theta_j^0 - \tau\widetilde{\theta}_j^0\right)\omega_j^2\gamma_j^2.
\end{cases}
$$

Now, we give the following theorem which describes an exact SDP reformulation for (FTP).

Theorem 4.1 *Consider the problem* (FTP) *with* $\mathcal{U}^i$, $i=1,\dots,l$, *given by* (1) *and its reformulation problem* (FSDP). *Then,*

$$\inf(\text{FTP}) = \inf(\text{FSDP}).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (FTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3\times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3\times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (FSDP).

Proof Obviously, (FTP) is equivalent to

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\\ \rho\in\mathbb{R}^m,W\in\mathbb{R}^{m\times m}}} \ \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[2ex]
\quad s.t. \quad f_i(x,u^i) + g_i(y(z)) \le 0, \ \forall u^i \in \mathcal{U}^i, \ i = 1,\ldots,l, \\[1ex]
\qquad\qquad \dfrac{\max\limits_{z\in\mathcal{U}_{box}} g_0(y(z))}{\min\limits_{z\in\mathcal{U}_{box}} \widetilde{g}_0(y(z))} \le \tau, \\[2ex]
\qquad\qquad y(z) = \rho + Wz, \ \forall z \in \mathcal{U}_{box}.
\end{cases}
\tag{18}
$$

The problem (18) is equivalent to

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},\\ \rho\in\mathbb{R}^m,W\in\mathbb{R}^{m\times m}}} \ \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[2ex]
\quad s.t. \quad f_i(x,u^i) + g_i(y(z)) \le 0, \ \forall u^i \in \mathcal{U}^i, \ i = 1,\ldots,l, \\[1ex]
\qquad\qquad g_0(y(z)) - \tau \widetilde{g}_0(y(z)) \le 0, \\[1ex]
\qquad\qquad y(z) = \rho + Wz, \ \forall z \in \mathcal{U}_{box},
\end{cases}
$$

which can be written as:

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\tau\in\mathbb{R},u\in\mathbb{R}^{pl},\\ \rho\in\mathbb{R}^m,W\in\mathbb{R}^{m\times m}}} \ \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[2ex]
\quad s.t. \quad x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^p u_k^i \left((q_k^i)^\top x + \xi_k^i \right) + (y(z))^\top B_i y(z) \\[1ex]
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + b_i^\top y(z) + t_i \le 0, \ i = 1,\ldots,l, \\[1ex]
\qquad\qquad A^i + \sum_{k=1}^p u_k^i A_k^i \succeq 0, \ i = 1,\ldots,l, \\[1ex]
\qquad\qquad (y(z))^\top B_0 y(z) + b_0^\top y(z) + t_0 - \tau \left((y(z))^\top \widetilde{B}_0 y(z) + \widetilde{b}_0^\top y(z) + \widetilde{t}_0 \right) \le 0, \\[1ex]
\qquad\qquad y(z) = \rho + Wz, \ \forall z \in \mathcal{U}_{box}.
\end{cases}
$$

Now, using similar arguments as in the proof of Theorem 3.1, it is easy to show that Theorem 4.1 holds.

Remark 4.1 Note that related results for SDP reformulations of fractional optimization problems have been investigated in [22, Theorem 3.2] and [20, Theorem 3.5]. However, Theorem 4.1 extends [22, Theorem 3.2] from deterministic models to the uncertain setting. Theorem 4.1 also extends the results obtained in [20, Theorem 3.5] from static to the adjustable setting.

In the special case that $B_0 = O_{m\times m}$, $b_0 = 0_m$ and $t_0 = 0$, (FTP) collapses to

$$
\text{(FTP}_0)\quad
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\rho\in\mathbb{R}^m,\\ W\in\mathbb{R}^{m\times m}}} \ \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} \\[2ex]
\quad s.t. \quad x^\top Q_i x + q_i^\top x + \xi_i + \sum_{k=1}^p u_k^i \left((q_k^i)^\top x + \xi_k^i \right) + (y(z))^\top B_i y(z) \\[1ex]
\qquad\qquad\qquad + b_i^\top y(z) + t_i \le 0, \ \forall (u_1^i,\ldots,u_p^i) \in \mathcal{U}^i, \ i = 1,\ldots,l, \\[1ex]
\qquad\qquad y(z) = \rho + Wz, \ \forall z \in \mathcal{U}_{box},
\end{cases}
$$

