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Nonemptiness and Compactness of Solution Sets to Weakly Homogeneous Generalized Variational Inequalities

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Abstract

In this paper, we deal with the weakly homogeneous generalized variational inequality, which provides a unified setting for several special variational inequalities and complementarity problems studied in recent years. By exploiting weakly homogeneous structures of involved map pairs and using degree theory, we establish a result which demonstrates the connection between weakly homogeneous generalized variational inequalities and weakly homogeneous generalized complementarity problems. Subsequently, we obtain a result on the nonemptiness and compactness of solution sets to weakly homogeneous generalized variational inequalities by utilizing Harker–Pang-type condition, which can lead to a Hartman–Stampacchia-type existence theorem. Last, we give several copositivity results for weakly homogeneous generalized variational inequalities, which can reduce to some existing ones.

Keywords Weakly homogeneous map · Generalized variational inequality · Harker–Pang-type condition · Copositivity · Hartman–Stampacchia-type theorem

Mathematics Subject Classification 65K10 · 90C33

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1 Introduction

Since it was proposed in 1960s, the finite-dimensional variational inequality (VI) has attracted widespread attention and developed into a very fruitful discipline in the field of mathematical programming with extensive applications and mature theory [2,3,9]. In 1988, Noor introduced a generalization of the VI, named as the general variational inequality problem (GVI) [21], which has been found to have plenty of applications in engineering and economics [1,23,29]. As one of the important theoretical topics, the existence of solutions to GVIs has been widely studied. For instance, Pang and Yao [24] proposed some sufficient conditions for the existence of solution of GVIs by using degree theory. Zhao and Yuan [30] gave an alternative theorem for GVIs with the help of a unified definition of exceptional family. Luo [18] established some existence theorems for GVIs from the perspective of generalizations of some known results for VIs.

Among various conditions for establishing the nonemptiness and compactness of solution sets to VIs or complementarity problems (CPs), Harker–Pang condition and copositivity are two important ones. In [9], Harker and Pang studied the existence and uniqueness theory of VIs and CPs in $\mathbb{R}^n$ and provided lots of important conditions, including the one named as Harker–Pang condition. Later, Isac et.al. [13,14] generalized Harker–Pang condition in $\mathbb{R}^n$ to an arbitrary Hilbert space. Copositivity of maps was extended from the classic concept of copositive matrix [20]. It is a condition that plays a critical role in obtaining rich existence results on VIs and CPs (see [2,3,9] for references), and at the same time, various extended definitions of copositivity were proposed, such as q-copositivity [19] and copositivity for multivalued maps [5].

Recently, the VI with the involved maps being weakly homogeneous, called the weakly homogeneous variational inequality (WHVI) [8,19], has been studied well. The WHVI contains the polynomial variational inequality (PVI) [10], the polynomial complementarity problem (PCP) [4,15], the tensor variational inequality (TVI) [27] and the tensor complementarity problem (TCP) [11,12,25] as its special subclasses. In [8], authors established a main result on the nonemptiness and compactness of solution sets to WHVIs with a degree-theoretic condition and an assumption that zero is the only solution of the corresponding recession cone complementarity problem. To our best knowledge, this degree-theoretic theorem covers a majority of existence results on the subcategory problems of WHVIs.

Encouraged by the achievements of WHVIs, we devote ourselves to investigating nonemptiness and compactness of solution sets to weakly homogeneous generalized variational inequalities (WHGVIs), which is a special class of GVIs with involved maps being weakly homogeneous. As a generalization of WHVIs, WHGVIs contains the recently studied generalized polynomial variational inequality (GPVI) [28] and the generalized polynomial complementarity problem (GPCP) [16,32] as special classes. Our results are established by utilizing Harker–Pang-type condition and copositivity, respectively. The contribution is threefold:

(I) We generalize the main result given for WHVIs in [8] to WHGVIs with the help of degree theory.

(II) We establish an existence result for WHGVIs under Harker–Pang-type condition and some additional assumptions, which is different from the corresponding version obtained for general GVIs in [18, Theorem 2.1]. In particular, we obtain a Hartman–Stampacchia-type existence theorem for WHGVIs.

(III) We achieve three copositivity results on the nonemptiness and compactness of solution sets to WHGVIs, which coincide with the corresponding ones for WHVIs when underlying problems are WHVIs.

Our paper is organized as follows. In Sect. 2, some notations and basic results are shown. In Sect. 3, we explore nonemptiness and compactness of solution sets to WHGVIs. In Sect. 3, we establish an existence result which builds a close connection between WHGVIs and WHGCPs. In Sect. 3, we give another existence result with the help of Harker–Pang-type condition. In Sect. 3, we derive some nonemptiness and compactness theorems by making use of the copositivity of maps. Finally, we sum up the conclusions in the last section.

2 Preliminaries

Throughout this paper, we assume that H is a finite-dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C is a closed convex cone in H and K is a closed convex set in H. We use K^* to denote the dual cone of K, which is defined as $K^* = \{u \in H \mid \langle u, x\rangle \geq 0, \forall x \in K\}$. For any nonempty set Ω in H, let $\mathrm{int}(\Omega)$, $\partial\Omega$ and $\mathrm{cl}\Omega$ denote the interior, boundary and closure of Ω, respectively. Let $\mathbb{R}$ be the set of real numbers. Denote $\mathbb{R}^n := \{x^\top = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \forall\, i \in [n]\}$, and $\mathbb{R}^n_+ (\mathbb{R}^n_{++}) := \{x^\top = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq (>) 0, \forall\, i \in [n]\}$, where $x^\top$ represent the transpose of x and $[n] := \{1, 2, \ldots, n\}$.

For any $z \in H$ and a closed convex set K in H, define $\Pi_K(z)$ as the orthogonal projection of z onto K, which is the unique vector $\bar{z} \in H$ such that $\langle y-\bar{z}, \bar{z}-z\rangle \geq 0$ for all $y \in K$. $\Pi_K(z)$ has nonexpansiveness, i.e., $\|\Pi_K(u) - \Pi_K(v)\| \leq \|u - v\|$ holds for any $u, v \in H$. Recall that for a closed convex set K, its recession cone [26], denoted by K^∞, is defined as $K^\infty := \{u \in H : \exists\, t_k \to \infty, \exists\, x_k \in K \text{ such that } u = \lim_{k\to\infty} \frac{x_k}{t_k}\}$, which satisfies $K^\infty + K \subseteq K$. Then, we have that the map

$$\mathscr{K}(t) = tK + K^\infty, \quad 0 \leq t \leq 1 \tag{1}$$

satisfies the following property:

$$\mathscr{K}(t) = tK + K^\infty = tK \ (t \neq 0) \ \text{ and } \ \mathscr{K}(0) = K^\infty,$$

where the first statement comes from the fact that K^∞ is a cone. In [8], the authors obtained the following result:

Lemma 2.1 *[8] Let $\mathscr{K}(\cdot)$ be defined as (1) and $\theta(\cdot,\cdot) : H \times [0, 1] \to H$ be continuous. Then, the map $(x, t) \mapsto \Pi_{\mathscr{K}(t)}\theta(x, t)$ is continuous.*

Given a nonempty closed convex set K in H and two continuous maps $f : C \to H$, $g : H \to H$ with $C \supseteq g^{-1}(K) := \{x \in H \mid g(x) \in K\}$. The generalized variational inequality (see Noor [21]), denoted by $\mathrm{GVI}(f, g, K)$, is to find an $x^* \in H$ such that

$$g(x^*) \in K, \quad \langle f(x^*), y - g(x^*)\rangle \geq 0, \quad \forall y \in K. \tag{2}$$

Let $\mathrm{SOL}(f, g, K)$ denote the solution set of $\mathrm{GVI}(f, g, K)$. When $g(x) = x$ for any $x \in H$, $\mathrm{GVI}(f, g, K)$ reduces to a variational inequality, denoted by $\mathrm{VI}(f, K)$ with its solution set being denoted by $\mathrm{SOL}(f, K)$.

