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Impulse Control of Conditional McKean–Vlasov Jump Diffusions

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Abstract

In this paper, we consider impulse control problems involving conditional McKean–Vlasov jump diffusions, with the common noise coming from the σ-algebra generated by the first components of a Brownian motion and an independent compensated Poisson random measure. We first study the well-posedness of the conditional McKean–Vlasov stochastic differential equations (SDEs) with jumps. Then, we prove the associated Fokker–Planck stochastic partial differential equation (SPDE) with jumps. Next, we establish a verification theorem for impulse control problems involving conditional McKean–Vlasov jump diffusions. We obtain a Markovian system by combining the state equation with the associated Fokker–Planck SPDE for the conditional law of the state. Then we derive sufficient variational inequalities for a function to be the value function of the impulse control problem, and for an impulse control to be the optimal control. We illustrate our results by applying them to the study of an optimal stream of dividends under transaction costs. We obtain the solution explicitly by finding a function and an associated impulse control, which satisfy the verification theorem.

Keywords Jump diffusion · Impulse control · Common noise

Mathematics Subject Classification 49J53 · 49K99

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1 Introduction

Consider a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ on which we are given a d-dimensional Brownian motion $B = (B_1, B_2, \dots, B_d)$, a k-dimensional compensated Poisson random measure $\widetilde{N}(\mathrm{d}t, \mathrm{d}z)$ such that

$$\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t,$$

where $N(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure and $\nu(\mathrm{d}z)$ is the Lévy measure of N, and a random variable $Z \in L^2(P)$ that is independent of $\mathbb{F}$. We denote by $L^2(P)$ the set of all the d-dimensional, $\mathbb{F}$-measurable random variables X such that $\mathbb{E}[X^2] < \infty$, where $\mathbb{E}$ denotes the expectation with respect to P. We consider the state process $X(t) \in \mathbb{R}^d$ given as the solution of the following *conditional McKean–Vlasov jump diffusion*

$$X(t) = Z + \int_0^t \alpha(s, X(s), \mu_s)\mathrm{d}s + \beta(s, X(s), \mu_s)\mathrm{d}B(s) + \int_0^t \int_{\mathbb{R}^k} \gamma(s, X(s^-), \mu_{s^-}, z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \tag{1}$$

where we denote by $\mu_t = \mathcal{L}(X(t)|\mathcal{G}_t)$ the conditional law of $X(t)$ given the natural filtration $\mathcal{G}_t$ generated by the first components $(B_1(t), \widetilde{N}_1(t, z))$ of the Brownian motion and an independent compensated Poisson random measure, respectively, up to time t. Loosely speaking, equation of the form (1) models a McKean–Vlasov dynamics which is subject to what is called a "common noise" coming from $(B_1(t), \widetilde{N}_1(t, z))$, which is observed and is influencing the dynamics of the system. This type of equation arises naturally in the framework of a particle system in the large-scale limit where the particles interact in the mean-field way with common noise. For instance, see Erny [13] for the unconditional case, where the well-posedness and the propagation of chaos for McKean–Vlasov SDE with jumps have been studied and with locally Lipschitz coefficients.

Conditional propagation of chaos (in continuous case) has been studied in the literature; for example, we refer to Carmona et al [9], Coghi and Flandoli [11]. There the common noise is represented by a common Brownian motion, which is already presented at the finite particle system. We refer also to the recent papers by Buckdahn et al. [7] and Lacker et al. [17], [18] for related results. However, in Erny et al. [14], the common noise is only presented at the limiting level. It comes from the joint action of the small jumps of the finite size particle system. They obtain a Fokker–Planck SPDE related to the one considered in Agram and Øksendal [1]. Since we are not concern about finite particle system in this paper, we work directly with the limit system (1). However, conditional propagation of chaos for a common noise presented by both the Brownian motion and the compensated Poisson random measure will be a purpose of future research.

For application of conditional propagation of chaos in a spatial stochastic epidemic model, we refer to Vuong et al. [22] in the continuous case.

To the best of our knowledge, none of the papers mentioned above deal with the common noise coming from both the Brownian and the Poissonian measure as we consider in the current paper. Moreover, the type of Fokker–Planck SPDE with jumps obtained, in this paper, as the equation for the conditional law of the state has not been considered in the literature.

Precisely, in the present paper, we are going to study impulse control problems for conditional McKean–Vlasov jump diffusions. In particular, we will define a performance criterion and then attempt to find a policy that maximizes performance within the admissible impulse strategies. Using a verification theorem approach, we establish a general form of quasi-variational inequalities and identify the sufficient conditions that lead to an optimal function. See precise formulation below. Standard impulse control problems can be solved by using the Dynkin formula. We refer to e.g. Bensoussan and Lions [4] in the continuous case and to Øksendal and Sulem [20] in the setting of jump diffusions.

Impulse control problems naturally arise in many concrete applications, in particular when an agent, because of the intervention costs, decides to control the system by intervening only at a discrete set of times with a chosen intervention size: a sequence of stopping times $(\tau_1, \tau_2, \ldots, \tau_k, \ldots)$ is chosen to intervene and exercise the control. At each time τ_k of the player's k^{th} intervention, the player chooses an intervention of size ζ_k. The impulse control consists of the sequence $\{(\tau_k, \zeta_k)\}_{k\geq 1}$.

Impulse control has sparked great interest in the financial field and beyond. See, for example, Korn [15] for portfolio theory applications, Basei [2] for energy markets, and Cadenillas et al. [8] for insurance. All of these works are based on quasi-variational inequalities and employ a verification approach.

Despite its adaptability to more realistic financial models, few papers have studied the case of mean-field problems with impulse control. We refer to Basei et al. [3] for a discussion of a more special type of impulse, where the only type of impulse is to add something to the system. Precisely, they consider a mean-field game (MFG) where the mean-field (only the empirical mean) appears as an approximation of a many-player game. Moreover, they use the smooth fit principle (as used in the present work) to solve a specific MFG explicitly.

We refer also to Christensen et al. [10] for a MFG impulse control approach. Specifically, a problem of optimal harvesting in natural resource management is addressed.

A maximum principle for regime switching control problem for mean-field jump diffusions is studied by Li et al. [19] but in that paper the problem considered is not really an impulse control problem because the intervention times are fixed in advance.

In our setting, we do not consider a MFG setup, as in the above-mentioned works. Instead, we consider a decision-maker who chooses the control to optimize a certain reward. Moreover, the mean-field appears as a conditional probability distribution, and to overcome the lack of the Markov property, we introduce the equation of the law, which is of stochastic Fokker–Planck type.

In Djehiche et al. [12], the authors could handle a non-Markovian dynamics. However, the impulse control is given in a particular compact form, and only a given number of impulses are allowed. They use a Snell envelope approach and related reflected backward stochastic differential equations.

The rest of the paper is organized as follows: In the next section, we study the well-posedness of the conditional McKean–Vlasov SDE (1). Section 3 is devoted to the Fokker–Planck SPDE with jumps. In Sect. 4, we state the optimal control problem and prove the verification theorem. In Sect. 5, we apply the previous results to solve an explicit problem of optimal dividend streams under transaction costs.

2 Conditional McKean–Vlasov SDEs with Jumps

Let us first study the well-posedness of the conditional McKean–Vlasov dynamics. We mean by $X_t \in \mathbb{R}^d$ the mean-field stochastic differential equation with jumps, from now on called a *McKean–Vlasov jump diffusion*, of the form

$$\begin{aligned} \mathrm{d}X_j(t) &= \alpha_j(t, X(t), \mu_t)\mathrm{d}t + \sum_{n=1}^{m} \beta_{j,n}(t, X(t), \mu_t)\mathrm{d}B_n(t) \\ &\quad + \sum_{\ell=1}^{k} \int_{\mathbb{R}^k} \gamma_{j,\ell}(t, X(t^-), \mu_{t^-}, z)\widetilde{N}_\ell(\mathrm{d}t, \mathrm{d}z); \quad j = 1, 2, ..., d \\ X_0 &= x \in \mathbb{R}^d, \end{aligned} \tag{2}$$

or, using matrix notation,

$$\mathrm{d}X(t) = \alpha(t, X(t), \mu_t)\mathrm{d}t + \beta(t, X(t), \mu_t)\mathrm{d}B(t) + \int_{\mathbb{R}^k} \gamma(t, X(t^-), \mu_t, z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z),$$

where $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T \in \mathbb{R}^m = \mathbb{R}^{m\times 1}$, $\widetilde{N} = (\widetilde{N}_1, ..., \widetilde{N}_k)^T \in \mathbb{R}^k = \mathbb{R}^{k\times 1}$ are, respectively, an m-dimensional Brownian motion and a k-dimensional compensated Poisson random measure on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$, and $\beta = (\beta_{j,n}) \in \mathbb{R}^{d\times m}$, $\gamma = (\gamma_{j,\ell}) \in \mathbb{R}^{d\times k}$.

We assume that for all ℓ; $1 \leq \ell \leq k$, the Lévy measure of N_ℓ, denoted by ν_ℓ, satisfies the condition $\int_{\mathbb{R}^k} z^2 \nu_\ell(\mathrm{d}z) < \infty$, which means that N_ℓ does not have many big jumps (but N_ℓ may still have infinite total variation near 0). This assumption allows us to use the version of the Itô formula for jump diffusion given, e.g. in Theorem 1.16 in Øksendal and Sulem [20].

We denote, by $\mathcal{G}_t$, the sub-filtrations of $\mathcal{F}_t$, generated by the first components of both the Brownian motion B and the compensated Poisson random measure $\widetilde{N}$. The sub-filtration $\mathcal{G}_t$ satisfies the usual conditions as $\mathcal{F}_t$.

We define μ_t^X to be regular conditional distribution of X_t given $\mathcal{G}_t$. This means that $\forall t \geq 0$, μ_t^X is a Borel probability measure on $\mathbb{R}^d$ and

$$\int_{\mathbb{R}^n} g(x)\mu_t^X(\mathrm{d}x) = \mathbb{E}[g(X_t)|\mathcal{G}_t](\omega) \tag{3}$$

for all functions g such that $\mathbb{E}[|g(X_t)|] < \infty$. We refer to Theorem 9 in Protter [21]. We shall define the special weighted Sobolev norm on the space of measures.

Definition 2.1 Let d be a given natural number. Then, let $\mathbb{M} = \mathbb{M}^d$ be the pre-Hilbert space of random measures μ on $\mathbb{R}^d$ equipped with the norm

$$\|\mu\|^2_{\mathbb{M}} := \mathbb{E}[\int_{\mathbb{R}^d} |\hat{\mu}(y)|^2 e^{-y^2} \mathrm{d}y], \tag{4}$$

where $y = (y_1, y_2, ..., y_d) \in \mathbb{R}^d$ and $\hat{\mu}$ is the Fourier transform of the measure μ, i.e.

$$\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{-ixy} \mu(\mathrm{d}x); \quad y \in \mathbb{R}^d,$$

where $xy = x \cdot y = x_1 y_1 + x_2 y_2 + ... + x_d y_d$ is the scalar product in $\mathbb{R}^d$.

