



Proximal Point Algorithms for Quasiconvex Pseudomonotone Equilibrium Problems

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Abstract

We propose a proximal point method for quasiconvex pseudomonotone equilibrium problems. The subproblems of the method are optimization problems whose objective is the sum of a strongly quasiconvex function plus the standard quadratic regularization term for optimization problems. We prove, under suitable additional assumptions, convergence of the generated sequence to a solution of the equilibrium problem, whenever the bifunction is strongly quasiconvex in its second argument, thus extending the validity of the convergence analysis of proximal point methods for equilibrium problems beyond the standard assumption of convexity of the bifunction in the second argument.

Keywords Proximal point algorithms · Equilibrium problems · Pseudomonotonicity · Quasiconvexity · Strong quasiconvexity

1 Introduction

Equilibrium problems were introduced in the pioneer work of Ky Fan [9] and deeply studied along the years (see [3,6,10,17,28,33] among others). They encompass several problems found in fixed point theory, optimization and nonlinear analysis, like minimization problems, linear complementary problems and variational inequalities.

Honoring Franco Giannessi on his 85th birthday.

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Equilibrium problems have been studied deeply in the last decades, with emphasis on existence results and optimality conditions, especially in the convex case. In the nonconvex case, some recent existence results and necessary and sufficient optimality conditions have been established (see [6,10,33]). Moreover, in [18], the authors prove an existence result for the pseudomonotone equilibrium problem in the general quasiconvex case.

Proximal point algorithms, introduced for dealing with maximal monotone operators and convex problems in [29,30,37], have been extended to equilibrium problems in the convex case (see [4,7,21,24–26,32,36] among others). Some of these algorithms assume a generalized monotonicity notion on the bifunction, but none of them has weakened the assumption of convexity of the bifunction in its second argument.

In this paper, we propose a proximal point type algorithms for solving pseudomonotone equilibrium problems, which allows us to relax the convexity assumption. More specifically, we demand strong quasiconvexity, rather than convexity, in the second argument of the bifunction. Additionally, we show that the convergence analysis of another well-known proximal point method for equilibrium problems, presented in [21], holds under the assumption of strong quasiconvexity, instead of convexity, in the second argument of the bifunction.

At this point, it is appropriate to discuss the relevance of extending the analysis of the proximal point algorithm to the case of quasiconvex bifunctions, rather than plainly convex ones.

In the first place, quasiconvexity is a notion with a very simple and geometric appeal: while convex functions are those with a convex epigraph, quasiconvex functions are those with convex sublevel sets, so that to some extent both notions are equally natural.

In the second place, a substantial number of applications deal with quasiconvex functions which are not convex. We mention three of them.

A typical class of quasiconvex functions appears in fractional min–max programming, where the objective function is the maximum of quotients with nonnegative convex numerators and positive concave denominators. These functions are quasiconvex, and no transformation is known which turns these problems into convex ones. Instances of problems of these types in Economic Theory, Financial Theory and Approximation Theory (in the sense of Chebyshev discrete rational approximation) can be found in [12, Sect. 4].

Another class of quasiconcave functions can be found in Economic Theory, and consists of the utility functions of consumers who maximize their utility under budget constraints. Such utility functions are quasiconcave under very natural assumptions on the consumer's preferences, while preferences which produce concave utility functions turn out to be quite artificial; see, e.g., [31].

A third class of quasiconvex functions appears in a very natural way in Location Theory, see [8, Chapter 6].

The structure of the paper is as follows. In Sect. 2, we set up notation, preliminaries and basic definitions related to generalized convexity, generalized monotonicity and asymptotic analysis. In Sect. 3, we analyze the assumptions that we will assume along the paper and we propose our proximal point algorithm, based on the usual regularization for the optimization case, applied to the bifunction in its second argument. After proving the usual properties of the generated sequence, we establish sufficient

conditions which ensure the convergence of the generated sequence to a solution of the pseudomonotone equilibrium problem, assuming strong quasiconvexity in the second argument of the bifunction. We also show that the convergence results for a method which uses another regularization term for the bifunction presented in [21], still hold, assuming strong quasiconvexity of the bifunction in its second argument. Finally, in Sect. 4 we present examples of equilibrium problems which satisfy our assumptions.

2 Preliminaries and Basic Definitions

The inner product of \mathbb{R}^n and the Euclidean norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty set in \mathbb{R}^n ; its closure is denoted by $\text{cl } K$, its boundary by $\text{bd } K$, its topological interior by $\text{int } K$ and its convex hull by $\text{conv } K$. We denote $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}_{++} =]0, +\infty[$.

Given any $x, y, z \in \mathbb{R}^n$ and any $\zeta \in \mathbb{R}$, the following relations hold:

$$\langle x - z, y - x \rangle = \frac{1}{2}\|z - y\|^2 - \frac{1}{2}\|x - z\|^2 - \frac{1}{2}\|y - x\|^2, \tag{2.1}$$

$$\|\zeta x + (1 - \zeta)y\|^2 = \zeta\|x\|^2 + (1 - \zeta)\|y\|^2 - \zeta(1 - \zeta)\|x - y\|^2. \tag{2.2}$$

Given any extended-valued function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of h is defined by $\text{dom } h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. It is said that h is proper if $\text{dom } h$ is nonempty and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$. The notion of properness is important when dealing with minimization problems.

We denote as $\text{epi } h := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$ the epigraph of h , as $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ the sublevel set of h at the height $\lambda \in \mathbb{R}$ and as $\text{argmin}_{\mathbb{R}^n} h$ the set of all minimal points of h . A function h is lower semicontinuous (lsc henceforth) at $\bar{x} \in \mathbb{R}^n$ if for any sequence $\{x_k\}_k \in \mathbb{R}^n$ with $x_k \rightarrow \bar{x}$, $h(\bar{x}) \leq \liminf_{k \rightarrow +\infty} h(x_k)$. Furthermore, the current convention $\sup_\emptyset h := -\infty$ and $\inf_\emptyset h := +\infty$ is adopted.

