Journal of Optimization Theory and Applications (2022) 195:647–665
https://doi.org/10.1007/s10957-022-02102-2

A Population Harvesting Model with Time and Size Competition Dependence Function

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Received: 21 October 2021 / Accepted: 23 August 2022 / Published online: 25 September 2022

Abstract
We consider a nonlinear model describing a forest harvesting of a size-structured trees population with intra-specific competition, where the population compete with trees of bigger size. Using a fixed point argument, we prove the existence of a unique solution to the problem. We also prove the existence of an optimal control where the objective functional includes the benefits from timber production. Then, we give the necessary condition of optimality for the optimal control and give its characterization as well.

Keywords Size-structured population · Competition function · Timber production · Fixed point theory · Maximum principle · Bang–bang optimal control

Mathematics Subject Classification 35L65 · 92D25 · 65M06

1 Introduction

Forests provide unique ecosystem services to people around the world, and their conservation is essential. Massive deforestation must be avoided and forest degradation reduced. In addition, the maintenance of sustainable forest management is important

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Communicated by Vincenzo Capasso.

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for the carbon balance; it contributes to the mitigation of climate change too. But due to demographic pressure, timber needs are increasing significantly to meet the demand for construction and other exploitation, and this poses a double challenge to sustainable forest management: how to increase the volumes of wood produced and adapt to current and future global changes?

We study a forest management problem in order to analyze the impact of different assumptions regarding vital rates (growth, mortality, recruitment), as well as different environmental conditions, on forest dynamics. Growth and mortality processes are influenced in a nonlinear way by competition for resources: light (mainly), water and nutrients in the soil. A synthesis of works on the modeling of these problems can be found among the following references: Angulo et al. [2], Calsina and Saldana [7] and [8], Chave [9], Goetz et al. [11] (see also the references therein).

We are interested in the optimal forest harvesting. Several models are used in the literature from the discrete, deterministic or statistics point of view. See, for example, Malo et al. [20] where reinforcement learning algorithms are used for a discrete stochastic forest model or Fer et al. [10], where they use Gaussian models. But discrete and statistical approaches are not entirely satisfactory because they do not give enough knowledge on the behavior of the system from the qualitative point of view.
We propose to study the problem from an approach based on the formulation and analysis of a system of partial differential equations (PDE) as in [7]. The PDE model permits to consider a larger scale of space and time, and to study analytically the temporal evolution (trajectories) of the dynamic system described by the size distribution for the more representative species of the composition of the forest. Here, the PDE model is a population dynamics system structured in size. Finally, an optimal control problem (timber cutting) will define the optimal cut to be respected. The objective functional includes the benefits from timber production plus an ecological term corresponding to the total population of individuals of small diameter size, in the goal to allow a regeneration of the forest. This second term in the objective function plays a similar role as the one in Hritonenko et al. [13] (see also [14]) where they take into account the expenses generated from planting new trees.

The existence of a solution to the PDE problem considered in this article is not trivial due to the non-local nature of the model and the nonlinear terms present in the renewal, competition and growth processes. It will require the use of methods such as the fixed point method. Moreover, we deal with the optimal control problem. In particular, we maximize a benefit function from the timber price parameter, under environmental constraints. The control function (wood cutting) is characterized by an optimality system. We seek here to preserve two objectives: to maintain the natural regeneration of the forest while meeting the demand for wood from the current and future population. Note that harvesting problems for age structured populations, i.e., the case where the velocity of the growth is equal to 1, have been previously studied in Ainseba et al. [1], Bernhard and Veliov [4], Brokate [5], Gurtin and MacCamy [12], Murphy and Smith [21]. In this case the characteristic lines of the age-structured model are linear, so that the solution can be viewed as a solution to an ordinary differential equation with delay and one can characterize the optimal system via an adapted Pontryagin's principle.

The paper is organized as follows. In Sect. 2, we present the problem we are studying and we give the main definitions. In Sect. 3, we prove the existence and uniqueness of the solution to the nonlinear problem. In Sect. 4, we study the optimal control and its characterization by an optimality necessary condition. We give a conclusion in a last section.

2 Position of the Problem

We investigate a nonlinear size-structured forest model with a harvesting trees function (control), including harvesting benefit (timber production) and plantation of new trees (forest regeneration). We follow the lines of the article by Calsina and Saldana [7], where they considered the case with a growth rate depending on the size as well as on the total population. See also the article by Tahvonen [24] and the papers by Kato [15], Kato et al. [16], Kato and Torikata [18], where some variants of the model in [7] are studied.

Denoting by $u := u(t, x)$, the density of trees population, at time t and of size (diameter) x which varies from ℓ_0 to ℓ_{max}, where ℓ_0 is the minimum diameter recruitment of a tree called the diameter at breast height (dbh), and which in general measures all the living trees of size dbh$>$ 10 cm in tropical forests. For simplicity, we denote by $\ell_0 = 0$ and $\ell_{max} = \ell$ for the maximum dbh.

Then, we consider the intra-species competition function $E_u(t, x)$, defined by:

$$E_u(t, x) = \frac{\pi}{4} \int_x^{\ell} y^2 \, u(t, y) \, \mathrm{d}y. \tag{1}$$

The function E_u is called the cumulative basal area of trees greater in size than x. It is considered as the index of the shading and it measures the effect of shading of larger trees on a tree of size x (see Kohyama [19] and the references therein). This definition seems to be realistic. Indeed, the assumption of equal availability for light resource is not really suitable, since individuals of a less size with respect to one individual are not competing with it for this resource.

We study the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(V(E_u(t, x), x)u\right) = -\mu(E_u(t, x), x)\, u - v(t)u & (t, x) \in Q :=]0, T[\times]0, \ell[, \\ u(t, 0)\, V(E_u(t, 0), 0) = \displaystyle\int_0^{\ell} \beta(E_u(t, x), x)\, u(t, x)\, \mathrm{d}x & t \in]0, T[, \\ u(0, x) = u_0(x) & x \in]0, \ell[. \end{cases} \tag{2}$$

In (2) we find the classical functions:

- $V(E_u(t, x), x)$: growth rate;
- $v(t)$: time dependent harvesting rate. It represents the control function;
- $\mu(E_u(t, x), x)$: the death rate;

- $B(t) := \int_0^\ell \beta(E_u(t,x),x)\, u(t,x)\, \mathrm{d}x$: the birth function, $\beta(E_u(t,x),x)$: the birth rate.

