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A Union Self-evaluation Approach to Associated Consistency for Cooperative Games

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Abstract

Xu et al. (Linear Algebra Appl 430(11):2896–2897, 2009) introduced the notion of associated consistency according to the idea of "individual self-evaluation". In this paper, we introduce a new type of associated consistency according to the idea of "union self-evaluation" instead of "individual self-evaluation". Adopting this type of associated consistency, we provide new axiomatizations of the equal allocation of non-separable contributions (EANSC) value and the center-of-gravity of the imputation set (CIS) value. Moreover, a dynamic process is given based on the type of associated games, which leads to the CIS value and EANSC value, starting from an arbitrary efficient payoff vector.

Keywords Cooperative game · EANSC value · CIS value · Associated consistency

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1 Introduction

A situation in which finite coalitions of players can achieve specified amounts of worths by cooperating can be described as a cooperative game with transferable utility, or simply a TU-game. A central question in TU-games is to find a method to distribute the benefits of cooperation among all these players. A solution of TU-games is a function that assigns to every TU-game a payoff for every player.

In the framework of solution theory for TU-games, consistency is an important axiom to characterize the viability and stability of a solution. A solution satisfies consistency if this solution distributes the same payoff to players in the original game as in a related, modified game. The two most common types of modified games are the reduced game and the associated game. Reduced game consistency is defined using a reduced game, while associated consistency is defined using an associated game. Both types of consistency axioms require the payoffs of players to be invariant for certain changes in the game.

Reduced games consider situations where one or more players leave the game, and after an appropriate modification of the game, taking account of the effect of the leaving players on the worths that can be obtained by the remaining players, require the payoffs of the remaining players not to change. The concept of reduced game consistency, firstly proposed by Davis [1], has been used to characterize various solutions of cooperative games, such as the Shapley value [7], nucleolus [16], and the efficient, symmetric and linear (ESL) values [14, 17]. More results about reduced game consistency can be found in the survey paper by Driessen [2].

In this paper, we focus on associated games. Associated games consider situations where the player set does not change, but coalitions revalue their worths by claiming part of the surplus in the game that is left after this coalition and the players outside the coalition get some initial share in the total worth. An advantage of the associated consistency axioms is that no players leave or enter the game, and thus, the player set does not change. The concept of associated consistency was firstly introduced by Hamiache [5] to characterize the Shapley value. Subsequently, a matrix approach is applied to associated games to characterize the Shapley value in Xu et al. [18], Hamiache [6]. Driessen [3] generalized Hamiache's associated game and characterized the class of the ESL values by a corresponding associated consistency. Hwang [8] showed that the EANSC value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to Hwang's associated game). Xu et al. [20] gave comparable axiomatizations of the EANSC and the CIS values using associated consistency. Xu et al. [21] showed that the CIS value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to the so-called C-individual associated game).

To define an associated game, Xu et al. [19] and Xu et al. [21] assumed that any coalition is formed by its members joining one by one and each coalition considers players in the coalition as isolated elements. They adopted "individual self-evaluation" to reevaluate the worths of coalitions. The worth of a coalition in the associated games differs from the initial worth, by taking into account the possible loss of benefits due to the departure of players in the coalition. In this paper, we introduce an alternative way to reevaluate the worth. Instead of considering the players in the coalition as isolated elements, we consider the players in the coalition as a whole. That is, we adopt “union self-evaluation" to reevaluate the worths of coalitions. In this paper, under “union self-evaluation" instead of “individual self-evaluation", two alternative associated games are constructed, namely the E-union-associated game and the C-union-associated game.

We continue to develop the works of Xu et al. [19] and Xu et al. [21] as follows. Firstly, we introduce the sequences of the E-union-associated games and the C-union-associated games and explore the convergence of the two sequences and their limit games by the matrix approach. Then, we characterize the EANSC value and the CIS value by associated consistency with respect to the E-union-associated game and the C-union-associated game, respectively. Specifically, we show that the EANSC value is the unique solution satisfying E-union-associated consistency, continuity, efficiency, symmetry and translation covariance, while the CIS value is the unique solution satisfying C-union-associated consistency, continuity, efficiency, symmetry and translation covariance. Moreover, we propose a dynamic process on the basis of the E-union-associated game (respectively the C-union-associated game) that leads to the CIS value and EANSC value, starting from an arbitrary efficient payoff vector. This follows from a more general result showing that the dynamic process can lead to any solution satisfying the inessential game property and continuity.

The paper is organized as follows: In Sect. 2, we introduce some basic definitions and notations. In Sect. 3, we define two different versions of the union-associated games based on the idea of “union self-evaluation". In Sect. 4, we explore the convergence of the sequences of the union-associated games and their limit games by the matrix approach. In Sect. 5, we characterize the EANSC value and the CIS value by the union-associated consistency axioms. In Sect. 6, we propose a dynamic approach that leads to the CIS value and EANSC value based on the union-associated consistency axioms. Section 7 concludes with a brief summary.

2 Definitions and Notations

A cooperative game with transferable utility, or simply a TU-game, is a pair $\langle N, v\rangle$, where N is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning to each coalition $S \in 2^N\backslash\{\emptyset\}$, the worth $v(S)$ with $v(\emptyset) = 0$. Denote the set of all TU-games on player set N by $\mathscr{G}^N$, and denote the set of all non-empty coalitions on player set N by Ω. The cardinality of a finite set S is denoted by s. Given $\langle N, v\rangle \in \mathscr{G}^N$ and $i, j \in N$, players i and j are symmetric players in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$. A TU-game $\langle N, v\rangle$ is an inessential game (also called an additive game) if $v(S) = \sum_{k\in S} v(\{k\})$ for all $S \in \Omega$. A TU-game $\langle N, v\rangle$ is an almost inessential game (also called an almost additive game) if $v(S) = \sum_{k\in S} v(\{k\})$ for all $S \subsetneq N$.

