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Robust Necessary Optimality Conditions for Nondifferentiable Complex Fractional Programming with Uncertain Data

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Abstract
In this paper, we study robust necessary optimality conditions for a nondifferentiable complex fractional programming with uncertain data. A robust counterpart of uncertain complex fractional programming is introduced in the worst-case scenario. The concept of robust optimal solution of the uncertain complex fractional programming is introduced by using robust counterpart. We give an equivalence between the optimal solutions of the robust counterpart and a minimax nonfractional parametric programming. Finally, Fritz John-type and Karush–Kuhn–Tucker-type robust necessary optimality conditions of the uncertain complex fractional programming are established under some suitable conditions.

Keywords Robust necessary optimality conditions · Uncertain complex fractional programming · Robust counterpart · Robust constraint qualification

Mathematics Subject Classification 49J53 · 65K10 · 90C29

1 Introduction

The linear programming and linear fractional programming in the setting of complex spaces were first studied by Levinson [22] and Swarup and Sharma [30], respectively. Subsequently, optimality conditions and duality of various complex programming including nonlinear fractional or nonfractional programming were extensively studied; see, e.g., [7,8,16–19]. In [26], Mond and Craven pointed out that many existed complex nonlinear programming problems are special cases of a complex programming problem whose objective function includes the square root of a quadratic form. Though the complex programming can be equivalently expressed as a real-valued

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bi-objective fractional programming, the solution concept of bi-objective fractional programming depends on some special partial order. However, it is not easy to choose the best suitable partial order of bi-objective fractional programming such that the solution of bi-objective fractional programming is that of complex programming. So, it is deserved to study complex programming directly. Besides, complex programming plays an important role in the field of electrical engineering, and it has also been applied to phase recovery, MaxCut, statistical signal processing, blind deconvolution, blind equalization, maximal kurtosis and minimal entropy; see, e.g., [4,9,13,18,31,32].

In 2005, Chen et al. [4] studied complex fractional programming by using Charnes–Cooper transformation and established an equivalence between the complex fractional programming and nonfractional programming. Inspired by [8], Lai et al. [21] introduced minimax complex fractional programming and studied the Kuhn–Tucker-type necessary optimality conditions, sufficient optimality conditions as well as weak (strong and strict converse) duality results for such programming under the generalized convexity conditions. Thereafter, Lai and Huang [16,17] considered the optimality conditions for nondifferentiable minimax fractional and nonfractional programming with complex variables. It is worth mentioning that the multipliers corresponding to the constrained functions in the necessary optimality conditions presented in [16,17,21] are required to be nonzero. For this reason, the obtained necessary optimality conditions [16,17,21] may not recover the existing necessary optimality conditions of nonlinear programming with strict inequality constraints. In addition, the minimax fractional programming and minimax nonfractional programming with complex variables can be regarded as the robust counterpart of complex fractional programming and nonfractional programming with respect to the uncertain parameters. As a matter of fact, the real-world problems are always affected by the uncertainty of data due to the prediction errors, measurement errors, the lack of complete information and major emergency (e.g., COVID-19). So, it is necessary to construct mathematical modeling with possible uncertain data to solve practical problems.

Robust optimization method is an important approach to deal with mathematical programming with uncertain data. It is based on the principle that the robust counterpart, which is also called robust optimization, of the uncertain programming has a feasible solution, where the uncertain constraints are forced to be satisfied for all possible parameter realizations within some uncertain sets. In 1937, Soyster [28] proposed a linear optimization model to construct a feasible solution for all data belonging to a convex set. It was the first step in the direction of robust optimization and the deterministic robust correspondence model given in the worst-case scenario. Recently, various robust optimization problems, such as robust linear optimization, robust quadratic optimization, robust semidefinite optimization, robust multistage optimization and robust fractional programming, are studied; see, e.g., [1–3,5,6,12,15,23,29]. To the best of our knowledge, there is no result on robust counterpart of complex fractional programming with uncertain data. Also, many practical problems, such as phase recovery, MaxCut, statistical signal processing, the currents and voltages of electrical networks, are always subject of uncertainty from calculation errors, incomplete information, the natural and social factors such as the extreme weather, earthquake, tsunami and social insurrection. Therefore, it is necessary and meaningful to investigate the complex fractional programming with uncertain data by the robust optimization method.

The present paper is organized as follows. In Sect. 2, we present some basic definitions, existing results as well as complex fractional programming with uncertain data. In Sect. 3, we give a minimax nonfractional parametric programming reformation for the robust counterpart of uncertain complex fractional programming and present the equivalence between optimal solutions of the robust counterpart and one of the minimax nonfractional parametric programming. In Sect. 4, Fritz John-type robust necessary optimality conditions and Karush–Kuhn–Tucker-type robust necessary optimality conditions for the robust optimal solution of uncertain complex fractional programming are established in both differentiable and nondifferentiable cases, respectively. The presented necessary optimality conditions improve the corresponding results in [16,17,21].

2 Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional vector space of complex numbers with inner product $\langle\cdot,\cdot\rangle$ is defined by $y^H z = \langle z, y\rangle = \overline{y}^\top z$ for all $z, y \in \mathbb{C}^n$, where $y^H = \overline{y}^\top$ is the conjugate transpose of y. Denote by $\mathbb{C}^{m\times n}$ the set of all $m \times n$ complex matrices. The transpose, conjugate and conjugate transpose of a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ are denoted by $A^\top = (a_{ji})$, $\overline{A} = (\overline{a_{ij}})$ and $A^H = \overline{A}^\top$, respectively. Set $Q = \{(z, y) \in \mathbb{C}^{2n} : y = \overline{z}\}$, where $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}^n$. Clearly, Q is a closed convex cone; see, e.g., [11]. A matrix A is called Hermitian iff $A^H = A$; it is called positive semidefinite iff all of its eigenvalues are absolutely positive. For a complex number $z = a + bi \in \mathbb{C}$, we denote the real part and the imaginary part of z by $Rez = a$ and $Imz = b$, respectively. For a polyhedral cone $S = \{\xi \in \mathbb{C}^p : Re(K\xi) \geq 0\}$ with $K \in \mathbb{C}^{k\times p}$, the dual cone of S is defined as $S^* = \{\mu \in \mathbb{C}^p : Re\langle\xi, \mu\rangle \geq 0,\ \forall\xi \in S\}$. Clearly, $S = S^{**}$. A convex subset $D \subseteq S$ is said to be a base of the polyhedral cone S if and only if $\mathbf{0} \notin \mathrm{cl}D$ and $S = \mathrm{cone}D := \{s : s = \lambda d,\ \lambda \geq 0,\ d \in D\}$, where $\mathrm{cl}D$ is the closure of D. In particular, D is called a compact base of S iff it is a base of S and compact set.

In this paper, we consider the following complex fractional programming with uncertain data:

$$\min_{\zeta} \frac{Re[f(\zeta,\eta) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma) - (z^H Bz)^{\frac{1}{2}}]} \qquad \text{(UCFP)}$$
$$\text{subject to } -h(\zeta,\omega) \in S,\ \zeta = (z, \overline{z}) \in \mathbb{C}^{2n},$$

where $A, B \in \mathbb{C}^{n\times n}$ are positive semidefinite Hermitian matrices, $S \subseteq \mathbb{C}^p$ is a polyhedral cone which is specified by $K \in \mathbb{C}^{k\times p}$, $\eta \in U$, $\gamma \in V$, $\omega \in W$ are uncertain parameters, the uncertain subsets U, V and W of $\mathbb{C}^{2m}$ are nonempty and compact, $f, g : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \to \mathbb{C}$ and $h : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \to \mathbb{C}^p$ are continuous with respect to the second argument, and $f(\cdot,\eta)$, $g(\cdot,\gamma)$ and $h(\cdot,\omega)$ are analytic at each $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$. We denote the feasible solutions set of (UCFP) by $X(\omega) = \{\zeta = (z, \overline{z}) \in \mathbb{C}^{2n} : -h(\zeta,\omega) \in S\}$.

If f and g are analytic with respect to the first argument, the problem (UCFP) is also nondifferentiable when either $z^H Az$ or $z^H Bz$ vanishes at some point $\zeta_0 = (z_0, \overline{z_0})$ with

$z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, because the term $(z^H A z)^{\frac{1}{2}}$ or $(z^H B z)^{\frac{1}{2}}$ is nondifferentiable in the neighborhood of ζ_0.