If the solution u is differentiable with respect to the x variable (see below), we can rewrite the first equation of (2) as follows:

$$\frac{\partial u}{\partial t} + V(E_u(t,x),x)\,\frac{\partial u}{\partial x} = -G_v(E_u(t,x),x)\,u, \tag{3}$$

where

$$G_v(E_u(t,x),x) = \mu_v(E_u(t,x),x) + V_x(E_u(t,x),x), \tag{4}$$

and where

$$\mu_v(E_u(t,x),x) = \mu(E_u(t,x),x) + v(t) \tag{5}$$

(notice that $\mu_0 = \mu$), and where

$$V_x(E_u(t,x),x) = \frac{\partial V}{\partial E}(E_u(t,x),x)\frac{\partial E_u}{\partial x}(t,x) + \frac{\partial V}{\partial x}(E_u(t,x),x). \tag{6}$$

Remark 1 In fact, the mortality rate $\mu(E_u(t,x),x)$ is the sum of the mortality function due to competition and the natural mortality function $\mu_{_N}(x)$, where we have:

$$\lim_{x\to\ell}\int_0^x \mu_{_N}(y)\,\mathrm{d}y = +\infty,$$

and which expresses that individuals die before the maximal size $x = \ell$ (see, for example, Gurtin and MacCamy [12]).

Definition 2.1 A non-negative function u is a solution of the IBVP (2) if $E_u(t,x)$ is a continuous function on Q and if it satisfies:

$$\begin{aligned} D\,u(t,x(t)) &= \lim_{h\to 0}\left(\frac{u(t+h,x(t+h)) - u(t,x(t))}{h}\right) \\ &= -G_v(E_u(t,x),x)\,u \qquad (t,x)\in Q, \end{aligned}$$

where the characteristic curve $x(t) = \varphi(t;t_0,x_0)$ passing through $(t_0,x_0) \in Q$, is solution of the ordinary differential equation (ODE):

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t}(t) = V(E_u(t,x(t)),x(t)), \quad t\in]0,T[, \\ &x(t_0) = x_0 \in]0,\ell[. \end{aligned} \tag{7}$$

In particular, a solution to (7) is given by:

$$x(t) = x_0 + \int_{t_0}^t V(E_u(s,x(s)),x(s))\,\mathrm{d}s. \tag{8}$$

Denote by $\varphi(t;\tau,\eta)$, the characteristic curve passing through the initial pair (τ,η), and by $\phi_t := \varphi(t;0,0)$ the characteristic curve through $(0,0)$, separating the trajectories of the individuals present at the initial time $\tau = 0$ from the trajectories of those individuals born after the initial time. In particular, if $u(t,x)$ is a solution of the problem (2), the initial condition is given by $u_0(x) = u_0(\varphi(0;0,x))$.
For any $(t,x) \in Q$ such that $x < \phi_t$, the individuals are born after $t = 0$, at an initial time $\tau := \tau(t,x) > 0$ have their size equal to zero (hence $\varphi(\tau;t,x) = 0$). By composition, it is equivalent to:

$$\varphi(t;\tau,0) = x. \tag{9}$$

We consider (9) as a definition for the initial time τ (when $x < \phi_t$). It is called the initial time of the cohort through (t,x). Then, integrating along the characteristics $x(t)$, we obtain an explicit formula of the solution. Indeed, we have the:

Lemma 2.1 *Let $u(t,x)$ be a solution of the problem (2). Then u has the following representation along the characteristic curves:*

$$u(t,x) = \begin{cases} \dfrac{B(\tau)}{V(E_u(\tau,0),0)} \\ \exp\Big(-\int_\tau^t G_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\Big), \textit{ for a.e. } x \in [0,\phi_t[, \\ u_0(x) \\ \exp\Big(-\int_0^t G_v(E_u(s,\varphi(s;0,x)),\varphi(s;0,x))\,\mathrm{d}s\Big), \textit{ for a.e. } x \in [\phi_t,\ell[, \end{cases} \tag{10}$$

for any time $t > 0$.

Proof Let $u(t,x)$ be a solution of the problem (2). Define $\overline{u}(t;t^*,x^*) = u(t,x(t;t^*,x^*))$, where $x(t;t^*,x^*)$ is the characteristic curve taking the value x^* at time t^*. Then in view of equations (3)–(4) and (7), $\overline{u}$ satisfies to the initial value problem:

$$\begin{aligned} &\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}(t;t^*,x^*) = -G_v(E_u(t,x(t;t^*,x^*)),x(t;t^*,x^*))\,\overline{u}(t;t^*,x^*) && t > t^*, \\ &\overline{u}(t^*;t^*,x^*) = u(t^*,x^*) && t = t^*. \end{aligned}$$

The solution of this problem is represented by the following formula:

$$\begin{aligned} \overline{u}(t;t^*,x^*) &= u(t,x(t;t^*,x^*)) \\ &= u(t^*,x^*)\,\exp\left(-\int_{t^*}^t G_v(E_u(s,x(s;t^*,x^*)),x(s;t^*,x^*))\,\mathrm{d}s\right). \end{aligned}$$

Considering the case $x := x(t;t^*,x^*) < \phi_t$ where the initial data is given by (2_b) and the case $x := x(t;t^*,x^*) > \phi_t$ where it is given by (2_c), we obtain (10). □

Remark 2 The above method of characteristics to nonlinear problems is classical. One can see the article by Gurtin and MacCamy [12]. The method is also detailed in the book by Webb [25].

3 Welposedness of the Problem

We recall some differentiability properties of the characteristics curves, which are used for changing variables.

Lemma 3.1 *Let $\varphi(t;\ \tau,\ \eta)$ be the characteristic curve through (τ, η) solution to the ODE (7) with $x(\tau) = \varphi(\tau; \tau, \eta) = \eta$. Then, φ is differentiable with respect to τ and η, and we have:*

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}(t;\ \tau,\ \eta) = -V(E_u(\tau, \eta),\ \eta)\ \exp\Big(\int_\tau^t V_x(E_u(s, \varphi(s; \tau, \eta)),\ \varphi(s; \tau, \eta))\,\mathrm{d}s\Big), \tag{11}$$

and,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}(t;\ \tau,\ \eta) = \exp\Big(\int_0^t V_x(E_u(s, \varphi(s; \tau, \eta)),\ \varphi(s; \tau, \eta))\,\mathrm{d}s\Big). \tag{12}$$

Proof For the proof one applies the definition of differentiability of ODE equations with respect to parameters and to the initial condition (see, for example, Pontryagin [23], Chap. 4.6). One can find a detailed proof in Kato [18] (Lemma 3.4). □