For any $\mathrm{GVI}(f, g, K)$, we recall that the *natural map* (see Facchinei and Pang [2] for more details) is defined as follows:

$$(F, g)_K^{nat}(x) := g(x) - \Pi_K[g(x) - F(x)], \tag{3}$$

where F is a continuous extension of f. When $g(x) = x$, we write $(F, g)_K^{nat}(x)$ as $f_K^{nat}(x)$. With the help of the natural map and the same technique in Facchinei and Pang [2, Proposition 1.5.8], an equivalent reformulation of $\mathrm{GVI}(f, g, K)$ can be easily established.

Lemma 2.2 *Let K be a closed convex set in H and $f : C \to H$, $g : H \to H$ be two continuous maps with $C \supseteq g^{-1}(K)$. Suppose that F is a continuous extension of f. Then, $x^* \in H$ is a solution of* $\mathrm{GVI}(f, g, K)$ *if and only if* $(F, g)_K^{nat}(x^*) = 0$.

Degree theory is a powerful tool in the study of the existence of a solution to VIs [2,5–8]. Fruitful results on degree theory can refer to [17,22]. Recall that the topological degree of a map is an integer. When map $\phi : \mathrm{cl}\Omega \to H$ is continuous with Ω being a bounded open set in H and $b \in H$ satisfying $b \notin \phi(\partial\Omega)$, the topological degree of ϕ over Ω with respect to b is well defined, denoted by $\deg(\phi, \Omega, b)$.

In addition, if $x^* \in \Omega$ and $\phi(x) = \phi(x^*)$ has a unique solution x^* in $\mathrm{cl}\Omega$, then, let $\Omega' \subseteq \Omega$ be any bounded open set containing x^*, $\deg(\phi, \Omega', \phi(x^*))$ remains a constant, which is called the index of ϕ at x^* and denoted by $\mathrm{ind}(\phi, x^*)$. In particular, when the continuous map $\varphi : H \to H$ satisfies $\varphi(x) = 0$ if and only if $x = 0$, then,

$$\mathrm{ind}(\varphi, 0) = \deg(\varphi, \Omega, \varphi(0)) = \deg(\varphi, \Omega, 0)$$

holds for any bounded open set Ω containing 0.

There are many important properties of the degree. Below, we only show some of them used in later discussion.

Lemma 2.3 (the homotopy invariance of the degree)*[2] Let Ω be a nonempty, bounded open subset in H. Then for any two continuous maps $\mathscr{H} : \mathrm{cl}\Omega \times [0, 1] \to \mathbb{R}^n$ and $p : [0, 1] \to \mathbb{R}^n$ such that $p(t) \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in [0, 1]$,* $\deg(\mathscr{H}(\cdot, t), \Omega, p(t))$ *is independent of $t \in [0, 1]$.*

Lemma 2.4 (the excision property of the degree)*[2] Let Ω be a nonempty, bounded open subset in H, $\phi : \mathrm{cl}\Omega \to H$ be continuous and $p \notin \phi(\partial\Omega)$. Then for every open subset Ω_1 of Ω such that $p \notin \phi(\Omega \setminus \Omega_1)$,* $\deg(\phi, \Omega, p) = \deg(\phi, \Omega_1, p)$.

Lemma 2.5 (the degree of an injective map)[2] *Let Ω be a nonempty, bounded open subset in H and let $\phi : \mathrm{cl}\Omega \to H$ be a continuous injective map. For every $p \in \phi(\Omega)$,* $\deg(\phi, \Omega, p) = \pm 1$.

In [18], the following result on the existence of solutions to GVIs was given.

Lemma 2.6 [18] *Let K be a closed convex set in H and $f : C \to H$ with $C \supseteq g^{-1}(K)$ and $g : H \to H$ be two continuous maps. Suppose that F is a continuous extension of f. If there exists a bounded open set U with* $\mathrm{cl}U \subseteq C$ *such that* $\deg((F, g)_K^{nat}, U, 0) \neq 0$*, then* $\mathrm{GVI}(f, g, K)$ *has a solution in U.*

3 Nonemptiness and Compactness of Solution Sets to WHGVIs

Let C be a closed convex cone in H. Recall that a continuous map $f : C \to H$ is said to be weakly homogeneous of degree $\gamma \geq 0$, if there are two continuous maps $h, g : C \to H$ satisfying $h(\lambda x) = \lambda^\gamma h(x)$ for all $x \in C$ and $\lambda > 0$, and $g(x) = o(\|x\|^\gamma)$ as $\|x\| \to \infty$ in K such that $f = h + g$ (see [8]). Due to the fact that $h(x) = \lim_{\lambda\to\infty} \frac{f(\lambda x)}{\lambda^\gamma}$ for all $x \in C$, h is often called the "leading term" of f, denoted by f^∞.

Given K being a nonempty closed convex set in cone C, when maps f and g in (2) are two weakly homogeneous maps with degrees $\delta_1 > 0$ and $\delta_2 > 0$, respectively, defined as: $f : C \to H$ and $g : H \to H$ with $g^{-1}(K) \subseteq C$, we call $\mathrm{GVI}(f, g, K)$ the *generalized variational inequality with weakly homogeneous maps of positive degrees*, denoted by $\mathrm{WHGVI}(f, g, K)$. In order to simplify the notation, we let

$$\Delta := \left\{ (K, C, f, g) \,\middle|\, \begin{array}{l} K, C, f \text{ and } g \text{ are defined as above;} \\ g^{-1}(C) \subseteq C. \end{array} \right\}.$$

3.1 A Generalization of the Degree-Theoretic Result for WHVIs

Very recently, Gowda and Sossa [8] have achieved many good theoretical results on the nonemptiness and compactness of the solution set to $\mathrm{WHVI}(f, K)$. In particular, they gave a degree-theoretic result for WHVIs as the main result shown in [8, Theorem 4.1], which covers a majority of existence results on the subcategory problems of WHVIs, including the Karamardian-type one in [8, Theorem 5.1] and the copositivity one in [8, Theorem 6.1]. Below, we generalize the degree-theoretic result from WHVIs to WHGVIs.

Theorem 3.1 *Let $(K, C, f, g) \in \Delta$ with $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$, F and F^∞ be any given continuous extensions of f and f^∞, respectively. If* $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$ *and* $\mathrm{ind}\big((F^\infty, g^\infty)_{K^\infty}^{nat}, 0\big) \neq 0$ *where* $(F^\infty, g^\infty)_{K^\infty}^{nat}(\cdot)$ *is defined in* (3)*, then for any bounded open set Ω containing* $\mathrm{SOL}(f, g, K)$, $\deg\big((F, g)_K^{nat}, \Omega, 0\big) \neq 0$*, and* $\mathrm{WHGVI}(f, g, K)$ *has a nonempty, compact solution set.*

Proof Let $\mathscr{K}(t)$ be defined in (1). Consider the following homotopy map:

$$\mathscr{H}(x,t) := [(1-t)g^{\infty}(x) + tg(x)] - \Pi_{\mathscr{K}(t)}\{[(1-t)g^{\infty}(x) + tg(x)] - [(1-t)F^{\infty}(x) + tF(x)]\},$$

where $(x,t) \in H \times [0,1]$. Then, we can obtain that $\mathscr{H}(x,1) = g(x) - \Pi_K[g(x) - F(x)]$ and $\mathscr{H}(x,0) = g^{\infty}(x) - \Pi_{K^{\infty}}[g^{\infty}(x) - F^{\infty}(x)]$. Denote the set of zeros of $\mathscr{H}(\cdot,t)$ by: $\mathbb{Z} := \{x \in H \mid \mathscr{H}(x,t) = 0 \text{ for some } t \in [0,1]\}$. We claim $\mathbb{Z}$ is uniformly bounded.