Throughout this paper, we assume that for each $(\zeta, \eta) \in X \times U$ and $(\zeta, \gamma) \in X \times V$, $Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}] \geq 0$ and $Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}] > 0$. We adopt the robust optimization method to deal with (UCFP) in the worst-case scenarios. The robust counterpart of (UCFP) can be formulated as

$$\begin{aligned} &\min_{\zeta} \max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]} \\ &\text{subject to} - h(\zeta, \omega) \in S, \quad \forall \omega \in W,\ \zeta = (z, \bar{z}) \in \mathbb{C}^{2n}. \end{aligned} \tag{RCFP}$$

The feasible solution set of (RCFP) is denoted by

$$F := \{\zeta = (z, \bar{z}) \in \mathbb{C}^{2n} : h(\zeta, \omega) \in -S,\ \forall \omega \in W\},$$

which is called the robust feasible set of (UCFP). Since h is analytic with respect to the first argument, $X(\omega)$ is closed for each $\omega \in W$ and so, $F = \bigcap_{\omega \in W} X(\omega)$ is closed. A point $\zeta_0 \in F$ is called a robust optimal solution of (UCFP) iff it is an optimal solution of (RCFP):

$$\max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]} \geq \max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta_0, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta_0, \gamma) - (z^H B z)^{\frac{1}{2}}]}, \quad \forall \zeta \in F.$$

Observed that the problem (RCFP) is nondifferentiable if either $z^H A z$ or $z^H B z$ vanishes at some point $\zeta_0 = (z_0, \overline{z_0})$ with $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, since the term $(z^H A z)^{\frac{1}{2}}$ or $(z^H B z)^{\frac{1}{2}}$ is nondifferentiable in the neighborhood of ζ_0.

We now recall some definitions and basic results which will be used in the sequel.

Definition 2.1 [27] Let $\xi_0 \in S$. The set $S(\xi_0)$ is defined to be the intersection of those closed half spaces which determines S and include ξ_0 in their boundaries or, equivalently,

$$S(\xi_0) = \{z \in \mathbb{C}^p : Re(K_1 \xi) \geq 0\},$$

where $K_1 \in \mathbb{C}^{d \times p}$ is an arbitrary submatrix of $K \in \mathbb{C}^{k \times p}$ and $d \leq k$.

Clearly, $S \subseteq S(\xi_0)$ for each $\xi_0 \in S$. In particular, $S(\xi_0) = \mathbb{C}^p$ when $\xi_0 \in \text{int}S$. This implies that $S^*(\xi_0) \subseteq S^*$, where $S^*(\xi_0)$ is the dual cone of $S(\xi_0)$.

Lemma 2.1 [20] *Let $\eta \in U \subseteq \mathbb{C}^{2m}, \omega \in W \subseteq \mathbb{C}^{2m}$ and the mapping $f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ and $h(\cdot, \omega) : \mathbb{C}^{2n} \to \mathbb{C}^p$ be analytic at each $\zeta = (z, \bar{z}) \in \mathbb{C}^{2n}$. Then, for $\zeta_0 = (z_0, \overline{z_0}) \in \mathbb{C}^{2n}$,*

$$f(\zeta, \eta) - f(\zeta_0, \eta) = f'_\zeta(\zeta_0, \eta)(\zeta - \zeta_0) + o(|\zeta - \zeta_0|), \quad \forall \zeta \in \mathbb{C}^{2n},$$

$$h(\zeta, \omega) - h(\zeta_0, \omega) = h'_{\zeta}(\zeta_0, \omega)(\zeta - \zeta_0) + o(|\zeta - \zeta_0|), \quad \forall \zeta \in \mathbb{C}^{2n},$$

where $|\zeta - \zeta_0|$ *means the norm of complex vector* $\zeta - \zeta_0$,

$$\begin{aligned} f'_{\zeta}(\zeta_0, \eta)(\zeta - \zeta_0) &= (\nabla_z f(\zeta_0, \eta), \nabla_{\overline{z}} f(\zeta_0, \eta)) \begin{pmatrix} z - z_0 \\ \overline{z} - \overline{z_0} \end{pmatrix} \\ &= \nabla_z f(\zeta_0, \eta)(z - z_0) + \nabla_{\overline{z}} f(\zeta_0, \eta)(\overline{z} - \overline{z_0}) \in \mathbb{C}, \end{aligned}$$

and

$$\begin{aligned} h'_{\zeta}(\zeta_0, \omega)(\zeta - \zeta_0) &= (\nabla_z h(\zeta_0, \omega), \nabla_{\overline{z}} h(\zeta_0, \omega)) \begin{pmatrix} z - z_0 \\ \overline{z} - \overline{z_0} \end{pmatrix} \\ &= \nabla_z h(\zeta_0, \omega)(z - z_0) + \nabla_{\overline{z}} h(\zeta_0, \omega)(\overline{z} - \overline{z_0}) \in \mathbb{C}^p. \end{aligned}$$

Lemma 2.2 *Let* $f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ *and* $h(\cdot, \omega) : \mathbb{C}^{2n} \to \mathbb{C}^p$ *be analytic at* $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$, $A \in \mathbb{C}^{n \times n}$ *be a positive semidefinite Hermitian matrix, and* $\eta \in U$ *and* $\omega \in W$ *be uncertain parameters. For each* $y \in \mathbb{C}^n$, $\mu \in \mathbb{C}^p$, *if the function*

$$\Phi(\zeta) = f(\zeta, \eta) + z^H A y + \langle h(\zeta, \omega), \mu \rangle$$

is differentiable at $\zeta_0 = (z_0, \overline{z_0}) \in \mathbb{C}^{2n}$, *then*

$$\begin{aligned} & Re[\Phi'(\zeta_0)(\zeta - \zeta_0)] \\ & = Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\overline{z}} f(\zeta_0, \eta) + Ay + \mu^{\top} \overline{\nabla_z h(\zeta_0, \omega)} + \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) \rangle]. \end{aligned}$$

Proof Noted that $\langle h(\zeta, \omega), \mu \rangle = \mu^H h(\zeta, \omega) = \overline{\mu}^{\top} h(\zeta, \omega)$ and $z^H A y = \overline{z}^{\top} A y$. Since Φ is differentiable at ζ_0, then

$$\begin{aligned} \Phi'(\zeta_0) &= f'_{\zeta}(\zeta_0, \eta) + (z^H A y)'_{\zeta} + \mu^H h'_{\zeta}(\zeta_0, \omega) \\ &= (\nabla_z f(\zeta_0, \eta), \nabla_{\overline{z}} f(\zeta_0, \eta)) + (\mathbf{0}, Ay) + \mu^H (\nabla_z h(\zeta_0, \omega), \nabla_{\overline{z}} h(\zeta_0, \omega)). \end{aligned}$$

From Definition 2.1, it follows that

$$\begin{aligned} & \Phi'(\zeta_0)(\zeta - \zeta_0) \\ & = (\nabla_z f(\zeta_0, \eta), \nabla_{\overline{z}} f(\zeta_0, \eta)) \begin{pmatrix} z - z_0 \\ \overline{z} - \overline{z_0} \end{pmatrix} + Ay(\overline{z} - \overline{z_0}) + \left\langle (\nabla_z h(\zeta_0, \omega), \nabla_{\overline{z}} h(\zeta_0, \omega)) \begin{pmatrix} z - z_0 \\ \overline{z} - \overline{z_0} \end{pmatrix}, \mu \right\rangle \\ & = (\nabla_z f(\zeta_0, \eta) + \mu^H \nabla_z h(\zeta_0, \omega))(z - z_0) + (\mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) + Ay + \nabla_{\overline{z}} f(\zeta_0, \eta))(\overline{z} - \overline{z_0}) \\ & = \left\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \mu^{\top} \overline{\nabla_z h(\zeta_0, \omega)} \right\rangle + \left\langle \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) + Ay + \nabla_{\overline{z}} f(\zeta_0, \eta), z - z_0 \right\rangle. \end{aligned}$$

For any $x, y \in \mathbb{C}^n$, $\langle x, y \rangle = \overline{y}^{\top} x$, $\overline{\langle x, y \rangle} = \overline{(\overline{y}^{\top} x)} = y^{\top} \overline{x} = (y^{\top} \overline{x})^{\top} = \overline{x}^{\top} y = \langle y, x \rangle$ and so,

$$Re\langle x, y \rangle = Re\overline{\langle y, x \rangle} = Re\langle y, x \rangle. \tag{1}$$

Therefore, one has

$$
\begin{aligned}
& Re[\Phi^{'}(\zeta_0)(\zeta - \zeta_0)] \\
&= Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \mu^\top \overline{\nabla_z h(\zeta_0, \omega)}\rangle + \langle \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) + Ay + \nabla_{\overline{z}} f(\zeta_0, \eta), z - z_0\rangle] \\
&= Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \mu^\top \overline{\nabla_z h(\zeta_0, \omega)}\rangle] + Re[\langle \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) + Ay + \nabla_{\overline{z}} f(\zeta_0, \eta), z - z_0\rangle] \\
&= Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \mu^\top \overline{\nabla_z h(\zeta_0, \omega)}\rangle] + Re[\langle z - z_0, \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) + Ay + \nabla_{\overline{z}} f(\zeta_0, \eta)\rangle] \\
&= Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\overline{z}} f(\zeta_0, \eta) + Ay + \mu^\top \overline{\nabla_z h(\zeta_0, \omega)} + \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega)\rangle],
\end{aligned}
$$

where the third equality follows from (1), and $Re(\nu + \upsilon) = Re\nu + Re\upsilon$ implies the second and fourth equalities, $\nu, \upsilon \in \mathbb{C}$. □

Remark 2.1 If h is uncertain-free with respect to the uncertain parameter $\omega \in W$, that is, for any $\tilde{\omega}, \hat{\omega} \in W$, $h(\zeta, \tilde{\omega}) = h(\zeta, \hat{\omega})$, then Lemma 2.2 reduces to Lemma 2 of [17].