If $\mu, \eta \in \mathbb{M}$, we define the inner product $\langle \mu, \eta \rangle_{\mathbb{M}}$ by

$$\langle \mu, \eta \rangle_{\mathbb{M}} = \mathbb{E}[\int_{\mathbb{R}^d} \mathrm{Re}(\overline{\hat{\mu}}(y)\hat{\eta}(y))|y|^2 e^{-y^2} \mathrm{d}y],$$

where $\mathrm{Re}(z)$ denotes the real part and $\bar{z}$ denotes the complex conjugate of the complex number z.

The space $\mathbb{M}$ equipped with the inner product $\langle \mu, \eta \rangle_{\mathbb{M}}$ is a pre-Hilbert space. For not having ambiguity, we will also use the notation $\mathbb{M}$ for the completion of this pre-Hilbert space.

Lemma 2.1 *Let X_1 and X_2 be two d-dimensional random variables in $L^2(\mathbb{P})$. Thus,*

$$\|\mathcal{L}(X_1|\mathcal{G}_t) - \mathcal{L}(X_2|\mathcal{G}_t)\|^2_{\mathbb{M}} \le \pi \ \mathbb{E}[(X_1 - X_2)^2].$$

Proof By using the relation (4), we have

$$\begin{aligned}
&\|\mathcal{L}(X_1|\mathcal{G}_t) - \mathcal{L}(X_2|\mathcal{G}_t)\|^2_{\mathbb{M}} \\
&= \mathbb{E}\int_{\mathbb{R}^d} \left|\widehat{\mathcal{L}}(X_1|\mathcal{G}_t)(y) - \widehat{\mathcal{L}}(X_2|\mathcal{G}_t)(y)\right|^2 e^{-y^2} \mathrm{d}y \\
&= \mathbb{E}\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} e^{-ix_1 y} \mathrm{d}\mathcal{L}(X_1|\mathcal{G}_t)(x_1) - \int_{\mathbb{R}^d} e^{-ix_2 y} \mathrm{d}\mathcal{L}(X_2|\mathcal{G}_t)(x_2)\right|^2 e^{-y^2} \mathrm{d}y \\
&= \mathbb{E}\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} (e^{-ix_1 y} - e^{-ix_2 y}) \mathrm{d}\mathcal{L}((X_1, X_2)|\mathcal{G}_t)(x_1, x_2)\right|^2 e^{-y^2} \mathrm{d}y \\
&= \mathbb{E}\int_{\mathbb{R}^d} \left|\mathbb{E}(e^{-iX_1 y} - e^{-iX_2 y})|\mathcal{G}_t)\right|^2 e^{-y^2} \mathrm{d}y.
\end{aligned}$$

Using the fact that conditioning is a contractive projection of L^2 spaces and by standard properties of the complex exponential function, we obtain

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^d} \left|\mathbb{E}(e^{-iX_1 y} - e^{-iX_2 y})|\mathcal{G}_t)\right|^2 e^{-y^2} \mathrm{d}y &\le \mathbb{E}\int_{\mathbb{R}^d} \mathbb{E}\Big[\left|e^{-iX_1 y} - e^{-iX_2 y}\right|^2\Big] e^{-y^2} \mathrm{d}y \\
&\le \int_{\mathbb{R}^d} y^2 e^{-y^2} \mathrm{d}y \mathbb{E}[|X_1 - X_2|^2].
\end{aligned}$$

Therefore, the desired result follows. □

To study the well-posedness of the conditional McKean–Vlasov SDE (2), we impose the following set of assumptions on the coefficients α, β and γ:

Assumption I

- $\alpha(t,x,\mu) : [0,T] \times \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}^d$, $\beta(t,x,\mu) : [0,T] \times \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}^{d\times m}$ and $\gamma(t,x,\mu,\zeta) : [0,T] \times \mathbb{R}^d \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are locally bounded and Borel-measurable functions.
- There exists a constant C, such that for all $t \in [0,T]$, x, x', μ, μ', we have

$$\begin{aligned}&\left|\alpha(t,x,\mu) - \alpha(t,x',\mu')\right| + \left|\beta(t,x,\mu) - \beta(t,x',\mu')\right|\\&+ \int_{\mathbb{R}^k} \left|\gamma(t,x,\mu,z) - \gamma(t,x',\mu',z)\right| \nu(\mathrm{d}z)\\&\leq C(\left|x-x'\right| + \left|\left|\mu-\mu'\right|\right|_{\mathbb{M}})\end{aligned}$$

and

$$\mathbb{E}\int_0^T [|\alpha(t,0,\delta_0)|^2 + |\beta(t,0,\delta_0)|^2 + \int_{\mathbb{R}^k} |\gamma(t,0,\delta_0,z)|^2 \,\nu(\mathrm{d}z)]\mathrm{d}t \leq \infty,$$

where δ_0 is the Dirac measure with mass at zero.

Theorem 2.1 (Existence and uniqueness) *Under the above assumptions, the conditional McKean–Vlasov SDE* (2) *has a unique strong solution.*

Proof The proof is based on the Banach fixed point argument. Let $\mathcal{S}^2$ be the space of cadlag, $\mathcal{F}_t$-progressively measurable processes equipped with the norm

$$||X||^2 := \mathbb{E}[\sup_{t\in[0,T]} |X(t)|^2] < \infty.$$

This space equipped with this norm is a Banach space. Define the mapping $\Phi : \mathcal{S}^2 \to \mathcal{S}^2$ by $\Phi(x) = X$. We want prove that Φ is contracting in $\mathcal{S}^2$ under the norm defined above. For two arbitrary elements (x_1, x_2) and (X_1, X_2), we denote their difference by $\widetilde{x} = x_1 - x_2$ and $\widetilde{X} = X_1 - X_2$, respectively. In the following, $C < \infty$ will denote a constant which is big enough for all the inequalities to hold. Applying the Itô formula to $\widetilde{X}^2(t)$, we get

$$\begin{aligned}\widetilde{X}^2(t) = 2\int_0^t \widetilde{X}(s)(\alpha(s,x_1(s),\mu_1(s)) - \alpha(s,x_2(s),\mu_2(s)))\mathrm{d}s \\ + 2\int_0^t \widetilde{X}(s)(\beta(s,x_1(s),\mu_1(s)) - \beta(s,x_2(s),\mu_2(s)))\mathrm{d}B(s) \\ + 2\int_0^t \widetilde{X}(s)\int_{\mathbb{R}_k}(\gamma(s,x_1(s),\mu_1(s),z) - \gamma(s,x_2(s),\mu_2(s),z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ + \int_0^t (\beta(s,x_1(s),\mu_1(s)) - \beta(s,x_2(s),\mu_2(s)))^2\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}_k}(\gamma(s,x_1(s),\mu_1(s),z) - \gamma(s,x_2(s),\mu_2(s),z))^2\nu(\mathrm{d}z)\mathrm{d}s.\end{aligned}$$

By the Lipschitz assumption combined with standard majorization of the square of a sum (resp. integral) via the sum (resp. integral) of the square (up to a constant), we get

$$\begin{aligned}\widetilde{X}^2(t) \le C\textstyle\int_0^t|\widetilde{X}(s)|\Delta_s\mathrm{d}s \\ + |\textstyle\int_0^t\widetilde{X}(s)\widetilde{\beta}(s)\mathrm{d}B(s)| + |\int_0^t\int_{\mathbb{R}_k}\widetilde{X}(s)\widetilde{\gamma}(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)| + C\int_0^t\Delta_s^2\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}\Delta_s &:= |\widetilde{x}_s| + ||\widetilde{\mu}_s||_{\mathbb{M}},\\ \Delta_s^2 &:= |\widetilde{x}_s|^2 + ||\widetilde{\mu}_s||_{\mathbb{M}}^2,\\ \widetilde{\beta}(s) &= \beta(s,x_1(s),\mu_1(s)) - \beta(s,x_2(s),\mu_2(s)),\\ \widetilde{\gamma}(s,z) &= \gamma(s,x_1(s),\mu_1(s),z) - \gamma(s,x_2(s),\mu_2(s),z),\end{aligned}$$

and the value of the constant C is allowed to vary from line to line.
By the Burkholder–Davis–Gundy inequality, for all $t \in [0,T]$:

$$\mathbb{E}[\sup_{0\le t_0\le t}|\int_0^{t_0}\widetilde{X}(s)\widetilde{\beta}(s)\mathrm{d}B(s)|] \le C\mathbb{E}[(\int_0^t\widetilde{X}^2(s)\widetilde{\beta}^2(s)\mathrm{d}s)^{\frac{1}{2}}] \le CT\mathbb{E}[\int_0^t|\widetilde{X}_s|\Delta_s\mathrm{d}s],$$

and we need to use Kunita's inequality for the jumps (see Corollary 2.12 in Kunita [16]):

$$\begin{aligned}&\mathbb{E}[\sup_{0\le t_0\le t}|\int_0^{t_0}\widetilde{X}(s)\widetilde{\gamma}(s)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)|]\\ &\le C\mathbb{E}[(\int_0^t\widetilde{X}^2(s)\widetilde{\gamma}^2(s)\nu(\mathrm{d}z)\mathrm{d}s)^{\frac{1}{2}}]\\ &\le CT\mathbb{E}[\int_0^t|\widetilde{X}_s|\Delta_s\mathrm{d}s].\end{aligned}$$

Combining the above and using that

$$|\widetilde{X}_s|\Delta_s \leq C(|\widetilde{X}_s|^2 + \Delta_s^2),$$

we obtain

$$\mathbb{E}[\sup_{0\leq t_0\leq t} \widetilde{X}^2(t_0)] \leq CT\mathbb{E}[\int_0^t (|\widetilde{X}_s|^2 + \Delta_s^2)\mathrm{d}s].$$

By definition of the norms, we have

$$\Delta_t^2 \leq C|\widetilde{x}_t|^2.$$

Iterating the above inequality, we get, for any integer $k > 1$:

$$\mathbb{E}[\sup_{0\leq t\leq T} \widetilde{X}^2(t)] \leq (CT)^k \int_0^T \frac{(T-s)^{k-1}}{(k-1)!}\mathbb{E}[|\widetilde{x}_s|^2]\mathrm{d}s \leq (CT)^k \frac{T^k}{k!}||\widetilde{x}||^2.$$

Hence, for k large enough Φ is a contraction on $\mathcal{S}^2$. Therefore, the equation has a unique solution up to T for any $T < \infty$. □

3 Fokker–Planck SPDE with Jumps

In this part of the paper, we will formulate the associated Fokker–Planck SPDE with jumps for McKean–Vlasov SDE driven by jumps.

Equation (1) is not in itself Markovian, so to be able to use the Dynkin formula, we extend the system to the process Y defined by

$$Y(t) = (s+t, X(t), \mu_t); \quad t \geq 0; \quad Y(0) = (s, Z, \mu_0) =: y,$$

for some arbitrary starting time $s \geq 0$, with state dynamics given by $X(t)$, conditional law of the state given by μ_t and with $X(0) = Z$, $\mu_0 = \mathcal{L}(X(0))$. This system is Markovian, in virtue of the following Fokker–Planck equation for the conditional law μ_t.

For fixed t, μ, ζ and $\ell = 1, 2, ...k$, we write for simplicity $\gamma^{(\ell)} = \gamma^{(\ell)}(t, x, \mu, \zeta)$ for column number ℓ of the $d \times k$-matrix γ. Then ν_ℓ represents the Lévy measure of N_ℓ for all ℓ.