and (FSDP) becomes

$$
\text{(FSDP}_0) \quad
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\delta\in\mathbb{R}^{lm},u\in\mathbb{R}^{pl},\\ \rho\in\mathbb{R}^m,X\in\mathbb{R}^{3\times 3lm},W\in\mathbb{R}^{m\times m}}} \quad \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0}\\[2em]
s.t. \quad \displaystyle\sum_{j=1}^m \delta_i^j \le -\left(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is}\right)x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i\right.\\[1.5em]
\qquad\qquad \left. + \sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i\right),\ i=1,\ldots,l,\\[1.5em]
\qquad A^i + \displaystyle\sum_{k=1}^p u_k^i A_k^i \succeq 0,\ i=1,\ldots,l,\\[1.5em]
\qquad X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij}\\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij}\\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij}\end{pmatrix}\succeq 0,\ i=1,\ldots,l,\ j=1,\ldots,m,\\[2.5em]
\qquad X_{11}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\beta_j - \theta_j^i\omega_j^2\beta_j^2,\\[0.5em]
\qquad X_{12}^{ij} = X_{23}^{ij} = 0,\\[0.5em]
\qquad 2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)(\beta_j+\gamma_j) - 2\theta_j^i\omega_j^2\beta_j\gamma_j,\\[0.5em]
\qquad X_{33}^{ij} = \delta_i^j - \left(2\rho_j\theta_j^i\omega_j + b_i^j\omega_j\right)\gamma_j - \theta_j^i\omega_j^2\gamma_j^2.
\end{cases}
$$

By virtue of Theorem 4.1, we can easily obtain the following result.

Corollary 4.1 *Consider the problem* (FTP$_0$) *with* $\mathcal{U}^i$, $i=1,\ldots,l$, *given by* (1) *and its reformulation problem* (FSDP$_0$). *Then,*

$$\inf(\text{FTP}_0) = \inf(\text{FSDP}_0).$$

Moreover, (x,ρ,W) *is an optimal solution of the problem* (FTP$_0$) *if and only if there exist* $\delta\in\mathbb{R}^{lm}$, $u\in\mathbb{R}^{pl}$ *and* $X\in\mathbb{R}^{3\times 3lm}$ *such that* (x,ρ,W,δ,u,X) *is an optimal solution of the problem* (FSDP$_0$).

In the special case when (FTP) with $t_0 = \widetilde{t}_0 = 0$ and $Q_i = \widetilde{Q}_0 = B_i = \widetilde{B}_0 = O_{m\times m}$, $i=0,1,\ldots,l$, (FTP) collapses to the following fractionally adjustable robust linear optimization

$$
\text{(FTP}_1) \quad
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n,\rho\in\mathbb{R}^m,\\ W\in\mathbb{R}^{m\times m}}} \quad \dfrac{q_0^\top x + \xi_0}{\widetilde{q}_0^\top x + \widetilde{\xi}_0} + \dfrac{\max\limits_{z\in\mathcal{U}_{box}} b_0^\top y(z)}{\min\limits_{z\in\mathcal{U}_{box}} \widetilde{b}_0^\top y(z)}\\[2em]
s.t. \quad q_i^\top x + \xi_i + \sum_{k=1}^p u_k^i\left((q_k^i)^\top x + \xi_k^i\right) + b_i^\top y(z)\\
\qquad\quad + t_i \le 0,\ \forall(u_1^i,\ldots,u_p^i)\in\mathcal{U}^i,\ i=1,\ldots,l,\\
\qquad y(z) = \rho + Wz,\ \forall z\in\mathcal{U}_{box},
\end{cases}
$$