Lemma 2.3 [24] *Let $E \in \mathbb{C}^{p \times n}$, $A \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^n$ and $\mu \in S^*$. Then the following assertions are equivalent:*

(a) *The system*

$$E^H \mu = Au + b, \quad u^H Au \leq 1,$$

has a solution $u \in \mathbb{C}^n$.

(b) *If $Ez \in S$ for $z \in \mathbb{C}^n$, then $Re[(z^H Az)^{\frac{1}{2}} + b^H z] \geq 0$.*

Lemma 2.4 [25] *Let $\xi_0 \in S$ and $\mu \in (S(\xi_0))^*$. Then $Re(\mu^H \xi_0) = 0$.*

Lemma 2.5 [24] *Let $A \in \mathbb{C}^{n \times n}$ and $z, u \in \mathbb{C}^n$. Then the following generalized Schwarz inequality in complex spaces holds:*

$$Re(z^H Au) \leq (z^H Az)^{\frac{1}{2}} (u^H Au)^{\frac{1}{2}},$$

and the equality holds whenever $Az = \lambda Au$ or $z = \lambda u$ for $\lambda \geq 0$.

In the rest of the paper, we assume that all complex functions f, g and h are defined on the linear manifold $Q = \{\zeta = (z, \overline{z}) \in \mathbb{C}^{2n} : z \in \mathbb{C}^n\}$ over the real field $\mathbb{R}$, and the robust feasible solution set F is nonempty.

3 Nonfractional Parametric Programming Reformulation of (RCFP)

In this section, we present an equivalent nonfractional parametric programming problem of (RCFP) and establish the relationship between the optimal solution of the nonfractional parametric programming problem and the robust optimal solution of (UCFP).

We introduce the following nonfractional parametric programming problem:

$$\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V} Re\{[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]-v[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]\}, \qquad (\text{RCFP}_v)$$

where the parameter v is defined as

$$v=\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]}.$$

Since $Re[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]\geq 0$ and $Re[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]>0$ for each $(\zeta,\eta)\in X\times U$ and $(\zeta,\gamma)\in X\times V$, one can conclude that $v\geq 0$. Since $f(\zeta,\cdot)$ and $g(\zeta,\cdot)$ are continuous, and U and V are compact sets, for each $\zeta=(z,\overline{z})\in F$, there exist $\tilde{\eta}\in U$ and $\tilde{\gamma}\in V$ such that

$$\frac{Re[f(\zeta,\tilde{\eta})+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\tilde{\gamma})-(z^HBz)^{\frac{1}{2}}]}=\max_{(\eta_1,\gamma_1)\in U\times V}\frac{Re[f(\zeta,\eta_1)+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma_1)-(z^HBz)^{\frac{1}{2}}]}.$$

For the sake of brevity, for each $\zeta\in Q$, we set

$$Y(\zeta)=\left\{(\eta,\gamma)\in U\times V:\frac{Re[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]}=\max_{(\eta_1,\gamma_1)\in U\times V}\frac{Re[f(\zeta,\eta_1)+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma_1)-(z^HBz)^{\frac{1}{2}}]}\right\},$$

and

$$\begin{aligned}Y_v(\zeta)=\Big\{(\eta,\gamma)\in U\times V:Re\left[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}\right]-vRe\left[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}\right]\\=\max_{(\eta_1,\gamma_1)\in U\times V}\left\{Re\left[f(\zeta,\eta_1)+(z^HAz)^{\frac{1}{2}}\right]-vRe\left[g(\zeta,\gamma_1)-(z^HBz)^{\frac{1}{2}}\right]\right\}\Big\}.\end{aligned}$$

It is easy to see that $Y(\zeta)$ and $Y_v(\zeta)$ are compact subsets of $U\times V$.

The next result shows the equivalence between the robust optimal solution of (UCFP) and (RCFP_v).

Theorem 3.1 (a) *$\zeta_0=(z_0,\overline{z_0})\in F$ is a robust optimal solution of* (UCFP) *with optimal value*

$$v^*=\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta_0,\eta)+(z_0^HAz_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma)-(z_0^HBz_0)^{\frac{1}{2}}]}=\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]},$$

if and only if

$$\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V}\left\{Re[f(\zeta,\eta)+(z^HAz)^{\frac{1}{2}}]-v^*Re[g(\zeta,\gamma)-(z^HBz)^{\frac{1}{2}}]\right\}$$

$$= \max_{(\eta,\gamma)\in U\times V} \left\{Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}] - v^* Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}]\right\} = 0,$$

i.e., ζ_0 is an optimal solution of (RCFP$_v$) *with $v = v^*$ and optimal value* 0.

(b) *If ζ_0 is a robust optimal solution of* (UCFP) *with optimal value v^*, then $Y(\zeta_0) = Y_{v^*}(\zeta_0)$.*

Proof (a) Let $\zeta_0 = (z_0, \overline{z_0})$ be a robust optimal solution of (UCFP) with optimal value

$$v^* = \max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}]} = \min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}]}.$$

Then, one has

$$v^* \geq \frac{Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}]}, \quad \forall \eta\in U,\ \gamma\in V,$$

which yields that $Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}] - v^* Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}] \leq 0$ for all $\eta\in U, \gamma\in V$. Consequently, we have

$$\begin{aligned}&\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}\big]\right\}\\ &\leq \max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}\big]\right\} \leq 0.\end{aligned}$$

Since $f(\cdot,\eta), g(\cdot,\gamma)$ are analytic on the closed linear manifold Q for any $\eta\in U, \gamma\in V$, F is closed and $F\subseteq Q$, we can assume that there exists $\zeta_1 = (z_1, \overline{z_1}) \in F$ such that

$$\begin{aligned}&\max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta_1,\eta)+(z_1^H Az_1)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_1,\gamma)-(z_1^H Bz_1)^{\frac{1}{2}}\big]\right\}\\ &= \min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}\big]\right\} \leq 0.\end{aligned} \tag{2}$$

Let us show that $\max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta_1,\eta)+(z_1^H Az_1)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_1,\gamma)-(z_1^H Bz_1)^{\frac{1}{2}}\big]\right\} = 0$. Suppose by contradiction that

$$\max_{(\eta,\gamma)\in U\times V} \left\{Re\big[f(\zeta_1,\eta)+(z_1^H Az_1)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_1,\gamma)-(z_1^H Bz_1)^{\frac{1}{2}}\big]\right\} < 0.$$

So, $Re\big[f(\zeta_1,\eta)+(z_1^H Az_1)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta_1,\gamma)-(z_1^H Bz_1)^{\frac{1}{2}}\big]<0$ for all $\eta\in U$, $\gamma\in V$. Then

$$\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta_1,\eta)+(z_1^H Az_1)^{\frac{1}{2}}]}{Re[g(\zeta_1,\gamma)-(z_1^H Bz_1)^{\frac{1}{2}}]}<v^*,$$

which contradicts the fact that v^* is the optimal value of (UCFP) at the robust optimal solution ζ_0. This together with (2) implies that

$$\begin{aligned}0&=\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V}\Big\{Re\big[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}\big]\Big\}\\&=\max_{(\eta,\gamma)\in U\times V}\Big\{Re\big[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}\big]\Big\}.\end{aligned}$$

Therefore, ζ_0 is an optimal solution of (RCFP$_v$) with $v=v^*$ and optimal value 0.

Conversely, assume that ζ_0 is an optimal solution of (RCFP$_v$) with $v=v^*$ and optimal value 0. Then, one has

$$\begin{aligned}&\min_{\zeta\in F}\max_{(\eta,\gamma)\in U\times V}\Big\{Re\big[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}\big]\Big\}\\&=\max_{(\eta,\gamma)\in U\times V}\Big\{Re\big[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}\big]\Big\}=0.\end{aligned}\tag{3}$$

Since $f(\zeta,\cdot)$ and $g(\zeta,\cdot)$ are continuous, U and V are compact sets, then there exist $\eta_1\in U$ and $\gamma_1\in V$ such that

$$Re\big[f(\zeta_0,\eta_1)+(z_0^H Az_0)^{\frac{1}{2}}\big]-v^*Re\big[g(\zeta_0,\gamma_1)-(z_0^H Bz_0)^{\frac{1}{2}}\big]=0.$$

Since $Re\big[g(\zeta_0,\gamma_1)-(z_0^H Bz_0)^{\frac{1}{2}}\big]>0$, we have

$$v^*=\frac{Re[f(\zeta_0,\eta_1)+(z_1^H Az_1)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma_1)-(z_1^H Bz_1)^{\frac{1}{2}}]}.\tag{4}$$

We next show that

$$v^*=\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}]}.\tag{5}$$

If (5) does not hold, then (4) implies

$$\max_{(\eta,\gamma)\in U\times V}\frac{Re[f(\zeta_0,\eta)+(z_0^H Az_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma)-(z_0^H Bz_0)^{\frac{1}{2}}]}>v^*.$$

Since $f(\zeta, \cdot)$ and $g(\zeta, \cdot)$ are continuous, and U and V are compact sets, there exist $\eta_2 \in U$ and $\gamma_2 \in V$ such that

$$\frac{Re[f(\zeta_0, \eta_2) + (z_0^H A z_0)^{\frac{1}{2}}]}{Re[g(\zeta_0, \gamma_2) - (z_0^H B z_0)^{\frac{1}{2}}]} > v^*,$$

and so, $Re\big[f(\zeta_0, \eta_2) + (z_0^H A z_0)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_0, \gamma_2) - (z_0^H B z_0)^{\frac{1}{2}}\big] > 0$, which contradicts (3). So, (5) holds.