A payoff vector for TU-game $\langle N, v\rangle \in \mathscr{G}^N$ is an n-dimensional vector $x \in \mathbb{R}^n$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. For notational convenience, denote $\sum_{i\in S} x_i$ by $x(S)$, $S \in \Omega$. A set solution on $\mathscr{G}^N$ is a function that assigns a set of payoff vectors to every game $\langle N, v\rangle \in \mathscr{G}^N$. The core is one of the most important set solutions in cooperative games and is given by

$$C(N, v) = \{x \in \mathbb{R}^n | x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in \Omega\},$$

for all $\langle N, v\rangle \in \mathscr{G}^N$. A solution on $\mathscr{G}^N$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^n$ to every game $\langle N, v\rangle \in \mathscr{G}^N$.

The center-of-gravity of the imputation set value (for short, CIS value), introduced by Driessen and Funaki [4], assigns to every player his individual worth and then distributes the remaining worth equally among all players. Formally, for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$,

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{n}\left[v(N) - \sum_{k\in N} v(\{k\})\right].$$

The equal allocation of non-separable contributions value (for short, EANSC value), introduced by Moulin [13], assigns to each player his separable contribution and then distributes the non-separable contributions equally among all players. The EANSC value is given by:

$$EANSC_i(N, v) = SC_i(N, v) + \frac{1}{n}\left[v(N) - \sum_{k\in N} SC_k(N, v)\right],$$

for all $\langle N, v\rangle \in \mathscr{G}^N$ and all $i \in N$, where the *separable contribution* of player i is given by $SC_i(N, v) \equiv v(N) - v(N\backslash\{i\})$ being the marginal contribution of player i to the grand coalition N. Obviously, for all $\langle N, v\rangle \in \mathscr{G}^N$ and $x \in C(N, v)$, it holds that $v(\{i\}) \leq x_i \leq SC_i(N, v)$ for all $i \in N$.

3 Union-Based Associated Games

In the framework of solution theory for TU-games, associated consistency is an important characteristic of viable and stable solutions. Associated consistency requires that the solution is invariant under the adaptation of the game into its associated game. Xu et al. [19] and Xu et al. [21] introduced the notion of the “individual associated game" to characterize the Shapley value and the CIS value by using two different associated consistency axioms. We review the two definitions of associated games as follows:

Definition 3.1 (Xu et al. [19]) Given $\langle N, v\rangle \in \mathscr{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the S-individual associated game $\langle N, v^*_{\lambda,Sh,I}\rangle$ is defined by:

$$v^*_{\lambda,Sh,I}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) - \lambda \sum_{j\in S}[v(S) - v(S\backslash\{j\}) - SC_j(N, v)], & \text{if } \emptyset \neq S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

Definition 3.2 (Xu et al. [21]) Given $\langle N, v\rangle \in \mathscr{G}^N$ and a real number λ, $0 \le \lambda \le 1$, the C-individual associated game $\langle N, v^*_{\lambda,C,I}\rangle$ is defined by:

$$v^*_{\lambda,C,I}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ v(S) - \lambda \sum_{j\in S}[v(S) - v(S\backslash\{j\}) - v(\{j\})], & \text{if } \emptyset \neq S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

A common interpretation of the two associated games is as follows. For a given TU-game, coalitions may reevaluate their worths by taking into consideration that they break down due to the departure of a player. Both associated games reflect a pessimistic self-evaluation of worths of coalitions. In the process of reevaluating worth, it is assumed that the departure of a player, say i, from coalition S causes a loss of benefits $v(S) - v(S\backslash\{i\}) - v(\{i\})$ according to the S-individual associated game (or $v(S) - v(S\backslash\{i\}) - SC_i(N, v)$ according to the C-individual associated game).

In the associated games above, each coalition S considers players in S as isolated elements. That is, they adopt "individual self-evaluation" to reevaluate the worths of coalitions. The goal of this paper is to see whether we can get similar results if, instead of an "individual self-evaluation" approach, we take a "union self-evaluation" approach, where, instead of adding the individual effects of players in a coalition, we look at the impact when coalitions reevaluate their worth as a whole. This seems a natural approach for TU-games, where coalitions are the units that act. Similar as in the two associated games above, we reevaluate based on the separable contributions and individual worths, respectively. But now each coalition S considers itself as a whole, and it will suffer a loss of benefits $v(S) - \sum_{i\in S} SC_i(N, v)$ (respectively $v(S) - \sum_{i\in S} v(\{i\})$) due to the departure of players in coalition S. That is, we adopt "union self-evaluation" to reevaluate the worths of coalitions. Similar as above, two different versions of such "union-associated games" can be defined as follows.

Definition 3.3 Given $\langle N, v\rangle \in \mathscr{G}^N$ and a real number λ, $0 \le \lambda \le 1$, the E-union-associated game $\langle N, v^*_{\lambda,E,U}\rangle$ is defined by:

$$v^*_{\lambda,E,U}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ v(S) - \lambda \left[v(S) - \sum_{j\in S} SC_j(N, v)\right], & \text{if } \emptyset \neq S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

Definition 3.4 Given $\langle N, v\rangle \in \mathscr{G}^N$ and a real number λ, $0 \le \lambda \le 1$, the C-union-associated game $\langle N, v^*_{\lambda,C,U}\rangle$ is defined by:

$$v^*_{\lambda,C,U}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ v(S) - \lambda \left[v(S) - \sum_{j\in S} v(\{j\})\right], & \text{if } \emptyset \neq S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

Remark 3.1 For all $\langle N, v\rangle \in \mathscr{G}^N$ and its E-union associated game $\langle N, v^*_{\lambda,E,U}\rangle$, it holds that $v^*_{\lambda,E,U}(N) = v(N)$, and for all $i \in N$,

$$\begin{aligned} v^*_{\lambda,E,U}(N\backslash\{i\}) =&v(N \setminus \{i\}) - \lambda \left[v(N \setminus \{i\}) + SC_i(N, v) - \sum_{k\in N} SC_k(N, v) \right] \\ =&v(N\backslash\{i\}) - \lambda \left[v(N) - \sum_{k\in N} SC_k(N, v) \right]. \end{aligned}$$

Thus, it is easy to obtain that $EANSC(N, v^*_{\lambda,E,U}) = EANSC(N, v)$.

For all $\langle N, v\rangle \in \mathscr{G}^N$ and its C-union associated game $\langle N, v^*_{\lambda,C,U}\rangle$, it holds that $v^*_{\lambda,C,U}(N) = v(N)$ and $v^*_{\lambda,C,U}(\{i\}) = v(\{i\})$ for all $i \in N$. Thus, it is easy to see that $CIS(N, v^*_{\lambda,C,U}) = CIS(N, v)$.