Define the operator $\mu^{(\gamma^{(\ell)})}$ on $C_0(\mathbb{R}^d)$ by

$$\langle \mu^{(\gamma^{(\ell)})}, g\rangle := \int_{\mathbb{R}^d} g(x)\mu^{(\gamma^{(\ell)})}(\mathrm{d}x) = \int_{\mathbb{R}^d} g(x+\gamma^{(\ell)})\mu(\mathrm{d}x), \text{ for all } g \in C_0(\mathbb{R}^d),$$

where $\langle \mu^{(\gamma^{(\ell)})}, g\rangle$ denotes the action of the measure $\mu^{(\gamma^{(\ell)})}$ on g. Then, we see that the Fourier transform of $\mu^{(\gamma^{(\ell)})}$ is

$$F[\mu^{(\gamma^{(\ell)})}](y) = \int_{\mathbb{R}^d} e^{-ixy}\mu^{(\gamma^{(\ell)})}(\mathrm{d}x) = \int_{\mathbb{R}^d} e^{-i(x+\gamma^{(\ell)})y}\mu(\mathrm{d}x) = e^{-i\gamma^{(\ell)}y}\widehat{\mu}(y).$$

Therefore, $\mu^{(\gamma^{(\ell)})} \in \mathbb{M}$ if μ is. We call $\mu^{(\gamma^{(\ell)})}$ the $\gamma^{(\ell)}$-shift of μ.
Assumption II The coefficients α, β and γ are C^1 with respect to x with bounded derivatives.
Let a test function $\psi \in C_b^2\left(\mathbb{R}^d\right)$, and with values in the complex plane $\mathbb{C}$.

Theorem 3.1 (Stochastic Fokker–Planck equation with jumps) *Let $X(t)$ be as in (1) and let $\mu_t = \mu_t^X$ be the regular conditional distribution of $X(t)$ given the sub-filtration $\mathcal{G}_t$ which is described by relation (3). Then, μ_t satisfies the following SPIDE (in the sense of distributions):*

$$d\mu_t = A_0^*\mu_t dt + A_1^*\mu_t dB_1(t) + \int_{\mathbb{R}^k} A_2^*\mu_t\widetilde{N}_1(dt, dz); \quad \mu_0 = \mathcal{L}(X(0)), \tag{5}$$

where A_0^* is the integro-differential operator

$$A_0^*\mu = -\sum_{j=1}^d D_j[\alpha_j\mu] + \frac{1}{2}\sum_{n,j=1}^d D_{n,j}[(\beta\beta^{(T)})_{n,j}\mu]$$
$$+\sum_{\ell=1}^k\int_{\mathbb{R}}\left\{\mu^{(\gamma^{(\ell)})} - \mu + \sum_{j=1}^d D_j[\gamma_j^{(\ell)}(s,\cdot,z)\mu]\right\}\nu_\ell(dz),$$

and

$$A_1^*\mu = -\sum_{j=1}^d D_j[\beta_{1,j}\mu], \quad A_2^*\mu = \mu^{(\gamma^{(1)})} - \mu,$$

where $\beta^{(T)}$ denotes the transposed of the $d\times m$ - matrix $\beta = \left[\beta_{j,k}\right]_{1\le j\le d, 1\le k\le m}$, $\gamma^{(\ell)}$ is column number ℓ of the matrix γ and $\nu_\ell(\cdot)$ is the Lévy measure of $N_\ell(\cdot,\cdot)$.

For notational simplicity, we use D_j, $D_{n,j}$ to denote $\frac{\partial}{\partial x_j}$ and $\frac{\partial^2}{\partial x_n \partial x_j}$ in the sense of distributions.

Proof We get by the Itô formula for jump diffusions (see, e.g. Theorem 1.16 in [20]):

$$\psi(X_t) - \psi(x) = \int_0^t A_0\psi(X_s)\,\mathrm{d}s + \sum_{n=1}^m\int_0^t A_{1,n}\psi(X_s)\mathrm{d}B_n(s)$$
$$+\sum_{\ell=1}^k\int_0^t\int_{\mathbb{R}^k} A_{2,\ell}\psi(X_s)\widetilde{N}_\ell(\mathrm{d}s, \mathrm{d}z),$$

where

$$
\begin{aligned}
A_0\psi(X_s) &= \sum_{j=1}^{d} \alpha_j(s, X_s, \mu_s) \frac{\partial \psi}{\partial x_j}(X_s) \\
&\quad + \frac{1}{2}\sum_{n,j=1}^{d} (\beta\beta^T)_{n,j}(s, X_s, \mu_s) \frac{\partial^2 \psi}{\partial x_n \partial x_j}(X_s) \\
&\quad + \sum_{\ell=1}^{k} \int_{\mathbb{R}} \Big\{ \psi\left(X_{s^-} + \gamma^{(\ell)}(s, X_{s^-}, \mu_{s^-}, z)\right) - \psi(X_s) \\
&\quad - \sum_{j=1}^{d} \frac{\partial \psi}{\partial x_j}(X_s)\, \gamma_j^{(\ell)}(s, X_{s^-}, \mu_{s^-}, z) \Big\} \nu_\ell(\mathrm{d}z),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{1,n}\psi(X_s) &= \sum_{j=1}^{d} \frac{\partial \psi}{\partial x_j}(X_s)\beta_{j,n}(s, X_s, \mu_s), \\
A_{2,\ell}\psi(X_s) &= \psi(X_{s^-} + \gamma^{(\ell)}(s, X_{s^-}, \mu_{s^-}, z)) - \psi(X_{s^-}),
\end{aligned}
$$

where β_j is the row number j of the $d \times m$ matrix β and $\gamma^{(\ell)}$ is column number ℓ of the $d \times k$ matrix γ. Since $(B_r, \widetilde{N}_r)_{r>1}$ are independent of $\mathcal{G}_t$, we get after conditioning

$$
\begin{aligned}
\mathbb{E}[\psi(X_t)\,|\mathcal{G}_t] - \psi(x) &= \mathbb{E}[\int_0^t A_0\psi(X_s)\,\mathrm{d}s \\
&+ \int_0^t A_{1,1}\psi(X_s)\mathrm{d}B_1(s) + \int_0^t \int_{\mathbb{R}^k} A_{2,1}\psi(X_s)\widetilde{N}_1(\mathrm{d}s, \mathrm{d}z)|\mathcal{G}_t] \\
&= \mathbb{E}[\int_0^t \mathbb{E}[A_0\psi(X_s)\,|\mathcal{G}_s]\mathrm{d}s + \int_0^t \mathbb{E}[A_{1,1}\psi(X_s)|\mathcal{G}_s]\mathrm{d}B_1(s) \\
&+ \int_0^t \int_{\mathbb{R}^k} \mathbb{E}[A_{2,1}\psi(X_s)|\mathcal{G}_s]\widetilde{N}_1(\mathrm{d}s, \mathrm{d}z)|\mathcal{G}_t] \\
&= \int_0^t \mathbb{E}[A_0\psi(X_s)\,|\mathcal{G}_s]\mathrm{d}s + \int_0^t \mathbb{E}[A_{1,1}\psi(X_s)|\mathcal{G}_s]\mathrm{d}B_1(s) \\
&+ \int_0^t \int_{\mathbb{R}^k} \mathbb{E}[A_{2,1}\psi(X_s)|\mathcal{G}_s]\widetilde{N}_1(\mathrm{d}s, \mathrm{d}z), \qquad (6)
\end{aligned}
$$

the last equality follows from the tower property of conditional expectation.

By the above, we see that

$$A_{1,1}\psi = \sum_{j=1}^{d} \frac{\partial \psi}{\partial x_j}(X_s)\beta_{j,1}(s, X_s, \mu_s),$$
$$A_{2,1}\psi = \Big\{\psi(X_{s^-} + \gamma^{(1)}(s, X_{s^-}, \mu_{s^-}, z)) - \psi\,(X_{s^-})\Big\}.$$

In particular, choosing, with $i = \sqrt{-1}$, $\psi\,(x) = \psi_y\,(x) = e^{-iyx}$; $\quad y, x \in \mathbb{R}^d$, we get

$$\begin{aligned} &A_0\psi\,(X_s)\\ &= \Bigg(-i\sum_{i=1}^{d} y_i\alpha_i\,(s, X_s, \mu_s) - \frac{1}{2}\sum_{n,j=1}^{d} y_n y_j(\beta\beta^{(T)})_{n.j}\,(s, X_s, \mu_s)\\ &+ \sum_{\ell=1}^{k}\int_{\mathbb{R}}\Big\{\exp\Big(-iy\gamma^{(\ell)}\,(s, X_{s^-}, \mu_{s^-}, z)\Big) - 1 + i\sum_{j=1}^{d} y_i\gamma_j^{(\ell)}\,(s, X_{s^-}, \mu_{s^-}, z)\Big\}\,\nu_\ell\,(\mathrm{d}z)\Bigg)\,e^{-iyX_s}. \end{aligned} \tag{7}$$

and

$$A_{1,1}\psi(X_s) = -i\sum_{j=1}^{\mathrm{d}} y_j\beta_{j,1}(s, X_s, \mu_s)e^{-iyX_s},$$
$$A_{2,1}\psi(X_s) = e^{-iy(X_{s^-}+\gamma^{(1)}(s,X_{s^-}),\mu_{s^-},z)} - e^{-iyX_{s^-}}.$$

In general, we have (see (3))

$$\mathbb{E}\Big[g\,(X_s)\,e^{-iyX_s}|\mathcal{G}_s\Big] = \int_{\mathbb{R}^d} g\,(x)\,e^{-iyx}\mu_s\,(\mathrm{d}x) = F\,[g\,(\cdot)\,\mu_s(\cdot)]\,(y)\,.$$

Therefore, we get

$$\begin{aligned} \mathbb{E}[e^{-iy\gamma(s,X_{s^-},\mu_{s^-},\zeta)}e^{-iyX_s}|\mathcal{G}_s] &= \int_{\mathbb{R}^d} e^{-iy\gamma(s,x,\mu_{s^-},\zeta)}e^{-ixy}\mu_s(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} e^{-iy(x+\gamma(s,x,\mu_{s^-},\zeta))}\mu_s(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} e^{-iyx}\mu_s^{(\gamma)}(\mathrm{d}x) = F[\mu_s^{(\gamma)}(\cdot)](y), \end{aligned} \tag{8}$$

where $\mu_s^{(\gamma)}(\cdot)$ is the γ-shift of μ_s. Recall that if $w \in S' = S'(\mathbb{R}^d)$ (the space of tempered distributions), using the notation $\frac{\partial}{\mathrm{d}x_j}w\,(t, x) =: D_j w\,(t, x)$, and similarly with higher order derivatives, we have, in the sense of distributions,

$$F\left[D_j w\,(t, \cdot)\right](y) = iy_j F\,[w\,(t, \cdot)]\,(y)\,.$$

Thus,

$$
\begin{aligned}
i y_j F[\alpha(s,\cdot)\mu_s](y) &= F[D_j(\alpha(s,\cdot)\mu_s)](y) \\
- y_n y_j F[\beta\beta^T(s,\cdot)\mu_s](y) &= F[D_{n,j}(\beta\beta^T(s,\cdot)\mu_s)](y).
\end{aligned}
$$