Lemma 3.2 *The intra-species competition function $E_u(t, x)$ is given by the integral equation:*

$$\begin{aligned} E_u(t, x) = \frac{\pi}{4}\int_{\varphi^{-1}(t;0,x)}^t &\varphi^2(t; \tau, 0)\, B(\tau) \\ &\exp\Big(-\textstyle\int_\tau^t \mu_v(E_u(s, \varphi(s; \tau, 0)), \varphi(s; \tau, 0))\,\mathrm{d}s\Big)\ \mathrm{d}\tau \\ &+\frac{\pi}{4}\int_0^\ell \varphi^2(t; 0, y)\, u_0(y) \\ &\exp\Big(-\textstyle\int_0^t \mu_v(E_u(s, \varphi(s; 0, y)), \varphi(s; 0, y))\,\mathrm{d}s\Big)\ \mathrm{d}y. \end{aligned} \tag{13}$$

Moreover, the birth function $B(t)$ is given by the integral equation:

$$\begin{aligned} B(t) = \int_0^t &\beta(E_u(\tau, \varphi(t; \tau, 0)), \varphi(t; \tau, 0)) B(\tau) \\ &\exp\Big(-\textstyle\int_\tau^t \mu_v(E_u(s, \varphi(s; \tau, 0)), \varphi(s; \tau, 0))\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ &+\int_0^\ell \beta(E_u(t, \varphi(t; 0, x)), \varphi(t; 0, x)) u_0(x) \\ &\exp\Big(-\textstyle\int_0^t \mu_v(E_u(s, \varphi(s; 0, x)), \varphi(s; 0, x))\,\mathrm{d}s\Big)\,\mathrm{d}x. \end{aligned} \tag{14}$$

Proof Indeed, we have:

$$E_u(t, x) = \frac{\pi}{4}\left[\int_x^{\phi_t} y^2 u(t, y)\,\mathrm{d}y + \int_{\phi_t}^\ell y^2 u(t, y)\,\mathrm{d}y\right].$$

Using (10), we obtain the following integral equation for $E_u(t, x)$:

$$E_u(t,x) = \frac{\pi}{4}\int_x^{\phi_t} \varphi^2(t;\tau,0)\frac{B(\tau)}{V(E_u(\tau,0),0)} \exp\Big(-\textstyle\int_\tau^t G_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\Big)\,\mathrm{d}\varphi$$

$$+\frac{\pi}{4}\int_{\phi_t}^{\ell} \varphi^2(t;0,y)\,u_0(y) \exp\Big(-\textstyle\int_0^t G_v(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\Big)\,\mathrm{d}\varphi$$

$$= \frac{\pi}{4}\,(I+J)\,.$$

We use (4) and (11), to have:

$$I = \int_x^{\phi_t} \varphi^2(t;\tau,0)\frac{B(\tau)}{V(E_u(\tau,0),0)} \exp\Big(-\textstyle\int_\tau^t G_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\Big)\,\mathrm{d}\varphi$$

$$= \int_x^{\phi_t} \varphi^2(t;\tau,0)B(\tau)\,\exp\left(-\int_\tau^t \mu_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\right)$$

$$\times\frac{d\varphi}{V(E_u(\tau,0),0)\,\exp\left(\int_\tau^t V_x(E_u(t,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\right)}$$

$$= \int_t^{\varphi^{-1}(t;0,x)} \varphi^2(t;\tau,0)B(\tau)\exp\left(-\int_\tau^t \mu_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\right) \times(-\mathrm{d}\tau).$$

For the second integral J, we use (12) and we have:

$$J = \int_{\phi_t}^{\ell} \varphi^2(t;0,y)\,u_0(y)\,\exp\left(-\int_0^t G_v(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\right)\,\mathrm{d}\varphi$$

$$= \int_{\phi_t}^{\ell} \varphi^2(t;0,y)\,u_0(y)\exp\left(-\int_0^t \mu_v(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\right)$$

$$\times\frac{\mathrm{d}\varphi}{\exp\left(-\int_0^t V_x(E_u(t,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\right)}$$

$$= \int_0^{\ell} \varphi^2(t;0,y)\,u_0(y)\exp\left(-\int_0^t \mu_v(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\right)\,\mathrm{d}y.$$

By adding $I + J$, we obtain the desired result.
For the proof of (14), we use the same decomposition. Indeed, as in above we have:

$$B(t) = \left[\int_0^{\phi_t} \beta(E_u(t,x),x)\,u(t,x)\,\mathrm{d}x + \int_{\phi_t}^{\ell} \beta(E_u(t,x),x)\,u(t,x)\,\mathrm{d}x\right]$$

$$= \int_0^{\phi_t} \frac{\beta(E_u(t,\varphi(t;\tau,0)),\varphi(t;\tau,0))\,B(\tau)}{V(E_u(\tau,0),0)} \exp\left(-\int_\tau^t G_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\right)\mathrm{d}\varphi$$

$$+ \int_{\phi_t}^{\ell} \beta(E_u(t,\varphi(t;0,x)),\varphi(t;0,x))\,u_0(x) \exp\left(-\int_0^t G_v(E_u(s,\varphi(s;0,x)),\varphi(s;0,x))\,\mathrm{d}s\right)\mathrm{d}\varphi.$$

Using (4), (11) and (12), we find the result. □

Remark 3 In equation (13), we integrate from $\tau = \varphi^{-1}(t;0,x)$, which corresponds to the initial time of the characteristic passing through zero, to t. In the following analysis we will consider non-negative time, since we work with the time interval $[0,T]$ only.

Next, we prove the existence of a weak solution to problem (2) using fixed point arguments. From Lemmas 3.1 and 3.2, we deduce that the existence of a solution $u(t,x)$ is equivalent to the existence of a couple of functions $E_u(t,x)$ and $B(t)$ (see [7]). We will first do the following hypothesis:
(H_1) : V is upper-bounded in E_u and x, $|V| \leq V_0$ (for some $V_0 > 0$), and V is Lipschitz with respect to E_u of Lipschitz constant V_L;
(H_2) : β is a non-negative function, upper-bounded by β_0, and is Lipschitz with respect to E_u and x of Lipschitz constant β_L;
(H_3) : μ_v is a non-negative Lipschitz function with respect to E_u and x of Lipschitz constant μ_L.
From (8), we have that:

$$\varphi(t;\tau,0) = \int_\tau^t V(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s, \tag{15}$$

where $x_0 = 0$, from which we deduce that $|\varphi^2(t;\tau,0)| \leq V_0^2\,|t-\tau|^2$ where V_0 is the upper bound of $|V|$ in $\mathbb{R}\times[0,\ell]$. And for $\tau = 0$ we have:

$$\varphi(t;0,y) = y + \int_0^t V(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s. \tag{16}$$