For the sake of contradiction, assume that $\mathbb{Z}$ is not uniformly bounded. Then there exist two sequences $\{t_k\} \subseteq [0,1]$ and $\{0 \neq x^k\} \subseteq H$ such that $\mathscr{H}(x^k, t_k) = 0$ for any k and $\|x^k\| \to \infty$. From $\mathscr{H}(x^k, t_k) = 0$, it follows that

$$(1-t_k)g^{\infty}(x^k) + t_k g(x^k) = \Pi_{\mathscr{K}(t_k)}\Big([(1-t_k)g^{\infty}(x^k) + t_k g(x^k)] - [(1-t_k)F^{\infty}(x^k) + t_k F(x^k)]\Big),$$

which means that $(1-t_k)g^{\infty}(x^k) + t_k g(x^k) \in \mathscr{K}(t_k) = t_k K \subseteq C$ and

$$\Big\langle (1-t_k)F^{\infty}(x^k) + t_k F(x^k), z - [(1-t_k)g^{\infty}(x^k) + t_k g(x^k)]\Big\rangle \geq 0 \tag{4}$$

holds for any $z \in \mathscr{K}(t_k)$. Since $g(x^k) - g^{\infty}(x^k) \in C$ as $\|x^k\| \to \infty$ and C is a convex cone, it follows that $(1-t_k)[g(x_k) - g^{\infty}(x_k)] \in C$ for sufficiently large k. Furthermore, we have that for sufficiently large k,

$$g(x^k) = (1-t_k)[g(x) - g^{\infty}(x)] + (1-t_k)g^{\infty}(x^k) + t_k g(x^k) \in C,$$

which, together with $g^{-1}(C) \subseteq C$, implies that $x^k \in C$. So, we have that $F(x^k) = f(x^k)$ and $F^{\infty}(x^k) = f^{\infty}(x^k)$ for sufficiently large k. Therefore, for sufficiently large k, (4) can be written as

$$\Big\langle (1-t_k)f^{\infty}(x^k) + t_k f(x^k), z - [(1-t_k)g^{\infty}(x^k) + t_k g(x^k)]\Big\rangle \geq 0, \quad \forall\, z \in \mathscr{K}(t_k).$$

Noting that for any $u \in K^{\infty}$ and (fixed) $g(x^0) \in K$, $t_k g(x^0) + \|x^k\|^{\delta_2} u \in \mathscr{K}(t_k)$ holds for sufficiently large k. Then, by choosing $z = t_k g(x^0) + \|x^k\|^{\delta_2} u$ and dividing the above relation by $\|x^k\|^{\delta_1+\delta_2}$, we get that

$$\left\langle \frac{(1-t_k)f^{\infty}(x^k) + t_k f(x^k)}{\|x^k\|^{\delta_1}}, u - \frac{(1-t_k)g^{\infty}(x^k) + t_k g(x^k) - t_k g(x^0)}{\|x^k\|^{\delta_2}} \right\rangle \geq 0 \tag{5}$$

holds for any $u \in K^{\infty}$ and sufficiently large k. Since $\{t_k\}$ and $\{\frac{x^k}{\|x^k\|}\}$ are bounded, we can assume that $\lim_{k\to\infty} t_k = \bar{t}$ and $\lim\limits_{k\to\infty} \frac{x^k}{\|x^k\|} = \bar{x}$. Then, let $k \to \infty$ in (5), we have that

$$\langle f^{\infty}(\bar{x}), u - g^{\infty}(\bar{x})\rangle \geq 0, \ \forall\, u \in K^{\infty}. \tag{6}$$

Next, we show $g^{\infty}(\bar{x}) \in K^{\infty}$. Noting that $(1 - t_k)g^{\infty}(x^k) + t_k g(x^k) \in \mathscr{K}(t_k) = t_k K + K^{\infty}$ for all k, if $t_k = 0$ for infinitely many k, then we can find a subsequence $\{x^{k'}\}$ of $\{x^k\}$ such that $t_{k'} = 0$; thus, we can easily get that

$$g^{\infty}(\bar{x}) = \lim_{k' \to \infty} \frac{(1 - t_{k'})g^{\infty}(x^{k'}) + t_{k'} g(x^{k'})}{\|x^{k'}\|^{\delta_2}} \in K^{\infty}.$$

Otherwise, there must exist infinitely many k such that $t_k > 0$, then we can find a subsequence $\{x^{k'}\}$ of $\{x^k\}$ such that $t_{k'} > 0$. Since

$$(1 - t_{k'})g^{\infty}(x^{k'}) + t_{k'} g(x^{k'}) \in \mathscr{K}(t_{k'}) = t_{k'} K + K^{\infty} = t_{k'} K,$$

each $(1 - t_{k'})g^{\infty}(x^{k'}) + t_{k'} g(x^{k'})$ can be written as $(1 - t_{k'})g^{\infty}(x^{k'}) + t_{k'} g(x^{k'}) = t_{k'} y^{k'}$ with $y^{k'} \in K$. Noting that $y^{k'} \in K$ and $\frac{\|x^{k'}\|}{t_{k'}} \to \infty$ as $k' \to \infty$, then by the definition of the recession cone, we have that

$$\begin{aligned} g^{\infty}(\bar{x}) &= \lim_{k' \to \infty} \frac{(1 - t_{k'})g^{\infty}(x^{k'}) + t_{k'} g(x^{k'})}{\|x^{k'}\|^{\delta_2}} \\ &= \lim_{k' \to \infty} \frac{t_{k'} y^{k'}}{\|x^{k'}\|^{\delta_2}} = \lim_{k' \to \infty} \frac{y^{k'}}{\|x^{k'}\|^{\delta_2}/t_{k'}} \in K^{\infty}. \end{aligned}$$

Therefore, both cases imply that $g^{\infty}(\bar{x}) \in K^{\infty}$. This, together with (6), implies that $0 \neq \bar{x} \in \mathrm{SOL}(f^{\infty}, g^{\infty}, K^{\infty})$, which contradicts $\mathrm{SOL}(f^{\infty}, g^{\infty}, K^{\infty}) = \{0\}$. Therefore, $\mathbb{Z}$ is uniformly bounded.

Now, let Ω be a bounded open set in H, which contains $\mathbb{Z}$, then, $0 \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in [0, 1]$. By Lemma 2.3, we have that,

$$\deg(\mathscr{H}(\cdot, 1), \Omega, 0) = \deg(\mathscr{H}(\cdot, 0), \Omega, 0) = \mathrm{ind}\left((F^{\infty}, g^{\infty})^{nat}_{K^{\infty}}, 0\right) \neq 0.$$

From the excision property of the degree shown in Lemma 2.4, we can obtain that $\deg\left((F, g)^{nat}_{K}, \Omega, 0\right) = \deg(\mathscr{H}(\cdot, 1), \Omega, 0) \neq 0$ for any bounded open set Ω which contains $\mathrm{SOL}(f, g, K)$. Furthermore, by Lemma 2.3, it follows that $\mathrm{SOL}(f, g, K)$ is nonempty. In addition, it follows that $\mathrm{SOL}(f, g, K)$ is bounded from the boundedness of $\mathbb{Z}$ which contains $\mathrm{SOL}(f, g, K)$. Moreover, the closeness of $\mathrm{SOL}(f, g, K)$ is obvious. Therefore, $\mathrm{SOL}(f, g, K)$ is nonempty and compact. □

Remark 3.1 When $g(x) = g^{\infty}(x) = x$, WHGVI(f, g, K) reduces to the variational inequality with weakly homogeneous maps of positive degrees, denoted by WHVI(f, K) with its solution set being written as SOL(f, K). In this case, it is obvious that these conditions "$g^{-1}(C) = C$ and $g(x) - g^{\infty}(x) = 0 \in C$ for any $x \in H$" hold naturally. Thus, Theorem 3.1 can reduce to [8, Theorem 4.1].