Applying this and (8) to (7), we get

$$
\begin{aligned}
&\mathbb{E}[A_0\psi(X_s)|\mathcal{G}_s] = \int_{\mathbb{R}^d} \Big(-i \sum_{j=1}^{d} y_j \alpha_j(s,x,\mu_s) - \frac{1}{2}\sum_{n,j=1}^{d} y_n y_j (\beta\beta^{(T)})_{n,j}(s,x,\mu_s) \\
&\quad + \sum_{\ell=1}^{k} \int_{\mathbb{R}} \Big\{ \exp\Big(-iy\gamma^{(\ell)}(s,x,\mu_{s^-},z)\Big) - 1 \\
&\quad + i \sum_{j=1}^{d} y_j \gamma_j^{(\ell)}(s,x,\mu_{s^-},z)\} \nu_\ell(\mathrm{d}z) \Big) e^{-iyx} \mu_s(\mathrm{d}x) = -i \sum_{j=1}^{d} y_j F[\alpha_j \mu_s](y) \\
&\quad - \frac{1}{2} \sum_{n,j=1}^{d} y_n y_j F[(\beta\beta^{(T)})_{n,j}\mu_s](y) + \sum_{\ell=1}^{k} F\Big[\int_{\mathbb{R}} \Big\{ \exp\Big(-iy\gamma^{(\ell)}(s,x,\mu_{s^-},z)\Big) \\
&\quad - 1 + i \sum_{j=1}^{d} y_j \gamma_j^{(\ell)}(s,x,\mu_{s^-},z) \Big\} \nu_\ell(\mathrm{d}z)\, \mu_s \Big](y) = F\Big[-\sum_{j=1}^{d} D_j(\alpha_j \mu_s) \\
&\quad + \frac{1}{2} \sum_{n,j=1}^{d} D_{n,j}((\beta\beta^{(T)})_{n,j}\mu_s) + \sum_{\ell=1}^{k} \int_{\mathbb{R}} \Big\{ \mu_s^{(\gamma^{(\ell)})} - \mu_s \\
&\quad + \sum_{j=1}^{d} D_j[\gamma_j^{(\ell)}(s,\cdot,z)\mu_s] \Big\} \nu_\ell(\mathrm{d}z) \Big](y) = F[A_0^*\mu_s](y),
\end{aligned}
$$

where A_0^* is the integro-differential operator

$$
\begin{aligned}
A_0^*\mu = &- \sum_{j=1}^{d} D_j[\alpha_j \mu] + \frac{1}{2}\sum_{n,j=1}^{d} D_{n,j}[(\beta\beta^{(T)})_{n,j}\mu] + \sum_{\ell=1}^{k} \int_{\mathbb{R}} \{\mu^{(\gamma^{(\ell)})} - \mu \\
&+ \sum_{j=1}^{d} D_j[\gamma_j^{(\ell)}(s,\cdot,\zeta)\mu]\} \nu_\ell(\mathrm{d}z)\,.
\end{aligned}
$$

Note that $A_0^*\mu_s$ exists in $\mathcal{S}'$.

Similarly, we get

$$\begin{aligned}\mathbb{E}[A_{1,1}\psi(X_s)|\mathcal{G}_s] &= \int_{\mathbb{R}^d} -i\sum_{j=1}^{d} y_j\beta_{j,1}(s,x,\mu_s)e^{-iyx}\mu_s(\mathrm{d}x)\\ &= F[-i\sum_{j=1}^{d} y_j\beta_{j,1}(s,x,\mu_s)\mu_s]\\ &= F[-\sum_{j=1}^{d} D_j(\beta_{j,1}\mu_s)](y) = F[A_1^*\mu_s](y),\end{aligned}$$

where A_1^* is the operator

$$A_1^*\mu_s = -\sum_{j=1}^{d} D_j[\beta_{j,1}\mu_s].$$

and

$$\begin{aligned}\mathbb{E}[A_{2,1}\psi(X_s)|\mathcal{G}_s] &= \mathbb{E}[e^{-iy(X_{s-}+\gamma^{(1)}(s,X_{s-},\mu_{s-},\zeta))} - e^{-iyX_{s-}}|\mathcal{G}_s]\\ &= \int_{\mathbb{R}^d}\Big(\exp(-iy\gamma^{(1)}(s,x,\mu_{s-},\zeta)) - 1\Big)e^{-iyx}\mu_s(\mathrm{d}x)\\ &= F[\mu_s^{(\gamma^{(1)})} - \mu_s](y) = F[A_2^*\mu_s](y),\end{aligned}$$

$\mathcal{S}$ where A_2^* is the operator given by

$$A_2^*\mu_s = \mu_s^{(\gamma^{(1)})} - \mu_s.$$

Substituting what we have obtained above into (6), leads

$$\begin{aligned}&\mathbb{E}\Big[\psi\left(X_t\right)|\mathcal{G}_t\Big]\\ &= \psi(x) + \int_0^t \mathbb{E}[A_0\psi\left(X_s\right)|\mathcal{G}_s]\mathrm{d}s + \int_0^t \mathbb{E}[A_{1,1}\psi(X_s)|\mathcal{G}_s]\mathrm{d}B_1(s)\\ &\quad + \int_0^t\int_{\mathbb{R}^k} \mathbb{E}[A_{2,1}\psi(X_s)|\mathcal{G}_s]\widetilde{N}_1(\mathrm{d}s,\mathrm{d}z)\\ &= \psi(x) + \int_0^t F[A_0^*\mu_s](y)\mathrm{d}s + \int_0^t F[A_1^*\mu_s](y)\mathrm{d}B_1(s)\\ &\quad + \int_0^t\int_{\mathbb{R}^k} F[A_2^*\mu_s](y)\widetilde{N}_1(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{9}$$

On the other hand,

$$\mathbb{E}[\psi\left(X_t\right)|\mathcal{G}_t] - \psi(x) = \mathbb{E}[e^{-iyX_t} - e^{-iyX_0}|\mathcal{G}_t] = \widehat{\mu}_t(y) - \widehat{\mu}_0(y). \tag{10}$$

Combining (9) and (10), we get

$$\widehat{\mu}_t(y) - \widehat{\mu}_0(y) = \int_0^t F[A_0^*\mu_s](y)\mathrm{d}s + \int_0^t F[A_1^*\mu_s](y)\mathrm{d}B_1(s) + \int_0^t \int_{\mathbb{R}^k} F[A_2^*\mu_s](y)\widetilde{N}_1(\mathrm{d}s, \mathrm{d}z).$$

Since the Fourier transform of a distribution determines the distribution uniquely, we deduce that

$$\mu_t - \mu_0 = \int_0^t A_0^*\mu_s \mathrm{d}s + \int_0^t A_1^*\mu_s \mathrm{d}B_1(s) + \int_0^t \int_{\mathbb{R}^k} A_2^*\mu_s \widetilde{N}_1(\mathrm{d}s, \mathrm{d}z),$$

or, in differential form,

$$\mathrm{d}\mu_t = A_0^*\mu_t \mathrm{d}t + A_1^*\mu_t \mathrm{d}B_1(t) + \int_{\mathbb{R}^k} A_2^*\mu_t \widetilde{N}_1(\mathrm{d}t, \mathrm{d}z); \quad \mu_0 = \mathcal{L}(X(0)),$$

as claimed. That completes the proof. □

Remark 3.1 If the common noise is coming from the Lévy noise, then the corresponding Fokker–Planck equation is an SPDE driven by this noise. Similarly, if the common noise is coming only from the Brownian motion, we get the Fokker–Planck SPDE as in Agram and Øksendal [1]. When the conditioning is with respect to the trivial filtration $\mathcal{F}_0$, meaning that there is no common noise available, we get the Fokker–Planck PDE studied, for example, in Bogachev et al. [6].

4 A General Formulation and a Verification Theorem

Since Eq. (1) is not Markovian, in the sense that it does not have the flow-property, we construct the following process Y:

$$\begin{aligned}\mathrm{d}Y(t) &= F(Y(t))\mathrm{d}t + G(Y(t))\mathrm{d}B(t) + \int_{\mathbb{R}^k} H(Y(t^-), z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \\ &:= \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}X(t) \\ \mathrm{d}\mu_t \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha(Y(t)) \\ A_0^*\mu_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0_{1\times m} \\ \beta(Y(t)) \\ A_1^*\mu_t, 0, 0..., 0 \end{bmatrix} \mathrm{d}B(t) \\ &+ \int_{\mathbb{R}^k} \begin{bmatrix} 0_{1\times k} \\ \gamma(Y(t^-), z) \\ A_2^*\mu_t, 0, 0, ..., 0 \end{bmatrix} \widetilde{N}(\mathrm{d}t, \mathrm{d}z), \quad s \le t \le T, \end{aligned} \tag{11}$$

where $X(t)$ and μ_t satisfy Eqs. (1) and (5), respectively. Moreover, we have used the shorthand notation

$$\alpha(Y(t)) = \alpha(s+t, X(t), \mu(t)),$$
$$\beta(Y(t)) = \beta(s+t, X(t), \mu(t)),$$
$$\gamma(Y(t^-), z) = \gamma(s+t, X(t^-), \mu(t^-), z).$$

The process $Y(t)$ starts at $y = (s, Z, \mu)$. We shall use the following notation:

Notation 4.1 *We use*

- *x to denote a generic value of the point $X(t,\omega) \in \mathbb{R}^d$, and*
- *X to denote a generic value of the random variable $X(t) \in L^2(P)$.*
- *When the meaning is clear from the context we use x in both situations.*
- *μ for either the initial probability distribution $\mathcal{L}(X(0))$ or the generic value of the conditional law $\mu_t := \mathcal{L}(X(t)|\mathcal{G}_t)$, when there is no ambiguity.*

The concept of impulse control is simple and intuitive: At any time, the agent can make an intervention ζ into the system. Due to the cost of each intervention, the agent can intervene only at discrete times $\tau_1, \tau_2, \ldots$. The impulse problem is to find out at what times it is optimal to intervene and what is the corresponding optimal intervention sizes. We now proceed to formulate precisely our impulse control problem for conditional McKean–Vlasov jump diffusions.

Suppose that—if there are no interventions—the process $Y(t) = (s+t, X(t), \mu_t)$ is the conditional McKean–Vlasov jump diffusion given by (11).

Suppose that at any time t and any state $y = (s, X, \mu)$ we are free to intervene and give the state X an impulse $\zeta \in \mathcal{Z} \subset \mathbb{R}^d$, where $\mathcal{Z}$ is a given set (the set of admissible impulse values). Suppose the result of giving the state X the impulse ζ is that the state jumps immediately from X to $\Gamma(X, \zeta)$, where $\Gamma(X, \zeta) : L^2(P) \times \mathcal{Z} \to L^2(P)$ is a given function. In many applications, the process shifts as a result of a simple translation, i.e. $\Gamma(y, \zeta) = y + \zeta$.
Simultaneously, the conditional law jumps from $\mu_t = \mathcal{L}(X(t)|\mathcal{G}_t)$ to

$$\mu_t^{\Gamma(X,\zeta)} := \mathcal{L}(\Gamma(X(t), \zeta)|\mathcal{G}_t). \tag{12}$$

An impulse control for this system is a double (possibly finite) sequence

$$v = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)_{j \le M}, \quad M \le \infty,$$

where $0 \le \tau_1 \le \tau_2 \le \cdots$ are $\mathcal{G}_t$-stopping times (the intervention times) and $\zeta_1, \zeta_2, \ldots$ are the corresponding impulses at these times. Mathematically, we assume that τ_j is a stopping time with respect to the filtration $\{\mathcal{G}_t\}_{t \ge 0}$, with $\tau_{j+1} > \tau_j$ and ζ_j is $\mathcal{G}_{\tau_j}$-measurable for all j. We let $\mathcal{V}$ denote the set of all impulse controls.