We deduce that:

$$\frac{\pi}{4}\int_0^{\ell} \varphi^2(t;0,y)u_0(y)\,\mathrm{d}y \leq E_0 + P_0V_0^2t^2 + 2\ell P_0V_0t \leq E_0 + C_0P_0V_0T, \tag{17}$$

where $C_0 = TV_0 + 2\ell$ and where we use the following notations:

$$E_0 := E_{u_0} = \frac{\pi}{4}\int_0^\ell y^2 u_0(y)\,\mathrm{d}y \quad \text{and} \quad P_0 := P_{u_0} = \frac{\pi}{4}\int_0^\ell u_0(y)\,\mathrm{d}y.$$

Now, let be $K > \max(E_0, P_0)$. Denote by:

$$M = \left\{ f \in \mathcal{C}\big(\overline{Q}\big);\quad f(0,0) = E_0,\quad \|f\| \le K \right\},$$

where $\|\cdot\| := \|\cdot\|_{\mathcal{C}(\overline{Q})}$ is the SupNorm on $\mathcal{C}\left(\overline{Q}\right)$ with $\overline{Q} := [0, T] \times [0, \ell]$ (the set M is a closed metric subset of $\mathcal{C}\left(\overline{Q}\right)$), and define the mapping $\mathcal{E} : M \to \mathcal{C}\left(\overline{Q}\right)$ such that for any fixed $E_u \in M$, compute $B(t)$ as the unique solution for the linear Volterra integral equation (14). Then, the operator $\mathcal{E}(E_u(t))$ corresponds to the right hand side of (13) for these $E_u(t)$ and $B(t)$.
And thus, we have the theorem:

Theorem 3.1 *Consider the hypothesis $(H_1) - (H_3)$ on the V, β and μ_v functions, respectively. Let the mapping $\mathcal{E} : M \to \mathcal{C}\left(\overline{Q}\right)$ be defined as above. Then $\mathcal{E}$ is a contraction from M into itself (i.e., $\mathcal{E}$ has a fixed point and there exists a unique solution to (2)).*

Proof *Step 1*. We show that $\mathcal{E}$ maps M into itself.
We follow the classical method by Gurtin and MacCamy [12] (see also [7]), beginning by obtaining a bound to $B(t)$. From (14) and from the hypothesis (H_2) and (H_3) on β and μ_v, respectively, we have $B(t) \le \frac{4}{\pi}\beta_0 P_0 + \beta_0 \int_0^t B(\tau)\,\mathrm{d}\tau$, where $\beta_0 = \sup_{(E_u,x)\in\mathbb{R}\times[0,\ell]} \{\beta(E_u(.,.), x)\}$. Using the Gronwall inequality, we obtain:

$$B(t) \le \frac{4}{\pi}\beta_0 P_0\, e^{\beta_0 t} \tag{18}$$

Now, we put this in (13) and we use the hypothesis (H_1) for V. Then, we obtain:

$$\begin{aligned}
|\mathcal{E}(E_u)(t,x)| \le\ & \beta_0 P_0 \int_0^t \varphi^2(t;\tau,0)\exp(\beta_0\tau) \\
& \exp\Big(-\textstyle\int_\tau^t \mu_v(E_u(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s\Big)\mathrm{d}\tau \\
& + \frac{\pi}{4}\int_0^\ell \varphi^2(t;0,y)\,u_0(y) \\
& \exp\Big(-\textstyle\int_0^t \mu_v(E_u(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s\Big)\,\mathrm{d}y \\
\le\ & P_0 V_0^2 T^2 e^{\beta_0 T} + E_0 + C_0 P_0 V_0 T \ \le K
\end{aligned}$$

up to small values of T if necessary. Hence, we have $\mathcal{E}(E_u) \in M$.
Step 2. We show that $\mathcal{E}$ is a contraction.

Let be E_{u_1} and E_{u_2} two elements of M corresponding to two functions u_1 and u_2 such that $u_1(0,x) = u_2(0,x) = u_0(x)$. From the expressions (15) and (16), we use the following notations:

$$\varphi_{i\tau}(t) := \varphi_{E_{u_i}}(t;\tau,0) = \int_\tau^t V(E_{u_i}(s,\varphi(s;\tau,0)),\varphi(s;\tau,0))\,\mathrm{d}s, \quad i=1,2,$$
$$\varphi_{iy}(t) := \varphi_{E_{u_i}}(t;0,y) = y + \int_0^t V(E_{u_i}(s,\varphi(s;0,y)),\varphi(s;0,y))\,\mathrm{d}s, \quad i=1,2. \tag{19}$$

For simplicity, we also use the notations:

$$\begin{aligned}
\mu_{i\tau}(t) &:= \mu_v\left(E_{u_i}(t,\varphi(t;\tau,0)),\varphi(t;\tau,0)\right),\\
\beta_{i\tau}(t) &:= \beta\left(E_{u_i}(t,\varphi(t;\tau,0)),\varphi(t;\tau,0)\right),\\
\mu_{iy}(t) &:= \mu_v\left(E_{u_i}(t,\varphi(t;0,y)),\varphi(t;0,y)\right),\\
\beta_{iy}(t) &:= \beta\left(E_{u_i}(t,\varphi(t;0,y)),\varphi(t;0,y)\right).
\end{aligned} \tag{20}$$

Then, we have:

$$\begin{aligned}
&\mathcal{E}(E_{u_1})(t,x) - \mathcal{E}(E_{u_2})(t,x)\\
&\quad= -\frac{\pi}{4}\int_{\varphi^{-1}(t;0,x)}^t \left(\varphi_{1\tau}^2\,\mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2\,\mathcal{B}(E_{u_2})(\tau)\right)\exp\left(-\int_\tau^t \mu_{1\tau}(s)\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\qquad-\frac{\pi}{4}\int_{\varphi^{-1}(t;0,x)}^t \varphi_{2\tau}^2\,\mathcal{B}(E_{u_2})(\tau)\\
&\left(\exp\left(-\textstyle\int_\tau^t \mu_{1\tau}(s)\,\mathrm{d}s\right) - \exp\left(-\textstyle\int_\tau^t \mu_{2\tau}(s)\,\mathrm{d}s\right)\right)\mathrm{d}\tau\\
&\qquad+\frac{\pi}{4}\int_0^\ell \left(\varphi_{1y}^2 - \varphi_{2y}^2\right) u_0(y)\,\exp\left(-\int_0^t \mu_{1y}(s)\,\mathrm{d}s\right)\mathrm{d}y\\
&\qquad+\frac{\pi}{4}\int_0^\ell \varphi_{2y}^2\,u_0(y)\,\left(\exp\left(-\int_0^t \mu_{1y}(s)\,\mathrm{d}s\right) - \exp\left(-\int_0^t \mu_{2y}(s)\,\mathrm{d}s\right)\right)\mathrm{d}y,
\end{aligned}$$