and (FSDP) becomes

$$
\text{(FSDP}_1) \quad
\begin{cases}
\displaystyle \inf_{\substack{x\in\mathbb{R}^n, \tau\in\mathbb{R}, \delta\in\mathbb{R}^{lm}, u\in\mathbb{R}^{pl}, \rho\in\mathbb{R}^m, \\ X\in\mathbb{R}^{3\times 3lm}, W\in\mathbb{R}^{m\times m}, \sigma\in\mathbb{R}^m, Y\in\mathbb{R}^{3\times 3m}}} \quad \dfrac{q_0^\top x + \xi_0}{\widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[1em]
s.t. \quad \displaystyle \sum_{j=1}^m \delta_i^j \leq -\left(\sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \right. \\[1em]
\qquad\qquad \left. + \displaystyle \sum_{j=1}^m b_i^j \rho_j + t_i \right), \; i = 1, \ldots, l, \\[1em]
A^i + \displaystyle \sum_{k=1}^p u_k^i A_k^i \succeq 0, \; i = 1, \ldots, l, \\[1em]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \; i = 1, \ldots, l, \; j = 1, \ldots, m, \\[1.5em]
X_{11}^{ij} = \delta_i^j - b_i^j \omega_j \beta_j, \\[0.5em]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[0.5em]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - b_i^j \omega_j \left(\beta_j + \gamma_j \right), \\[0.5em]
X_{33}^{ij} = \delta_i^j - b_i^j \omega_j \gamma_j, \\[0.5em]
\displaystyle \sum_{j=1}^m \sigma_j \leq \tau - \sum_{j=1}^m \left(b_0^j - \tau \widetilde{b}_0^j \right) \rho_j, \\[1em]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \; j = 1, \ldots, m, \\[1.5em]
Y_{11}^j = \sigma_j - \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \beta_j, \\[0.5em]
Y_{12}^j = Y_{23}^j = 0, \\[0.5em]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \left(\beta_j + \gamma_j \right) \\[0.5em]
Y_{33}^j = \sigma_j - \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \gamma_j.
\end{cases}
$$

Corollary 4.2 *Consider the problem* (FTP$_1$) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (1) *and its reformulation problem* (FSDP$_1$). *Then,*

$$\inf(\text{FTP}_1) = \inf(\text{FSDP}_1).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (FTP$_1$) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3\times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3\times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (FSDP$_1$).

At the end of this section, we examine the category of uncertain sets such that (FTP) has the exact SDP reformulation. We first consider the case when $\mathcal{U}^i$, $i = 1, \ldots, l$, are ellipsoids in (12). In this case, its SDP reformulation becomes

$$
\begin{cases}
\displaystyle\inf_{\substack{x\in\mathbb{R}^n, \tau\in\mathbb{R}, \delta\in\mathbb{R}^{lm}, u\in\mathbb{R}^{pl}, \rho\in\mathbb{R}^m, \\ X\in\mathbb{R}^{3\times 3lm}, W\in\mathbb{R}^{m\times m}, \sigma\in\mathbb{R}^m, Y\in\mathbb{R}^{3\times 3m}}} \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[2em]
s.t. \quad \displaystyle\sum_{j=1}^m \delta_i^j \le -\Bigg(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \Big(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \Big) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \\[1.5em]
\qquad\qquad + \displaystyle\sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i \Bigg), \; i = 1, \ldots, l, \\[1.5em]
\begin{pmatrix} (E^i)^{-1} & u^i \\ (u^i)^\top & 1 \end{pmatrix} \succeq 0, \; i = 1, \ldots, l, \\[1.5em]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \; i = 1, \ldots, l, \; j = 1, \ldots, m, \\[2em]
X_{11}^{ij} = \delta_i^j - \Big(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \Big) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[1em]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[1em]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \Big(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \Big)(\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[1em]
X_{33}^{ij} = \delta_i^j - \Big(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \Big) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\[1em]
\displaystyle\sum_{j=1}^m \sigma_j \le \tau - \sum_{j=1}^m \Big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \Big) \rho_j^2 - \sum_{j=1}^m \Big(b_0^j - \tau \widetilde{b}_0^j \Big) \rho_j - \Big(t_0 - \tau \widetilde{t}_0 \Big), \\[1.5em]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \; j = 1, \ldots, m, \\[2em]
Y_{11}^j = \sigma_j - \Big(2 \big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \big) \rho_j \omega_j + \big(b_0^j - \tau \widetilde{b}_0^j \big) \omega_j \Big) \beta_j - \Big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \Big) \omega_j^2 \beta_j^2, \\[1em]
Y_{12}^j = Y_{23}^j = 0, \\[1em]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \Big(2 \big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \big) \rho_j \omega_j + \big(b_0^j - \tau \widetilde{b}_0^j \big) \omega_j \Big)(\beta_j + \gamma_j) \\[1em]
\qquad\qquad - 2 \Big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \Big) \omega_j^2 \beta_j \gamma_j, \\[1em]
Y_{33}^j = \sigma_j - \Big(2 \big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \big) \rho_j \omega_j + \big(b_0^j - \tau \widetilde{b}_0^j \big) \omega_j \Big) \gamma_j - \Big(\theta_j^0 - \tau \widetilde{\theta}_j^0 \Big) \omega_j^2 \gamma_j^2.
\end{cases}
$$