A function h with convex domain is said to be:

- (a) convex if, given any $x, y \in \text{dom } h$, then

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y), \quad \forall \lambda \in [0, 1]; \tag{2.3}$$

- (b) semistrictly quasiconvex if, given any $x, y \in \text{dom } h$, with $h(x) \neq h(y)$, then

$$h(\lambda x + (1 - \lambda)y) < \max\{h(x), h(y)\}, \quad \forall \lambda \in]0, 1[; \tag{2.4}$$

- (c) quasiconvex if, given any $x, y \in \text{dom } h$, then

$$h(\lambda x + (1 - \lambda)y) \leq \max\{h(x), h(y)\}, \quad \forall \lambda \in [0, 1]. \tag{2.5}$$

It is said that h is *strictly convex* (resp. *strictly quasiconvex*) if the inequality in (2.3) (resp. (2.5)) is strict.

Note that every (strictly) convex function is (strictly) quasiconvex and semistrictly quasiconvex, and every lsc and semistrictly quasiconvex function is quasiconvex (see [5, Theorem 2.3.2]). The continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$, with $h(x) := \min\{|x|, 1\}$, is quasiconvex without being semistrictly quasiconvex. Recall that

$$\begin{aligned} h \text{ is convex} &\iff \text{epi } h \text{ is a convex set;} \\ h \text{ is quasiconvex} &\iff S_\lambda(h) \text{ is a convex set for all } \lambda \in \mathbb{R}. \end{aligned}$$

For algorithmic purposes, the following notions from [35] are very useful (see also [1,38,39]). We use the notation of [2, Definition 5.16] for simplicity.

A function h with a convex domain is said to be:

- (a) strongly convex on $\text{dom } h$ if there exists $\gamma \in]0, +\infty[$ such that for all $x, y \in \text{dom } h$ and all $\lambda \in [0, 1]$, we have

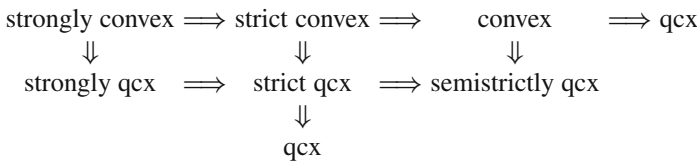
$$h(\lambda y + (1 - \lambda)x) \leq \lambda h(y) + (1 - \lambda)h(x) - \lambda(1 - \lambda)\frac{\gamma}{2}\|x - y\|^2, \quad (2.6)$$

- (b) strongly quasiconvex on $\text{dom } h$ if there exists $\gamma \in]0, +\infty[$ such that for all $x, y \in \text{dom } h$ and all $\lambda \in [0, 1]$, we have

$$h(\lambda y + (1 - \lambda)x) \leq \max\{h(y), h(x)\} - \lambda(1 - \lambda)\frac{\gamma}{2}\|x - y\|^2. \quad (2.7)$$

In these cases, it is said that h is strongly convex (resp. quasiconvex) with modulus $\gamma > 0$. Note that every strongly convex function is strongly quasiconvex with the same modulus, and every strongly quasiconvex function is strictly quasiconvex. The Euclidean norm $h_1(x) = \|x\|$ is strongly quasiconvex on any bounded convex set $K \subseteq \mathbb{R}^n$ (see [23, Theorem 2]) without being strongly convex, and the function $h_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by $h_2(x) = \max\{\sqrt{|x|}, 2\}$ is semistrictly quasiconvex without being strongly quasiconvex.

Summarizing (quasiconvex is denoted by qcx):



The function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $h(x) = \sqrt{\|x\|}$ is strongly quasiconvex on every bounded and convex set (see [27, Theorem 16]).

A proper function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be:

- (i) 2-supercoercive, if

$$\liminf_{\|x\| \rightarrow +\infty} \frac{h(x)}{\|x\|^2} > 0, \quad (2.8)$$

(ii) supercoercive, if

$$\lim_{\|x\| \rightarrow +\infty} \frac{h(x)}{\|x\|} = +\infty, \tag{2.9}$$

(iii) 1-supercoercive, if

$$\liminf_{\|x\| \rightarrow +\infty} \frac{h(x)}{\|x\|} > 0, \tag{2.10}$$

(iv) coercive, if

$$\lim_{\|x\| \rightarrow +\infty} h(x) = +\infty. \tag{2.11}$$

or equivalently, if $S_\lambda(h)$ is bounded for all $\lambda \in \mathbb{R}$.

Clearly, (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv), but the converse statements do not hold. Indeed, the function $h(x) = |x|^{3/2}$ is supercoercive without being 2-supercoercive, the function $h(x) = |x|$ is 1-supercoercive without being supercoercive and the function $h(x) = \sqrt{x}$ is coercive without being 1-supercoercive.

Recently, it was proved in [27] that every strongly quasiconvex function is 2-supercoercive, namely,

Lemma 2.1 ([27, Theorem 1]) *Let K be a convex set in \mathbb{R}^n and $h : K \rightarrow \mathbb{R}$ be a strongly quasiconvex function. Then, h is 2-supercoercive (in particular, supercoercive).*

As a consequence, every lsc and strongly quasiconvex function has exactly one minimizer on every closed and convex subset K of \mathbb{R}^n (see [27, Corollary 3]). Therefore, Lemma 2.1 is useful for analyzing proximal point algorithms for classes of quasiconvex functions (see [27]).

Let K be a closed and convex set in \mathbb{R}^n and $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a proper function with $K \subseteq \text{dom } h$. Then, the proximity operator on K of parameter $\beta > 0$ of h at $x \in \mathbb{R}^n$ is defined as $\text{Prox}_{\beta h} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ where

$$\text{Prox}_{\beta h}(K, x) = \operatorname{argmin}_{y \in K} \left\{ h(y) + \frac{1}{2\beta} \|y - x\|^2 \right\}. \tag{2.12}$$

If $K = \mathbb{R}^n$, then $\text{Prox}_{\beta h}(\mathbb{R}^n, x) = \text{Prox}_{\beta h}(x)$.

The properties of proximity operators are essential for studying proximal point algorithms. In the quasiconvex case, the following result was obtained in [27]. We remark that if h is strongly quasiconvex, then $\gamma > 0$, and if h is quasiconvex, then $\gamma = 0$.