where here we used the property $a_1b_1 - a_2b_2 = a_2(b_1-b_2) + b_1(a_1-a_2)$, valid for any functions or reals a_1, a_2, b_1 and b_2.
Recall that we consider non-negative time variable $t \in [0,T]$ (see Remark 3). That is, we have the bounds:

$$\begin{aligned}
&\left|\mathcal{E}(E_{u_1})(t,x) - \mathcal{E}(E_{u_2})(t,x)\right|\\
&\quad\le \frac{\pi}{4}\int_0^t \left|\varphi_{1\tau}^2\,\mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2\,\mathcal{B}(E_{u_2})(\tau)\right|\,\mathrm{d}\tau\\
&\qquad+\frac{\pi}{4}\int_0^t \varphi_{2\tau}^2\,\left|\mathcal{B}(E_{u_2})(\tau)\right|\left(\int_\tau^t \left|\mu_{1\tau}(s) - \mu_{2\tau}(s)\right|\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\qquad+\frac{\pi}{4}\int_0^\ell \left(\left|\varphi_{1y} - \varphi_{2y}\right|\left|\varphi_{1y} + \varphi_{2y}\right|\left|u_0(y)\right|\right)\mathrm{d}y
\end{aligned}$$

$$+ \frac{\pi}{4} \int_0^{\ell} \varphi_{2y}^2 \, |u_0(y)| \left(\int_0^t \left| \mu_{1y}(s) - \mu_{2y}(s) \right| \, \mathrm{d}s \right) \mathrm{d}y = \frac{\pi}{4} \left(I_1 + I_2 + I_3 + I_4 \right),$$

using in particular the fact that $|e^{-x} - e^{-y}| \le |x - y|$ for every $x, y > 0$. For the first integral I_1, we have:

$$\begin{aligned}
&\left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(\tau) \right| \\
&\le \beta_0 \int_0^{\tau} \left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(s) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(s) \right| \, \mathrm{d}s \\
&\quad + \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 \tau} \int_0^{\tau} \varphi_{2\tau}^2 \\
&\quad \times \left| \beta_{1\tau} \exp\left(- \int_s^{\tau} \mu_{1s}(\alpha) \, \mathrm{d}s\alpha \right) - \beta_{2\tau} \exp\left(- \int_s^{\tau} \mu_{2s}(\alpha) \, \mathrm{d}s\alpha \right) \right| \mathrm{d}s \\
&\quad + \int_0^{\ell} |\varphi_{1\tau} - \varphi_{2\tau}| \, |\varphi_{1\tau} + \varphi_{2\tau}| \, u_0(y) \, \beta_{1y} \exp\left(- \int_0^{\tau} \mu_{1y}(\alpha) \, \mathrm{d}s\alpha \right) \mathrm{d}y \\
&\quad + \int_0^{\ell} \varphi_{2\tau}^2 \, u_0(y) \left| \beta_{1y} \exp\left(- \int_0^{\tau} \mu_{1y}(\alpha) \, \mathrm{d}\alpha \right) - \beta_{2y} \exp\left(- \int_0^{\tau} \mu_{2y}(\alpha) \, \mathrm{d}\alpha \right) \right| \mathrm{d}y
\end{aligned}$$

that we write as:

$$\begin{aligned}
&\left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(\tau) \right| \\
&\le \beta_0 \int_0^{\tau} \left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(s) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(s) \right| \, \mathrm{d}s \\
&\quad + \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 \tau} \int_0^{\tau} \varphi_{2\tau}^2 \, \beta_{1\tau} \left| \exp\left(- \int_s^{\tau} \mu_{1s}(\alpha) \, \mathrm{d}\alpha \right) - \exp\left(- \int_s^{\tau} \mu_{2s}(\alpha) \, \mathrm{d}\alpha \right) \right| \mathrm{d}s \\
&\quad + \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 \tau} \int_0^{\tau} \varphi_{2\tau}^2 \, |\beta_{1\tau} - \beta_{2\tau}| \exp\left(- \int_s^{\tau} \mu_{2s}(\alpha) \, \mathrm{d}\alpha \right) \mathrm{d}s \\
&\quad + \int_0^{\ell} |\varphi_{1\tau} - \varphi_{2\tau}| \, |\varphi_{1\tau} + \varphi_{2\tau}| \, u_0(y) \, \beta_{1y} \exp\left(- \int_0^{\tau} \mu_{1y}(\alpha) \, \mathrm{d}\alpha \right) \mathrm{d}y \\
&\quad + \int_0^{\ell} \varphi_{2\tau}^2 \, u_0(y) \, \beta_{1y} \left| \exp\left(- \int_0^{\tau} \mu_{1y}(\alpha) \, \mathrm{d}\alpha \right) - \exp\left(- \int_0^{\tau} \mu_{2y}(\alpha) \, \mathrm{d}\alpha \right) \right| \mathrm{d}y \\
&\quad + \int_0^{\ell} \varphi_{2\tau}^2 \, u_0(y) \, \left| \beta_{1y} - \beta_{2y} \right| \exp\left(- \int_0^{\tau} \mu_{2y}(\alpha) \, \mathrm{d}\alpha \right) \mathrm{d}y \\
&\le \beta_0 \int_0^{\tau} \left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(s) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(s) \right| \, \mathrm{d}s + J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

In J_3, we have from (19) the following majoration:

$$\begin{aligned}
&|\varphi_{1\tau} - \varphi_{2\tau}| \\
&= \left| \int_{\tau}^{t} \big(V(E_{u_1}(s, \varphi(s; \tau, 0)), \varphi(s; \tau, 0)) - V(E_{u_2}(s, \varphi(s; \tau, 0)), \varphi(s; \tau, 0)) \big) \, \mathrm{d}s \right| \\
&\le V_L \int_0^t \left| E_{u_1}(s, \varphi(s; \tau, 0)) - E_{u_2}(s, \varphi(s; \tau, 0)) \right| \, \mathrm{d}s \le T \, V_L \left\| E_{u_1} - E_{u_2} \right\|,
\end{aligned}$$

where V_L is the Lipschitz constant on V. Thus,

$$\begin{aligned} J_3 &= \int_0^\ell |\varphi_{1\tau} - \varphi_{2\tau}| \, |\varphi_{1\tau} + \varphi_{2\tau}| \; u_0(y) \, \beta_{1y} \exp\left(-\int_0^\tau \mu_{1y}(\alpha) \, \mathrm{d}\alpha\right) \mathrm{d}y \\ &\le \frac{8}{\pi} \beta_0 P_0 V_0 V_L T^2 \left\| E_{u_1} - E_{u_2} \right\|. \end{aligned}$$