If $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$, the corresponding state process $Y^{(v)}(t)$ is defined by

$$Y^{(v)}(0^-) = y \quad \text{and} \quad Y^{(v)}(t) = Y(t); \quad 0 < t \leq \tau_1, \tag{13}$$

$$Y^{(v)}(\tau_j) = \Big(\tau_j, \Gamma[\check{X}^{(v)}(\tau_j^-), \zeta_j], \mathcal{L}(\Gamma[\check{X}^{(v)}(\tau_j^-), \zeta_j]|\mathcal{G}_t)\Big), \quad j = 1, 2, \ldots \tag{14}$$

$$\mathrm{d}Y^{(v)}(t) = F(Y^{(v)}(t))\mathrm{d}t + G(Y^{(v)}(t))\mathrm{d}B(t) + \int_{\mathbb{R}^k} H(Y^{(v)}(t^-), z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \quad \text{for } \tau_j < t < \tau_{j+1} \wedge \tau^*, \tag{15}$$

where we have used the notation

$$\check{X}^{(v)}(\tau_j^-) = X^{(v)}(\tau_j^-) + \Delta_N X(\tau_j),$$

$\Delta_N X^{(v)}(t)$ being the jump of $X^{(v)}$ stemming from the jump of the random measure $N(t, \cdot)$. Note that we distinguish between the (possible) jump of $X^{(v)}(\tau_j)$ stemming from the random measure N, denoted by $\Delta_N X^{(v)}(\tau_j)$ and the jump caused by the intervention v, given by

$$\Delta_v X^{(v)}(\tau_j) := \Gamma(\check{X}^{(v)}(\tau_j^-), \zeta) - \check{X}^{(v)}(\tau_j^-).$$

Accordingly, at the time $t = \tau_j$, $X^{(v)}(t)$ jumps from $\check{X}^{(v)}(\tau_j^-)$ to $\Gamma[\check{X}^{(v)}(\tau_j^-), \zeta_j]$ and $\mu_{\tau_j^-}$ jumps to

$$\mu_{\tau_j} = \mathcal{L}(\Gamma[\check{X}^{(v)}(\tau_j^-), \zeta_j]|\mathcal{G}_{\tau_j}).$$

Consider a fixed open set (called the solvency region) $\mathcal{S} \subset [0, \infty) \times \mathbb{R}^d \times \mathbb{M}$. It represents the set in which the game takes place since it will end once the controlled process leaves $\mathcal{S}$. In portfolio optimization problems, for instance, the game ends in case of bankruptcy, which may be modelled by choosing $\mathcal{S}$ to be the set of states where the capital is above a certain threshold. Define

$$\tau_{\mathcal{S}} = \inf\{t \in (0, \infty); Y^{(v)}(t) \notin \mathcal{S}\},$$

and

$$\mathcal{T} = \{\tau \text{ stopping time}, 0 \leq \tau \leq \tau_{\mathcal{S}}\}.$$

Suppose we are given a continuous *profit function* $f : \mathcal{S} \to \mathbb{R}$ and a continuous *bequest function* $g : \mathcal{S} \to \mathbb{R}$. Moreover, suppose the profit/utility of making an intervention with impulse $\zeta \in \mathcal{Z}$ when the state is y is $K(y, \zeta)$, where $K : \mathcal{S} \times \mathcal{Z} \to \mathbb{R}$ is a given continuous function.

We assume we are given a set $\mathcal{V}$ of *admissible impulse controls* which is included in the set of $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ such that a unique solution $Y^{(v)}$ of (13)–(15)

exist, for all $v \in \mathcal{V}$, and the following additional properties hold, assuring that the performance functional below is well-defined:

$$\mathbb{E}^y\Big[\int_0^{\tau_{\mathcal{S}}} f^-(Y^{(v)}(s))\mathrm{d}s\Big] < \infty, \quad \text{for all } y \in \mathcal{S},\ v \in \mathcal{V},$$
$$\mathbb{E}^y\left[g^-(Y^{(v)}(\tau_{\mathcal{S}}))\mathbb{1}_{[\tau_{\mathcal{S}}<\infty]}\right] < \infty, \quad \text{for all } y \in \mathcal{S},\ v \in \mathcal{V},$$

and

$$\mathbb{E}^y\left[\sum_{\tau_j \le \tau_{\mathcal{S}}} K^-(\check{Y}^{(v)}(\tau_j^-), \zeta_j)\right] < \infty, \quad \text{for all } y \in \mathcal{S}, v \in \mathcal{V},$$

where $\mathbb{E}^y$ denotes expectation, given that $Y(0) = y$, and we mean by $a^- = -min\{a, 0\}$ for $a = f, g, K$.
We now define the performance criterion, which consists of three parts: A continuous time running profit in $[0, \tau_{\mathcal{S}}]$, a terminal bequest value if the game ends, and a discrete-time intervention profit, namely

$$J^{(v)}(y) = \mathbb{E}^y\Big[\int_0^{\tau_{\mathcal{S}}} f(Y^{(v)}(t))\mathrm{d}t + g(Y^{(v)}(\tau_{\mathcal{S}}))\mathbb{1}_{[\tau_{\mathcal{S}}<\infty]} + \sum_{\tau_j \le \tau_{\mathcal{S}}} K(\check{Y}^{(v)}(\tau_j^-), \zeta_j)\Big].$$

We consider the following *impulse control problem*:

Problem 4.1 Find $\Phi(y)$ and $v^* \in \mathcal{V}$ such that

$$\Phi(y) = \sup\{J^{(v)}(y); v \in \mathcal{V}\} = J^{(v^*)}(y), \quad y \in \mathcal{S}.$$

The function $\Phi(y)$ is called the value function and v^* is called an optimal control.

The following concept is crucial for the solution to this problem.

Definition 4.1 Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$. The *intervention operator* $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{M}h(s, X, \mu) = \sup_{\zeta \in \mathcal{Z}}\{h(s, \Gamma(X, \zeta), \mu^{\Gamma(X,\zeta)}) + K(y, \zeta); (s, \Gamma(X, \zeta), \mu^{\Gamma(X,\zeta)}) \in \mathcal{S}\}, \tag{16}$$

where $\mu^{\Gamma(X,\zeta)}$ is given by (12).

Let $C^{(1,2,2)}(\mathcal{S})$ denote the family of functions $\varphi(s, x, \mu) : \mathcal{S} \to \mathbb{R}$ which are continuously differentiable w.r.t. s and twice continuously Fréchet differentiable with respect to $x \in \mathbb{R}^d$ and $\mu \in \mathbb{M}$. We let $\nabla_\mu \varphi \in \mathbb{L}(\mathbb{M}, \mathbb{R})$ (the set of bounded linear functionals on $\mathbb{M}$) denote the Fréchet derivative (gradient) of φ with respect to $\mu \in \mathbb{M}$. Similarly, $D^2_\mu \varphi$ denotes the double derivative of φ with respect to μ and it belongs to $\mathbb{L}(\mathbb{M} \times \mathbb{M}, \mathbb{R})$ (see Appendix for further details).

The infinitesimal generator L of the Markov jump-diffusion process $Y(t)$ is defined on functions $\varphi \in C^{(1,2,2)}(\mathcal{S})$ by

$$
\begin{aligned}
L\varphi &= \frac{\partial \varphi}{\partial s} + \sum_{j=1}^{d} \alpha_j \frac{\partial \varphi}{\partial x_j} + \langle \nabla_\mu \varphi, A_0^* \mu \rangle + \tfrac{1}{2} \sum_{j,n=1}^{d} (\beta\beta^T)_{j,n} \frac{\partial^2 \varphi}{\partial x_j \partial x_n} \\
&\quad + \tfrac{1}{2} \sum_{j=1}^{d} \beta_{j,1} \frac{\partial}{\partial x_j} \langle \nabla_\mu \varphi, A_1^* \mu \rangle + \tfrac{1}{2} \langle A_1^* \mu, \langle D_\mu^2 \varphi, A_1^* \mu \rangle\rangle \\
&\quad + \int_{\mathbb{R}^k} \Big\{ \varphi(s, x + \gamma^{(1)}, \mu + A_2^* \mu) - \varphi(s, x, \mu) \\
&\quad - \sum_{j=1}^{d} \gamma_j^{(1)} \tfrac{\partial}{\partial x_j} \varphi(s, x, \mu) - \langle A_2^* \mu, D_\mu \varphi \rangle \Big\} \nu_1(\mathrm{d}z) \\
&\quad + \sum_{\ell=2}^{k} \int_{\mathbb{R}^k} \Big\{ \varphi(s, x + \gamma^{(\ell)}, \mu)) - \varphi(s, x, \mu) - \sum_{j=1}^{d} \gamma_j^{(\ell)} \tfrac{\partial}{\partial x_j} \varphi(s, x, \mu) \Big\} \nu_\ell(\mathrm{d}z),
\end{aligned}
$$

where, as before, A_0^* is the integro-differential operator

$$
\begin{aligned}
A_0^* \mu = &- \sum_{j=1}^{d} D_j[\alpha_j \mu] + \frac{1}{2} \sum_{n,j=1}^{d} D_{n,j}[(\beta\beta^{(T)})_{n,j} \mu] \\
&+ \sum_{\ell=1}^{k} \int_{\mathbb{R}^k} \Big\{ \mu^{(\gamma^{(\ell)})} - \mu + \sum_{j=1}^{d} D_j[\gamma_j^{(\ell)}(s, \cdot, z)\mu] \Big\} \nu_\ell\,(\mathrm{d}z)\,,
\end{aligned}
$$

and

$$A_1^* \mu = - \sum_{j=1}^{d} D_j[\beta_{1,j} \mu].$$

and

$$A_2^* \mu = \mu^{(\gamma^{(1)})} - \mu.$$

We can now state a verification theorem for conditional McKean–Vlasov impulse control problems, providing sufficient conditions that a given function is the value function and a given impulse control is optimal. The verification theorem links the impulse control problem to a suitable system of quasi-variational inequalities.
Since the process $Y(t)$ is Markovian, we can, with appropriate modifications, use the approach in Chapter 9 in [20].
For simplicity of notation, we will in the following write

$$\overline{\Gamma}(y, \zeta) = (s, \Gamma(x, \zeta), \mu^{\Gamma(x,\zeta)}), \text{ when } y = (s, x, \mu) \in [0, \infty) \times L^2(P) \times \mathbb{M}.$$

Theorem 4.2 *Variational inequalities for conditional McKean–Vlasov impulse control*

(a) *Suppose we can find* $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ *such that*

(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$.

(ii) $\phi \geq \mathcal{M}\phi$ *on* $\mathcal{S}$. *Define*

$$D = \{y \in \mathcal{S}; \phi(y) > \mathcal{M}\phi(y)\} \quad \textit{(the continuation region).}$$

Assume

(iii) $\mathbb{E}^y\left[\int_0^{\tau_\mathcal{S}} Y^{(v)}(t)\mathbb{1}_{\partial D}\mathrm{d}t\right] = 0$ *for all* $y \in \mathcal{S},\ v \in \mathcal{V}$, *i.e. the amount of time* $Y(t)$ *spends on* ∂D *has Lebesgue measure zero.*

(iv) ∂D *is a Lipschitz surface.*

(v) $\phi \in C^{(1,2,2)}(\mathcal{S} \setminus \partial D)$ *with locally bounded derivatives near* ∂D. *((iv)-(v) are needed for the approximation argument in the proof).*

(vi) $L\phi + f \leq 0$ *on* $\mathcal{S} \setminus \partial D$.