Lemma 2.2 ([27, Proposition 7]) *Let K be a closed and convex set in \mathbb{R}^n , $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a proper, lsc, strongly quasiconvex function with modulus $\gamma \geq 0$ and such that*

$K \subseteq \text{dom } h, \beta > 0$ and $x \in K$. If $\bar{x} \in \text{Prox}_{\beta h}(K, x)$, then

$$h(\bar{x}) - \max\{h(y), h(\bar{x})\} \leq \frac{\lambda}{\beta} \|\bar{x} - x, y - \bar{x}\| + \frac{\lambda}{2} \left(\frac{\lambda}{\beta} - \gamma + \lambda\gamma \right) \|y - \bar{x}\|^2, \quad \forall y \in K, \forall \lambda \in [0, 1]. \tag{2.13}$$

For analyzing our algorithms, the following result due to Opial [34] will be useful:

Lemma 2.3 *Let $\{x^k\}_k$ be a sequence in \mathbb{R}^n . If there exists a nonempty closed set $S \subseteq \mathbb{R}^n$ such that:*

- (a) *for every $z \in S$, we have $\lim_{k \rightarrow +\infty} \|x^k - z^k\|$ exists;*
- (b) *any cluster point of $\{x^k\}_k$ belongs to S ;*

Then, there exists $\bar{x} \in S$ such that $x^k \rightarrow \bar{x}$.

Given a nonempty set C in \mathbb{R}^n and a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, f is said to be:

- (i) *monotone on C , if for all $x, y \in C$, we have*

$$f(x, y) + f(y, x) \leq 0. \tag{2.14}$$

- (ii) *pseudomonotone on C , if for all $x, y \in C$, we have*

$$f(x, y) \geq 0 \implies f(y, x) \leq 0. \tag{2.15}$$

- (iii) *quasimonotone on C , if for all $x, y \in C$, we have*

$$f(x, y) > 0 \implies f(y, x) \leq 0. \tag{2.16}$$

Every monotone bifunction is pseudomonotone, and every pseudomonotone bifunction is quasimonotone. The reverse statements are not true in general (see, for instance, [15]).

Now, we recall the following asymptotic function, introduced in [18] for proving existence results for pseudomonotone equilibrium problems in the quasiconvex case (see [11,16] for the minimization problem). Note that the definition refers to the second argument of f .

Definition 2.1 Assume that the bifunction f satisfies that $f(x, x) = 0$ for all $x \in K$. The q -asymptotic function of f at $u \in K^\infty$ is defined as:

$$f_q^\infty(u) := \sup_{y \in K} \sup_{t > 0} \frac{f(y, y + tu)}{t}. \tag{2.17}$$

Clearly, $f_q^\infty(0) = 0$. If $f(y, y + tu) = h(y + tu) - h(y)$, then for every $u \in \mathbb{R}^n$ we have $f_q^\infty = h_q^\infty$, which is the q -asymptotic function for the scalar case (see [11,16]).

If h is lsc and convex, then $f_q^\infty = h_q^\infty = h^\infty$ coincides with the usual asymptotic (recession) function from convex analysis:

$$h^\infty(u) := \sup_{t>0} \frac{h(y + tu) - h(y)}{t} = \lim_{t \rightarrow +\infty} \frac{h(y + tu) - h(y)}{t}. \tag{2.18}$$

For each $x \in K$, we extend $f(x, \cdot)$ to the whole \mathbb{R}^n by setting $f(x, y) = +\infty$ if $y \in \mathbb{R}^n \setminus K$. Hence, if $u \notin K^\infty$, then $f_q^\infty(u) = +\infty$.

For additional results on strongly quasiconvex functions, generalized convexity and generalized monotonicity, refer to [5,15,23,27,38,39].

3 A Proximal Point Algorithm for Equilibrium Problems

Let K be a closed and convex set in \mathbb{R}^n and $f : K \times K \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem is defined by

$$\text{find } \bar{x} \in K : f(\bar{x}, y) \geq 0, \forall y \in K. \tag{3.1}$$

Its solution set is denoted by $S(K, f)$.

We discuss next the assumptions on the equilibrium problem which will be used in the analysis of our algorithm.

- (A1) For every $x \in K$, the function $f(x, \cdot)$ is lsc, and for every $y \in K$, the function $f(\cdot, y)$ is usc.
- (A2) f is pseudomonotone on K .
- (A3) f is lsc (jointly in both arguments).
- (A4) For every $x \in K$, the function $f(x, \cdot)$ is strongly quasiconvex on K with modulus $\gamma > 0$.
- (A5) f satisfies the following Lipschitz condition: There exists $\eta > 0$ such that

$$f(x, z) - f(x, y) - f(y, z) \leq \eta \left(\|x - y\|^2 + \|y - z\|^2 \right), \forall x, y, z \in K. \tag{3.2}$$

- (A6) The Lipschitz constant η and the modulus of quasiconvexity γ are such that $12\eta < \gamma$.

We will state an additional assumption, which is standard in the theory of equilibrium problems. In our case, it is a consequence of (A2) and (A5), as shown in Remark 3.1(i), so that we chose to exclude it from the basic list, in order to avoid redundant hypotheses. We'll use it in some intermediate results which do not require (A5).

$$(A0) f(x, x) = 0 \text{ for all } x \in K.$$

We mention that (A1), (A2) and (A3) are rather standard assumptions in the literature on our subject. On the other hand, (A4), (A5) and (A6) substitute for the

standard hypothesis of convexity of f in its second argument, and are specific to our presentation.

We discuss next these assumptions.

Remark 3.1 (i) Note that (A2) and (A5) imply (A0). Indeed, setting $x = y = z$ in (A5) gives $f(x, x) \geq 0$. On the other hand, (A2) implies $f(x, x) \leq 0$, so that $f(x, x) = 0$ for all $x \in K$.

(ii) If $f(x, y) = \langle T(x), y - x \rangle$, where $T : K \rightarrow \mathbb{R}^n$ is a L -Lipschitz and continuous operator with $L > 0$ (i.e., $\|T(x) - T(y)\| \leq L\|x - y\|$ for all $x, y \in K$), then f satisfies (3.2) with constant $\eta = \frac{L}{2}$. Indeed, for all $x, y, z \in K$, we have

$$f(x, y) + f(y, z) - f(x, z) = -\langle T(y) - T(x), y - z \rangle \geq -2\eta\|y - x\|\|y - z\| \geq -\eta\left(\|y - x\|^2 + \|y - z\|^2\right).$$

(iii) It is relevant to exhibit instances of equilibrium problems which satisfy (Ai) ($i = 1, 2, 3, 4, 5, 6$). We present them in Sect. 4.