3.2 An Existence Result Under Harker–Pang-Type Condition

In [9], Harker and Pang studied the existence and uniqueness theory of VIs and CPs in $\mathbb{R}^n$ and provided lots of important conditions, including the one named as Harker–Pang condition, which was generalized to an arbitrary Hilbert space by Isac et.al. [13] as follows:

Definition 3.1 [13] We say that a map $f : H \to H$ satisfies Harker–Pang condition on K, if there exists some $x^* \in K$ such that the set $K(x^*) = \{x \in K : \langle f(x), x - x^*\rangle < 0\}$ is bounded (or empty).

Now, we give a generalization of above Harker–Pang condition for a pair of maps.

Definition 3.2 Given two subset D, K in H, and two maps $f : D \to H$, $g : H \to H$ with $g^{-1}(K) \subseteq D$. We say that f satisfies Harker–Pang condition with respect to g on K, if there is some x^* satisfying $g(x^*) \in K$ such that the set

$$K(x^*) = \{g(x) \in K : \langle f(x), g(x) - g(x^*)\rangle < 0\}$$

is bounded (or empty).

Below, we give an existence result for WHGVIs with the help of Theorem 3.1 and above Harker–Pang-type condition.

Theorem 3.2 *Let $(K, C, f, g) \in \Delta$ with $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$, $(g^\infty)^{-1}(K^\infty) \subseteq C$, F be any given continuous extension of f. Suppose that g^∞ is injective, f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞ and* $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$, *then* $\deg\big((F, g)_K^{nat}, \Omega, 0\big) \neq 0$ *for any bounded open set Ω containing* $\mathrm{SOL}(f, g, K)$, *and* $\mathrm{WHGVI}(f, g, K)$ *has a nonempty, compact solution set.*

Proof Let $F^\infty : H \to H$ be a continuous extension of f^∞. Since f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞, there exists some x^* satisfying $g^\infty(x^*) \in K^\infty$ such that the set

$$K^\infty(x^*) = \{g^\infty(x) \in K^\infty : \langle f^\infty(x), g^\infty(x) - g^\infty(x^*)\rangle < 0\} \tag{7}$$

is bounded (or empty). Thus, we can find a bounded open set Ω containing 0, which satisfies that $x^* \in \Omega$ and $K^\infty(x^*) \subseteq \Omega$. From $K^\infty(x^*) \subseteq \Omega$ and Ω being open, it follows that $K^\infty(x^*) \cap \partial\Omega = \emptyset$, which immediately implies that

$$\langle F^\infty(x), g^\infty(x) - g^\infty(x^*)\rangle \geq 0 \quad \forall\, x \in (g^\infty)^{-1}(K^\infty) \cap \partial\Omega. \tag{8}$$

Below, we show that $\mathrm{ind}\big((F^\infty, g^\infty)_{K^\infty}^{nat}, 0\big) \neq 0$. Since $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$ and $0 \in \Omega$, we have that $[(f^\infty, g^\infty)_{K^\infty}^{nat}]^{-1}(0) \cap \partial\Omega = \emptyset$. Thus, $\deg((f^\infty, g^\infty)_{K^\infty}^{nat}, \Omega, 0)$ is well defined. Consider the following homotopy map:

$$\mathscr{H}(x, t) := g^\infty(x) - \Pi_{K^\infty}(t(g^\infty(x) - F^\infty(x)) + (1 - t)g^\infty(x^*)),$$

where $(x,t) \in \mathrm{cl}\Omega \times [0,1]$. Then we have that

$$\mathscr{H}(x,0) = g^\infty(x) - \Pi_{K^\infty}(g^\infty(x^*)) \text{ and } \mathscr{H}(x,1) = g^\infty(x) - \Pi_{K^\infty}(g^\infty(x) - F^\infty(x)).$$

Noting that g^∞ is an injective map on H and $x^* \in \Omega$, thus $\deg(g^\infty, \Omega, g^\infty(x^*)) \neq 0$ and $g^\infty(x) \neq g^\infty(x^*)$ for any $x \in (g^\infty)^{-1}(K^\infty) \setminus \{x^*\}$. In addition, from $g^\infty(x^*) \in K^\infty$, it follows that $\mathscr{H}(x,0) = g^\infty(x) - g^\infty(x^*)$. Furthermore, since $x \in \Omega$ and g^∞ is an injective map on H, $\deg(\mathscr{H}(\cdot,0), \Omega, 0)$ is well defined and

$$\deg(\mathscr{H}(\cdot,0), \Omega, 0) = \deg(g^\infty - g^\infty(x^*), \Omega, 0) = \deg(g^\infty, \Omega, g^\infty(x^*)) \neq 0.$$

If $0 \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in [0,1]$, then it follows from Lemma 2.3 that

$$\mathrm{ind}\left((F^\infty, g^\infty)^{nat}_{K^\infty}, 0\right) = \deg(\mathscr{H}(\cdot,1), \Omega, 0) = \deg(\mathscr{H}(\cdot,0), \Omega, 0) \neq 0.$$

Therefore, to show $\mathrm{ind}\left((F^\infty, g^\infty)^{nat}_{K^\infty}, 0\right) \neq 0$, we only need to prove that $0 \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in [0,1]$.

Obviously, $0 \notin \mathscr{H}(\partial\Omega, t)$ when $t = 0, 1$, now we show that $0 \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in (0,1)$. Suppose that $\mathscr{H}(x,t) = 0$ for some $t \in (0,1)$. We divide the proof into the following two cases:

- Case 1: $x = x^*$, then $x \notin \partial\Omega$ from $x^* \in \Omega$;
- Case 2: $x \neq x^*$. Since $\mathscr{H}(x,t) = 0$, it follows that $g^\infty(x) \in K^\infty$ and

  $$\langle y - g^\infty(x), g^\infty(x) - t(g^\infty(x) - F^\infty(x)) - (1-t)g^\infty(x^*)\rangle \geq 0$$

  for all $y \in K^\infty$. Taking $y = g^\infty(x^*)$, then we have $y \in K^\infty$ and

  $$\langle g^\infty(x^*) - g^\infty(x), tF^\infty(x) + (1-t)(g^\infty(x) - g^\infty(x^*))\rangle \geq 0,$$

  which, together with $g^\infty(x) \neq g^\infty(x^*)$ for any $x \in (g^\infty)^{-1}(K^\infty) \setminus \{x^*\}$ and $t \in (0,1)$, implies that $\langle F^\infty(x), g^\infty(x^*) - g^\infty(x)\rangle \geq \frac{1-t}{t}||g^\infty(x^*) - g^\infty(x)||_2^2 > 0$. From (8), we obtain that $x \notin \partial\Omega$.

Thus, these two cases together mean that $0 \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in (0,1)$. Therefore, we obtain that $\mathrm{ind}\left((F^\infty, g^\infty)^{nat}_{K^\infty}, 0\right) \neq 0$, which, together with $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$, implies that $\mathrm{SOL}(f, g, K)$ is nonempty and compact from Theorem 3.1. □

In [18], a related result for GVIs was given as follows.

Theorem 3.3 [18] *Let K be a closed convex set in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, injective. If the set*

$$L_{\leq} := \{g(x) \in K : \langle f(x), g(x) - g(x^*)\rangle \leq 0\}$$

is bounded (or empty), then $\mathrm{SOL}(f, g, K)$ *is nonempty and compact.*

Here, we show an example, which satisfies all conditions in Theorem 3.2 but do not satisfy the conditions given in Theorem 3.3 when the maps involved in GVIs are weakly homogeneous.