4 Matrix Approach and The Limit Game

In this section, we consider the sequences of the E-union-associated games and the C-union-associated games, respectively, where, starting with the original game, we take its associated game, the associated game of this associated game, etc. We show that these sequences converge to a special type of games. For all $\langle N, v\rangle \in \mathscr{G}^N$, the sequence of the E-union-associated games, $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}_{m=0}^{\infty}$, is defined by $v^{0*}_{\lambda,E,U} = v$, and $v^{(m+1)*}_{\lambda,E,U} = (v^{m*}_{\lambda,E,U})^*_{\lambda,E,U}, m = 0, 1, \cdots$. Similarly, the sequence of the C-union-associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$, is defined by $v^{0*}_{\lambda,C,U} = v$, and $v^{(m+1)*}_{\lambda,C,U} = (v^{m*}_{\lambda,C,U})^*_{\lambda,C,U}, m = 0, 1, \cdots$. Next, we will explore the convergence of the two sequences and their limit games by the matrix approach.

The set $\mathscr{G}^N$ of all n-person TU-games with player set N is identified with the $(2^n - 1)$-dimensional vector space $\mathbb{R}^{2^n-1}$. The components of a $(2^n - 1)$-dimensional vector represent the worths of the $(2^n - 1)$ non-empty coalitions in Ω. A linear solution for TU-games is a linear operator in the TU-games space $\mathscr{G}^N$ that can be represented as a matrix multiplication. Xu et al. [18] introduced some concepts of coalitional matrices to analyze linear operators on $\mathscr{G}^N$ that will be used below. A matrix M is row-coalitional (column-coalitional) if the number of rows (columns) is $2^n - 1$ and each row (column) is indexed by coalition $S \in \Omega$. A $(2^n - 1)$-dimensional vector x is row-inessential if $x_S = \sum_{i\in S} x_i$ for all $S \in \Omega$, and is almost-inessential if $x_S = \sum_{i\in S} x_i$ for all $S \subsetneq N$, where each component x_S of x is indexed by coalition $S \in \Omega$. A $(2^n - 1) \times m$ row-coalitional matrix M is row-inessential if the row of M indexed by coalition $S \in \Omega$ is the sum of all rows of M indexed by $i \in S$, that is, $M_S = \sum_{i\in S} M_i$ for all $S \in \Omega$.

Linear solutions can be written as the product of an $n \times (2^n - 1)$-dimensional matrix M and the vector v representing the TU-game. Specifically, for all $\langle N, v\rangle \in \mathscr{G}^N$, the EANSC value can be rewritten in matrix form as

$$EANSC(N, v) = M^E v,$$

where $M^E = [M^E_{i,S}]_{i\in N, S\in\Omega}$ is a $n \times (2^n - 1)$ column-coalitional matrix, which component $M^E_{i,S}$ is given by

$$M^E_{i,S} = \begin{cases} \frac{1}{n}, & \text{if } S = N; \\ -1 + \frac{1}{n}, & \text{if } S = N\backslash\{i\}; \\ \frac{1}{n}, & \text{if } S = N\backslash\{j\},\ j \in N\backslash\{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

Associated games can be expressed as a linear transformation of TU-games, which is written by multiplication of an $(2^n - 1) \times (2^n - 1)$-dimensional matrix M with the TU-game vector v. Specifically, for all $\langle N, v\rangle \in \mathscr{G}^N$, the E-union-associated game $\langle N, v^*_{\lambda,E,U}\rangle$ can be rewritten in matrix form as:

$$v^*_{\lambda,E,U} = M^{E,U} \cdot v,$$

where $M^{E,U} = [M^{E,U}_{S,T}]_{S\in\Omega, T\in\Omega}$ is a $(2^n - 1) \times (2^n - 1)$ row-coalitional matrix, which component $M^{E,U}_{S,T}$ is given by

$$M^{E,U}_{S,T} = \begin{cases} 1-\lambda, & \text{if } T = S \subsetneq N; \\ 1, & \text{if } T = S = N; \\ -\lambda, & \text{if } T = N\backslash\{k\},\ k \in S \subsetneq N; \\ s\lambda, & \text{if } T = N,\ S \subsetneq N; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.1 *For the row-coalitional matrix $M^{E,U}$, the following three statements hold.*

(i) *1 is an eigenvalue of $M^{E,U}$, the dimension of the corresponding eigenspace is equal to n and the corresponding eigenvectors are row-inessential.*

(ii) *$1-\lambda$ is an eigenvalue of $M^{E,U}$ and the dimension of the corresponding eigenspace is equal to* $2^n - n - 2$.

(iii) *$1-n\lambda$ is an eigenvalue of $M^{E,U}$ and the dimension of the corresponding eigenspace is equal to* 1.

Proof (i) Let I be the $(2^n-1)\times(2^n-1)$ identity matrix. It is easy to verify that the last row of matrix $(M^{E,U} - I)$ is the zero vector. Thus, 1 is an eigenvalue of $M^{E,U}$. Let x be the eigenvector corresponding to eigenvalue 1 indexed by coalitions $S \in \Omega$. Since $(M^{E,U} - I)x = \mathbf{0}$ and $0 < \lambda < 1$, it follows that

$$-x_S - \sum_{k\in S} x_{N\backslash\{k\}} + s x_N = 0, \tag{1}$$

for all $S \subsetneq N$. If $s = 1$, then we have $x_N = x_k + x_{N\backslash\{k\}}$ for all $k \in N$. Together with Eq. (1), we can obtain that $x_S = \sum_{k\in S} x_k$ for all $S \in \Omega$. Therefore, any eigenvector x corresponding to eigenvalue 1 is row-inessential and the dimension of the corresponding eigenspace is equal to n.