We recall the following existence result, which will be useful in the sequel; other existence results for classes of nonconvex equilibrium problems may be found in [6,18,28,33].

Lemma 3.1 ([18, Theorem 3.1]) *Suppose that K is a closed and convex set, that f satisfies (Ai) with $i = 0, 1, 2, 4$ and that for every $x \in K$, $f(x, \cdot)$ is quasiconvex (in particular, strongly quasiconvex) on K . If $f_q^\infty(u) > 0$ for all $u \neq 0$, then $S(K, f) \neq \emptyset$ and compact.*

We emphasize that $f_q^\infty(u) > 0$ for all $u \neq 0$ does not imply that f is coercive on the second argument as the function $f(x, y) = h(y) - h(x)$ with $h : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $h(x) = \frac{\|x\|}{1+\|x\|}$ shows.

Next we present our algorithm, where at each step we use the standard proximal point method for optimization, applied for minimizing the bifunction f in its second argument, with the first argument taken as the previous iterate.

The precise statement of Algorithm 1 is given below:

Algorithm 1 PPA-1 for Strongly Quasiconvex EP's

Step 0. Take $x^1 \in K, k = 0$ and a sequence $\{\beta_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_{++}$ bounded away from 0.

Step 1. Take $k = k + 1$, and

$$x^{k+1} \in \operatorname{argmin}_{x \in K} \left(f(x^k, x) + \frac{1}{2\beta_k} \|x^k - x\|^2 \right). \tag{3.3}$$

Step 2. If $x^{k+1} = x^k$, then STOP: $\{x^k\} = S(K, f)$. Otherwise, go to Step 1.

We mention that (3.3) is just the iterative step of the standard proximal point method for optimization applied to the function f^k defined as $f^k(y) = f(x^k, y)$. As a first result, we have:

Proposition 3.1 *Let K be a closed and convex set in \mathbb{R}^n and suppose that f satisfies (A0), (A1), (A2) and (A4). Then, $S(f, K)$ is a singleton.*

Proof By (A1) and (A4), f is lsc and strongly quasiconvex in its second argument. We conclude from Lemma 2.2 that f is lsc and 2-supercoercive in its second argument. Hence, $f_q^\infty(u) > 0$ for all $u \neq 0$. Therefore, $S(K, f)$ is nonempty and compact by Lemma 3.1.

Now, take $x_1, x_2 \in S(K, f)$ with $x_1 \neq x_2$, then $f(x_1, x_2) \geq 0$ and $f(x_2, x_1) \geq 0$. By pseudomonotonicity, $f(x_2, x_1) = 0$. Using (A0) and strong quasiconvexity of $f(x_2, \cdot)$, we get $f(x_2, \frac{x_2+x_1}{2}) < 0$, a contradiction. Therefore, $S(f, K)$ is a singleton. \square

Now, we validate the stopping criteria.

Proposition 3.2 *Let K be a closed and convex set in \mathbb{R}^n , $\{\beta_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ be the sequence generated by Algorithm 1 and suppose that f satisfies (A0), (A1) and (A4). If $x^{k+1} = x^k$, then $x^k \in S(K, f)$.*

Proof If $x^{k+1} = x^k$, then it follows from (3.3) that

$$\begin{aligned} f(x^k, x^{k+1}) + \frac{1}{2\beta_k} \|x^k - x^{k+1}\|^2 &\leq f(x^k, x) + \frac{1}{2\beta_k} \|x^k - x\|^2, \forall x \in K \\ \implies 0 &\leq f(x^k, x) + \frac{1}{2\beta_k} \|x^k - x\|^2, \forall x \in K. \end{aligned} \tag{3.4}$$

Take $x = \lambda y + (1 - \lambda)x^k$ with $y \in K \setminus \{x^k\}$ and $\lambda \in [0, 1]$. Since f is strongly quasiconvex with modulus $\gamma > 0$ in its second argument,

$$\begin{aligned} 0 &\leq f(x^k, \lambda y + (1 - \lambda)x^k) + \frac{1}{2\beta_k} \|\lambda(x^k - y)\|^2 \\ &\leq \max\{f(x^k, y), 0\} + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^k - y\|^2, \forall y \in K, \forall \lambda \in]0, 1]. \end{aligned}$$

Since $\{\beta_k\}_k$ is a positive sequence bounded away from zero, we take λ such that $0 < \lambda < \frac{\beta_k\gamma}{1+\beta_k\gamma}$ for all $k \in \mathbb{N}$. Then, $\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma < 0$ for all k , thus,

$$0 < \max\{f(x^k, y), 0\} = f(x^k, y), \forall y \in K \setminus \{x^k\},$$

i.e., $x^k \in S(K, f)$. \square

In the case when $x^{k+1} \neq x^k$, we have the following result.

Proposition 3.3 *Let K be a closed and convex set in \mathbb{R}^n , $\{\beta_k\}_k$ a sequence of positive numbers, $\{x^k\}_k$ the sequence generated by Algorithm 1, and suppose that f satisfies (Ai) with $i = 0, 1, 2$. If $x^{k+1} \neq x^k$, then $f(x^k, x^{k+1}) < 0$.*

Proof Take $x = x^k$ in (3.4) and the result follows. \square

Remark 3.2 If $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a lsc and strongly quasiconvex function with modulus $\gamma > 0$, then taking $f(x, y) = h(y) - h(x)$, we recover [27, Proposition 8] and [27, Theorem 15], from Propositions 3.2 and 3.3, respectively.

For establishing the boundedness of the sequence $\{x^k\}_k$ generated by Algorithm 1, we start with the following result.