Example 3.1 Consider WHGVI(f, g, K) where $K := \{x \in \mathbb{R}^2 : -1 \le x_1 \le 0, x_2 \ge 0\}$, $f(x) = (x_1^3 + x_1^2, -x_1^2 x_2 - x_2^2)^\top$, $g(x) = (x_1^3 - x_1, x_2^3)^\top$, $\forall x \in C := \{x \in \mathbb{R}^2 : x_2 \ge 0\}$.

Obviously, K is a convex set in C with $K^\infty := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \ge 0\}$ and $(K^\infty)^* := \{x \in \mathbb{R}^2 : x_2 \ge 0\}$, and f, g are weakly homogeneous maps of degree 3.

- It is easy to see that g^∞ is an injective map, $(g^\infty)^{-1}(K^\infty) = K^\infty \subseteq C, g^{-1}(C) = C$ and $g(x) - g^\infty(x) \in C$ for any $x \in \mathbb{R}^2$.
- Taking $x^* = (0, 0)^\top$, we have $g^\infty(x^*) = (0, 0)^\top \in K^\infty$ and for any $g^\infty(x) \in K^\infty$, i.e., $x_1 = 0, x_2 \ge 0$, it follows that $\langle f^\infty(x), g^\infty(x) - g^\infty(x^*)\rangle = x_1^6 - x_1^2 x_2^4 = 0$, which means that f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞.
- Consider WHGCP$(f^\infty, g^\infty, K^\infty)$. Suppose that $x^* \in$ SOL$(f^\infty, g^\infty, K^\infty)$. On the one hand, from $g^\infty(x^*) \in K^\infty$, it follows that $x_1^* = 0$ and $x_2^* \ge 0$. On the other hand, from $f^\infty(x^*) \in (K^\infty)^*$, it follows that $x_2^* \le 0$. Thus, we get SOL$(f^\infty, g^\infty, K^\infty) = \{0\}$.

So, all conditions of Theorem 3.2 are satisfied. Noting that $g(y) = g(z) = (0, 0)^\top$ where $y := (0, 0)^\top$ and $z := (1, 0)^\top$, thus g is not an injective map. In addition, taking $g(x^*) \in K$, then for any $g(x) \in K$ with $x_1 = -1$, we can obtain that

$$\langle f(x), g(x) - g(x^*)\rangle = (-x_2 - x_2^2)[x_2^3 - (x_2^*)^3] \to -\infty \text{ as } x_2 \to +\infty,$$

which means that $L_\le$ are unbounded. So, conditions in Theorem 3.3 do not hold.

Recall that for VIs, there are two useful consequences of the existence result established under Harker–Pang condition, where one is the well-known coercivity theorem and the other is the famous Hartman–Stampacchia's theorem. Similarly, for WHGVIs, we can obtain corresponding versions of these two important results. First, we show the coercivity one, which turns out to utilize a different coercivity condition from the corresponding one for VIs when underlying problems reduce to WHVIs.

Theorem 3.4 *Let* $(K, C, f, g) \in \Delta$ *with* $g(x) - g^\infty(x) \in C$ *as* $\|x\| \to \infty$, $(g^\infty)^{-1}(K^\infty) \subseteq C$. *Suppose that* g^∞ *is an injective map on* H *and* SOL$(f^\infty, g^\infty, K^\infty) = \{0\}$. *If there exists some* $x^* \in K^\infty$, $c \ge 0$ *and* $\xi \ge 0$ *such that*

$$\langle f^\infty(x), g^\infty(x) - g^\infty(x^*)\rangle \ge c\|x\|^\xi, \quad \forall\, g^\infty(x) \in K^\infty \text{ and } \|x\| \to \infty \tag{9}$$

holds, then WHGVI(f, g, K) *has a nonempty, compact solution set.*

Proof Since (9) implies that $\langle f^\infty(x), g^\infty(x) - g^\infty(x^*)\rangle \ge 0$ for any $g^\infty(x) \in K^\infty$ and $\|x\| \to \infty$, that is to say that f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞. Therefore, from Theorem 3.2, we get that WHGVI(f, g, K) has a nonempty, compact solution set. □

When $g(x) = x$, Theorem 3.4 yields the following result on the nonemptiness and compactness of the solution set to WHVIs:

Corollary 3.1 *Let K be a closed convex set in cone C and $f : C \to \mathbb{R}^n$ be a weakly homogeneous map. Suppose that* $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$. *If there exists some $x^* \in K^\infty$, $c \geq 0$ and $\xi \geq 0$ such that*

$$\langle f^\infty(x), x - x^* \rangle \geq c\|x\|^\xi, \quad \forall\, x \in K^\infty \text{ and } \|x\| \to \infty \tag{10}$$

holds, then $\mathrm{WHVI}(f, g, K)$ *has a nonempty, compact solution set.*

Below, we compare the conditions in Corollary 3.1 with the coercivity condition for general VIs with the considered problem being restricted to WHVIs. To this end, we recall that the corresponding coercivity result for VIs:

Theorem 3.5 [2] *Let K be a closed convex set in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ be continuous. If f is coercive on K, that is there exists some $x^{ref} \in K$, $c > 0$ and $\xi \geq 0$ such that*

$$\langle f(x), x - x^{ref} \rangle \geq c\|x\|^\xi, \quad \forall\, x \in K \text{ and } \|x\| \to \infty,$$

then VI(f, K) has a nonempty, compact solution set.

Now, we construct an example to show the coercivity condition used for WHVIs in Corollary 3.1 is different from the classic one for VIs, where all conditions in Corollary 3.1 are satisfied, but the coercivity condition of Theorem 3.5 does not hold.

Example 3.2 Consider $\mathrm{WHVI}(f, K)$ where $K := \{x \in \mathbb{R}^2 : x_1 \geq x_2\}$ and for any $x \in K$, $f(x) = (x_1^3 + x_2^3 - x_1, -x_1^3 - 2x_1x_2^2 + x_2^3 - x_2)^\top$.

Obviously, K is convex with $K^* := \{0\}$, and f is weakly homogeneous of degree 3. First, we show that all conditions of Corollary 3.1 are satisfied.

- Taking $x^* = (0, 0)^\top \in K^\infty = K$, then it follows that

  $$\langle f^\infty(x), x - x^* \rangle = x_1(x_1^3 + x_2^3) + x_2(-x_1^3 - 2x_1x_2^2 + x_2^3) = (x_1 - x_2)(x_1^3 - x_2^3) \geq 0$$

  for any $x \in K^\infty$. Thus, (10) holds.
- Consider $\mathrm{WHCP}(f^\infty, K^\infty)$. Suppose that $\bar{x} \in \mathrm{SOL}(f^\infty, K^\infty)$, then it follows from $f^\infty(\bar{x}) \in (K^\infty)^*$ that $\bar{x}_1^3 + \bar{x}_2^3 = 0$ and $-\bar{x}_1^3 - 2\bar{x}_1\bar{x}_2^2 + \bar{x}_2^3 = 0$, i.e. $\bar{x}_1 = -\bar{x}_2$ and $4\bar{x}_2^3 = 0$. So, we get that $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$.

Therefore, all conditions of Corollary 3.1 are satisfied.

Second, we show the coercivity condition of Theorem 3.5 does not hold.

- For any $x^{ref} \in K$ with $x_1^{ref} + x_2^{ref} \geq 0$, by taking $x \in K$ with $x_1 = x_2 \leq 0$, we have that $\langle f(x), x - x^{ref} \rangle = 2x_2^3(x_1^{ref} + x_2^{ref}) - x_2(2x_2 + x_1^{ref} + x_2^{ref}) \to -\infty$, as $x_2 \to -\infty$.
- For any $x^{ref} \in K$ with $x_1^{ref} + x_2^{ref} < 0$, by taking $x \in K$ with $x_1 = x_2 \geq 0$, we have that $\langle f(x), x - x^{ref} \rangle = 2x_2^3(x_1^{ref} + x_2^{ref}) - x_2(2x_2 + x_1^{ref} + x_2^{ref}) \to -\infty$, as $x_2 \to +\infty$.