(ii) Let $A = M^{E,U} - (1-\lambda)I$. Denote the columns of matrix A by A_T, $T \in \Omega$. It is easy to verify that all columns A_T with $1 \le t \le n-2$ are zero vectors. Thus, $1-\lambda$ is an eigenvalue of $M^{E,U}$. Let x be the eigenvector corresponding to eigenvalue $1-\lambda$ indexed by coalitions $S \in \Omega$. Since $Ax = 0$ and $0 < \lambda < 1$, we have

$$sx_N - \sum_{k \in S} x_{N \setminus \{k\}} = 0, \tag{2}$$

for all $S \subsetneq N$. If $S = N$, we have $x_N = 0$. Together with Eq. (2), we can obtain that $\sum_{k \in S} x_{N\setminus\{k\}} = 0$ for all $S \subsetneq N$. Then, we have $x_N = 0$ and $x_{N\setminus\{k\}} = 0$ for all $k \in N$. Therefore, the variables x_S with $1 \le s \le n-2$ are free variables, and the dimension of the corresponding eigenspace is equal to $2^n - n - 2$.

(iii) Let $x = [x_S]_{S\in\Omega}$ be a (2^n-1)-dimensional vector with $x_N = 0$ and $x_S = s$ for all $S \subsetneq N$. Denote the rows of matrix $M^{E,U}$ by $M_S^{E,U}$, $S \in \Omega$. Then, we have $M_N^{E,U} x = 0$, and for all $S \subsetneq N$,

$$M_S^{E,U} x = (1-\lambda)s - \lambda(n-1)s = (1-n\lambda)s.$$

Thus, we have $M^{E,U}x = (1-n\lambda)x$, and $1-n\lambda$ is an eigenvalue of $M^{E,U}$. Suppose the multiplicities of the eigenvalue $1-n\lambda$ equal to m. Then, we have

$$1 \le m \le 2^n - 1 - n - (2^n - n - 2) = 1,$$

which implies that $m = 1$. Therefore, 1, $1-\lambda$, $1-n\lambda$ are all eigenvalues of $M^{E,U}$ since the sum of their dimensions of the corresponding eigenspace equals to $2^n - 1$, and $M^{E,U}$ is diagonalizable.

□

Lemma 4.2 (See Xu et al. [18]) *Let A be a matrix and M be a row-coalitional matrix.*

(i) *If M is row-inessential, then the matrix MA is also row-inessential.*
(ii) *If A is an invertible matrix, then MA is row-inessential if and only if M is row-inessential.*
(iii) *If M is an row-inessential matrix, then the TU-game $\langle N, Mv\rangle$ is inessential.*

Now, we state our first main result on the convergence of the sequence of E-union-associated games.

Proposition 4.1 *Let $0 < \lambda < \frac{1}{n}$. Then for all $\langle N, v\rangle \in \mathcal{G}^N$, the sequence of the E-union-associated games $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}_{m=0}^{\infty}$ converges, and its limit game $\langle N, \hat{v}\rangle$ is inessential.*

Proof By Lemma 4.1, the matrix $M^{E,U}$ is diagonalizable and $M^{E,U} = PDP^{-1}$, where $D_\lambda = diag(1, \ldots, 1, 1-\lambda, \ldots, 1-\lambda, 1-n\lambda)$ and P consists of eigenvectors of $M^{E,U}$ corresponding to eigenvalues 1, $1-\lambda$ and $1-n\lambda$. Since $0 < \lambda < \frac{1}{n}$, then we have

$$\lim_{k\to\infty} (M^{E,U})^k = \lim_{k\to\infty} P(D_\lambda)^k P^{-1} = PDP^{-1},$$

where $D = diag(1, \ldots, 1, 0, \ldots, 0)$. Then, we have $PD = [x^1, x^2, \ldots, x^n, \mathbf{0}, \ldots, \mathbf{0}]$, where the column vectors x^i, $i = 1, \ldots, n$, are the corresponding eigenvectors of eigenvalue 1. Since x^i, $i = 1, \ldots, n$, are row-inessential by Lemma 4.1, PD is also row-inessential. Thus, by Lemma 4.2, PDP^{-1} is also row-inessential, and the TU-game $\langle N, PDP^{-1}v\rangle$ is inessential. Since $\hat{v} = \lim_{k\to\infty}(M^{E,U})^k \cdot v = PDP^{-1}v$, the limit game $\langle N, \hat{v}\rangle$ is inessential. □

Next, we consider the sequence of C-union-associated games. For all $\langle N, v\rangle \in \mathcal{G}^N$, the C-union-associated game $\langle N, v^*_{\lambda,C,U}\rangle$ can be rewritten in matrix form as:

$$v^*_{\lambda,C,U} = M^{C,U} \cdot v,$$

where $M^{C,U} = [M^{C,U}_{S,T}]_{S\in\Omega,T\in\Omega}$ is a $(2^n - 1) \times (2^n - 1)$ row-coalitional matrix, and its component $M^{C,U}_{S,T}$ is given by:

$$M^{C,U}_{S,T} = \begin{cases} 1-\lambda, & \text{if } T = S \subsetneq N; \\ 1, & \text{if } T = S = N; \\ \lambda, & \text{if } T = \{k\},\ k \in S \subsetneq N; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.3 *Eigenvalues of the row-coalitional matrix $M^{C,U}$ are equal to 1 or $1-\lambda$. Moreover, the eigenspace corresponding to eigenvalue 1 has dimension $(n+1)$ (the only free variables are x_N and x_k, $k \in N$, and every eigenvector is almost-inessential). The eigenspace corresponding to eigenvalue $1-\lambda$ has dimension $(2^n - n - 2)$ (the only free variables are x_S, $2 \le s \le n-1$).*

Proof Let I be the $(2^n - 1) \times (2^n - 1)$ identity matrix. It is easy to verify that the last row of matrix $(M^{C,U} - I)$ is the zero vector. Thus, 1 is an eigenvalue of $M^{C,U}$. Let x be the eigenvector corresponding to eigenvalue 1 indexed by coalitions $S \in \Omega$. Since $(M^{C,U} - I)x = \mathbf{0}$ and $0 < \lambda < 1$, this implies that $x_S = \sum_{k\in S} x_k$ for all $S \subsetneq N$. Thus, the only free variables are x_N and x_k, $k \in N$. Therefore, any eigenvector x corresponding to eigenvalue 1 is almost-inessential, and the dimension of the corresponding eigenspace is equal to $n+1$.