Corollary 4.3 *Consider the problem* (FTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (12) *and its reformulation problem* (FSDP). *Then,*

$$
\inf(\text{FTP}) = \inf(\text{FSDP}).
$$

Moreover, (x, ρ, W) is an optimal solution of the problem (FTP) *if and only if there exist $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ and $Y \in \mathbb{R}^{3 \times 3m}$ such that $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ is an optimal solution of the problem* (FSDP).

Proof Together with (13) and Theorem 4.1, we can easily get the desired result. $\qed$

We now consider the case where $\mathcal{U}^i$, $i = 1, \ldots, l$, are cross-polytopes in (14). In this case, the SDP reformulation of (FTP) becomes

$$
\begin{aligned}
&\inf_{\substack{x \in \mathbb{R}^n, \tau \in \mathbb{R}, \delta \in \mathbb{R}^{lm}, u \in \mathbb{R}^{pl}, \rho \in \mathbb{R}^m, \\ X \in \mathbb{R}^{3 \times 3lm}, W \in \mathbb{R}^{m \times m}, \sigma \in \mathbb{R}^m, Y \in \mathbb{R}^{3 \times 3m}}} \quad \frac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\
&s.t. \quad \sum_{j=1}^m \delta_i^j \leq - \left(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \right. \\
&\qquad\qquad \left. + \sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i \right), \; i = 1, \ldots, l, \\
&\qquad \lambda_i + \sum_{k=1}^p u_k^i \geq 0, \; \lambda_i - \sum_{k=1}^p u_k^i \geq 0, \; i = 1, \ldots, l, \\
&\qquad X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \; i = 1, \ldots, l, \; j = 1, \ldots, m, \\
&\qquad X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\
&\qquad X_{12}^{ij} = X_{23}^{ij} = 0, \\
&\qquad 2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\
&\qquad X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\
&\qquad \sum_{j=1}^m \sigma_j \leq \tau - \sum_{j=1}^m \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j^2 - \sum_{j=1}^m \left(b_0^j - \tau \widetilde{b}_0^j \right) \rho_j - (t_0 - \tau \widetilde{t}_0), \\
&\qquad Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \; j = 1, \ldots, m, \\
&\qquad Y_{11}^j = \sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) \beta_j - \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \beta_j^2, \\
&\qquad Y_{12}^j = Y_{23}^j = 0, \\
&\qquad 2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) (\beta_j + \gamma_j) \\
&\qquad\qquad - 2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \beta_j \gamma_j, \\
&\qquad Y_{33}^j = \sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) \gamma_j - \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \gamma_j^2.
\end{aligned}
$$

Corollary 4.4 *Consider the problem* (FTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (14) *and its reformulation problem* (FSDP). *Then,*

$$\inf(\text{FTP}) = \inf(\text{FSDP}).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (FTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3 \times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (FSDP).

Proof From (15) and Theorem 4.1, it is easy to show the validity of the corollary.

Let us now consider the case when $\mathcal{U}^i$, $i = 1, \ldots, l$, are boxes in (16). In this setting, the SDP reformulation of (FSDP) becomes