Thus, we cannot find some $x^{ref} \in K$, $c > 0$ and $\xi \geq 0$ such that $\langle f(x), x - x^{ref}\rangle \geq c\|x\|^{\xi}$ for any $x \in K$ satisfying $\|x\| \to \infty$, which implies that f is not coercive on K.

Hartman–Stampacchia's theorem is a basic result on the existence of solutions to VIs, which plays a critical role in deriving many existence results for VIs. Besides, it is well known that Hartman–Stampacchia's theorem is another important consequences of the fundamental existence result under Harker–Pang condition. In the following, we investigate the corresponding version for WHGVIs, which is the basis for establishing the first copositivity result in the next section.

Theorem 3.6 *Let* $(K, C, f, g) \in \Delta$ *with* $g(x) - g^{\infty}(x) \in C$ *as* $\|x\| \to \infty$, $(g^{\infty})^{-1}(K^{\infty}) \subseteq C$ *and* K *being compact. If* g^{∞} *is injective on* H, *then* WHGVI(f, g, K) *has a nonempty, compact solution set.*

Proof Since K is compact, $K^{\infty} = \{0\}$. Take $x^* = 0$, then it follows that $K^{\infty}(x^*)$ defined by (7) is empty, which implies that f^{∞} satisfies Harker–Pang condition with respect to g^{∞} on K^{∞}. In addition, since $g^{\infty}(x) \in K^{\infty}$ if and only if $g^{\infty}(x) = 0$ if and only if $x = 0$, it follows that SOL$(f^{\infty}, g^{\infty}, K^{\infty}) = \{0\}$ holds. Hence, by Theorem 3.2, we know that SOL(f, g, K) is nonempty and compact. □

3.3 Several Copositivity Results

Let $\psi : H \supseteq D \to H$ be a continuous map. Recall that ψ is said to be copositive on D [9], if $\langle \psi(x) - \psi(0), x\rangle \geq 0$ holds for any $x \in D$, and ψ is said to be q-copositive on D [19], if there is some $q \in H$ such that $\langle \psi(x) - q, x\rangle \geq 0$ holds for any $x \in D$. They are two useful conditions for deriving existence results for VIs. Below, we show three copositivity results for WHGVIs. To this end, we give the following definitions.

Definition 3.3 Given two maps $f : C \to H$ and $g : H \to H$. f is said to be

(i) copositive with respective to g on K, if $g^{-1}(K) \subseteq C$ and $\langle f(x) - f(0), g(x)\rangle \geq 0$ holds for any $g(x) \in K$.
(ii) q-copositive with respective to g on K, if $g^{-1}(K) \subseteq C$ and there exists a vector $q \in H$ such that $\langle f(x) - q, g(x)\rangle \geq 0$ holds for any $g(x) \in K$.

The first copositivity result on the nonemptiness and compactness of solution sets to WHGVIs is obtained with the help of the specific Hartman–Stampacchia's theorem for WHGVIs given as Theorem 3.6 in subsection 3.1.

Theorem 3.7 *Let* $(K, C, f, g) \in \Delta$ *with* $g(x) - g^{\infty}(x) \in C$ *as* $\|x\| \to \infty$, $(g^{\infty})^{-1}(K^{\infty}) \subseteq C$, *and* $p \in H$. *Suppose that* g^{∞} *is injective on* H, $g^{\infty}(x) \neq 0$ *for any* $x \in H$ *satisfying* $\|x\| = 1$ *and the following conditions hold:*

(a) f *is* q*-copositive with respect to* g *on* K;
(b) *there is a vector* $\hat{x} \in K$ *such that* $\langle f(x), \hat{x}\rangle \leq 0$ *for all* x *satisfying* $g(x) \in K$;
(c) $p + q \in \mathrm{int}((g^{\infty}(\mathscr{S}))^*)$ *with* $\mathscr{S}$:=SOL$(f^{\infty}, g^{\infty}, K^{\infty})$, $g^{\infty}(\mathscr{S}) := \{g^{\infty}(x) : x \in \mathscr{S}\}$,

then WHGVI$(f+p,g,K)$ *has a nonempty, compact solution set.*

Proof Denote $K_k = \{x \in H : x \in K, \|x\| \le k\}$, $\forall\ k = 1, 2, \ldots$. Obviously, for every fixed k, K_k is a compact and convex set. Then it follows from Theorem 3.6, for sufficiently large k, K_k is nonempty and WHGVI(f, g, K_k) has a solution, denoted by x^k. That is to say, for sufficiently large k, there is some $x^k \in H$ such that $g(x^k) \in K_k$ and

$$\langle f(x^k) + p, y - g(x^k)\rangle \ge 0, \quad \forall\ y \in K_k. \tag{11}$$

Below, we show that the sequence $\{x^k\}$ is bounded, which means that $\lim_{k\to\infty} x^k = \tilde{x}$ solves VI(f, g, K); thus, SOL(f, g, K) is nonempty and compact.

For the sake of contradiction, assume that the sequence $\{x^k\}$ is unbounded and $\frac{x^k}{\|x^k\|} \to \bar{x}$ as $k \to \infty$. Taking $y = \hat{x}$ in (11) (without loss of generality, assume that $\|\hat{x}\| \le k$), we obtain $\langle f(x^k) + p, \hat{x} - g(x^k)\rangle \ge 0$, i.e.,

$$-\langle f(x^k), \hat{x}\rangle + \langle f(x^k) - q, g(x^k)\rangle \le -\langle p + q, g(x^k)\rangle + \langle p, \hat{x}\rangle,$$

which, together with conditions (a) and (b), implies that $\langle p+q, g(x^k)\rangle - \langle p, \hat{x}\rangle \le 0$. Dividing it by $\|x^k\|^{\gamma_2}$ and letting $k \to \infty$, we have that $\langle p + q, g^\infty(\bar{x})\rangle \le 0$, which implies that $\bar{x} \notin \mathscr{S}$ from the condition that $p + q \in \mathrm{int}((g^\infty(\mathscr{S}))^*)$. Thus, for the sake of contradiction, to show the boundedness of $\{x^k\}$, we only need to show that $\bar{x} \in \mathscr{S}$, i.e., $\bar{x} \in$ SOL$(f^\infty, g^\infty, K^\infty)$.

For every $v \in K^\infty$ with $v \ne 0$ and every k, since $K^\infty + K \subseteq K$, $\frac{\|g(x^k)\|}{\|v\|} v \in K^\infty$, and $x^1 \in K$, it follows that $y_k := \frac{\|g(x^k)\|}{\|v\|} v + x^1 \in K$. Noting that

$$\|y_k\| \le \|g(x^k)\| + \|x^1\| \le k + 1,$$

which means that $y_k \in K_{k+1}$. By substituting y_k for y in (11), we have that

$$\left\langle f(x^{k+1}) + p, \frac{\|g(x^k)\|}{\|v\|} v + x^1 - g(x^{k+1})\right\rangle \ge 0.$$

Dividing it by $\|x^{k+1}\|^{\gamma_1+\gamma_2}$, we obtain

$$\left\langle \frac{f(x^{k+1}) + p}{\|x^{k+1}\|^{\gamma_1}}, \frac{\|g(x^k)\|}{\|x^k\|^{\gamma_2}} \frac{\|x^k\|^{\gamma_2}}{\|x^{k+1}\|^{\gamma_2}} \frac{v}{\|v\|} + \frac{x^1}{\|x^{k+1}\|^{\gamma_2}} - \frac{g(x^{k+1})}{\|x^{k+1}\|^{\gamma_2}}\right\rangle \ge 0. \tag{12}$$