Let $A = M^{C,U} - (1-\lambda)I$. Denote the columns of matrix A by A_T, $T \in \Omega$. It is easy to verify that all columns A_T with $2 \le t \le n-1$ are zero vectors. Thus, $1-\lambda$ is an eigenvalue of $M^{C,U}$. Let x be the eigenvector corresponding to eigenvalue $1-\lambda$ indexed by coalitions $S \in \Omega$. Since $Ax = 0$ and $0 < \lambda < 1$, we have $x_N = 0$ and $x_k = 0$ for all $k \in N$. Therefore, the variables x_S with $2 \le s \le n-1$ are free variables, and the dimension of the corresponding eigenspace is equal to $2^n - n - 2$. □

Lemma 4.4 (See Xu et al. [21]) *Let A be a matrix and M be a row-coalitional matrix.*

(i) *If M is almost-inessential, then the matrix MA is also almost-inessential.*
(ii) *If A is an invertible matrix, then MA is almost-inessential if and only if M is almost-inessential.*

(iii) *If M is an almost-inessential matrix, then the TU-game $\langle N, Mv\rangle$ is an almost inessential game.*

Next, we state our result on the convergence of the sequence of C-union-associated games.

Proposition 4.2 *Let $0 < \lambda < 1$. Then for all $\langle N, v\rangle \in \mathscr{G}^N$, the sequence of the C-union-associated games $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$ converges, and its limit game $\langle N, \bar{v}\rangle$ is an almost inessential game.*

Proof By Lemma 4.3, the matrix $M^{C,U}$ is diagonalizable and $M^{C,U} = PDP^{-1}$, where $D_\lambda = diag(1, \ldots, 1, 1-\lambda, \ldots, 1-\lambda)$ and P consists of eigenvectors of $M^{C,U}$ corresponding to eigenvalues 1 and $1-\lambda$. Since $0 < \lambda < 1$, then we have

$$\lim_{k\to\infty} (M^{C,U})^k = \lim_{k\to\infty} P(D_\lambda)^k P^{-1} = PDP^{-1},$$

where $D = diag(1, \ldots, 1, 0, \ldots, 0)$. Then, we have $PD = [x^1, x^2, \ldots, x^n, x^{n+1}, \mathbf{0}, \ldots, \mathbf{0}]$, where the column vectors x^i, $i = 1, \ldots, n, n+1$, are the eigenvectors corresponding to eigenvalue 1. Since x^i, $i = 1, \ldots, n, n+1$, are almost-inessential by Lemma 4.3, then PD is also almost-inessential. Thus, by Lemma 4.4, PDP^{-1} is also almost-inessential, and the TU-game $\langle N, PDP^{-1}v\rangle$ is an almost inessential game. Since $\bar{v} = \lim_{k\to\infty}(M^{C,U})^k \cdot v = PDP^{-1}v$, the limit game $\langle N, \bar{v}\rangle$ is an almost inessential game. □

Remark 4.1 As mentioned, the convergence of the sequences of the two union associated games and the limit games is revealed by using the matrix approach. An alternative approach to prove convergence of the sequences is as follows. Let us take the sequence of the C-union associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$, as an example. Given $\langle N, v\rangle \in \mathscr{G}^N$ and $S \subsetneq N$, the term $v^{m*}_{\lambda,C,U}(S)$ can be expressed as a linear combination of $v(S)$ and $v(\{i\})$, $i \in S$, that is,

$$v^{m*}_{\lambda,C,U}(S) = \alpha_m v(S) + \beta_m \sum_{j\in S} v(\{j\}),$$

where $\alpha_m \in \mathbb{R}$ and $\beta_m \in \mathbb{R}$. We can obtain the following three facts: (a) the coefficients α_m and β_m satisfy the recursive relationships, $\alpha_{m+1} = (1-\lambda)\alpha_m$ and $\beta_{m+1} = (1-\lambda)\beta_m + \lambda$; (b) the coefficients α_m and β_m are given by $\alpha_m = (1-\lambda)^m$ and $\beta_m = 1-(1-\lambda)^m$ for all $m \geq 1$; (c) the sequence of the C-union-associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$, converges and its limit game is an almost inessential game. These results are coincident with the conclusions in Proposition 4.2.[1]

5 Axiomatizations of the EANSC Value and the CIS Value Using the New Associated Games

Hwang [8] showed that the EANSC value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with

[1] The detailed proofs of these results can be obtained from the authors on request.

respect to Hwang's associated game). Xu et al. [21] showed that the CIS value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to the C-individual associated game). In Sect. 3, we introduced two new associated games that are based on union self-evaluation: the E-union-associated game and the C-union-associated game. In this section, we will characterize the EANSC value and the CIS value by associated consistency with respect to the E-union-associated game and the C-union-associated game, respectively.

Let us first recall the following axioms of solutions for TU-games. The first two axioms are standard and introduced by Shapley [15] to characterize the Shapley value. Translation covariance requires that the solution should behave in a natural way with respect to changes in scales, which are comparable with affine transformations. Continuity and the inessential game property do not need any further explanation and are usually combined with associated consistency to implement solutions for TU-games, such as the Shapley value [5, 19], the CIS value [20] and the EANSC value [12]. The almost inessential game property is introduced by Xu et al. [21] to characterize the CIS value. A solution φ on $\mathscr{G}^N$ satisfies

(i) efficiency, if $\sum_{k\in N}\varphi_k(N,v)=v(N)$ for all $\langle N,v\rangle\in\mathscr{G}^N$.
(ii) symmetry, if $\varphi_i(N,v)=\varphi_j(N,v)$ whenever i and j are symmetric players in TU-game $\langle N,v\rangle$ for all $\langle N,v\rangle\in\mathscr{G}^N$.
(iii) translation covariance, if $\varphi(N,v+\alpha)=\varphi(N,v)+\alpha$ for all $\langle N,v\rangle\in\mathscr{G}^N$ and $\alpha\in\mathbb{R}^N$, where $\langle N,v+\alpha\rangle$ is defined by $(v+\alpha)(S)=v(S)+\sum_{k\in S}\alpha_i$ for all $S\in\Omega$.
(iv) continuity, if for any convergent sequence of TU-games $\{\langle N,v^k\rangle\}_{k=1}^{\infty}$ and its limit game $\langle N,\tilde{v}\rangle$ (i.e., for all $S\subseteq N$, $\lim_{k\to\infty}v^k(S)=\tilde{v}(S)$), the corresponding sequence of the solution outcomes $\{\varphi(N,v^k)\}_{k=1}^{\infty}$ converges to the payoff vector $\varphi(N,\tilde{v})$.
(v) the inessential game property, if $\varphi_i(N,v)=v(\{i\})$ for all inessential games $\langle N,v\rangle\in\mathscr{G}^N$ and $i\in N$.
(vi) the almost inessential game property, if $\varphi_i(N,v)=v(\{i\})+a[v(N)-\sum_{j\in N}v(\{j\})]$ for all almost inessential games $\langle N,v\rangle\in\mathscr{G}^N$, $i\in N$ and some $a\in[0,1]$.