We claim that

$$\min_{\zeta \in F} \max_{(\eta,\gamma) \in U \times V} \frac{Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]} = v^*.$$

If the above equality does not hold, then (5) implies that

$$\min_{\zeta \in F} \max_{(\eta,\gamma) \in U \times V} \frac{Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]} < v^*.$$

Since $f(\cdot, \eta)$ and $g(\cdot, \gamma)$ are analytic on the closed linear manifold Q for any $\eta \in U, \gamma \in V$, and $F \subseteq Q$ is closed, we can assume that there exists $\zeta_2 \in F$ such that

$$\max_{(\eta,\gamma) \in U \times V} \frac{Re[f(\zeta_2, \eta) + (z_2^H A z_2)^{\frac{1}{2}}]}{Re[g(\zeta_2, \gamma) - (z_2^H B z_2)^{\frac{1}{2}}]} = \min_{\zeta \in F} \max_{(\eta,\gamma) \in U \times V} \frac{Re[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]} < v^*.$$

This implies that

$$\frac{Re[f(\zeta_2, \eta) + (z_2^H A z_2)^{\frac{1}{2}}]}{Re[g(\zeta_2, \gamma) - (z_2^H B z_2)^{\frac{1}{2}}]} < v^*, \quad \forall (\eta, \gamma) \in U \times V.$$

So, $Re\big[f(\zeta_2, \eta) + (z_2^H A z_2)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_2, \gamma) - (z_2^H B z_2)^{\frac{1}{2}}\big] < 0$ for all $(\eta, \gamma) \in U \times V$. This together with the continuity of f and g and compactness of U and V yields that

$$\max_{(\eta,\gamma) \in U \times V} Re\big[f(\zeta_2, \eta) + (z_2^H A z_2)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta_2, \gamma) - (z_2^H B z_2)^{\frac{1}{2}}\big] < 0,$$

which contradicts that

$$\min_{\zeta \in F} \max_{(\eta,\gamma) \in U \times V} \Big\{Re\big[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}\big] - v^* Re\big[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}\big]\Big\} = 0.$$

Consequently, we have

$$v^* = \min_{\zeta \in F} \max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta,\eta) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma) - (z^H Bz)^{\frac{1}{2}}]} = \max_{(\eta,\gamma)\in U\times V} \frac{Re[f(\zeta_0,\eta) + (z_0^H Az_0)^{\frac{1}{2}}]}{Re[g(\zeta_0,\gamma) - (z_0^H Bz_0)^{\frac{1}{2}}]}.$$

So, $\zeta_0 = (z_0, \overline{z_0}) \in F$ is the robust optimal solution of (UCFP) with optimal value v^*.

(b) It directly follows from (a) that $Y(\zeta_0) = Y_{v^*}(\zeta_0)$. □

Remark 3.1 If f and g are perturbed by the same uncertain parameter, h is uncertain-free and $U = V$, then Theorem 4.5 reduces Theorem 1 in [16, p. 233]. Moreover, if $A = B = \mathbf{0}$, $U = V$ and h is uncertain-free, then Theorem 4.5 reduces Lemma 2.2 in [21, p. 177]. In particular, if f, g and h are uncertain-free, f and g are also continuous and real-valued functions, $A = B = \mathbf{0}$ and F is a compact and connected subset of $\mathbb{R}^{2n}$, then the Dinkelbach's result [10, Theorem, p.494] can be recovered from Theorem 3.1 (a).

4 Robust Necessary Optimality Conditions of (UCFP)

In this section, we study the Fritz John-type/Karush–Kuhn–Tucker-type robust necessary optimality conditions for the robust optimal solution of (UCFP) in both differentiable and nondifferentiable cases.

We first give the Fritz John-type robust necessary optimality conditions for the robust optimal solution of (UCFP) in the differentiable case.

Theorem 4.1 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of (UCFP) with optimal value v^*, $z_0^H Az_0 > 0$ and $z_0^H Bz_0 > 0$. Then there exist $(\hat{\alpha}, \hat{l}) \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$, $\widehat{\mu} \in S^*$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and $\widehat{\omega} \in W$ such that*

$$\hat{\alpha}\left\{\left[\overline{\nabla_z f(\zeta_0,\widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0,\widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0,\widehat{\gamma})\right] + A\hat{u}_1 + v^* B\hat{u}_2\right\}$$
$$+ \left((\hat{l}\widehat{\mu})^\top \overline{\nabla_z h(\zeta_0,\widehat{\omega})} + (\hat{l}\widehat{\mu})^H \nabla_{\overline{z}} h(\zeta_0,\widehat{\omega})\right) = \mathbf{0}, \tag{6}$$

$$Re\langle h(\zeta_0,\widehat{\omega}), \hat{l}\widehat{\mu}\rangle = 0, \quad \hat{u}_1^H A\hat{u}_1 = 1, \quad \hat{u}_2^H B\hat{u}_2 = 1, \tag{7}$$

$$Re(z_0^H Az_0)^{\frac{1}{2}} = Re(z_0^H A\hat{u}_1), \quad Re(z_0^H Bz_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2). \tag{8}$$

Proof It follows from Theorem 3.1 (a) that $\zeta_0 = (z_0, \overline{z_0}) \in F$ is an optimal solution of (RCFP_v) with $v = v^*$ and optimal value 0. For each $\eta \in U$, $\gamma \in V$ and $\omega \in W$, $f(\cdot,\eta)$, $g(\cdot,\gamma)$ and $h(\cdot,\omega)$ are analytic at each $\zeta = (z, \overline{z}) \in Q$, and A, B are positive semidefinite Hermitian matrices; we deduce that for each $\mu \in S^*$, $Re[f(\cdot,\eta) + (z^H Az)^{\frac{1}{2}}] - v^* Re[g(\cdot,\gamma) - (z^H Bz)^{\frac{1}{2}}]$ and $Re\langle h(\cdot,\omega), \mu\rangle$ are analytic at ζ_0. We observe that

$$\zeta \in F \;\Leftrightarrow\; \max_{(\omega,\mu)\in W\times S^*} Re\langle h(\zeta,\omega), \mu\rangle \le 0 \;\Leftrightarrow\; \max_{(\omega,\mu)\in W\times S_1^*} Re\langle h(\zeta,\omega), \mu\rangle \le 0,$$

where $S_1^* = \{\mu \in S^* : |\mu| \leq 1\}$. So, (RCFP$_v$) with $v = v^*$ is equivalent to the following optimization problem:

$$\begin{aligned} &\min_{\zeta} \max_{(\eta,\gamma)\in U\times V} Re\{[f(\zeta,\eta) + (z^H Az)^{\frac{1}{2}}] - v^*[g(\zeta,\gamma) - (z^H Bz)^{\frac{1}{2}}]\}, \\ &\text{subject to },\ \psi(\zeta) \leq 0, \quad \zeta = (z,\overline{z}) \in \mathbb{C}^{2n}, \end{aligned} \tag{9}$$

where $\psi(\zeta) = \max_{(\omega,\mu)\in W\times S_1^*} Re\langle h(\zeta,\omega),\mu\rangle$. Since h is analytic on Q with respect to the first argument, and W and S_1^* are nonempty compact sets, for each $\zeta \in Q$, there exists $(\omega,\mu) \in W \times S_1^*$ such that $\psi(\zeta) = Re\langle h(\zeta,\omega),\mu\rangle$, ψ is differentiable at $\zeta \in Q$ and so,

$$\psi'(\zeta) = \mu^\top \overline{\nabla_z h(\zeta,\omega)} + \mu^H \nabla_{\overline{z}} h(\zeta,\omega). \tag{10}$$

We define the generalized Lagrangian function of the problem (9) as follows:

$$L(\zeta,\alpha,l) = \alpha \max_{(\eta,\gamma)\in U\times V} \left\{Re[f(\zeta,\eta) + (z^H Az)^{\frac{1}{2}}] - v^* Re[g(\zeta,\gamma) - (z^H Bz)^{\frac{1}{2}}]\right\} + l\psi(\zeta),$$

where $\zeta = (z,\overline{z}) \in \mathbb{C}^{2n}, \alpha \geq 0$ and $l \geq 0$. Since U and V are compact, for each $\zeta \in F$, there exist $\eta \in U$ and $\gamma \in V$ such that

$$L(\zeta,\alpha,l) = \alpha Re[f(\zeta,\eta) + (z^H Az)^{\frac{1}{2}}] - v^* Re[g(\zeta,\gamma) - (z^H Bz)^{\frac{1}{2}}] + l\psi(\zeta).$$