$$
\begin{cases}
\displaystyle \inf_{\substack{x \in \mathbb{R}^n, \tau \in \mathbb{R}, \delta \in \mathbb{R}^{lm}, u \in \mathbb{R}^{pl}, \rho \in \mathbb{R}^m, \\ X \in \mathbb{R}^{3 \times 3lm}, W \in \mathbb{R}^{m \times m}, \sigma \in \mathbb{R}^m, Y \in \mathbb{R}^{3 \times 3m}}} \dfrac{x^\top Q_0 x + q_0^\top x + \xi_0}{x^\top \widetilde{Q}_0 x + \widetilde{q}_0^\top x + \widetilde{\xi}_0} + \tau \\[4mm]
s.t. \quad \displaystyle\sum_{j=1}^m \delta_i^j \le -\left(\sum_{s=1}^n a_s^i x_s^2 + \sum_{s=1}^n \left(q_i^s + \sum_{k=1}^p u_k^i q_k^{is} \right) x_s + \xi_i + \sum_{k=1}^p u_k^i \xi_k^i \right. \\[3mm]
\qquad\qquad \left. + \displaystyle\sum_{j=1}^m \theta_j^i \rho_j^2 + \sum_{j=1}^m b_i^j \rho_j + t_i \right), \ i = 1, \ldots, l, \\[3mm]
\lambda_i + u_k^i \ge 0, \ \lambda_i - u_k^i \ge 0, \ k = 1, \ldots, p, \ i = 1, \ldots, l, \\[2mm]
X^{ij} := \begin{pmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ X_{12}^{ij} & X_{22}^{ij} & X_{23}^{ij} \\ X_{13}^{ij} & X_{23}^{ij} & X_{33}^{ij} \end{pmatrix} \succeq 0, \ i = 1, \ldots, l, \ j = 1, \ldots, m, \\[4mm]
X_{11}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \beta_j - \theta_j^i \omega_j^2 \beta_j^2, \\[2mm]
X_{12}^{ij} = X_{23}^{ij} = 0, \\[2mm]
2X_{13}^{ij} + X_{22}^{ij} = 2\delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) (\beta_j + \gamma_j) - 2\theta_j^i \omega_j^2 \beta_j \gamma_j, \\[2mm]
X_{33}^{ij} = \delta_i^j - \left(2\rho_j \theta_j^i \omega_j + b_i^j \omega_j \right) \gamma_j - \theta_j^i \omega_j^2 \gamma_j^2, \\[2mm]
\displaystyle\sum_{j=1}^m \sigma_j \le \tau - \sum_{j=1}^m \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j^2 - \sum_{j=1}^m \left(b_0^j - \tau \widetilde{b}_0^j \right) \rho_j - \left(t_0 - \tau \widetilde{t}_0 \right), \\[3mm]
Y^j := \begin{pmatrix} Y_{11}^j & Y_{12}^j & Y_{13}^j \\ Y_{12}^j & Y_{22}^j & Y_{23}^j \\ Y_{13}^j & Y_{23}^j & Y_{33}^j \end{pmatrix} \succeq 0, \ j = 1, \ldots, m, \\[4mm]
Y_{11}^j = \sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) \beta_j - \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \beta_j^2, \\[2mm]
Y_{12}^j = Y_{23}^j = 0, \\[2mm]
2Y_{13}^j + Y_{22}^j = 2\sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) (\beta_j + \gamma_j) \\[2mm]
\qquad\qquad - 2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \beta_j \gamma_j, \\[2mm]
Y_{33}^j = \sigma_j - \left(2 \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \rho_j \omega_j + \left(b_0^j - \tau \widetilde{b}_0^j \right) \omega_j \right) \gamma_j - \left(\theta_j^0 - \tau \widetilde{\theta}_j^0 \right) \omega_j^2 \gamma_j^2.
\end{cases}
$$

Corollary 4.5 *Consider the problem* (FTP) *with* $\mathcal{U}^i$, $i = 1, \ldots, l$, *given by* (16) *and its reformulation problem* (FSDP). *Then,*

$$\inf(\text{FTP}) = \inf(\text{FSDP}).$$

Moreover, (x, ρ, W) *is an optimal solution of the problem* (FTP) *if and only if there exist* $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^{lm}$, $u \in \mathbb{R}^{pl}$, $X \in \mathbb{R}^{3 \times 3lm}$, $\sigma \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^{3 \times 3m}$ *such that* $(x, \rho, W, \tau, \delta, u, X, \sigma, Y)$ *is an optimal solution of the problem* (FSDP).

Proof By (17) and Theorem 4.1, it is easy to obtain the desired result.

5 Conclusions

In this paper, under the framework of adjustable robust optimization approach, a class of adjustable robust quadratic optimization problems, where both objective and constraint functions involve uncertain data, is considered. We first obtain exact SDP reformulations for adjustable robust quadratic optimization problems with an affine decision rule on spectrahedral uncertain sets. For applications, we also establish exact SDP reformulations for fractionally adjustable robust quadratic optimization problems on spectrahedral uncertain sets.

In the future, further studies on SDP problems and optimality conditions for adjustable robust quadratic optimization problems are still needed. For example, similar to [11], can we obtain some SDP dual results for adjustable robust quadratic optimization problems with an affine decision rule. On the other hand, it is important to consider how the proposed approach can be extended to handle adjustable robust nonlinear optimization problems.

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