Besides these six well-known axioms, we introduce two new associated consistencies based on the new associated games. Associated consistency shows stability with respect to a specific way that coalitions reevaluate their worth when players in the coalition stop cooperation. If a solution violates associated consistency, then players might not respect the original compromise but revise the payoff distribution. E-union-associated consistency, respectively, C-union-associated consistency say that a solution gives the same payments to players in the original game as it does to players of the E-union-associated game, respectively, the C-union-associated game. Take $0\le\lambda\le 1$. A solution φ on $\mathscr{G}^N$ satisfies

(vii) E-union-associated consistency for λ, if $\varphi(N,v)=\varphi(N,v^*_{\lambda,E,U})$ for all $\langle N,v\rangle\in\mathscr{G}^N$ and its E-union-associated game $\langle N,v^*_{\lambda,E,U}\rangle$.
(viii) C-union-associated consistency for λ, if $\varphi(N,v)=\varphi(N,v^*_{\lambda,C,U})$ for all $\langle N,v\rangle\in\mathscr{G}^N$ and its C-union-associated game $\langle N,v^*_{\lambda,C,U}\rangle$.

Next, we characterize the EANSC value and the CIS value by E-union-associated consistency and C-union-associated consistency, respectively.

Theorem 5.1 *Let $0 < \lambda < \frac{1}{n}$. The EANSC value is the unique solution satisfying E-union-associated consistency for λ, continuity and the inessential game property.*

Proof It is straightforward to verify that the EANSC value satisfies continuity and the inessential game property. E-union-associated consistency follows from Remark 3.1. It is left to show the uniqueness.

Suppose that a solution φ on $\mathscr{G}^N$ satisfies E-union-associated consistency, continuity and the inessential game property. For all $\langle N, v\rangle \in \mathscr{G}^N$, by Proposition 4.1, the sequence of repeated E-union associated games $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}_{m=0}^{\infty}$ converges to an inessential game $\langle N, \hat{v}\rangle$. By E-union-associated consistency and continuity, we have

$$\varphi(N, v) = \varphi(N, v^{1*}_{\lambda,E,U}) = \varphi(N, v^{2*}_{\lambda,E,U}) = \cdots = \varphi(N, \hat{v}).$$

By the inessential game property, it holds that $\varphi_i(\hat{v}) = \hat{v}(\{i\})$ for all $i \in N$. From this, φ is uniquely determined by these three axioms. Therefore, $\varphi(N, v) = EANSC(N, v)$. □

Logical independence of the axioms used in Theorem 5.1 can be shown by the following alternative solutions.

(i) The solution φ, defined by $\varphi(N, v) = CIS(N, v)$ for all $\langle N, v\rangle \in \mathscr{G}^N$, satisfies all axioms of Theorem 5.1 except E-union-associated consistency for λ.

(ii) The solution φ, defined by

$$\varphi_i(N, v) = \begin{cases} v(\{i\}), & \text{if } \langle N, v\rangle \text{ is an inessential game;} \\ \frac{v(N)}{n}, & \text{otherwise,} \end{cases}$$

for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$, satisfies all axioms of Theorem 5.1 except continuity.

(iii) The solution φ, defined by $\varphi_i(N, v) = \frac{v(N)}{n}$ for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$, satisfies all axioms of Theorem 5.1 except the inessential game property.

An alternative axiomatization is provided by replacing the inessential game property with efficiency, symmetry and translation covariance. It is well known that, if a solution satisfies efficiency, symmetry and translation covariance, then it satisfies the inessential game property. Thus, we can draw the following conclusion directly.

Corollary 5.1 *Let $0 < \lambda < \frac{1}{n}$. The EANSC value is the unique solution satisfying E-union-associated consistency for λ, continuity, efficiency, symmetry and translation covariance.*

Next, we give an axiomatization of the CIS value using C-union-associated consistency. As mentioned in Sect. 4, the sequence of the E-union-associated games converges to an inessential game, while the sequence of the C-union-associated games converges to an almost inessential game. Replacing in Theorem 5.1, E-union-associated consistency with C-union-associated consistency, and replacing the inessential game property with the almost inessential game property and efficiency, characterizes the CIS value. The proof is similar to that of Theorem 5.1, and we omit it.

Theorem 5.2 *Let* $0 < \lambda < 1$. *The CIS value is the unique solution satisfying C-union-associated consistency for* λ, *continuity, the almost inessential game property and efficiency.*

Logical independence of the axioms used in Theorem 5.2 can be shown by the following alternative solutions.

(i) The solution φ, defined by $\varphi(N, v) = EANSC(N, v)$ for all $\langle N, v\rangle \in \mathscr{G}^N$, satisfies all axioms of Theorem 5.2 except C-union-associated consistency for λ.
(ii) The solution φ, defined by

$$\varphi_i(N, v) = \begin{cases} CIS_i(N, v), & \text{if } \langle N, v\rangle \text{ is an almost inessential game;} \\ \frac{v(N)}{n}, & \text{otherwise,} \end{cases}$$

for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$, satisfies all axioms of Theorem 5.2 except continuity.
(iii) The solution φ, defined by $\varphi_i(N, v) = \frac{v(N)}{n}$ for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$, satisfies all axioms of Theorem 5.2 except the almost inessential game property.
(iv) The solution φ, defined by $\varphi_i(N, v) = v(\{i\})$ for all $\langle N, v\rangle \in \mathscr{G}^N$ and $i \in N$, satisfies all axioms of Theorem 5.2 except efficiency.