Now, using the fact that β and μ are Lipschitzian functions with respect to E and x, and using (20) we obtain successively:

$$\begin{aligned} J_1 &= \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 \tau} \int_0^\tau \varphi_{2\tau}^2 \, \beta_{1\tau} \left| \exp\left(-\int_s^\tau \mu_{1s}(\alpha) \, \mathrm{d}\alpha\right) - \exp\left(-\int_s^\tau \mu_{2s}(\alpha) \, \mathrm{d}\alpha\right) \right| \mathrm{d}s \\ &\le \frac{4}{\pi} \beta_0^2 P_0 e^{\beta_0 T} V_0^2 T^3 \mu_L \left\| E_{u_1} - E_{u_2} \right\|, \\ J_2 &= \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 \tau} \int_0^\tau \varphi_{2\tau}^2 \; |\beta_{1\tau} - \beta_{2\tau}| \exp\left(-\int_s^\tau \mu_{2s}(\alpha) \, \mathrm{d}\alpha\right) \mathrm{d}s \\ &\le \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 T} V_0^2 T^3 \beta_L \left\| E_{u_1} - E_{u_2} \right\|, \\ J_4 &= \int_0^\ell \varphi_{2\tau}^2 \, u_0(y) \, \beta_{1y} \left| \exp\left(-\int_0^\tau \mu_{1y}(\alpha) \, \mathrm{d}\alpha\right) - \exp\left(-\int_0^\tau \mu_{2y}(\alpha) \, \mathrm{d}\alpha\right) \right| \mathrm{d}y \\ &\le \frac{4}{\pi} \beta_0 P_0 V_0^2 T^2 \mu_L \left\| E_{u_1} - E_{u_2} \right\|, \end{aligned}$$

and,

$$J_5 = \int_0^\ell \varphi_{2\tau}^2 \, u_0(y) \; \left|\beta_{1y} - \beta_{2y}\right| \exp\left(-\int_0^\tau \mu_{2y}(\alpha) \, \mathrm{d}\alpha\right) \mathrm{d}y \le \frac{4}{\pi} P_0 V_0^2 T^2 \beta_L \left\| E_{u_1} - E_{u_2} \right\|.$$

We resume by:

$$\begin{aligned} &\left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(\tau) \right| \le \beta_0 \int_0^\tau \left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(s) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(s) \right| \mathrm{d}s \\ &\quad + \frac{4}{\pi} \widetilde{C} T^2 \left\| E_{u_1} - E_{u_2} \right\|, \end{aligned}$$

where

$$\widetilde{C} = P_0 V_0 \left(\beta_0^2 e^{\beta_0 T} V_0 T \mu_L + \beta_0 e^{\beta_0 T} V_0 T \beta_L + 2\beta_0 V_L + \beta_0 V_0 \mu_L + V_0 \beta_L \right).$$

Using Gronwall's inequality, we obtain:

$$\left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(\tau) \right| \le \frac{4}{\pi} \widetilde{C} T^2 e^{\beta_0 \tau} \left\| E_{u_1} - E_{u_2} \right\|.$$

We deduce that:

$$I_1 = \int_0^t \left| \varphi_{1\tau}^2 \, \mathcal{B}(E_{u_1})(\tau) - \varphi_{2\tau}^2 \, \mathcal{B}(E_{u_2})(\tau) \right| \mathrm{d}\tau \le \frac{4}{\pi} \widetilde{C} T^3 e^{\beta_0 T} \; \left\| E_{u_1} - E_{u_2} \right\|.$$

For the remaining integrals, we have:

$$
\begin{aligned}
I_2 &= \int_0^t \varphi_{2\tau}^2 \, \left|\mathcal{B}(E_{u_2})(\tau)\right| \left(\int_\tau^t |\mu_{1\tau}(s) - \mu_{2\tau}(s)| \, \mathrm{d}s \right) \mathrm{d}\tau \\
&\leq \frac{4}{\pi} \beta_0 P_0 e^{\beta_0 T} T^4 V_0^2 \mu_L \left\| E_{u_1} - E_{u_2} \right\| , \\
I_3 &= \int_0^\ell \left(\left|\varphi_{1y} - \varphi_{2y}\right| \left|\varphi_{1y} + \varphi_{2y}\right| |u_0(y)| \right) \mathrm{d}y \leq \frac{8}{\pi} P_0 T V_L (T V_0 + \ell) \left\| E_{u_1} - E_{u_2} \right\| , \\
I_4 &= \int_0^\ell \varphi_{2y}^2 \, |u_0(y)| \left(\int_0^t \left|\mu_{1y}(s) - \mu_{2y}(s)\right| \, \mathrm{d}s \right) \mathrm{d}y \\
&\leq \frac{4}{\pi} \left(E_0 + C_0 P_0 V_0 T\right) T \mu_L \left\| E_{u_1} - E_{u_2} \right\| ,
\end{aligned}
$$

where here we use in particular the majoration (17).
Finally:

$$
\left\| \mathcal{E}(E_{u_1}) - \mathcal{E}(E_{u_2}) \right\| \leq \frac{\pi}{4} \left(I_1 + I_2 + I_3 + I_4 \right) = \widetilde{C}' T \left\| E_{u_1} - E_{u_2} \right\| ,
$$

which is contractant for small T. Here,

$$
\widetilde{C}' = \widetilde{C} T^2 e^{\beta_0 T} + \beta_0 P_0 e^{\beta_0 T} V_0^2 T^3 \mu_L + 2 P_0 V_L (T V_0 + \ell) + \left(E_0 + C_0 P_0 V_0 T\right) \mu_L .
$$

This ends the proof of the theorem. □

4 The Optimal Control Problem

We study the optimal timber production where the control is the amount cut of trees. The objective functional we would like to maximize includes two terms. The first one corresponds to the net benefits from timber production and the second one corresponds to the total number of individuals of size x small, $x \in [0, \ell_1]$, where the size ℓ_1 is the minimum diameter cutting in forestry, also called diameter-limit cutting (DLC) (see Nyland [22]). These trees play the same role as new trees which are planted in order to replace those that have been cut down as in Hritonenko et al. [13] and [14] (see also Kato [17]).