(vii) $\phi(y) = g(y)$ *for all* $y \notin \mathcal{S}$.

(viii) $\{\phi^-(Y^{(v)}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable, for all* $y \in \mathcal{S}$, $v \in \mathcal{V}$.

(ix) $\mathbb{E}^y\left[|\phi(Y^{(v)}(\tau))| + \int_0^{\tau_\mathcal{S}} |L\phi(Y^{(v)}(t))|\mathrm{d}t\right] < \infty$ *for all* $\tau \in \mathcal{T}$, $v \in \mathcal{V}$, $y \in \mathcal{S}$.

Then,

$$\phi(y) \geq \Phi(y) \quad \textit{for all } y \in \mathcal{S}.$$

(b) *Suppose in addition that*

(x) $L\phi + f = 0$ *in* D.

(xi) $\hat{\zeta}(y) \in \mathrm{Argmax}\{\phi(\overline{\Gamma}(y, \cdot)) + K(y, \cdot)\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\zeta}(\cdot)$ *is a Borel measurable selection.*
Put $\hat{\tau}_0 = 0$ *and define* $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)$ *inductively by* $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j; Y^{(\hat{v}_j)}(t) \notin D\} \wedge \tau_\mathcal{S}$ *and* $\hat{\zeta}_{j+1} = \hat{\zeta}(Y^{(\hat{v}_j)}(\hat{\tau}_{j+1}^-))$ *if* $\hat{\tau}_{j+1} < \tau_\mathcal{S}$, *where* $Y^{(\hat{v}_j)}$ *is the result of applying* $\hat{v}_j := (\hat{\tau}_1, \ldots, \hat{\tau}_j; \hat{\zeta}_1, \ldots, \hat{\zeta}_j)$ *to* Y.
Suppose

(xii) $\hat{\tau}_{j+1} > \hat{\tau}_j$ *for all* j, $\hat{v} \in \mathcal{V}$ *and* $\{\phi(Y^{(\hat{v})}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable.*

Then,

$$\phi(y) = \Phi(y) \quad \textit{and } \hat{v} \textit{ is an optimal impulse control .}$$

Remark 4.1 We give the intuitive idea behind intervention operator as in (16):

$$\mathcal{M}\Phi(y) = \sup_{\zeta \in \mathcal{Z}}\{\Phi(\overline{\Gamma}(y, \zeta)) + K(y, \zeta),\ \zeta \in \mathcal{Z} \text{ and } \overline{\Gamma}(y, \zeta) \in \mathcal{S}\}.$$

Assume that the value function Φ is known. If $y = (s, x, \mu)$ is the current state of the process, and the agent intervenes with impulse of size ζ, the resulting value can

be represented as $\Phi(\overline{\Gamma}(y,\zeta)) + K(y,\zeta)$, consisting of the sum of the value of Φ in the new state $\overline{\Gamma}(y,\zeta)$ and the intervention profit K. Therefore, $\mathcal{M}\Phi(y)$ represents the optimal new value if the agent decides to make an intervention at y.
Note that by (ii), $\Phi \geq \mathcal{M}\Phi$ on $\mathcal{S}$, so it is always not optimal to intervene. At the time $\hat{\tau}_j$, the operator should intervene with impulse $\hat{\zeta}_j$ when the controlled process leaves the continuation region, that is when $\Phi(Y^{\hat{v}_j}) \leq \mathcal{M}\Phi(Y^{\hat{v}_j})$.

Proof (a) By an approximation argument (see, e.g. Theorem 3.1 in [20]) and (iii)–(v), we may assume that $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$.
Choose $v = (\tau_1, \tau_2, \dots; \zeta_1, \zeta_2, \dots) \in \mathcal{V}$ and set $\tau_0 = 0$. By approximating the stopping times τ_j by stopping times with finite expectation, we may assume that we can apply the Dynkin formula to the stopping times τ_j. Then for $j = 0, 1, 2, \dots$, with $Y = Y^{(v)}$

$$\mathbb{E}^y[\phi(Y(\tau_j))] - \mathbb{E}^y[\phi(\check{Y}(\tau_{j+1}^-))] = -\mathbb{E}^y\left[\int_{\tau_j}^{\tau_{j+1}} L\phi(Y(t))\mathrm{d}t\right],$$

where $\check{Y}(\tau_{j+1}^-) = Y(\tau_{j+1}^-) + \Delta_N Y(\tau_{j+1})$, as before. Summing this from $j = 0$ to $j = m$, we get

$$\begin{aligned}\phi(y) &+ \sum_{j=1}^{m} \mathbb{E}^y[\phi(Y(\tau_j)) - \phi(\check{Y}(\tau_j^-))] - \mathbb{E}^y[\phi(\check{Y}(\tau_{m+1}^-))] \\ &= -\mathbb{E}^y\left[\int_0^{\tau_{m+1}} L\phi(Y(t))\mathrm{d}t\right] \geq \mathbb{E}^y\left[\int_0^{\tau_{m+1}} f(Y(t))\mathrm{d}t\right].\end{aligned} \tag{17}$$

Now

$$\begin{aligned}\phi(Y(\tau_j)) &= \phi(\Gamma(\check{Y}(\tau_j^-), \zeta_j)) \\ &\leq \mathcal{M}\phi(\check{Y}(\tau_j^-)) - K(\check{Y}(\tau_j^-), \zeta_j); \quad \text{if } \tau_j < \tau_{\mathcal{S}} \text{ by (16)},\end{aligned}$$

and

$$\phi(Y(\tau_j)) = \phi(\check{Y}(\tau_j^-)); \quad \text{if } \tau_j = \tau_{\mathcal{S}} \text{ by } (vii).$$

Therefore,

$$\mathcal{M}\phi(\check{Y}(\tau_j^-)) - \phi(\check{Y}(\tau_j^-)) \geq \phi(Y(\tau_j)) - \phi(\check{Y}(\tau_j^-)) + K(\check{Y}(\tau_j^-), \zeta_j),$$

and

$$\begin{aligned}\phi(y) &+ \sum_{j=1}^{m} \mathbb{E}^y[\{\mathcal{M}\phi(\check{Y}(\tau_j^-)) - \phi(\check{Y}(\tau_j^-))\}\mathbb{1}_{[\tau_j < \tau_{\mathcal{S}}]}] \\ &\geq \mathbb{E}^y\left[\int_0^{\tau_{m+1}} f(Y(t))\mathrm{d}t + \phi(\check{Y}(\tau_{m+1}^-)) + \sum_{j=1}^{m} K(\check{Y}(\tau_j^-), \zeta_j)\right].\end{aligned}$$

Letting $m \to M$ and using quasi-left continuity, i.e. left continuity along increasing sequences of stopping times, of $Y(\cdot)$, we get

$$\phi(y) \geq \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\mathbb{1}_{[\tau_{\mathcal{S}}<\infty]} + \sum_{j=1}^{M} K(\check{Y}(\tau_j^-), \zeta_j)\right] = J^{(v)}(y). \tag{18}$$

Hence, $\phi(y) \geq \Phi(y)$.
(b) Next assume (x)–(xii) also hold.
Apply the above argument to $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)$. Then by (x), we get *equality* in (17) and by our choice of $\zeta_j = \hat{\zeta}_j$, we have *equality* in (18). Hence

$$\phi(y) = J^{(\hat{v})}(y),$$

which combined with (a) completes the proof. □

5 Example: Optimal Stream of Dividends Under Transaction Costs

One of the motivations for studying conditional McKean–Vlasov equations is that they represent natural models for stochastic systems where there is an underlying common noise, observable by all. This common noise could come from various types of uncertainty, e.g. uncertainty in the information available or unpredictable mechanical disturbances, e.g. market volatility, interest rate fluctuations, etc. Conditional McKean–Vlasov equations may therefore be regarded as a special class of systems subject to noisy observations, and they represent an alternative to filtering theory approaches.
In particular, such equations may be relevant the modelling of systems with noise in economics and finance.

Consider a company that wants to find the best way to distribute dividends to its shareholders while taking into account transaction costs. In this section, we illustrate our general results above by solving explicitly a financial problem, consisting of optimizing the overall anticipated dividend sum while considering the impact of transaction costs.
To this end, for $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ with $\zeta_i \in \mathbb{R}_+$, we define

$$Y^{(v)}(t) = (s + t, X^{(v)}(t), \mu_t^{(v)})$$

where $X(t) = X^{(v)}(t)$ denotes the value at time t of a company. We assume that between the intervention (dividend payout) times τ_i, τ_{i+1} the growth of the value is proportional to the conditional current value $\mathbb{E}[X(t)|\mathcal{G}_t]$ given the common information $\mathcal{G}_t$, and satisfies the following conditional McKean–Vlasov equation:

$$\begin{aligned} &\mathrm{d}X(t) = \mathbb{E}[X(t) \mid \mathcal{G}_t]\Big(\alpha_0 \mathrm{d}t + \sigma_1 \mathrm{d}B_1(t) \\ &\quad + \sigma_2 \mathrm{d}B_2(t) + \int_{\mathbb{R}} \kappa_1(z)\widetilde{N}_1(\mathrm{d}t, \mathrm{d}z) + \int_{\mathbb{R}} \kappa_2(z)\widetilde{N}_2(\mathrm{d}t, \mathrm{d}z)\Big), \\ &\mu_t^{(v)} = \mathcal{L}(X^{(v)}(t)|\mathcal{G}_t); \quad \tau_i < t < \tau_{i+1}, \\ &X^{(v)}(\tau_{i+1}) = \check{X}^{(v)}(\tau_{i+1}^-) - (1+\lambda)\zeta_{i+1} - c, \\ &\mu_{\tau_{i+1}}^{(v)} = \mathcal{L}(X^{(v)}(\tau_{i+1})|\mathcal{G}_{\tau_{i+1}}); \quad i = 0, 1, 2, \ldots, \\ &X^{(v)}(0^-) = x > 0; \text{ a.s.} \end{aligned}$$

Here $\alpha_0, \sigma_1 \neq 0, \sigma_2 \neq 0$ are given constants, and we assume that the functions κ_i satisfy $-1 < \kappa_i(z)$ a.s. $\nu_i(\mathrm{d}z)$ and $\int_{\mathbb{R}} |\kappa_i(z)|^2 \nu_i(\mathrm{d}z) < \infty; \quad i = 1, 2$.
Note that at any time $\tau_i,\ i = 0, 1, 2, \ldots$, the system jumps from $\check{X}^{(v)}(\tau_i^-)$ to

$$X^{(v)}(\tau_i) = \Gamma[\check{X}^{(v)}(\tau_i^-), \zeta_i] = \check{X}^{(v)}(\tau_i^-) - (1+\lambda)\zeta_i - c,$$

where $\lambda \geq 0$, and $c > 0$ and the quantity $c + \lambda\zeta_i$ represents the transaction cost with a *fixed* part c and a *proportional* part $\lambda\zeta_i$, while ζ_i is the amount we decide to take out at time τ_i.
At the same time $\mu_{\tau_i^-}$ jumps to

$$\mu_{\tau_i} = \mathcal{L}(\check{X}^{(v)}(\tau_i^-)|\mathcal{G}_{\tau_i}).$$

Problem 5.1 We want to find Φ and $v^* \in \mathcal{V}$, such that

$$\Phi(s, x, \mu) = \sup_v J^{(v)}(s, x, \mu) = J^{(v^*)}(s, x, \mu),$$

where

$$J^{(v)}(s, x, \mu) = J^{(v)}(y) = \mathbb{E}^y\left[\sum_{\tau_k < \tau_{\mathcal{S}}} \mathrm{e}^{-\rho(s+\tau_k)}\zeta_k\right] \qquad (\rho > 0 \text{ constant})$$

is the expected discounted total dividend up to time $\tau_{\mathcal{S}}$, where

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}(\omega) = \inf\{t > 0;\ P^y[\mathbb{E}^y[X^{(v)}(t)|\mathcal{G}_t] \leq 0] > 0\}$$

is the time of bankruptcy.