Similar as Corollary 5.1 for the EANSC value, since efficiency, symmetry and translation covariance of a solution imply that it satisfies the almost inessential game property, another axiomatization of the CIS value can be obtained by replacing the almost inessential game property with symmetry and translation covariance.

Corollary 5.2 *Let* $0 < \lambda < 1$. *The CIS value is the unique solution satisfying C-union-associated consistency for* λ, *continuity, efficiency, symmetry and translation covariance.*

Remark 5.1 Associated consistency is a requirement of “stability" in the sense that it expresses how payoffs of players are invariant if the worth of coalitions are reevaluated because (the expectation that) some players might not cooperate. The EANSC value and the CIS value have been characterized by different associated consistency axioms before in, e.g., Hwang [8], Hwang et al. [10] and Xu et al. [21]. Different associated games take a different angle in ‘revaluating’ the worth of coalitions. Some associated games in the literature [5, 12] focus on reevaluating the worth of a coalition by considering what the coalition expects from the surplus it can obtain from cooperation with players outside the coalition. However, in the E-union-associated game and the C-union-associated game considered in this paper, the worths of coalitions are reevaluated in view of expecting that some players inside the coalition might not fully contribute. Besides the difference between focusing on gains or losses, the associated games in this paper take a union self-evaluation approach, while other associated games consider individual self-evaluation (see [19, 21]).

We also want to remark that the introduction of the union-associated consistency greatly simplifies the proof of the axiomatizations of the EANSC value and CIS value.

Remark 5.2 The method of characterizing the EANSC value and CIS value in Theorems 5.1 and 5.2 can be generalized to other solutions. Let f be a solution on $\mathscr{G}^N$ satisfying linearity[2] and the inessential game property. We can define a new associated game by taking the solution f instead of the separable contributions in the E-union-associated game (or the individual worths in the C-union-associated game). Given $\langle N, v\rangle \in \mathscr{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the f-union-associated game $\langle N, v^*_{\lambda,f,U}\rangle$ is defined by:

$$v^*_{\lambda,f,U}(S) = v(S) - \lambda\left[v(S) - \sum_{j\in S} f_j(N, v)\right].$$

A solution f satisfies f-union-associated consistency for λ, if $f(N, v) = f(N, v^*_{\lambda,f,U})$ for all $\langle N, v\rangle \in \mathscr{G}^N$. We can prove that the solution f is the unique solution satisfying f-union-associated consistency for λ $(0 < \lambda < 1)$, continuity and the inessential game property.

6 A Dynamic Approach to the EANSC Value and the CIS Value

Hwang et al. [11] introduced a dynamic process based on Hamiache's associated game [5] and proved that this dynamic process leads to any solution satisfying both the inessential game property and continuity. Similar as we used the convergence results in Section 4 to obtain new axiomatic characterizations of the EANSC and CIS values, in this section we use this convergence to provide a dynamic process on the basis of the E-union-associated game (or the C-union-associated game) that leads to the CIS value and the EANSC value, starting from an arbitrary efficient payoff vector.

Given $\langle N, v\rangle \in \mathscr{G}^N$, let the set of efficient payoff vectors $X(N, v)$ be given by $X(N, v) = \{x \in \mathbb{R}^N \mid \sum_{k\in N} x_k = v(N)\}$. For all $\langle N, v\rangle \in \mathscr{G}^N$, $x \in X(N, v)$ and $0 \leq \lambda \leq 1$, we define the x-associated game $\langle N, v^*_{\lambda,x}\rangle$ by $v^*_{\lambda,x}(\emptyset) = 0$ and

$$v^*_{\lambda,x}(S) = v(S) - \lambda[v(S) - x(S)], \tag{3}$$

for all $S \in \Omega$. Note that the x-associated game is constructed by replacing "$\sum_{i\in S} SC_i(N, v)$" in the E-union associated game (or "$\sum_{i\in S} v(\{i\})$" in the C-union associated game) by the payoff "$x(S)$". Then, the sequence of the x-associated games, $\{\langle N, v^{m*}_{\lambda,x}\rangle\}_{m=0}^{\infty}$, is inductively defined by $v^{0*}_{\lambda,x} = v$, and $v^{(m+1)*}_{\lambda,x} = (v^{m*}_{\lambda,x})^*_{\lambda,x}$. In view of the representation (3) of the x-associated game, the general representation of the m-fold x-associated game $\langle N, v^{m*}_{\lambda,x}\rangle$ can be written as:

$$v^{m*}_{\lambda,x}(S) = a^S_m v(S) + b^S_m x(S) \tag{4}$$

[2] A solution φ satisfies linearity if $\varphi(N, av + bw) = a\varphi(N, v) + b\varphi(N, w)$ for all $\langle N, v\rangle, \langle N, w\rangle \in \mathscr{G}^N$ and $a, b \in \mathbb{R}$

for all $S \in \Omega$, where a_m^S and b_m^S are certain coefficients with respect to λ.
The next lemma identifies these coefficients.

Lemma 6.1 *The coefficients a_m^S and b_m^S in expression (4) of the m-fold x-associated game $\langle N, v_{\lambda,x}^{m*}\rangle$ satisfy the following recursive formulas:*

$$a_m^S = (1-\lambda)^m \quad \textit{and} \quad b_m^S = 1-(1-\lambda)^m.$$

Proof For all $\langle N, v\rangle \in \mathscr{G}^N$ and $S \in \Omega$, combining Eqs. (3) and (4), we have

$$\begin{aligned} v_{\lambda,x}^{(m+1)*}(S) =&(v_{\lambda,x}^{m*})_{\lambda,x}^{*}(S) \\ =&(1-\lambda)v_{\lambda,x}^{m*}(S)+\lambda x(S) \\ =&(1-\lambda)[a_m^S v(S)+b_m^S x(S)]+\lambda x(S). \end{aligned}$$

Since this must hold for every $v(S)$, $S \subseteq N$, we have $a_{m+1}^S = (1-\lambda)a_m^S$ and $b_{m+1}^S = (1-\lambda)b_m^S + \lambda$, where $a_1^S = 1-\lambda$ and $b_1^S = \lambda$. From this, we have the following recursive formulas:

$$\frac{a_{m+1}^S}{a_m^S} = 1-\lambda \text{ and } \frac{b_{m+1}^S - 1}{b_m^S - 1} = 1-\lambda.$$

Therefore, the coefficients a_m^S and b_m^S of the m-fold x-associated game $\langle N, v_{\lambda,x}^{m*}\rangle$ satisfy $a_m^S = (1-\lambda)^m$ and $b_m^S = 1-(1-\lambda)^m$, $m = 1, 2, \cdots$. □

The next lemma shows that, updating the worths of coalitions by assigning to every coalition S its worth minus a fraction of its excess according to the proposed payoff vector x, converges to an inessential game that is described by the payoff vector x.