Since $\zeta_0 = (z_0,\overline{z_0}) \in F$ is an optimal solution of (RCFP$_v$) with $v = v^*$, ζ_0 is also an optimal solution of (9). This together with (10) yields that there exist $(\hat{\alpha},\hat{l}) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, $\widehat{\eta} \in U, \widehat{\gamma} \in V, \widehat{\omega} \in W$ and $\widehat{\mu} \in S_1^* \subseteq S^*$ such that

$$\begin{aligned} &\hat{\alpha}\Big\{\big[\overline{\nabla_z f(\zeta_0,\widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0,\widehat{\eta})\big] - v^*\big[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0,\widehat{\gamma})\big] + \frac{Az_0}{\langle Az_0,z_0\rangle^{\frac{1}{2}}} + \frac{v^* Bz_0}{\langle Bz_0,z_0\rangle^{\frac{1}{2}}}\Big\} \\ &\quad + (\hat{l}\widehat{\mu})^\top \overline{\nabla_z h(\zeta_0,\widehat{\omega})} + (\hat{l}\widehat{\mu})^H \nabla_{\overline{z}} h(\zeta_0,\widehat{\omega}) \\ &= \hat{\alpha}\Big\{\big[\overline{\nabla_z f(\zeta_0,\widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0,\widehat{\eta})\big] - v^*\big[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0,\widehat{\gamma})\big] + \frac{Az_0}{\langle Az_0,z_0\rangle^{\frac{1}{2}}} + \frac{v^* Bz_0}{\langle Bz_0,z_0\rangle^{\frac{1}{2}}}\Big\} \\ &\quad + \hat{l}\Big\{\widehat{\mu}^\top \overline{\nabla_z h(\zeta_0,\widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0,\widehat{\omega})\Big\} \\ &= \hat{\alpha}\Big\{\big[\overline{\nabla_z f(\zeta_0,\widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0,\widehat{\eta})\big] - v^*\big[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0,\widehat{\gamma})\big] + \frac{Az_0}{\langle Az_0,z_0\rangle^{\frac{1}{2}}} + \frac{v^* Bz_0}{\langle Bz_0,z_0\rangle^{\frac{1}{2}}}\Big\} \\ &\quad + \hat{l}\psi'(\zeta_0) \\ &= \mathbf{0} \end{aligned}$$

and $Re\langle h(\zeta_0,\widehat{\omega}),\hat{l}\widehat{\mu}\rangle = \hat{l} Re\langle h(\zeta_0,\widehat{\omega}),\widehat{\mu}\rangle = \hat{l}\psi(\zeta_0) = 0$. Since $z_0^H Az_0 = \langle Az_0,z_0\rangle > 0$ and $z_0^H Bz_0 = \langle Bz_0,z_0\rangle > 0$, we set $\hat{u}_1 = \frac{z_0}{\langle Az_0,z_0\rangle^{\frac{1}{2}}}$ and $\hat{u}_2 = \frac{z_0}{\langle Bz_0,z_0\rangle^{\frac{1}{2}}}$.

Then, it shows that

$$\hat{\alpha}\Big\{\Big[\overline{\nabla_z f(\zeta_0,\widehat{\eta})}+\nabla_{\bar{z}} f(\zeta_0,\widehat{\eta})\Big]-v^*\Big[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})}+\nabla_{\bar{z}} g(\zeta_0,\widehat{\gamma})\Big]+A\hat{u}_1+v^*B\hat{u}_2\Big\}$$
$$+(\hat{l}\widehat{\mu})^\top\overline{\nabla_z h(\zeta_0,\widehat{\omega})}+(\hat{l}\widehat{\mu})^H\nabla_{\bar{z}} h(\zeta_0,\widehat{\omega})=\mathbf{0}.$$

Moreover, we have

$$\hat{u}_1^H A\hat{u}_1=\frac{z_0^H A z_0}{(z_0^H A z_0)^{\frac{1}{2}}(z_0^H A z_0)^{\frac{1}{2}}}=1,\;\; \hat{u}_2^H B\hat{u}_2=\frac{z_0^H B z_0}{(z_0^H B z_0)^{\frac{1}{2}}(z_0^H B z_0)^{\frac{1}{2}}}=1,$$
$$Re(z_0^H A\hat{u}_1)=Re\Big(z_0^H A\frac{z_0}{\langle Az_0,z_0\rangle^{\frac{1}{2}}}\Big)=Re\Big(\frac{z_0^H Az_0}{(z_0^H Az_0)^{\frac{1}{2}}}\Big)=Re(z_0^H Az_0)^{\frac{1}{2}},$$

and $Re(z_0^H B\hat{u}_2)=Re\Big(z_0^H B\frac{z_0}{\langle Bz_0,z_0\rangle^{\frac{1}{2}}}\Big)=Re\Big(\frac{z_0^H Bz_0}{(z_0^H Bz_0)^{\frac{1}{2}}}\Big)=Re(z_0^H Bz_0)^{\frac{1}{2}}$, as required. □

Remark 4.1 (a) In Theorem 4.1, $\widehat{\mu}\in S^*$ and $\hat{l}\geq 0$ imply $\hat{l}\widehat{\mu}\in S^*$ since S^* is a closed and convex cone. So, Theorem 4.1 implies that there exist $\big(\hat{\alpha},\hat{l}\big)\in\mathbb{R}_+^2\setminus\{\mathbf{0}\}$, $\tilde{\mu}=\hat{l}\widehat{\mu}\in S^*$, $\hat{u}_1,\hat{u}_2\in\mathbb{C}^n$ and $\widehat{\omega}\in W$ such that

$$\hat{\alpha}\,\Big\{\Big[\overline{\nabla_z f(\zeta_0,\widehat{\eta})}+\nabla_{\bar{z}} f(\zeta_0,\widehat{\eta})\Big]-v^*\Big[\overline{\nabla_z g(\zeta_0,\widehat{\gamma})}+\nabla_{\bar{z}} g(\zeta_0,\widehat{\gamma})\Big]+A\hat{u}_1+v^*B\hat{u}_2\Big\}$$
$$+\,\tilde{\mu}^\top\overline{\nabla_z h(\zeta_0,\widehat{\omega})}+\tilde{\mu}\nabla_{\bar{z}} h(\zeta_0,\widehat{\omega})=\mathbf{0}, \tag{11}$$
$$Re\langle h(\zeta_0,\widehat{\omega}),\tilde{\mu}\rangle=0,\quad \hat{u}_1^H A\hat{u}_1=1,\quad \hat{u}_2^H B\hat{u}_2=1, \tag{12}$$
$$Re(z_0^H Az_0)^{\frac{1}{2}}=Re(z_0^H A\hat{u}_1),\quad Re(z_0^H Bz_0)^{\frac{1}{2}}=Re(z_0^H B\hat{u}_2). \tag{13}$$

(b) Since $\mathbf{0}\in S_1^*\subseteq S^*$, for each $\zeta\in F$, $(\omega,\mathbf{0})\in W\times S_1^*$ is a trivial solution of the problem: $\psi(\zeta):=\max\limits_{(\omega,\mu)\in W\times S_1^*} Re\langle h(\zeta,\omega),\mu\rangle=0$, that is, $W\times\{\mathbf{0}\}\subseteq\psi_\zeta^{-1}(0)$, where

$$\psi_\zeta^{-1}(0)=\Big\{(\bar{\omega},\bar{\mu})\in W\times S_1^*\;:\;Re\langle h(\zeta,\bar{\omega}),\bar{\mu}\rangle=\max_{(\omega,\mu)\in W\times S_1^*} Re\langle h(\zeta,\omega),\mu\rangle\Big\}.$$

If $\zeta\notin F$, then there exist $\mu\in S^*,\omega\in W$ such that $Re\langle h(\zeta,\omega),\mu\rangle>0$ due to $S=(S^*)^*$. This yields that $\psi(\zeta)=\max_{(\omega,\mu)\in W\times S_1^*} Re\langle h(\zeta,\omega),\mu\rangle>0$. In Theorem 4.1, if $\widehat{\mu}=\mathbf{0}$ and $\hat{\alpha}=0$, then one can easily check that (6)–(8) still hold for all $\hat{l}>0$. In this case, for any $\hat{l}>0$, $(\hat{\alpha},\tilde{\mu})=\mathbf{0}$, $\hat{\alpha}$ and $\tilde{\mu}$ also satisfy (11)–(13), where $\tilde{\mu}=\hat{l}\widehat{\mu}$.
In order to avoid the trivial case $(\hat{\alpha},\tilde{\mu})=\mathbf{0}$, let us analyze the set $\psi_\zeta^{-1}(0)$.

(i) If $\psi_\zeta^{-1}(0)=W\times\{\mathbf{0}\}$ for each $\zeta\in F$, then for each $\zeta=(z,\bar{z})\in\mathbb{C}^{2n}$,

$$\max_{\omega\in W} Re\langle h(\zeta,\omega),\mu\rangle<0,\quad\forall\mu\in S_1^*\setminus\{\mathbf{0}\}.$$

It implies $F = Q$, i.e., (UCFP), (RCFP$_v$) with $v = v^*$ and the optimization problem (9) are unconstrained. Then, there exist $\hat{\alpha} = 1, \widehat{\mu} = \mathbf{0}$ and $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ such that for each $\hat{l} \geq 0, \widehat{\omega} \in W, \tilde{\mu} = \hat{l}\widehat{\mu} = \mathbf{0} \in S^*$, and the robust necessary optimality conditions (11)–(13) hold.