Without loss of generality, we can assume that $\|x^{k+1}\| \ge \|x^k\|$ for all sufficiently large k, then we have that, $k - 1 \le \|x^k\| \le k$, $\quad k \le \|x^{k+1}\| \le k + 1$, which means that $\lim\limits_{k\to\infty} \frac{\|x^k\|^{\gamma_2}}{\|x^{k+1}\|^{\gamma_2}} = 1$. By letting $k \to \infty$ in (12), we can further obtain that

$$\langle f^\infty(\bar{x}), \|g^\infty(\bar{x})\| \frac{v}{\|v\|} - g^\infty(\bar{x})\rangle \ge 0, \quad \forall\ v \in K^\infty. \tag{13}$$

Noting that $\|g^\infty(\bar{x})\| \neq 0$ as $g^\infty(x) \neq 0$ for any x satisfying $\|x\| = 1$, and K^∞ is a cone, thus (13) is equivalent to $\langle f^\infty(\bar{x}), v - g^\infty(\bar{x})\rangle \geq 0$ for any $v \in K^\infty$, which, together with $g^\infty(\bar{x}) = \lim_{k\to\infty} \frac{g(x_k)}{\|x_k\|^{\gamma_2}} \in K^\infty$, means that $\bar{x} \in \mathscr{S}$. This is a contradiction to condition (c) and thus further a contradiction to the assumption that the sequence $\{x^k\}$ is unbounded.

The proof is complete. □

Remark 3.2 It is easy to see that Theorem 3.7 becomes the main result recently established in [19] when the underlying problem reduces to a WHVI.

The second copositivity result can be seen as a generalization of the one given in [8, Theorem 6.1]. But it should be noticed that this new existence theorem is derived in a different way from [8, Theorem 6.1].

Theorem 3.8 *Let $(K, C, f, g) \in \Delta$ with $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$, $(g^\infty)^{-1}(K^\infty) \subseteq C$, and F be any given continuous extension of f. If f^∞ is copositive with respect to g^∞ on K^∞, g^∞ is injective on H and* $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$*, then* $\deg\left((F, g)_K^{nat}, \Omega, 0\right) \neq 0$ *for any bounded open set Ω containing* $\mathrm{SOL}(f, g, K)$*, and* $\mathrm{SOL}(f, g, K)$ *is nonempty and compact.*

Proof It is easy to see that if f^∞ is copositive with respect to g^∞ on K^∞, then f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞. From Theorem 3.2, we have that $\deg\left((F, g)_K^{nat}, \Omega, 0\right) \neq 0$ for any bounded open set Ω containing $\mathrm{SOL}(f, g, K)$, and $\mathrm{SOL}(f, g, K)$ is nonempty and compact. □

Remark 3.3 When $g(x) = x$, WHGVI(f, g, K) reduces to WHVI(f, K), and Theorem 3.8 reduces to Theorem 6.1 (a) in [8].

Next, we show the third copositivity result. To this aim, an "alternative theorem" is established first.

Theorem 3.9 *Let $(K, C, f, g) \in \Delta$ with $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$, $(g^\infty)^{-1}(K^\infty) \subseteq C$. Suppose that g^∞ is an injective map on H. If f^∞ is copositive with respect to g^∞ on K^∞, then either the* WHGVI(f, g, K) *has a solution or there exists an unbounded sequence $\{x_k\}$ and a positive sequence $\{t_k\} \subseteq (0, 1)$ such that $g(x_k) \in K$ and for each k,*

$$\langle f^\infty(x_k) + t_k g(x_k) + (1 - t_k)(f(x_k) - f^\infty(x_k)), y - g(x_k)\rangle \geq 0, \quad \forall y \in K. \quad (14)$$

Proof Let F and F^∞ be any given continuous extensions of f and f^∞, respectively. For the sake of contradiction, we assume that the set

$$\bigcup_{0<t<1} \mathrm{SOL}(f^\infty + tg + (1 - t)(f - f^\infty), g, K)$$

is bounded and $\mathrm{SOL}(f, g, K) = \emptyset$. Then, consider the homotopy:

$$\mathscr{H}(x, t) = g(x) - \Pi_K(g(x) - (F^\infty(x) + tg(x) + (1 - t)(F(x) - F^\infty(x)))),$$

where $(x, t) \in H \times [0, 1]$. It is easy to see that $\mathscr{H}(\cdot, t)$ is just the natural map of WHGVI$(f^\infty + tg + (1-t)(f - f^\infty), g, K)$ for each $t \in [0, 1]$. Denote the set of zeros of $\mathscr{H}(\cdot, t)$ by: $\mathbb{Z} := \{x \in H \mid \mathscr{H}(x, t) = 0 \text{ for some } t \in [0, 1]\}$.

Since $\mathrm{SOL}(f, g, K) = \emptyset$, it follows that $\{x \in H \mid \mathscr{H}(x, 0) = 0\}$ is bounded, which, together with another assumption, implies $\{x \in H \mid \mathscr{H}(x, t) = 0 \text{ for some } t \in [0, 1)\}$ is bounded. Now, we consider the set

$$\{x \in H \mid \mathscr{H}(x, 1) = g(x) - \Pi_K(g(x) - (F^\infty(x) + g(x))) = 0\}.$$

Since f^∞ is copositive with respect to g^∞ on K^∞, it follows that $\langle f^\infty(x), g^\infty(x)\rangle \geq 0$ for all $g^\infty(x) \in K^\infty$. In addition, noting that $\langle g^\infty(x), g^\infty(x)\rangle > 0$ for any $g^\infty(x) \neq 0$ and g^∞ is injective, thus $\mathrm{SOL}((f^\infty + g)^\infty, g^\infty, K^\infty) = \{0\}$. Besides, it is easy to see that $(f^\infty + g)^\infty$ is copositive with respect to g^∞ on K^∞. So from Theorem 3.8, it follows that $\mathrm{SOL}(f^\infty + g, g, K)$ is nonempty and compact. Hence, the set $\mathbb{Z}$ is uniformly bounded.

Let Ω be a bounded open set in H, which contains $\mathbb{Z}$, then for all $t \in [0, 1]$, $0 \notin \mathscr{H}(\partial\Omega, t)$. By Lemma 2.3, it follows that,

$$\deg(\mathscr{H}(\cdot, 0), \Omega, 0) = \deg(\mathscr{H}(\cdot, 1), \Omega, 0) = \deg((F^\infty + g, g)_K^{nat}, \Omega, 0). \quad (15)$$

Since $f^\infty + g^\infty$ is copositive with respect to g^∞ on K^∞, g^∞ is injective on H and $\mathrm{SOL}(f^\infty + g^\infty, g^\infty, K^\infty) = \{0\}$, it follows that $\deg((F^\infty + g, g)_K^{nat}(x), \Omega, 0) \neq 0$ by Theorem 3.8. Hence, from (15), we have that $\deg(\mathscr{H}(\cdot, 0), \Omega, 0) \neq 0$, which means that $\mathrm{SOL}(f, g, K)$ is nonempty. Contradiction! □

Recall that in [8], the authors investigated the nonemptiness and compactness of the solution set of WHVI(f, K) well, and established some good results based on a condition that $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$. With Theorem 3.9, we can obtain the third copositivity result for WHGVI(f, g, K), which does not need the assumption that $\mathrm{SOL}(f^\infty, g^\infty, K^\infty) = \{0\}$.