Lemma 6.2 *For all $\langle N, v\rangle \in \mathscr{G}^N$, $x \in X(N, v)$ and $0 < \lambda < 1$, the sequence of m-fold x-associated games $\{\langle N, v_{\lambda,x}^{m*}\rangle\}_{m=0}^{\infty}$ converges to the limit game $\langle N, \hat{v}_x\rangle$, which is given by $\hat{v}_x(S) = x(S)$ for all $S \in \Omega$.*

Proof For all $\langle N, v\rangle \in \mathscr{G}^N$, $x \in X(N, v)$ and $0 < \lambda < 1$, by Lemma 6.1, we have

$$\begin{aligned} \lim_{m\to\infty} v_{\lambda,x}^{m*}(S) =& \lim_{m\to\infty}\{(1-\lambda)^m v(S) + [1-(1-\lambda)^m]x(S)\} \\ =&x(S). \end{aligned}$$

for all $S \in \Omega$. □

Next, we introduce a dynamic process that leads to any solution satisfying the inessential game property and continuity. Let φ be a solution satisfying both the inessential game property and continuity. Given $\langle N, v\rangle \in \mathscr{G}^N$ and $x \in X(N, v)$, we define the dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and

$$x^m = x^{m-1} + [\varphi(N, v_{\lambda,x}^{(m-1)*}) - \varphi(N, v_{\lambda,x}^{m*})]. \tag{5}$$

The dynamic sequence defined by expression (5) is inspired by Hwang [9] and Hwang et al. [11]. Hwang [9] and Hwang et al. [11] proposed two dynamic sequences on the basis of Hamiache's associated game [5] and the complement-associated game of Hwang et al. [10], respectively. Compared with these dynamic sequences of Hwang [9] and Hwang et al. [11], our dynamic sequence uses a different associated game (the E-union-associated game or the C-union-associated game).

The dynamic sequence $\{x^t\}_{t=0}^{\infty}$ can be regarded as a reappraised process. Starting from an arbitrary payoff vector $x \in X(N, v)$, using any solution φ satisfying both the inessential game property and continuity, this dynamic sequence converges to $\varphi(N, v)$. Consider a situation in which there is an arbitrator and some players. Every player follows the arbitrator's suggestion, and then, the arbitrator will lead these players to a reasonable allocation by using a fair rule.

Theorem 6.1 *Let $0 < \lambda < 1$. Given $\langle N, v\rangle \in \mathscr{G}^N$ and $x \in X(N, v)$, the dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and x^m described by (5) converges to $\varphi(N, v)$ if the solution φ satisfies both the inessential game property and continuity..*

Proof Let φ be a solution satisfying both the inessential game property and continuity. Then for all $\langle N, v\rangle \in \mathscr{G}^N$ and $x \in X(N, v)$, consider the dynamic sequence

$$x^m = x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x}) - \varphi(N, v^{m*}_{\lambda,x})].$$

By recursion, we have

$$\begin{aligned} x^m &= x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x}) - \varphi(N, v^{m*}_{\lambda,x})] \\ &= x^{m-2} + [\varphi(N, v^{(m-2)*}_{\lambda,x}) - \varphi(N, v^{m*}_{\lambda,x})] \\ &= \cdots \\ &= x^0 + [\varphi(N, v^{0*}_{\lambda,x}) - \varphi(N, v^{m*}_{\lambda,x})] \\ &= x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x})]. \end{aligned}$$

By Lemma 6.2, the inessential game property and continuity, we obtain that $\lim_{m\to\infty} \varphi(N, v^{m*}_{\lambda,x}) = \varphi(N, \hat{v}_x) = x$. Therefore,

$$\begin{aligned} \lim_{m\to\infty} x^m &= \lim_{m\to\infty} \{x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x})]\} \\ &= x + [\varphi(N, v) - x] = \varphi(N, v). \end{aligned}$$

□

The following two corollaries follow from Theorem 6.1 and the fact that the EANSC value, respectively the CIS value satisfy the inessential game property and continuity. They state that the EANSC value and the CIS value can be implemented by a dynamic process as above, respectively, starting from an arbitrary efficient payoff vector.

Corollary 6.1 *Let* $0 < \lambda < 1$. *Given* $\langle N, v\rangle \in \mathscr{G}^N$ *and* $x \in X(N, v)$, *the dynamic sequence* $\{x^m\}_{t=m}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [EANSC(N, v_{\lambda,x}^{(m-1)*}) - EANSC(N, v_{\lambda,x}^{m*})],$$

converges to the EANSC value.

Corollary 6.2 *Let* $0 < \lambda < 1$. *Given* $\langle N, v\rangle \in \mathscr{G}^N$ *and* $x \in X(N, v)$, *the dynamic sequence* $\{x^m\}_{t=m}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [CIS(N, v_{\lambda,x}^{(m-1)*}) - CIS(N, v_{\lambda,x}^{m*})],$$

converges to the CIS value.

7 Summary

This work belongs to the growing literature on associated consistency. Different associated games take a different angle in revaluating the worth of coalitions. Some associated games in the literature [19, 21] focus on reevaluating the worth of a coalition by considering “individual self-evaluation". In this paper, we introduce an alternative way to reevaluate the worth. Instead of considering the players in the coalition as isolated elements, we consider the players in the coalition as a whole. We define two different associated games according to the idea of “union self-evaluation" instead of “individual self-evaluation" and provide new axiomatizations of the EANSC value and the CIS value using associated consistency. Moreover, we also propose a dynamic process on the basis of the “union self-evaluation" associated games that leads to the EANSC value and the CIS value, starting from an arbitrary efficient payoff vector.

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