To put this problem into the context above, we define

$$
\begin{aligned}
&Y^{(v)}(t) = \begin{bmatrix} s+t \\ X^{(v)}(t) \\ \mu_t^{(v)} \end{bmatrix}, \quad Y^{(v)}(0^-) = \begin{bmatrix} s \\ x \\ \mu \end{bmatrix} = y,\\
&\Gamma(y,\zeta) = \Gamma(s,x,\mu) = (s, x-c-(1+\lambda)\zeta, \mathcal{L}(x-c-(1+\lambda)\zeta)|\mathcal{G}), \quad x \in L^2(P),\\
&K(y,\zeta) = \mathrm{e}^{-\rho s}\zeta,\\
&f \equiv g \equiv 0,\\
&\mathcal{S} = \left\{ y = (s,x,\mu); \mathbb{E}^y[X^{(v)}(s)|\mathcal{G}_s] > 0 \text{ a.s.} \right\}.
\end{aligned}
$$

Comparing with our theorem, we see that in this case we have $d = 1, m = 2, k = 1$ and

$$
\begin{aligned}
&\alpha_1 = \alpha_0 \langle \mu, q \rangle, \beta_1 = \sigma_1 \langle \mu, q \rangle,\\
&\beta_2 = \sigma_2 \langle \mu, q \rangle, \gamma_1(s,x,\mu,z) = \kappa_1(z)\langle \mu, q\rangle, \gamma_2(s,x,\mu,z)\\
&\quad = \kappa_2(z)\langle \mu, q \rangle,
\end{aligned}
$$

where we have put $q(x) = x$ so that $\langle \mu_t, q \rangle = \mathbb{E}[X(t) \mid \mathcal{G}_t]$.
Therefore, the operator L takes the form

$$
\begin{aligned}
L\varphi(s,x,\mu) &= \frac{\partial \varphi}{\partial s} + \alpha_0 \langle \mu, q \rangle \frac{\partial \varphi}{\partial x} + \left\langle \nabla_\mu \varphi, A_0^* \mu \right\rangle\\
&+ \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2) \langle \mu, q \rangle^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{2}\sigma_1 \langle \mu, q \rangle \frac{\partial}{\partial x} \left\langle \nabla_\mu \varphi, A_1^* \mu \right\rangle\\
&+ \tfrac{1}{2} \left\langle A_1^*\mu, \left\langle D_\mu^2 \varphi, A_1^* \mu \right\rangle \right\rangle\\
&+ \int_{\mathbb{R}} \{ \varphi(s, x + \kappa_1(z)\langle \mu, q\rangle, \mu + A_2^*\mu) - \varphi(s,x,\mu)\\
&- \kappa_1(z) \langle \mu, q \rangle \frac{\partial}{\partial x} \varphi(s,x,\mu)\\
&- \langle A_2^*\mu, D_\mu \varphi \rangle \} \nu_1(\mathrm{d}z) + \int_{\mathbb{R}} \Big\{ \varphi(s, x + \kappa_2(z)\langle \mu, q \rangle, \mu) - \varphi(s,x,\mu)\\
&- \kappa_2(z)\langle \mu, q\rangle \tfrac{\partial}{\partial x}\varphi(s,x,\mu) \Big\} \nu_2(\mathrm{d}z),
\end{aligned}
$$

where

$$
\begin{aligned}
A_0^*\mu &= -D[\alpha_0 \langle \mu, q \rangle \mu] + \tfrac{1}{2} D^2[(\sigma_1^2 + \sigma_2^2) \langle \mu, q \rangle^2 \mu],\\
A_1^*\mu &= -D[\sigma_1 \langle \mu, q \rangle \mu]
\end{aligned}
$$

and

$$
A_2^*\mu = \mu^{(\kappa_1 \langle \mu, q \rangle)} - \mu.
$$

The adjoints of the first two operators are

$$A_0\mu = \alpha_0 \langle \mu, q \rangle D\mu + \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2) \langle \mu, q \rangle^2 D^2\mu,$$

and

$$A_1\mu = \sigma_1 \langle \mu, q \rangle D\mu.$$

In this case, the intervention operator becomes

$$\mathcal{M}h(s, x, \mu) = \sup_{\zeta} \left\{ h(s, x - c - (1+\lambda)\zeta, \mu^{x-c-(1+\lambda)\zeta}) + e^{-\rho t}\zeta; \quad 0 \le \zeta \le \frac{x-c}{1+\lambda} \right\}.$$

Note that the condition on ζ is due to the fact that the impulse must be positive and $x - c - (1+\lambda)\zeta$ must belong to $\mathcal{S}$. We distinguish between two cases:
Case 1. $\alpha_0 > \rho$.
In this case, suppose we wait until some time t_1 and then take out

$$\zeta_1 = \frac{X(t_1) - c}{1+\lambda}.$$

Noting that $\mathbb{E}^y[X(t)] = x \exp(\alpha_0 t)$ for $t < t_1$, we see that the corresponding performance is:

$$\begin{aligned} J^{(v_1)}(s, x, \mu) &= \mathbb{E}^y \left[\frac{\mathrm{e}^{-\rho(t_1+s)}}{1+\lambda} (X(t_1) - c) \right] \\ &= \mathbb{E}^x \left[\frac{1}{1+\lambda} \left(x\mathrm{e}^{-\rho s} \mathrm{e}^{(\alpha_0-\rho)t_1} - c\, \mathrm{e}^{-\rho(s+t_1)} \right) \right] \\ &\to \infty \text{ as } t_1 \to \infty. \end{aligned}$$

Therefore, we obtain $\Phi(s, x, \mu) = +\infty$ in this case.
Case 2. $\alpha_0 < \rho$.
We look for a solution by using the results of Theorem 4.2.
We guess that the continuation region is of the form

$$D = \{(s, x, \mu) : 0 < \langle \mu, q \rangle < \bar{x}\}$$

for some $\bar{x} > 0$ (to be determined), and in D we try a value function of the form

$$\varphi(s, x, \mu) = e^{-\rho s} \psi(\langle \mu, q \rangle).$$

This gives

$L\varphi(s,x,\mu) = e^{-\rho s} L_0\psi(\langle\mu,q\rangle)$, where

$$L_0\psi(\langle\mu,q\rangle) = -\rho\psi(\langle\mu,q\rangle) + \left\langle\nabla_\mu\psi, A_0^*\mu\right\rangle + \frac{1}{2}\sigma_1\langle\mu,q\rangle\frac{\partial}{\partial x}\left\langle\nabla_\mu\psi, A_1^*\mu\right\rangle$$
$$+ \frac{1}{2}\left\langle A_1^*\mu, \left\langle D_\mu^2\psi, A_1^*\mu\right\rangle\right\rangle$$
$$+ \int_{\mathbb{R}}\left\{\psi(\langle\mu^{(\kappa_1\langle\mu,q\rangle)},q\rangle) - \psi(\langle\mu,q\rangle) - \langle\mu^{(\kappa_1\langle\mu,q\rangle)} - \mu, D_\mu\psi\rangle\right\}\nu_1(d\zeta).$$

By the chain rule for Fréchet derivatives (see Appendix), we have

$$\nabla_\mu\psi(h) = \psi'(\langle\mu,q\rangle)\langle h,q\rangle \quad \text{and} \quad D_\mu^2\psi(h,k) = \psi''(\langle\mu,q\rangle)\langle h,q\rangle\langle k,q\rangle.$$

Therefore,

$$\langle\nabla_\mu\psi, A_0^*\mu\rangle = \psi'(\langle\mu,q\rangle)\langle A_0^*\mu,q\rangle = \psi'(\langle\mu,q\rangle)\langle\mu,A_0q\rangle = \psi'(\langle\mu,q\rangle)\alpha_0\langle\mu,q\rangle,$$

and similarly

$$\tfrac{1}{2}\langle A_1^*\mu, \langle D_\mu^2\psi, A_1^*\mu\rangle\rangle = \tfrac{1}{2}\psi''(\langle\mu,q\rangle\langle A_1^*\mu,q\rangle\langle A_1^*\mu,q\rangle = \tfrac{1}{2}\psi''(\langle\mu,q\rangle)\langle\mu,A_1q\rangle\langle\mu,A_1q\rangle$$
$$= \tfrac{1}{2}\psi''(\langle\mu,q\rangle)\sigma_1^2\langle\mu,q\rangle^2.$$

Moreover, since ψ does not depend on x we see that

$$\int_{\mathbb{R}}\left\{\varphi(s,x+\gamma_0\langle\mu,q\rangle,\mu) - \varphi(s,x,\mu) - \gamma_0\langle\mu,q\rangle\frac{\partial\varphi}{\partial x}(s,x,\mu)\right\}\nu_1(\mathrm{d}z) = 0.$$

Substituting this into the expression for $L_0\psi$ we get, with $u = \langle\mu,q\rangle$,

$$L_0\psi(u) = -\rho\psi(u) + \alpha_0 u\psi'(u) + \tfrac{1}{2}\sigma_1^2u^2\psi''(u)$$
$$+ \int_{\mathbb{R}}\left\{\psi(\langle\mu^{(\kappa_1\langle\mu,q\rangle)},q\rangle) - \psi(\langle\mu,q\rangle) - \psi'(u)\langle\mu^{(\kappa_1\langle\mu,q\rangle)} - \mu, q\rangle\right\}\nu_1(\mathrm{d}z)$$
$$= -\rho\psi(u) + \alpha_0u\psi'(u) + \tfrac{1}{2}\sigma_1^2u^2\psi''(u)$$
$$+ \int_{\mathbb{R}}\left\{\psi((1+\kappa_1)u) - \psi(u) - \kappa_1u\psi'(u)\right\}\nu_1(\mathrm{d}z), \tag{19}$$

where we have used that

$$\langle\mu^{(\kappa_1\langle\mu,q\rangle)},q\rangle = \int_{\mathbb{R}} q(x+\kappa_1\langle\mu,q\rangle)\mu(\mathrm{d}x)$$
$$= \int_{\mathbb{R}} x\mu(\mathrm{d}x) + \kappa_1\langle\mu,q\rangle = \langle\mu,q\rangle + \kappa_1\langle\mu,q\rangle = (1+\kappa_1)\langle\mu,q\rangle,$$

so that

$$\langle\mu^{(\kappa_1\langle\mu,q\rangle)} - \mu, q\rangle = \kappa_1\langle\mu,q\rangle.$$