Theorem 3.10 *Let $(K, C, f, g) \in \Delta$ with $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$, $0 \in K$ and $(g^\infty)^{-1}(K^\infty) \subseteq C$. Suppose that g^∞ is an injective map on H and the following conditions hold:*

(a) *f^∞ is copositive with respect to g^∞ on K^∞;*
(b) *f^∞ is copositive with respect to g on K;*
(c) *for any x satisfying $g(x) \in K$ and $\|x\| \to \infty$, $g(x) \neq 0$ and there exists no $c > 0$ such that $-g(x) = c(f(x) - f^\infty(x))$;*
(d) *$\lim\limits_{g(x)\in K, \|x\|\to\infty} \|(F, g)_K^{nat}(x)\| = \infty$ and $\|f(x) - f^\infty(x)\| \leq \|(F, g)_K^{nat}(x)\|$ for any x satisfying $g(x) \in K$ and $\|x\| \to \infty$, where $(F, g)_K^{nat}(x) := g(x) - \Pi_K(g(x) - F(x))$ is the natural map of* GVI(f, g, K) *with F being a given continuous extension of f;*

then WHGVI(f, g, K) *has a nonempty, compact solution set.*

Proof First, we show that $\mathrm{SOL}(f, g, K) \neq \emptyset$. Suppose that $\mathrm{SOL}(f, g, K) = \emptyset$, then from Theorem 3.9, there exists an unbounded sequence $\{x_k\}$ and a positive sequence

$\{t_k\} \subseteq (0, 1)$ such that $g(x_k) \in K$ and (14) holds for each k. Noting that $0 \in K$, thus we have that

$$\langle g(x_k), f^\infty(x_k) + t_k g(x_k) + (1 - t_k)(f(x_k) - f^\infty(x_k))\rangle \le 0,$$

which, together with the condition that f^∞ is copositive with respect to g on K and $g(x_k) \in K$, implies that

$$\langle g(x_k), t_k g(x_k) + (1 - t_k)(f(x_k) - f^\infty(x_k))\rangle \le 0.$$

This, together with Cauchy–Schwarz inequality, means that

$$\begin{aligned} t_k\|g(x_k)\|^2 &= \langle g(x_k), t_k g(x_k)\rangle \\ &\le \langle g(x_k), -(1 - t_k)(f(x_k) - f^\infty(x_k))\rangle \\ &\le (1 - t_k)\|g(x_k)\|\|f(x_k) - f^\infty(x_k)\|; \end{aligned}$$

thus, it follows that

$$t_k\|g(x_k)\| \le (1 - t_k)\|f(x_k) - f^\infty(x_k)\|, \quad \text{if } \|g(x_k)\| \ne 0.$$

Since when $\|g(x_k)\| = 0$, $t_k\|g(x_k)\| \le (1 - t_k)\|f(x_k) - f^\infty(x_k)\|$ holds naturally, we have that

$$t_k\|g(x_k)\| \le (1 - t_k)\|f(x_k) - f^\infty(x_k)\|, \tag{16}$$

which, together with condition (c), implies that $f(x_k) - f^\infty(x_k) \ne 0$ for sufficiently large k. Further, by $t_k > 0$, condition (c) and the trigonometric inequality of the norm, we have that

$$\| - t_k g(x_k) + t_k(f(x_k) - f^\infty(x_k))\| < t_k\|g(x_k)\| + t_k\|f(x_k) - f^\infty(x_k)\| \tag{17}$$

for sufficiently large k. In addition, by Lemma 2.2 and (14), it follows that

$$g(x_k) = \Pi_K(g(x_k) - f^\infty(x_k) - t_k g(x_k) - (1 - t_k)(f(x_k) - f^\infty(x_k))),$$

which, together with $F(x_k) = f(x_k)$ since $x_k \in K$, implies that for sufficiently large k,

$$\begin{aligned} &\|(F, g)^{nat}_K(x_k)\| \\ &\quad = \|g(x_k) - \Pi_K(g(x_k) - F(x_k))\| \\ &\quad = \|\Pi_K(g(x_k) - f^\infty(x_k) - t_k g(x_k) - (1 - t_k)(f(x_k) - f^\infty(x_k))) - \Pi_K(g(x_k) - f(x_k))\| \\ &\quad \le \| - t_k g(x_k) + t_k(f(x_k) - f^\infty(x_k))\| \\ &\quad < t_k\|g(x_k)\| + t_k\|f(x_k) - f^\infty(x_k)\| \\ &\quad \le (1 - t_k)\|f(x_k) - f^\infty(x_k)\| + t_k\|f(x_k) - f^\infty(x_k)\| \\ &\quad = \|f(x_k) - f^\infty(x_k)\|, \end{aligned}$$

where the first inequality follows from nonexpansiveness of Euclidean projector and the second inequality follows from (17) and the third inequality follows from (16).

On the one hand, it follows from

$$\|(F, g)_K^{nat}(x_k)\| < \|f(x_k) - f^\infty(x_k)\|$$

for sufficiently large k. On the other hand, by condition (d), $g(x_k) \in K$ and $\|x_k\| \to \infty$ as $k \to \infty$, we have

$$\|f(x_k) - f^\infty(x_k)\| \leq \|(F, g)_K^{nat}(x_k)\|$$

for sufficiently large k. A contradiction yields! So, it follows that $\mathrm{SOL}(f, g, K) \neq \emptyset$.

Obviously, from $\lim\limits_{g(x)\in K, \|x\|\to\infty} \|(F, g)_K^{nat}(x)\| = \infty$, it follow that $\mathrm{SOL}(f, g, K)$ is bounded. Therefore, we obtain that $\mathrm{SOL}(f, g, K)$ is nonempty and compact. □

Remark 3.4 (i) It has been shown in [31] that the condition (b) can imply the condition (a) of Theorem 3.10 when $g(x) = x$. In addition, when $g(x) = x$, these constraints that $g^{-1}(C) \subseteq C$, $g(x) - g^\infty(x) \in C$ as $\|x\| \to \infty$ and g^∞ is an injective map on H are satisfied immediately. So, when $g(x) = x$, Theorem 3.10 reduces to [31, Theorem 2]

(ii) However, it should be noticed if $g(x) \neq x$, then condition (b) may not imply condition (a). For example, take $K := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$, $f(x) = (x_1 - \sqrt[3]{x_1}, x_2 - \sqrt[3]{x_2})^\top$ and $g(x) = (-x_2^2, -x_2^3 + x_1^2)^\top$. Then for any $g(x) \in K$, it follows that $x_2 = 0$; thus, we have $\langle f^\infty(x), g(x)\rangle = -x_2^2 x_1 - x_2^4 + x_1^2 x_2 = 0$, which means f^∞ is copositive with respect to g on K. But for any $g^\infty(x) = (0, -x_2^3)^\top \in K^\infty = K$, it follows that $x \in \mathbb{R}^2$; thus, we have that $\langle f^\infty(x), g^\infty(x)\rangle = -x_2^4 \leq 0$, which implies f^∞ is not copositive with respect to g^∞ on K^∞.

4 Conclusions

In this paper, we investigated the nonemptiness and compactness of the solution set to the WHGVI, which contains a lot of recently well-studied special VIs and CPs as its subclasses. First, we obtained a result for demonstrating the relationship between the solution set of $\mathrm{WHGVI}(f, g, K)$ and the solution set of $\mathrm{WHGCP}(f^\infty, g^\infty, K^\infty)$. Second, we proved that $\mathrm{WHGVI}(f, g, K)$ has a nonempty, compact solution set under some assumptions and the condition that f^∞ satisfies Harker–Pang condition with respect to g^∞ on K^∞, which is different from the corresponding one shown in [18]. Third, we established three copositivity results for WHGVIs. In particular, one copositivity result does not need to the common restriction on the solution set of corresponding recession cone complementarity problem. If underlying problems reduce to WHVIs, some of our results coincide with the corresponding ones.

In our opinions, the efficient algorithms and the applications of WHGVIs in equilibrium theory and engineering deserve to further study in the future work.

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