(ii) If there exists $\zeta \in F$ such that $\psi_{\zeta}^{-1}(0) \neq W \times \{\mathbf{0}\}$, then, there exists $\widehat{\mu} \in S_1^* \setminus \{\mathbf{0}\}$ such that $\max_{\omega \in W} Re\langle h(\zeta, \omega), \widehat{\mu}\rangle = 0$. In particular, if all conditions of Theorem 4.1 hold and $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$, then there exist $\hat{\omega} \in W$ and a nonzero $\hat{\mu} \in S_1^* \subseteq S^*$ such that for $(\hat{\alpha}, \hat{l}) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ given in Theorem 4.1, $(\hat{\alpha}, \tilde{\mu}) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$, where $\tilde{\mu} = \hat{l}\hat{\mu}$.

The following Fritz John-type robust necessary optimality conditions for the robust optimal solution of (UCFP) can be obtained directly from Theorem 4.1 and Remark 4.1(b)(ii).

Theorem 4.2 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of (UCFP) with optimal value v^*, $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$, $z_0^H A z_0 > 0$ and $z_0^H B z_0 > 0$. Then there exist $\widehat{\omega} \in W$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and $(\hat{\alpha}, \widehat{\mu}) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$ such that*

$$\hat{\alpha}\left\{\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] + A\hat{u}_1 + v^* B\hat{u}_2\right\} + \widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}, \tag{14}$$

$$Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu}\rangle = 0, \quad \hat{u}_1^H A\hat{u}_1 = 1, \quad \hat{u}_2^H B\hat{u}_2 = 1, \tag{15}$$

$$Re(z_0^H A z_0)^{\frac{1}{2}} = Re(z_0^H A\hat{u}_1), \quad Re(z_0^H B z_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2). \tag{16}$$

We next present another Fritz John-type robust necessary optimality conditions for the robust optimal solution of (UCFP) when S^* has a compact base.

Theorem 4.3 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of (UCFP) with optimal value v^*, S^* have a compact base, and let $z_0^H A z_0 > 0$ and $z_0^H B z_0 > 0$. Then there exist $\widehat{\omega} \in W$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and $(\hat{\alpha}, \widehat{\mu}) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$ such that (14)–(16) hold.*

Proof Since S^* has a compact base, we assume that $\mathscr{D}$ is a compact base of S^*. Then

$$\zeta \in F \Leftrightarrow \max_{(\omega,\mu)\in W\times S^*} Re\langle h(\zeta, \omega), \mu\rangle \leq 0 \Leftrightarrow \max_{(\omega,\mu)\in W\times \mathscr{D}} Re\langle h(\zeta, \omega), \mu\rangle \leq 0.$$

Consequently, (RCFP$_v$) with $v = v^*$ is equivalent to the following optimization problem:

$$\begin{aligned} &\min_{\zeta} \max_{(\eta,\gamma)\in U\times V} Re\{[f(\zeta, \eta) + (z^H A z)^{\frac{1}{2}}] - v^*[g(\zeta, \gamma) - (z^H B z)^{\frac{1}{2}}]\}, \\ &\text{subject to } \varphi(\zeta) \leq 0, \quad \zeta = (z, \overline{z}) \in \mathbb{C}^{2n}, \end{aligned} \tag{17}$$

where $\varphi(\zeta) = \max_{(\omega,\mu)\in W\times\mathscr{D}} Re\langle h(\zeta, \omega), \mu\rangle$. Since f, g and h are analytic on Q with respect to the first argument, and U, V, W and $\mathscr{D}$ are nonempty compact sets, for each $\zeta \in Q$, there exist $\eta_1 \in U$, $\gamma_1 \in V$ and $(\omega, \mu) \in W \times \mathscr{D}$ such that

$$\begin{aligned} &Re\{[f(\zeta, \eta_1) + (z^H Az)^{\frac{1}{2}}] - v^*[g(\zeta, \gamma_1) - (z^H Bz)^{\frac{1}{2}}]\} \\ &\quad = \max_{(\eta,\gamma)\in U\times V} Re\{[f(\zeta, \eta) + (z^H Az)^{\frac{1}{2}}] - v^*[g(\zeta, \gamma) - (z^H Bz)^{\frac{1}{2}}]\}, \end{aligned}$$

$\varphi(\zeta) = Re\langle h(\zeta, \omega), \mu\rangle$ and so, $\max_{(\eta,\gamma)\in U\times V} Re\{[f(\zeta, \eta) + (z^H Az)^{\frac{1}{2}}] - v^*[g(\zeta, \gamma) - (z^H Bz)^{\frac{1}{2}}]\}$ and $\varphi(\zeta)$ are differentiable at $\zeta \in Q$. Moreover, one has

$$\varphi'(\zeta) = \mu^\top \overline{\nabla_z h(\zeta, \omega)} + \mu^H \nabla_{\overline{z}} h(\zeta, \omega). \tag{18}$$

Since $\zeta_0 = (z_0, \overline{z_0}) \in F$ is an optimal solution of (RCFP$_v$) with $v = v^*$, ζ_0 is also an optimal solution of (17). This together with (18) yields that there exist $(\hat{\alpha}, \hat{l}) \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$, $\widehat{\eta} \in U$, $\widehat{\gamma} \in V$, $\widehat{\omega} \in W$ and $\tilde{\mu} \in \mathscr{D} \subseteq S^*$ such that

$$\begin{aligned} &\hat{\alpha}\left\{\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] + \frac{Az_0}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} + \frac{v^* Bz_0}{\langle Bz_0, z_0\rangle^{\frac{1}{2}}}\right\} \\ &+ (\hat{l}\tilde{\mu})^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + (\hat{l}\tilde{\mu})^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0} \end{aligned}$$

and $Re\langle h(\zeta_0, \widehat{\omega}), \hat{l}\tilde{\mu}\rangle = \hat{l} Re\langle h(\zeta_0, \widehat{\omega}), \tilde{\mu}\rangle = \hat{l}\varphi(\zeta_0) = 0$. Since $\tilde{\mu} \in \mathscr{D}$ and $\mathscr{D}$ is a compact base of S^*, then $\tilde{\mu} \neq \mathbf{0}$. Using $(\hat{\alpha}, \hat{l}) \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ yields $(\hat{\alpha}, \hat{l}\tilde{\mu}) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$. Set $\widehat{\mu} := \hat{l}\tilde{\mu}$, $\hat{u}_1 = \frac{z_0}{\langle Az_0, z_0\rangle^{\frac{1}{2}}}$ and $\hat{u}_2 = \frac{z_0}{\langle Bz_0, z_0\rangle^{\frac{1}{2}}}$. By the similar proof of Theorem 4.1, one can conclude that (14)–(16) hold. □

We now introduce the robust constraint qualification.

Definition 4.1 The problem (UCFP) satisfies the robust constraint qualification (shortly, RCQ) at $\zeta_0 = (z_0, \overline{z_0})$ if and only if for any nonzero $\mu \in S^* \subset \mathbb{C}^p$ and for any $\omega \in W$,

$$Re\langle h'_\zeta(\zeta_0, \omega)(\zeta - \zeta_0), \mu\rangle \neq 0, \quad \zeta \neq \zeta_0.$$

It is easy to verify that if (UCFP) satisfies RCQ at ζ_0, then

$$\mu^\top \overline{\nabla_z h(\zeta_0, \omega)} + \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) \neq \mathbf{0}, \quad \forall \mu \in S^* \setminus \{\mathbf{0}\}, \omega \in W,$$

and for any $\omega \in W$, $\mu^\top \overline{\nabla_z h(\zeta_0, \omega)} + \mu^H \nabla_{\overline{z}} h(\zeta_0, \omega) = \mathbf{0}$ implies $\mu = \mathbf{0}$.

If $Re[g(\zeta, \gamma) - (z^H Bz)^{\frac{1}{2}}] \equiv 1$ and h is uncertain-free with respect to the uncertain parameter $\omega \in W$, then RCQ reduces to the constraint qualification defined by Definition 3 in [17].

It is known that the RCQ corresponds to quasinormality condition [14] is weaker than Mangasarian–Fromovitz constraint qualification as well as positively linearly independent constraint qualification in nonlinear programming.

If $S = \mathbb{R}^p_+$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable and uncertain-free, then RCQ reduces to the following positively linearly independent constraint qualification (in short, PLICQ): For a feasible point $z_0 \in \mathbb{R}^n$ with $h(z_0) = \mathbf{0}$, there exists no nonzero $\mu = (\mu_1, \mu_2, \ldots, \mu_p) \in \mathbb{R}^p_+$ such that $\sum_{i=1}^p \mu_i \nabla h_i(z_0) = 0$.

For a feasible point $z_0 \in \mathbb{R}^n$, RCQ is slightly weaker than the following quasi-normality condition: There exist no nonzero $\mu = (\mu_1, \mu_2, \ldots, \mu_p) \in \mathbb{R}^p_+$ and no sequence $\{z_k\} \to z_0$ such that $\sum_{i=1}^p \mu_i \nabla h_i(z_0) = 0$ and for all k, $\mu_i h_i(z_k) > 0$ for all i with $\mu_i \neq 0$.