Proposition 3.4 *Let K be a closed and convex set in \mathbb{R}^n , $\{\beta_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ be the sequence generated by Algorithm 1 and suppose that f satisfies (Ai) with $i = 1, 2, 4, 5$. Take $\bar{x} \in S(K, f)$. Then, for every $k \in \mathbb{N}$, at least one of the following inequalities holds:*

$$\left(\frac{1 + \gamma\beta_k}{2}\right) \|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \|x^{k+1} - x^k\|^2, \quad (3.5)$$

$$\left(\frac{1 + \gamma\beta_k}{8\beta_k} - \eta\right) \|x^{k+1} - \bar{x}\|^2 \leq \frac{1}{4\beta_k} \|x^k - \bar{x}\|^2 - \left(\frac{1}{4\beta_k} - \eta\right) \|x^{k+1} - x^k\|^2. \quad (3.6)$$

Proof Since f satisfies (Ai) with $i = 1, 2, 4, 5$, f satisfies (A0), thus $S(K, f)$ is a singleton by Proposition 3.1. Then, applying Lemma 2.2 with $\gamma > 0$ to f as a function of its second argument, in step (3.3) for $k + 1$, we have

$$\begin{aligned} f(x^k, x^{k+1}) - \max\{f(x^k, x^{k+1}), f(x^k, x)\} &\leq \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, x - x^{k+1} \rangle \\ &+ \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right) \|x - x^{k+1}\|^2, \quad \forall x \in K, \quad \forall \lambda \in [0, 1]. \end{aligned} \quad (3.7)$$

Taking $x = \bar{x} \in S(K, f)$ in Eq. (3.7) with $\gamma > 0$, we have

$$\begin{aligned} f(x^k, x^{k+1}) - \max\{f(x^k, x^{k+1}), f(x^k, \bar{x})\} &\leq \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle \\ &+ \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right) \|x^{k+1} - \bar{x}\|^2, \quad \forall \lambda \in [0, 1]. \end{aligned}$$

We consider two cases:

(i) If $f(x^k, x^{k+1}) \geq f(x^k, \bar{x})$, then

$$0 \leq \frac{1}{\beta_k} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle + \frac{1}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right) \|x^{k+1} - \bar{x}\|^2, \quad \forall \lambda \in]0, 1].$$

Taking $\lambda = \frac{1}{2}$ and using (2.1),

$$\begin{aligned} \frac{1 + \gamma\beta_k}{2\beta_k} \|x^{k+1} - \bar{x}\|^2 &\leq \frac{1}{\beta_k} \|x^k - \bar{x}\|^2 - \frac{1}{\beta_k} \|x^{k+1} - x^k\|^2, \\ \implies \frac{1 + \gamma\beta_k}{2} \|x^{k+1} - \bar{x}\|^2 &\leq \|x^k - \bar{x}\|^2 - \|x^{k+1} - x^k\|^2, \end{aligned}$$

and the inequality (3.5) hold.

(ii) If $f(x^k, x^{k+1}) < f(x^k, \bar{x})$, then

$$\begin{aligned} 0 &\leq \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^{k+1} - \bar{x}\|^2 \\ &\quad + f(x^k, \bar{x}) - f(x^k, x^{k+1}) \\ &\leq \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^{k+1} - \bar{x}\|^2 \\ &\quad + f(x^{k+1}, \bar{x}) + \eta \left(\|x^{k+1} - x^k\|^2 + \|x^{k+1} - \bar{x}\|^2 \right), \end{aligned}$$

where the last inequality follows from assumption (A5). Furthermore, since $\bar{x} \in S(K, f)$ and f is pseudomonotone, $f(x^{k+1}, \bar{x}) \leq 0$, so that

$$\begin{aligned} 0 &\leq \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^{k+1} - \bar{x}\|^2 \\ &\quad + \eta \left(\|x^{k+1} - x^k\|^2 + \|x^{k+1} - \bar{x}\|^2 \right), \quad \forall \lambda \in [0, 1]. \end{aligned}$$

Using (2.1), we have

$$\begin{aligned} 0 &\leq \frac{\lambda}{2\beta_k} \|x^k - \bar{x}\|^2 + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma - \frac{1}{\beta_k} \right) \|x^{k+1} - \bar{x}\|^2 \\ &\quad - \frac{\lambda}{2\beta_k} \|x^{k+1} - x^k\|^2 + \eta \left(\|x^{k+1} - x^k\|^2 + \|x^{k+1} - \bar{x}\|^2 \right), \quad \forall \lambda \in [0, 1]. \end{aligned}$$

Take $\lambda = \frac{1}{2}$. Then,

$$\left(\frac{1 + \gamma\beta_k}{8\beta_k} - \eta \right) \|x^{k+1} - \bar{x}\|^2 \leq \frac{1}{4\beta_k} \|x^k - \bar{x}\|^2 - \left(\frac{1}{4\beta_k} - \eta \right) \|x^{k+1} - x^k\|^2,$$

and (3.6) hold.

Therefore, for every $k \in \mathbb{N}$, at least one among (3.5) and (3.6) holds. □

In the following proposition, we present sufficient conditions for the boundedness of the sequence $\{x^k\}_k$ generated by Algorithm 1.

Proposition 3.5 *Let $K \subset \mathbb{R}^n$ be a closed and convex set and suppose that assumptions (Ai) with $i = 1, 2, 4, 5, 6$ hold. Assume that $\{\beta_k\}_k \subset]\frac{1}{\gamma-8\eta}, \frac{1}{4\eta}[$. Then, the sequence $\{x^k\}_k$ generated by Algorithm 1 is bounded.*

Proof First note that Assumption (A6) ensures that $\frac{1}{\gamma-8\eta} < \frac{1}{4\eta}$, so that the prescribed choice of the regularization parameters β_k is possible. In view of Proposition 3.1, $S(K, f)$ is a singleton. Take $\{\bar{x}\} = S(K, f)$. Then, for every $k \in \mathbb{N}$, by Proposition 3.4, one among (3.5) and (3.6) holds. We have two cases:

(i) Suppose that Eq. (3.5) hold. Since $\beta_k > \frac{1}{\gamma-8\eta} > \frac{1}{\gamma}$ for all k , we have

$$\|x^{k+1} - \bar{x}\|^2 < \frac{1 + \gamma\beta_k}{2} \|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2. \tag{3.8}$$

(ii) Suppose that Eq. (3.6) hold. Since $\beta_k < \frac{1}{4\eta}$, we have

$$\begin{aligned} \left(\frac{1 + \gamma\beta_k}{8\beta_k} - \eta\right) \|x^{k+1} - \bar{x}\|^2 &\leq \frac{1}{4\beta_k} \|x^k - \bar{x}\|^2 - \left(\frac{1}{4\beta_k} - \eta\right) \|x^{k+1} - x^k\|^2 \\ &\leq \frac{1}{4\beta_k} \|x^k - \bar{x}\|^2. \end{aligned}$$

Since $\beta_k > \frac{1}{\gamma-8\eta} > \frac{1}{\gamma}$ for all k , it follows that

$$\begin{aligned} \frac{1}{4\beta_k} \|x^k - \bar{x}\|^2 &\geq \left(\frac{1 + \gamma\beta_k}{8\beta_k} - \eta\right) \|x^{k+1} - \bar{x}\|^2 \\ &\geq \left(\frac{1 + \gamma\beta_k}{8\beta_k} - \frac{\gamma\beta_k - 1}{8\beta_k}\right) \|x^{k+1} - \bar{x}\|^2 = \frac{1}{4\beta_k} \|x^{k+1} - \bar{x}\|^2. \end{aligned}$$

Then,

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2. \tag{3.9}$$

Therefore, in both cases the sequence $\{\|x^k - \bar{x}\|^2\}_k$ is nonnegative and nonincreasing, hence is bounded. It follows that the sequence $\{x^k\}_k$ is also bounded. \square

Our first main result, which shows that the sequence generated by Algorithm 1 converges to a solution of problem (3.1), is given next.