The optimal harvesting problem consists in maximizing the function J defined by:

$$
J(v) = \int_Q k(x) \, v(t) \, u(t, x) \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_0^{\ell_1} \rho(t) \, u(t, x) \, \mathrm{d}x \mathrm{d}t, \tag{21}
$$

where u is the solution of (2), $k(x)$ is the price function such that:

$$
k \in \mathcal{C}^1([0, \ell]), \quad 0 < k_0 = k(0) \leq k(x) \leq k(\ell) = k_M \quad \text{and}
$$

$$0 \le k'(x) \le k_2 \quad \text{a.e. in } [0, \ell], \tag{22}$$

k_2 being a constant, and where

$$\rho \in \mathcal{C}^1([0, T]), \quad \rho(t) > 0 \ \text{ a.e. in } [0, T]. \tag{23}$$

The control function $v := v(t)$ is to be find in the set of controls:

$$\mathcal{U} = \{v(t) \in \mathcal{C}[0, T]\,;\ 0 \le v(t) \le v_{\mathrm{M}}, \quad \forall\, t \in [0, T]\}\,, \tag{24}$$

where v_{M} is the maximum harvesting rate.
We use—in a different way—the method of separable models introduced by Busenberg and Iannelli [6] (see also Anita [3]). We write the solution $u(t, x)$ in the form:

$$u(t, x) = z(t)\, \tilde{u}(t, x), \tag{25}$$

so that we obtain the two problems (26) and (27) below. The first one is the PDE problem corresponding to the state of the forest in the case of no harvest ($v = 0$) :

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + V(E_u(t, x), x)\, \frac{\partial \tilde{u}}{\partial x} = -G_0(E_u(t, x), x)\tilde{u} && \text{in } Q :=]0, T[\times]0, \ell[, \\ &\tilde{u}(t, 0)\, V(E_u(t, 0), 0) = \int_0^{\ell} \beta(E_u(t, x), x)\, \tilde{u}(t, x)\, \mathrm{d}x && \text{in }]0, T[, \\ &\tilde{u}(0, x) = u_0(x) && \text{in }]0, \ell[. \end{aligned} \tag{26}$$

The second problem is the simple ODE given by :

$$\begin{aligned} &\dot{z}(t) + v(t)\, z(t) = 0 \ \text{ in }]0, T[, \\ &z(0) = 1, \end{aligned} \tag{27}$$

of solution:

$$z(t) = \exp\left(-\int_0^t v(s)\, \mathrm{d}s\right), \qquad \forall\, t \in [0, T]. \tag{28}$$

With this, the criterion J changes to the following. Maximize:

$$\widetilde{J}(v) = \int_0^T v(t)\, z(t)\, \tilde{q}_1(t)\, \mathrm{d}t + \int_0^T \rho(t)\, z(t)\, \tilde{q}_2(t)\, \mathrm{d}t, \tag{29}$$

where the functions $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$ depend on the state of the forest in the case of no harvesting:

$$\tilde{q}_1(t) = \int_0^{\ell} k(x)\tilde{u}(t, x)\, \mathrm{d}x, \qquad \tilde{q}_2(t) = \int_0^{\ell_1} \tilde{u}(t, x)\, \mathrm{d}x. \tag{30}$$

Hence, the study of the maximization problem (21) is equivalent to the one of (29)–(30). We notice that the control function appears only in the ODE problem (27), but u depending on v appears in the velocity of the PDE (26).

Proposition 4.1 *Let $(\tilde{u}(t,x), z(t))$ be the solution pair of problems (26)–(27). Then, there exists at least one optimal control $v^* \in \mathcal{U}$ to the problem (29)–(30) (and then for problem (21)).*

Proof Let be $d := \sup_{v\in\mathcal{U}} J(v) = \sup_{v\in\mathcal{U}} \int_0^T z(t)\zeta(t)\,\mathrm{d}t$ where

$$\zeta(t) = v(t)\tilde{q}_1(t) + \rho(t)\tilde{q}_2(t). \tag{31}$$

Then, there is $(v_n)_{n\in\mathbb{N}^*}$, a minimizing sequence such that:

$$d - \frac{1}{n} < \int_0^T z_n(t)\zeta_n(t)\,\mathrm{d}t \le d, \tag{32}$$

where $z_n := z(v_n)$. Since (v_n) is bounded (hypothesis), we deduce from (28) that (z_n) and its time derivative $(\dot{z}_n)$ are bounded. Thus, $(z_n) \subset \mathcal{C}[0,T]$ is precompact and there is a sequence—still denoted (z_n)—which strongly converges to $z^* \in \mathcal{C}([0,T])$.
Now, we have to prove that:

$$\int_0^T z_n(t)\zeta_n(t)\,\mathrm{d}t \longrightarrow \int_0^T z^*(t)\zeta^*(t)\,\mathrm{d}t,$$

where $\zeta^*(t) = v^*(t)\tilde{q}_1^*(t) + \rho^*(t)\tilde{q}_2^*(t)$.
As in the above section, we write $z_n\zeta_n - z^*\zeta^* = z_n(\zeta_n - \zeta^*) + \zeta^*(z_n - z^*)$. We then have to show the weak convergence of $\zeta_n(t)$ to $\zeta^*(t)$. According to (31) we have $\zeta_n(t) = v_n(t)\tilde{q}_{1n}(t) + \rho(t)\tilde{q}_{2n}(t)$. Since $\tilde{u}$ is bounded, $\tilde{q}_{2n}(t)$ weakly converges to $\tilde{q}_2^*(t) = \int_0^\ell \tilde{u}^*(t,x)\,\mathrm{d}x$. It remains to show that $\tilde{q}_{1n}(t)$ converges strongly. That is to show that the time derivative sequence $(\dot{\tilde{q}}_{1n})$ is bounded. We have:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{q}_{1n}}{\mathrm{d}t}(t) &= \int_0^\ell k(x)\,\frac{\partial \tilde{u}_n}{\partial t}(t,x)\,\mathrm{d}x \\
&= -\int_0^\ell k(x)\,V(E_n(t,x),x)\,\frac{\partial \tilde{u}_n}{\partial x}(t,x)\,\mathrm{d}x \\
&\quad - \int_0^\ell k(x)\,G_0(E_n(t,x),x)\,\tilde{u}_n(t,x)\,\mathrm{d}x \\
&= -k(\ell)\,V(E_n(t,\ell),\ell)\,\tilde{u}_n(t,\ell) + k_0\,V(E_n(t,0),0)\,\tilde{u}_n(t,0) \\
&\quad + \int_0^\ell k'(x)\,V(E_n(t,x),x)\,\tilde{u}_n(t,x)\,\mathrm{d}x \\
&\quad + \int_0^\ell k(x)\,V_x(E_n(t,x),x)\,\tilde{u}_n(t,x)\,\mathrm{d}x \\
&\quad - \int_0^\ell k(x)\,G_0(E_n(t,x),x)\,\tilde{u}_n(t,x)\,\mathrm{d}x.
\end{aligned}$$

We use the hypothesis $(H_1) - (H_3)$, hence:

$$\left| \frac{d\tilde{q}_{1n}}{dt}(t) \right| \leq (k_0\, \beta_0 + k_2\, V_0 + k_M\, V_L)\ |\tilde{u}_n(t,.)|_{L^1(0,\ell)} \qquad \forall\, t \in [0,T]. \quad \square$$

4.1 Necessary Conditions

We now give the necessary conditions of the optimal control v^*. For simplicity, we suppose that:

$$V := V(t,x), \qquad \mu := \mu(t,x) \qquad \beta := \beta(t,x). \tag{33}$$

We have the following result.