By condition (x), we are required to have $L_0\psi(u) = 0$ for all $u \in (0, \bar{x})$. Note that by (19) we get that if we try the function $\psi(u) = u^a$ for some $a \in \mathbb{R}$, then

$$L_0\psi(u) = \Big[-\rho + a\alpha_0 + \tfrac{1}{2}a(a-1)\sigma_1^2 + \int_{\mathbb{R}} \{(1+\kappa_1(z))^a - 1 - a\kappa_1(z)\}\nu_1(\mathrm{d}z)\Big]u^a.$$

Therefore, if we choose $a = \hat{a}$ such that

$$-\rho + \hat{a}\alpha_0 + \tfrac{1}{2}\hat{a}(\hat{a}-1)\sigma_1^2 + \int_{\mathbb{R}} \{(1+\kappa_1(z))^{\hat{a}} - 1 - \hat{a}\kappa_1(z)\}\nu_1(\mathrm{d}z) = 0 \tag{20}$$

then for all constants C the function

$$\psi(u) = Cu^{\hat{a}} \tag{21}$$

is a solution of the equation

$$L_0\psi(u) = 0.$$

Define

$$F(a) = -\rho + a\alpha_0 + \tfrac{1}{2}a(a-1)\sigma_1^2 + \int_{\mathbb{R}} \{(1+\kappa_1(z))^a - 1 - a\kappa_1(z)\}\nu_1(\mathrm{d}z); \quad a \in \mathbb{R}.$$

Then we see that $F(1) = \alpha_0 - \rho < 0$, $F'(a) > 0$ for $a \geq 1$ and $F(a) \to \infty$ when $a \to \infty$. Therefore, there exists a unique $a = \hat{a} > 1$ such that $F(\hat{a}) = 0$. Since we expect ψ to be bounded near 0, we choose this exponent $\hat{a}$ in the Definition (21) of ψ. It remains to determine C.

We guess that it is optimal to wait till $u = \langle \mu_t, q\rangle = \mathbb{E}^y[X(t)|\mathcal{G}_t]$ reaches or exceeds a value $u = \bar{u} > c$ and then take out as much as possible, i.e. reduce $\mathbb{E}^y[X(t)|\mathcal{G}_t]$ to 0. Taking the transaction costs into account, this means that we should take out

$$\hat{\zeta}(u) = \frac{u-c}{1+\lambda} \text{ for } u \geq \bar{u}.$$

We therefore propose that $\psi(u)$ has the form

$$\psi(u) = \begin{cases} Cu^{\hat{a}} \text{for } 0 < u < \bar{u}, \\ \dfrac{u-c}{1+\lambda} \text{ for } u \geq \bar{u}. \end{cases} \tag{22}$$

Continuity and differentiability of $\psi(u)$ at $u = \bar{u}$ give the equations

$$C\bar{u}^{\hat{a}} = \frac{\bar{u}-c}{1+\lambda} \quad \text{and} \quad C\hat{a}\bar{u}^{\hat{a}-1} = \frac{1}{1+\lambda}.$$

Combining these, we get

$$\bar{u} = \frac{\hat{a}c}{\hat{a}-1} \quad \text{and} \quad C = \frac{\bar{u}-c}{1+\lambda}\bar{u}^{-\hat{a}}.$$

With these values of $\bar{u}$ and C, we have to verify that

$$\varphi(s, x, \mu) = e^{-\rho s}\psi(\langle \mu, q\rangle)$$

with ψ given by (22) satisfies all the requirements of Theorem 4.2. We check some of them:
(ii) $\varphi \geq \mathcal{M}\varphi$ on $\mathcal{S}$:
In our case, we have

$$\Gamma(s, X, \mu) = (s, X - c - (1+\lambda)\zeta, \mu^{X-c-(1+\lambda)\zeta}),$$

and hence we get

$$\begin{aligned}
\mathcal{M}\varphi(s, X, \mu) &= \sup_{\zeta}\left\{\varphi(s, X - c - (1+\lambda)\zeta), \mu^{X-c-(1+\lambda)\zeta}) + e^{-\rho s}\zeta;\ 0 \leq \zeta \leq \frac{\bar{u}-c}{1+\lambda}\right\} \\
&= e^{-\rho s}\sup_{\zeta}\left\{C\langle \mu^{X-c-(1+\lambda)\zeta}, q\rangle^{\hat{a}} + \zeta;\ 0 \leq \zeta \leq \frac{\bar{u}-c}{1+\lambda}\right\} \\
&= e^{-\rho s}\sup_{\zeta}\left\{C(\langle \mu, q(x) - c - (1+\lambda)\zeta\rangle^{\hat{a}} + \zeta;\ 0 \leq \zeta \leq \frac{\bar{u}-c}{1+\lambda}\right\} \\
&= e^{-\rho s}\sup_{\zeta}\left\{C(\langle \mu, q\rangle - c - (1+\lambda)\zeta)^{\hat{a}} + \zeta;\ 0 \leq \zeta \leq \frac{\bar{u}-c}{1+\lambda}\right\}.
\end{aligned}$$

If $u - c - (1+\lambda)\zeta \geq \bar{u}$, then

$$\psi(u - c - (1+\lambda)\zeta) + \zeta = \frac{u-2c}{1+\lambda} < \frac{u-c}{1+\lambda} = \psi(u),$$

and if $u - c - (1+\lambda)\zeta < \bar{u}$ then

$$h(\zeta) := \psi(u - c - (1+\lambda)\zeta) + \zeta = C(u - c - (1+\lambda)\zeta)^{\hat{a}} + \zeta.$$

Since

$$h'\left(\frac{u-c}{1+\lambda}\right) = 1 \ \text{ and } \ h''(\zeta) > 0,$$

we see that the maximum value of $h(\zeta);\ 0 \leq \zeta \leq \dfrac{u-c}{1+\lambda}$, is attained at $\zeta = \hat{\zeta}(u) = \dfrac{u-c}{1+\lambda}$.

Therefore,

$$\mathcal{M}\psi(u) = \max\left(\frac{x-2c}{1+\lambda}, \frac{u-c}{1+\lambda}\right) = \frac{u-c}{1+\lambda} \text{ for all } u > c.$$

Hence, $\mathcal{M}\psi(u) = \psi(u)$ for $u \geq \bar{u}$.
For $0 < u < \bar{u}$, consider

$$k(u) := Cu^{\hat{a}} - \frac{u-c}{1+\lambda}.$$

Since

$$k(\bar{u}) = k'(\bar{u}) = 0 \quad \text{and} \quad k''(u) > 0 \text{ for all } u,$$

we conclude that

$$k(u) > 0 \quad \text{for } 0 < u < \bar{u}.$$

Hence,

$$\psi(u) > \mathcal{M}\psi(u) \quad \text{for } 0 < u < \bar{u}.$$

(vi) $L_0\psi(u) \leq 0$ for $u \in \mathcal{S}\backslash\bar{D}$, i.e. for $u > \bar{u}$:
For $u > \bar{u}$, we have $\psi(u) = \frac{u-c}{1+\lambda}$, and therefore, since $\alpha_0 < \rho$,

$$\begin{aligned} L_0\psi(u) &= -\rho\frac{u-c}{1+\lambda} + \alpha_0 u \frac{1}{1+\lambda} = \frac{(\alpha_0-\rho)u+\rho c}{1+\lambda} \leq \frac{(\alpha_0-\rho)\bar{u}+\rho c}{1+\lambda} \\ &= \frac{(\alpha_0-\rho)\frac{\hat{a}c}{(\hat{a}-1}+\rho c}{1+\lambda} = \frac{(\alpha_0\hat{a}-\rho)c}{(1+\lambda)(\hat{a}-1)} < 0, \end{aligned}$$

since $1 < \hat{a} < \frac{\rho}{\alpha_0}$.
Therefore, we have proved the following:

Theorem 5.1 *The value function for* ***Problem*** *5.1 is*

$$\Phi(s,x,\mu) = \begin{cases} e^{-\rho s}Cu^{\hat{a}} & \text{for} \quad 0 < u < \bar{u}, \\ e^{-\rho s}\dfrac{u-c}{1+\lambda} & \text{for} \quad u \geq \bar{u}, \end{cases}$$

where $u = \langle\mu, q\rangle = \mathbb{E}[X(t)|\mathcal{G}_t]$ *and*

$$\bar{u} = \frac{\hat{a}c}{\hat{a}-1} \quad \text{and} \quad C = \frac{\bar{u}-c}{1+\lambda}\bar{u}^{-\hat{a}},$$

$\hat{a}$ being the positive solution of Eq. (20). The optimal impulse control is to do nothing while $u = \mathbb{E}[X(t)|\mathcal{G}_t] < \bar{u}$ and take out immediately

$$\hat{\zeta}(u) = \frac{u - c}{1 + \lambda} \text{ when } u \geq \bar{u}.$$

This brings $\mathbb{E}[X(t)|\mathcal{G}_t]$ down to 0, and the system stops. Hence, the optimal impulse consists of at most one intervention.

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Appendix: Double Fréchet Derivatives

In this section, we recall some basic facts we are using about the Fréchet derivatives of a function $f : V \mapsto W$, where V, W are given Banach spaces.

Definition 1 We say that f has a Fréchet derivative $\nabla_x f = Df(x)$ at $x \in V$ if there exists a bounded linear map $A : V \mapsto W$ such that

$$\lim_{h \to 0} \frac{||f(x + h) - f(x) - A(h)||_W}{||h||_V} = 0.$$

Then we call A the Fréchet derivative of f at x and we put $Df(x) = A$.

Note that $Df(x) \in L(V, W)$ (the space of bounded linear functions from V to W), for each x.

Definition 2 We say that f has a double Fréchet derivative $D^2 f(x)$ at x if there exists a bounded bilinear map $A(h, k) : V \times V \mapsto W$ such that

$$\lim_{k \to 0} \frac{||Df(x + k)(h) - Df(x)(h) - A(h, k)||_W}{||h||_V} = 0.$$

Example 1 – Suppose $f : \mathbb{M} \mapsto \mathbb{R}$ is given by

$$f(\mu) = \langle \mu, q \rangle^2, \text{ where } q(x) = x.$$

Then

$$\begin{aligned} f(\mu + h) - f(\mu) &= \langle \mu + h, q \rangle^2 - \langle \mu, q \rangle^2 \\ &= 2\langle \mu, q \rangle \langle h, q \rangle + \langle h, q \rangle^2, \end{aligned}$$

so we see that

$$Df(\mu)(h) = 2\langle \mu, q \rangle \langle h, q \rangle.$$

To find the double derivative, we consider

$$\begin{aligned} &Df(\mu + k)(h) - Df(\mu)(h) \\ &= 2\langle \mu + k, q \rangle \langle h, q \rangle - 2\langle \mu, q \rangle \langle h, q \rangle \\ &= 2\langle k, q \rangle \langle h, q \rangle, \end{aligned}$$

and we conclude that

$$D^2 f(\mu)(h, k) = 2\langle k, q \rangle \langle h, q \rangle.$$

– Next assume that $g : \mathbb{M} \mapsto \mathbb{R}$ is given by $g(\mu) = \langle \mu, q \rangle$. Then, proceeding as above we find that

$$Dg(\mu)(h) = \langle h, q \rangle \text{ (independent of } \mu)$$

and

$$D^2 g(\mu) = 0.$$

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