Theorem 3.1 *Let $K \subset \mathbb{R}^n$ be a closed and convex set in \mathbb{R}^n . Assume that $\{\beta_k\}_k \subset]\frac{1}{\gamma-8\eta}, \frac{1}{4\eta}[$ and that assumptions (Ai) with $i = 1, 2, 3, 4, 5, 6$ hold. Let $\{x^k\}_k$ be the sequence generated by Algorithm 1. Then, $\{x^k\}_k$ converges to a point $\bar{x} \in S(K, f)$.*

Proof The sequence $\{x^k\}_k$ is bounded by Proposition 3.5. Let $\hat{x} \in K$ be any cluster point of $\{x^k\}_k$, so that there exists a subsequence $\{x^{k_\ell}\}_\ell \subseteq \{x^k\}_k$ such that $x^{k_\ell} \rightarrow \hat{x}$ as $\ell \rightarrow +\infty$. It follows from step (3.3) that

$$\begin{aligned} f(x^k, x^{k+1}) + \frac{1}{2\beta_k} \|x^k - x^{k+1}\|^2 &\leq f(x^k, x) + \frac{1}{2\beta_k} \|x^k - x\|^2, \quad \forall x \in K \\ \implies f(x^k, x^{k+1}) &\leq f(x^k, x) + \frac{1}{2\beta_k} \|x^k - x\|^2, \quad \forall x \in K. \end{aligned}$$

Take $x = \lambda y + (1 - \lambda)\widehat{x}$ with $y \in K$ and $\lambda \in [0, 1]$. Then,

$$\begin{aligned} f(x^k, x^{k+1}) &\leq f(x^k, \lambda y + (1 - \lambda)\widehat{x}) + \frac{1}{2\beta_k} \|\lambda(x^k - y) + (1 - \lambda)(x^k - \widehat{x})\|^2 \\ &\leq \max\{f(x^k, y), f(x^k, \widehat{x})\} - \frac{\lambda(1 - \lambda)\gamma}{2} \|y - \widehat{x}\|^2 \\ &\quad + \frac{\lambda}{2\beta_k} \|x^k - y\|^2 + \frac{(1 - \lambda)}{2\beta_k} \|x^k - \widehat{x}\|^2 - \frac{\lambda(1 - \lambda)}{2\beta_k} \|y - \widehat{x}\|^2 \\ &= \max\{f(x^k, y), f(x^k, \widehat{x})\} + \frac{\lambda}{\beta_k} \langle \widehat{x} - x^k, y - \widehat{x} \rangle + \frac{1}{2\beta_k} \|x^k - \widehat{x}\|^2 \\ &\quad + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|y - \widehat{x}\|^2, \forall y \in K, \forall \lambda \in [0, 1]. \end{aligned}$$

For the sake of a shorter notation, define $\widehat{\beta} := \frac{1}{\gamma - 8\eta}$. Replace k by k_ℓ in the previous inequalities. Then, taking $\limsup_{\ell \rightarrow +\infty}$ and using Assumption (A3), we have

$$\begin{aligned} 0 = f(\widehat{x}, \widehat{x}) &\leq \liminf_{\ell \rightarrow +\infty} f(x^{k_\ell}, x^{k_\ell+1}) \leq \limsup_{\ell \rightarrow +\infty} f(x^{k_\ell}, x^{k_\ell+1}) \\ &\leq \max\{f(\widehat{x}, y), 0\} + \frac{\lambda}{2} \left(\frac{\lambda}{\widehat{\beta}} - \gamma + \lambda\gamma \right) \|y - \widehat{x}\|^2, \forall y \in K, \forall \lambda \in [0, 1]. \end{aligned}$$

Since $\gamma > 0$, we take $\lambda < \frac{\widehat{\beta}\gamma}{1 + \widehat{\beta}\gamma}$. Then, $(\frac{\lambda}{\widehat{\beta}} - \gamma + \lambda\gamma) < 0$. Thus,

$$0 < \max\{f(\widehat{x}, y), 0\} = f(\widehat{x}, y), \forall y \in K \setminus \{\widehat{x}\}.$$

Therefore, $\widehat{x} \in S(K, f)$.

Since f is strongly quasiconvex in its second argument, $S(K, f)$ is a singleton by Proposition 3.1. Thus, the whole sequence $\{x^k\}_k$ converges to $\bar{x} \in S(K, f)$ and the proof is complete. \square

In particular, we recover [27, Theorem 15], which establishes convergence of the proximal point algorithm for minimizing problem of strongly quasiconvex functions, as we show next.

Corollary 3.1 *Let K be a convex and closed set in \mathbb{R}^n , $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a lsc and strongly quasiconvex function on K with modulus $\gamma > 0$ and $\{\beta_k\}_k$ be a sequence of positive numbers bounded away from 0. Let us consider the problem:*

$$\min_{x \in K} h(x), \tag{3.10}$$

and the bifunction $f_h : K \times K \rightarrow \mathbb{R}$ given by:

$$f_h(x, y) := h(y) - h(x), \forall x, y \in K. \tag{3.11}$$

Then, the sequence $\{x^k\}_k$, generated by Algorithm 1 applied to f_h , converges to the unique minimizer of h in K .