Proposition 4.2 *Consider the problems (26) and (27) with the cost functional $\widetilde{J}$ given by (29)–(30). If $v(t)$ maximizes (29), then there exists a continuous function $\lambda \in \mathcal{C}(0,T;\mathbb{R})$ such that the optimal control is bang–bang:*

$$v = \begin{cases} 0 & \text{if } \tilde{q}_1(t) - \lambda(t) < 0, \\ v_M & \text{if } \tilde{q}_1(t) - \lambda(t) > 0. \end{cases}$$

Moreover, if $\tilde{q}_1'(t) + \rho(t)\tilde{q}_2(t) > 0$ and,

$$\tilde{q}_1(0) < \int_{t^*}^{T} \exp(v_M(t^* - s)(-v_M\tilde{q}_1(s) - \rho(s)\tilde{q}_2(s))ds + \int_0^{t^*} \rho(s)\tilde{q}_2(s)ds,$$

then we have one switch from the minimal value to the maximal one for the optimal control.

Proof Applying Pontryagin's maximum principle theorem, there exist an adjoint variable $\lambda(t)$ and a control v that we should determine, such that the Hamiltonian

$$H(z,v;\lambda) = v(t)z(t)\tilde{q}_1(t) + \rho(t)z(t)\tilde{q}_2(t) + \lambda(t)(-v(t)z(t)),$$

is constant along the optimal trajectories. The adjoint equation is given by:

$$\dot{\lambda} = -\frac{\partial H}{\partial z} = -v(t)\tilde{q}_1(t) - \rho(t)\tilde{q}_2(t) + \lambda(t)v(t), \tag{34}$$

with the transversality condition $\lambda(T) = 0$. The derivative of the Hamiltonian with respect to v gives:

$$\frac{\partial H}{\partial v} = z(t)\tilde{q}_1(t) - \lambda(t)z(t). \tag{35}$$

Thus, $\frac{\partial H}{\partial v}$ does not depend on v. Since $z(t) \geq 0$, the sign of $\frac{\partial H}{\partial v}$ depends on the sign of $\tilde{q}_1(t) - \lambda(t)$. The control is then bang–bang and we have:

$$v = \begin{cases} v_M & \text{if } \tilde{q}_1(t) - \lambda(t) > 0, \\ 0 & \text{if } \tilde{q}_1(t) - \lambda(t) < 0. \end{cases}$$

Note that we can easily see by contradiction that we have no singular controls under our hypothesis. Assume that v^* is singular, thus there exists an interval I such that $z(t)(\tilde{q}_1(t)-\lambda(t)) = 0$ for $t \in I$. As z cannot be zero thus we have that $\tilde{q}_1(t)-\lambda(t) = 0$ for $t \in I$ and by derivation $\lambda'(t) = \tilde{q}_1'(t)$ for $t \in I$. Using the adjoint equation one gets $\lambda'(t) = -\rho(t)\tilde{q}_2(t)$ for $t \in I$ so that $\tilde{q}_1'(t) = -\rho(t)\tilde{q}_2(t)$ for $t \in I$ and this is not possible.

In order to compute the number of switches in the optimal control function, we define the function $\Psi(t) = z(t)(\tilde{q}_1(t) - \lambda(t))$ and we compute its derivative with respect to t. Thus, using the state and the adjoint equations (27), (34) one gets:

$$\Psi'(t) = -v(t)z(t)(\tilde{q}_1(t) - \lambda(t)) - z(t)(\tilde{q}_1'(t) - v(t)\tilde{q}_1(t) - \rho(t)\tilde{q}_2(t) + \lambda(t)v(t)),$$

and after simplification:

$$\Psi'(t) = z(t)(\tilde{q}_1'(t) + \rho(t)\tilde{q}_2(t)).$$

Thanks to the transversality condition we have that $\Psi(T) = z(T)\tilde{q}_1(T)$ is non-negative. This means that close to the final time T the optimal control is at its maximal value.

Then, if $\tilde{q}_1'(t)+\rho(t)\tilde{q}_2(t) > 0$, we have $\Psi'(t) > 0$ and the function Ψ is increasing. It is clear now that if $\Psi(0) < 0$, then we have a unique switch in the optimal control function and that the optimal control strategy consists in not doing anything until a time t^* and then switch to the maximal effort until the final time T. The time t^* can be calculated explicitly in this case. The solution of the state equation is given by:

$$z = \begin{cases} 1 & \text{if } t < t^*, \\ \exp(-v_M(t-t^*)) & \text{if } t^* < t < T. \end{cases}$$

One can also compute the solution of the adjoint equation and the condition $\Psi(0) < 0$ is then equivalent to $\tilde{q}_1(0) < (1 - \exp(-v_M T)) \int_0^T (-v_M\tilde{q}_1(s) - \rho(s)\tilde{q}_2(s)) \exp(-v_M s) \mathrm{d}s$.

If $\Psi(0) > 0$, then the function Ψ cannot vanish in $[0, T]$ and thus the optimal strategy consists in taking the control $v = v_M$ during the whole harvesting period. □

Remark 4 The assumptions (33) on V, μ and β in the above proposition can be more general for establishing the necessary conditions to the optimal control, but they are necessary for its characterization.

5 Conclusion

In this work, we considered an optimal forest harvesting problem where trees are in competition for light. The functional we are maximizing includes the benefits from timber production with a penalization taking into account the regeneration of the forest. We proved the existence of a solution to the size and time non-local and nonlinear structured problem using the fixed point theorem. Splitting the control problem allowed us to use Pontryagin's maximum principle, and we proved the existence of an optimal control of bang–bang type with one switch under some conditions on the state of the forest when no man action is considered. The optimal control program always finishes with the maximum of harvest rate to insure a maximum benefit to the forestman. Depending on the state of the forest when no control acts, one can consider other optimal strategies with multiple switches or without any switch.

Declarations

Conflict of interests The authors declare that they have no conflict of interests.

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