We present the Karush–Kuhn–Tucker-type robust necessary optimality conditions for (UCFP) by using RCQ.

Theorem 4.4 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of* (UCFP) *with optimal value v^*. Assume that* (UCFP) *satisfies* RCQ *at ζ_0 with $z_0^H A z_0 > 0$ and $z_0^H B z_0 > 0$. If $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$ or S^* has a compact base, then there exist $\widehat{\mu} \in S^*$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and $\widehat{\omega} \in W$ such that*

$$\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] + A\hat{u}_1 + v^* B\hat{u}_2 + \widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}, \tag{19}$$

$$Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu}\rangle = 0, \tag{20}$$

$$\hat{u}_1^H A \hat{u}_1 = 1, \quad \hat{u}_2^H B \hat{u}_2 = 1, \tag{21}$$

$$Re(z_0^H A z_0)^{\frac{1}{2}} = Re(z_0^H A \hat{u}_1), \quad Re(z_0^H B z_0)^{\frac{1}{2}} = Re(z_0^H B \hat{u}_2). \tag{22}$$

Proof It follows from Theorems 4.2 and 4.3 that there exist $\left(\hat{\alpha}, \tilde{\mu}\right) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and $\widehat{\omega} \in W$ such that (21), (22) hold and

$$\hat{\alpha}\left\{\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] + A\hat{u}_1 + v^* B\hat{u}_2\right\} + \tilde{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \tilde{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}, \tag{23}$$

and $Re\langle h(\zeta_0, \widehat{\omega}), \tilde{\mu}\rangle = 0$.

If $\hat{\alpha} = 0$ in (23), then

$$\tilde{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \tilde{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}. \tag{24}$$

Since (UCFP) satisfies RCQ at ζ_0, then (24) implies $\tilde{\mu} = \mathbf{0}$, which contradicts the fact that $\left(\hat{\alpha}, \tilde{\mu}\right) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$. So, $\hat{\alpha} > 0$. Divided both the sides of equation (23) by $\hat{\alpha}$ and set $\widehat{\mu} = \frac{\tilde{\mu}}{\hat{\alpha}} \in S^*$, we have

$$\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] + A\hat{u}_1 + v^* B\hat{u}_2 + \widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0},$$

and $Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu}\rangle = \frac{1}{\hat{\alpha}} Re\langle h(\zeta_0, \widehat{\omega}), \tilde{\mu}\rangle = 0$. Consequently, (19) and (20) hold. □

We next give another form of Karush–Kuhn–Tucker-type robust necessary optimality conditions for (UCFP) by using RCQ.

Theorem 4.5 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of (UCFP) with optimal value v^*. Assume that (UCFP) satisfies* RCQ *at ζ_0 with $z_0^H A z_0 > 0$ and $z_0^H B z_0 > 0$. If $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$ or S^* has a compact base, then there exist $\widehat{\mu} \in S^*$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$, $\widehat{\omega} \in W$ and a positive integer k such that*

(i) $(\widehat{\eta_i}, \widehat{\gamma_i}) \in Y(\zeta_0)$, $i = 1, 2, \ldots, k$;
(ii) *there exist multipliers $\hat{\lambda}_i > 0$ for $i = 1, 2, \ldots, k$ with $\sum_{i=1}^k \hat{\lambda}_i = 1$ such that*

$$\sum_{i=1}^k \hat{\lambda}_i \left\{ \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i}) \right] - v^* \left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i}) \right] \right\} + \left(A\hat{u}_1 + v^* B\hat{u}_2 \right) + \left(\widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) \right) = \mathbf{0}, \tag{25}$$

$$Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = 0, \tag{26}$$

$$\hat{u}_1^H A\hat{u}_1 = 1, \quad Re(z_0^H A z_0)^{\frac{1}{2}} = Re(z_0^H A\hat{u}_1), \tag{27}$$

$$\hat{u}_2^H B\hat{u}_2 = 1, \quad Re(z_0^H B z_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2). \tag{28}$$

Proof It follows from Theorem 3.1 that $\zeta_0 = (z_0, \overline{z_0}) \in F$ is an optimal solution of (RCFP_v) with $v = v^*$ and optimal value 0, and $Y_{v^*}(\zeta_0) = Y(\zeta_0)$. For each $\eta \in U$, $\gamma \in V$ and $\omega \in W$, $f(\cdot, \eta)$, $g(\cdot, \gamma)$ and $h(\cdot, \omega)$ are analytic at each $\zeta = (z, \overline{z}) \in Q$, and A, B are positive definite Hermitian matrices; we deduce that for a nonzero vector $\mu \in S^*$, $Re\langle h(\cdot, \omega), \mu \rangle$ and $Re[f(\cdot, \eta) + (z^H A z)^{\frac{1}{2}}] - v^* Re[g(\cdot, \gamma) - (z^H B z)^{\frac{1}{2}}]$ are analytic at ζ. As in the proof of Theorem 4.1, we conclude from the compactness of U and V and Remark 4.1 (b)(ii) that for the robust optimal solution $\zeta_0 = (z_0, \overline{z_0}) \in F$, there exist $(\widehat{\alpha}, \tilde{\mu}) \in (\mathbb{R}_+ \times S^*) \setminus \{\mathbf{0}\}$, $\widehat{\omega} \in W$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^n$ and a positive integer k with $(\widehat{\eta_i}, \widehat{\gamma_i}) \in Y_{v^*}(\zeta_0)$, $\widehat{\lambda}_i > 0$ and $\sum_{i=1}^k \widehat{\lambda}_i = 1$, $i = 1, 2, \ldots, k$ such that

$$\widehat{\alpha} \sum_{i=1}^k \widehat{\lambda}_i \left\{ \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i}) \right] - v^* \left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i}) \right] + A\hat{u}_1 + v^* B\hat{u}_2 \right\} + \left(\tilde{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \tilde{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) \right) = \mathbf{0},$$

$Re\langle h(\zeta_0, \widehat{\omega}), \tilde{\mu} \rangle = 0$ and (27) and (28) hold.

Since $Y_{v^*}(\zeta_0) = Y(\zeta_0)$, we have $(\widehat{\eta_i}, \widehat{\gamma_i}) \in Y(\zeta_0)$, $i = 1, 2, \ldots, k$. Since (UCFP) satisfies RCQ at ζ_0, as in the proof of Theorem 4.4, we have $\widehat{\alpha} > 0$. Set $\widehat{\mu} = \frac{\tilde{\mu}}{\widehat{\alpha}} \in S^*$. Then

$$\sum_{i=1}^k \widehat{\lambda}_i \left\{ \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i}) \right] - v^* \left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i}) \right] \right\} + A\hat{u}_1 + v^* B\hat{u}_2 + \left(\widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) \right) = \mathbf{0}$$

and $Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = \frac{1}{\widehat{\alpha}} Re\langle h(\zeta_0, \widehat{\omega}), \tilde{\mu} \rangle = 0$, namely, (25) and (26) are true. □

Remark 4.2 Generally, the multipliers $\mu \in S^*$ corresponding to the constrained conditions are not necessary to be nonzero. In Theorem 4.5, we do not require the multiplier corresponding to constrained function h to be a nonzero $\widehat{\mu} \in S^*$. So, Theorem 4.5 is distinct from Theorem 3.1 in [21, p.177] even if f, g are perturbed by the same uncertain parameter, h is uncertain-free, $A = B = \mathbf{0}$ and $U = V$, and Theorem 2 in [16, p. 234] even though f, g are perturbed by the same uncertain parameter, h is uncertain-free and $U = V$. Moreover, if h is uncertain-free with respect to the uncertain parameter $\omega \in W$ and $Re[g(\zeta, \gamma) - (z^H Bz)^{\frac{1}{2}}] \equiv 1$, then Theorem 4.5 is different from Theorem 1 in [17, p. 1207].