Proof Note that for f_h , assumptions (A2) and (A4) hold immediately. Furthermore, since h is lsc, f_h is lsc on its first argument and usc on its second argument, so assumption (A1) holds, which easily entails that (A3) also holds. Finally, by definition of f_h , we have

$$f_h(x, z) - f_h(x, y) - f_h(y, z) = h(z) - h(x) - (h(y) - h(x)) - (h(z) - h(y)) = 0,$$

i.e., assumption (A5) holds with $\eta = 0$, and hence assumption (A6) also holds, since $\gamma > 0$. Hence, all assumptions needed for our previous results are valid in this case.

Therefore, it follows from Propositions 3.2 and 3.3 that if $x^{k+1} = x^k$, then $x^k \in \operatorname{argmin}_K h$, and that if $x^{k+1} \neq x^k$, then $h(x^{k+1}) < h(x^k)$. Moreover, since h is lsc and strongly quasiconvex and K is closed and convex, $\operatorname{argmin}_K h$ is nonempty by Lemma 2.1. Therefore, by Theorem 3.1, $x^k \rightarrow \bar{x} = \operatorname{argmin}_K h$. \square

Remark 3.3 Strongly quasiconvex functions are not restricted to “prox-convex” functions (see [13]). Take $K := [-\rho, \rho]$ with $\rho \geq 2$, and $h : K \rightarrow \mathbb{R}$ with $h(x) = \sqrt{|x|}$. Here h is strongly quasiconvex by [27, Proposition 15] without being prox-convex. Indeed, take $\alpha = 1$ and $z = 1, 5$, then $\operatorname{Prox}_{\alpha h}(K, z) = \{0, 1\}$ is not a singleton, thus h is not prox-convex.

In [32] (see also [4,21,28]), another regularization procedure was introduced for defining a proximal point algorithm. Instead of regularizing the function $f(x, \cdot)$, we use a regularization bifunction on the bifunction f (see Algorithm 2). In [21], convergence of this algorithm to the solution of the equilibrium problem was proved under (A0), (A1), (A2), (A3) and convexity of $f(x, \cdot)$ for all $x \in K$. We will show that the result holds assuming (A1)-(A6). The precise statement of this method, to be denoted as Algorithm 2, follows.

Algorithm 2 Iusem-Sosa [21] for Strongly Quasiconvex EP’s

Step 0. Take $x^1 \in K, k = 0, \theta > 0$ and a sequence $\{\beta_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_{++}$ such that $\beta_k \geq \theta$ for all k .

Step 1. Take $k = k + 1$, define $f_k(x, y) = f(x, y) + \frac{1}{\beta_k} \langle x - x^k, y - x \rangle$ and take

$$x^{k+1} \in S(K, f_k). \tag{3.12}$$

Step 2. If $x^{k+1} = x^k$, then STOP, $x^k \in S(K, f)$. Otherwise, go to Step 1.

We prove next that under our assumptions the sequence generated by Algorithm 2 is well defined.

Proposition 3.6 Take $\theta \in \mathbb{R}_{++}$. Let K be a convex and closed set in \mathbb{R}^n and $\{\beta_k\}_k$ be a sequence such that $\beta_k \geq \theta$ for all k . Suppose that assumptions (Ai) with $i = 1, 2, 3, 4, 5$ are satisfied. Then, $S(K, f)$ is nonempty and the sequence $\{x^k\}_k$ generated by Algorithm 2 is well defined.

Proof By Lemma 2.1, (A4) implies that $f(x, y)$ is 2-supercoercive in y , and since (A1) holds, $S(K, f)$ is nonempty. It remains to check that $S(K, f_k) \neq \emptyset$ for all $k \in \mathbb{N}$. Since the regularization term $\frac{1}{\beta} \langle x - z, y - x \rangle$ is linear in y for all $z \in K, \beta > 0$, it follows easily that $f(x, y) + \frac{1}{\beta} \langle x - z, y - x \rangle$ is also 2-supercoercive in y for all $z \in K, \beta > 0$, so that $f_{z,\beta}(x, \cdot)$ attains its minimum on K and hence $S(K, f_{z,\beta}) \neq \emptyset$ for all $z \in K$ and all $\beta > 0$. Hence, $S(K, f_k) \neq \emptyset$ for all $k \in \mathbb{N}$. \square

As a corollary, a convergence result for Algorithm 2 in the strongly quasiconvex case is obtained.

Corollary 3.2 *Take $\theta \in \mathbb{R}_{++}$. Let K be a convex and closed set in \mathbb{R}^n and $\{\beta_k\}_k$ be a sequence such that $\beta_k > \theta$ for all k and $\{x^k\}_k$ be the sequence generated by Algorithm 2. Suppose that assumptions (Ai) with $i = 1, 2, 3, 4, 5$ hold. Then, $x^k \rightarrow \bar{x} \in S(K, f)$.*

Proof In [21, Theorem 1], convergence of the sequence generated by Algorithm 2 to point in $S(K, f)$ was proved under the following assumptions: (A0), (A1), (A2), and in addition existence of solutions of the equilibrium problems and convexity of f in its second argument. In our setting, (A0) follows from (A2) and (A5), as shown in Remark 3.1(i), and nonemptiness of $S(K, f)$ follows from Proposition 3.6, so that we only need to deal with the assumption of convexity of $f(x, \cdot)$. A careful perusal of the analysis dealing to [21, Theorem 1] shows that such assumption was used only for proving existence of solutions of the subproblems, i.e., nonemptiness of $S(K, f_k)$ for all k , which in our case holds by Proposition 3.6. \square

Another consequence of our analysis is the following convergence result for the classical proximal point algorithm for optimization, applied to strongly quasiconvex minimization problems.

Corollary 3.3 *Take $\theta \in \mathbb{R}_{++}$. Let K be a convex and closed set in \mathbb{R}^n , $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a lsc and strongly quasiconvex function on K with modulus $\gamma > 0$ and $\{\beta_k\}_k$ be a sequence such that $\beta_k \geq \theta$ for all k . Let us consider the problem:*

$$\min_{x \in K} h(x), \tag{3.13}$$

and the bifunction $f_h : K \times K \rightarrow \mathbb{R}$ given by:

$$f_h(x, y) := h(y) - h(x), \forall x \in K, \forall y \in K. \tag{3.14}$$

Then, the sequence $\{x^k\}_k$, generated by Algorithm 2 applied to f_h , converges to a minimizer of h in K .

Proof Observe that assumptions (Ai) with $i = 0, 1, 2$ and (A4) hold for f_h . By Corollary 3.2, $x^k \rightarrow \bar{x} \in S(K, f) = \operatorname{argmin}_K h$. In addition, it follows from [21, Theorem 1, equation (16)] that $\|x^{k+1} - \bar{x}\| \leq \|x^k - \bar{x}\|$.

It follows from Proposition 3.6 that $h(x^{k+1}) \leq h(x^k)$ for all $k \in \mathbb{N}$ and that if $x^{k+1} = x^k$, then $x^k \in \operatorname{argmin}_K h$. Furthermore, applying Corollary 3.2, $x^k \rightarrow \bar{x} \in S(K, f_h) = \operatorname{argmin}_K h$. \square

Remark 3.4 Even when f is assumed to be strongly quasiconvex in its second argument, we cannot use Proposition 3.1 in the proofs of Proposition 3.6 and Corollary 3.2 because f_k is the sum of two quasiconvex functions, which in general is not quasiconvex. Instances in which sums of quasiconvex functions are quasiconvex can be found in [22].

4 Examples of Equilibrium Problems Satisfying Our Assumptions

In this section presenting instances of equilibrium problems which satisfy the hypotheses required in the convergence analysis of our algorithm, namely (Ai) with $i = 1, 2, 3, 4, 5, 6$. All these instances are indeed *mixed variational inequalities*, i.e., the bifunction f is of the form $f(x, y) = h(y) - h(x) + \langle T(x), y - x \rangle$ with $h : K \rightarrow \mathbb{R}$ and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. See, e.g., [19,20] for more information on mixed variational inequalities.

Example 4.1 Take symmetric and positive definite matrices $A, B \in \mathbb{R}^{n \times n}$, and define

$$f(x, y) = y^t A y - x^t A x + x^t B (y - x). \quad (4.1)$$

It is easy to check that f satisfies (Ai) with $i = 1, 2, 3$. Let now $\rho(B)$ be the largest eigenvalue of B (i.e., its spectral radius), and $\mu(A)$ the smallest eigenvalue of A . By the positive definiteness assumption, both $\rho(B)$ and $\mu(A)$ are positive. Since $f(x, \cdot)$ is a quadratic function with matrix A , it is strongly convex with modulus $\mu(A)$, and hence strongly quasiconvex with the same modulus, and (A4) holds. An easy computation shows that f is Lipschitz continuous with constant $\rho(B)/2$, so that (A5) holds. If we assume that $6\rho(B) < \mu(A)$, then (A6) holds.

We mention that the equilibrium problem in Example 4.1 is not a variational inequality one, because f is not affine on y ; as mentioned above, it is indeed a mixed variational inequality problem.

We continue with a nonconvex (albeit one-dimensional) example, meaning that $f(x, \cdot)$ is strongly quasiconvex but not convex.

Example 4.2 Take $\alpha, \delta \in \mathbb{R}_{++}$, $K = [0, \delta]$. Let $T : K \rightarrow \mathbb{R}$ be given by $T(x) = \alpha x$ and $h : K \rightarrow \mathbb{R}$ be the continuous function given by $h(x) = \sqrt{x}$. Define the bifunction $f_h^T : K \times K \rightarrow \mathbb{R}$ by

$$f_h^T(x, y) := h(y) - h(x) + \langle T(x), y - x \rangle. \quad (4.2)$$

Clearly, $S(f_h^T, K) = \{0\}$ for all $\alpha > 0$ and all $\delta > 0$.

Note that assumptions (A1) and (A3) trivially hold.

For (A2), take $x, y \in K$ be such that

$$f_h^T(x, y) = \alpha x(y - x) + \sqrt{y} - \sqrt{x} \geq 0 \implies -\alpha x(y - x) - \sqrt{y} + \sqrt{x} \leq 0. \quad (4.3)$$

Since $\alpha > 0$, we have $-\alpha(y - x)^2 \leq 0$, so that (4.3) implies

$$\alpha y(x - y) - h(y) + h(x) \leq -\alpha x(y - x) - h(y) + h(x) \leq 0.$$

Therefore, for every $\alpha > 0$, the bifunction f_h^T is pseudomonotone on K , i.e., assumption (A2) holds.

For (A5), observe that f_h^T satisfies (A5) with parameter $\eta = \frac{\alpha}{2} > 0$ because T is Lipschitz continuous with constant $\alpha > 0$.

Regarding (A4), since the functions $y \mapsto \langle T(x), y - x \rangle$ and h are increasing on K , its sum $y \mapsto \langle T(x), y - x \rangle + h(y)$ is increasing too, so that f_h^T is quasiconvex in its second argument by [14, Proposition 4.9]. We claim that f_h^T is strongly quasiconvex in its second argument with modulus $\gamma = \frac{1}{2\sqrt{\delta^3}} > 0$. Indeed, take $t_1, t_2 \in K$ with $t_1 < t_2$ and $t = \lambda t_1 + (1 - \lambda)t_2$ with $\lambda \in]0, 1[$. Then, for all $x \in K$, we have

$$\begin{aligned} \max\{f_h^T(x, t_1), f_h^T(x, t_2)\} - f_h^T(x, t) &= \alpha x(t_2 - x) + \sqrt{t_2} - \sqrt{x} - (\alpha x(t - x) + \sqrt{t} - \sqrt{x}) \\ &= \alpha x\lambda(t_2 - t_1) + (\sqrt{t_2} - \sqrt{t}) \geq (\sqrt{t_2} - \sqrt{t}), \end{aligned} \tag{4.4}$$

because $x \geq 0, \alpha > 0, \lambda > 0$ and $t_2 > t_1$.

Now, since h is strongly quasiconvex with modulus $\gamma = \frac{1}{2\sqrt{\delta^3}}$ by [27, Proposition 15], it follows from (4.4) that

$$\max\{f_h^T(x, t_1), f_h^T(x, t_2)\} - f_h^T(x, t) \geq (\sqrt{t_2} - \sqrt{t}) \geq \frac{\gamma}{2}\lambda(1 - \lambda)(t_2 - t_1)^2.$$

Therefore, f_h^T is strongly quasiconvex in its second argument with modulus $\gamma = \frac{1}{2\sqrt{\delta^3}} > 0$ and assumption (A4) holds.

Finally, if $0 < \alpha < \frac{1}{12\sqrt{\delta^3}}$, then (A6) holds and all the assumptions needed in the convergence analysis of Algorithm 1 are satisfied.

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