We consider the robust necessary optimality conditions for the robust optimal solution ζ_0 of (UCFP) when $\langle Az_0, z_0\rangle = 0$ or $\langle Bz_0, z_0\rangle = 0$, i.e., $\langle Az, z\rangle^{\frac{1}{2}}$ or $\langle Bz, z\rangle^{\frac{1}{2}}$ is nondifferentiable at $z = z_0$. By the compactness of $Y(\zeta_0)$, one can verify that Theorem 4.5 (a) and (b) are satisfied. Let $(\widehat{\eta_i}, \widehat{\gamma_i}) \in Y(\zeta_0)$, $\hat{\lambda}_i > 0$, $i = 1, 2, \dots k$ be the same as in Theorem 4.5. Since W is a nonempty and compact set, there exists $\widehat{\omega} \in W$ such that

$$\max_{\mu \in S_1^*} Re\langle h(\zeta_0, \widehat{\omega}), \mu\rangle = \max_{\omega \in W} \max_{\mu \in S_1^*} Re\langle h(\zeta_0, \omega), \mu\rangle = \max_{(\omega,\mu) \in W \times S_1^*} Re\langle h(\zeta_0, \omega), \mu\rangle.$$

Set $v^* = \min_{\zeta \in F} \max_{(\eta,\gamma) \in U \times V} \frac{Re[f(\zeta,\eta)+(z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta,\gamma)-(z^H Bz)^{\frac{1}{2}}]}$. Motivated by the works [16,17,27], we introduce a subset $Z(\zeta_0) \subseteq \mathbb{C}^{2n}$ as follows:

$$Z(\zeta_0) = \left\{\zeta \in Q \ : \ -h_\zeta^{'}(\zeta_0, \widehat{\omega})\zeta \in S(-h(\zeta_0, \widehat{\omega})), \text{ if any one of (c1), (c2) and (c3) holds}\right\},$$

where the conditions (c1), (c2) and (c3) are defined as:

(c1) $Re\left\{ \sum_{i=1}^k \hat{\lambda}_i [f_\zeta^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_\zeta^{'}(\zeta_0, \widehat{\gamma_i})]\zeta + \frac{\langle Az_0, z\rangle}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} + \langle (v^*)^2 Bz, z\rangle^{\frac{1}{2}} \right\} < 0$, if $z_0^H Az_0 > 0$ and $z_0^H Bz_0 = 0$;

(c2) $Re\left\{ \sum_{i=1}^k \hat{\lambda}_i [f_\zeta^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_\zeta^{'}(\zeta_0, \widehat{\gamma_i})]\zeta + \langle Az, z\rangle^{\frac{1}{2}} + \frac{\langle v^* Bz_0, z\rangle}{\langle Bz_0, z_0\rangle^{\frac{1}{2}}} \right\} < 0$, if $z_0^H Az_0 = 0$ and $z_0^H Bz_0 > 0$;

(c3) $Re\left\{ \sum_{i=1}^k \hat{\lambda}_i [f_\zeta^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_\zeta^{'}(\zeta_0, \widehat{\gamma_i})]\zeta + \langle [A + (v^*)^2 B]z, z\rangle^{\frac{1}{2}} \right\} < 0$, if $z_0^H Az_0 = 0$ and $z_0^H Bz_0 = 0$.

We next present Karush–Kuhn–Tucker-type robust necessary optimality conditions for (UCFP) without the assumptions $\langle Az_0, z_0\rangle > 0$, $\langle Bz_0, z_0\rangle > 0$ as well as the assumptions that $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$ and S^* has a compact base.

Theorem 4.6 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of* (UCFP) *with optimal value v^*. Assume that* (UCFP) *satisfies* RCQ *at ζ_0 and $Z(\zeta_0) = \emptyset$. Then there* *exist* $\widehat{\mu} \in S^*$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^p$ *such that*

$$\sum_{i=1}^{k} \hat{\lambda}_i \left\{ \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i}) \right] - v^* \left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i}) \right] \right\}$$
$$+ A\hat{u}_1 + v^* B\hat{u}_2 + \widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}, \tag{29}$$

$$Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = 0, \tag{30}$$

$$\hat{u}_1^H A\hat{u}_1 \leq 1, \quad Re(z_0^H A z_0)^{\frac{1}{2}} = Re(z_0^H A\hat{u}_1), \tag{31}$$

$$\hat{u}_2^H B\hat{u}_2 \leq 1, \quad Re(z_0^H B z_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2). \tag{32}$$

Proof Observe that $\langle Az, z \rangle = z^H Az = \zeta^H \widehat{A} \zeta = \langle \widehat{A}\zeta, \zeta \rangle$, $\langle Bz, z \rangle = z^H Bz = \zeta^H \widehat{B} \zeta = \langle \widehat{B}\zeta, \zeta \rangle$, where $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$ and

$$\widehat{A} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{C}^{2n \times 2n}, \quad \widehat{B} = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{C}^{2n \times 2n}.$$

We split the proof into three cases.

(1) If $z_0^H A z_0 > 0$ and $z_0^H B z_0 = 0$, then we deduce from $Z(\zeta_0) = \emptyset$ that for $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$ and $-h_\zeta^{'}(\zeta_0, \widehat{\omega})\zeta \in S(-h(\zeta_0, \widehat{\omega}))$,

$$Re \left\{ \sum_{i=1}^{k} \hat{\lambda}_i \left[f_\zeta^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_\zeta^{'}(\zeta_0, \widehat{\gamma_i}) \right] \zeta + \frac{\langle Az_0, z \rangle}{\langle Az_0, z_0 \rangle^{\frac{1}{2}}} + \langle (v^*)^2 Bz, z \rangle^{\frac{1}{2}} \right\} \geq 0.$$

It can be equivalently reformulated as

$$Re \left\{ \sum_{i=1}^{k} \hat{\lambda}_i \left[f_\zeta^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_\zeta^{'}(\zeta_0, \widehat{\gamma_i}) + \frac{\widehat{A}\zeta_0}{\langle Az_0, z_0 \rangle^{\frac{1}{2}}} \right] \zeta + \langle (v^*)^2 \widehat{B}\zeta, \zeta \rangle^{\frac{1}{2}} \right\} \geq 0. \tag{33}$$

Note that $-h(\zeta_0, \widehat{\omega}) \in S$. By Lemma 2.4, it implies that there exists $\widehat{\mu} \in S^*(-h(\zeta_0, \widehat{\omega}))$ such that $-Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = Re(\widehat{\mu}^H(-h(\zeta_0, \widehat{\omega}))) = 0$ and so, $Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = 0$. Hence, we conclude from $S^*(-h(\zeta_0, \widehat{\omega})) \subseteq S^*$ that $\widehat{\mu} \in S^*$ and

$$\max_{(\omega,\mu) \in W \times S^*} Re\langle h(\zeta_0, \omega), \mu \rangle = \max_{(\omega,\mu) \in W \times S_1^*} Re\langle h(\zeta_0, \omega), \mu \rangle = Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu} \rangle = 0.$$

Set $E = -h_{\zeta}^{'}(\zeta_0, \widehat{\omega})$, $b = \sum_{i=1}^{k} \hat{\lambda}_i \left[f_{\zeta}^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_{\zeta}^{'}(\zeta_0, \widehat{\gamma_i}) + \frac{\widehat{A}\zeta_0}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} \right]$, $z = \zeta$, $A = (v^*)^2 \widehat{B}$ in Lemma 2.3. It follows from Lemma 2.3(a) that there exists $\xi = (\alpha, \overline{\alpha})^{\top} \in \mathbb{C}^{2n}$ such that $\alpha^H (v^*)^2 B\alpha = \xi^H (v^*)^2 \widehat{B}\xi \leq 1$, and

$$(-h^{'}(\zeta_0, \widehat{\omega}))^H \widehat{\mu} = (v^*)^2 \widehat{B}\xi + \left(\sum_{i=1}^{k} \hat{\lambda}_i \left[f_{\zeta}^{'}(\zeta_0, \widehat{\eta_i}) - v^* g_{\zeta}^{'}(\zeta_0, \widehat{\gamma_i}) + \frac{\widehat{A}\zeta_0}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} \right] \right). \tag{34}$$

Set $\hat{u}_2 = v^*\alpha$. Then (34) implies that

$$\begin{pmatrix} -\overline{\nabla_z h(\zeta_0, \widehat{\omega})\widehat{\mu}} \\ -\overline{\nabla_{\overline{z}} h(\zeta_0, \widehat{\omega})\widehat{\mu}} \end{pmatrix} = \begin{pmatrix} v^* B\hat{u}_2 + \sum_{i=1}^{k} \hat{\lambda}_i \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} - v^* \overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} \right] + \frac{(Az_0)^H}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} \\ \sum_{i=1}^{k} \hat{\lambda}_i \left[\overline{\nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i})} - v^* \overline{\nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i})} \right] \end{pmatrix},$$

and so,

$$v^* B\hat{u}_2 + \sum_{i=1}^{k} \hat{\lambda}_i \left[\overline{\nabla_z f(\zeta_0, \widehat{\eta_i})} - v^* \overline{\nabla_z g(\zeta_0, \widehat{\gamma_i})} \right] + \frac{(Az_0)^H}{\langle Az_0, z_0\rangle^{\frac{1}{2}}} + \overline{\nabla_z h(\zeta_0, \widehat{\omega})\widehat{\mu}} = \mathbf{0}, \tag{35}$$

$$\sum_{i=1}^{k} \hat{\lambda}_i \left[\overline{\nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i})} - v^* \overline{\nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i})} \right] + \overline{\nabla_{\overline{z}} h(\zeta_0, \widehat{\omega})\widehat{\mu}} = \mathbf{0}. \tag{36}$$

Clearly, (36) can be equivalently expressed as

$$\sum_{i=1}^{k} \hat{\lambda}_i \left[\nabla_{\overline{z}} f(\zeta_0, \widehat{\eta_i}) - v^* \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma_i}) \right] + \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega})\overline{\widehat{\mu}} = \mathbf{0}. \tag{37}$$

Since $A \in \mathbb{C}^{n\times n}$ is positive semidefinite Hermitian matrix, we have $(Az_0)^H = z_0^H A$. Due to $z_0^H Az_0 > 0$, set $\hat{u}_1 = \frac{z_0}{\langle Az_0, z_0\rangle^{\frac{1}{2}}}$. So, (29) is obtained by summing (35) and (37). By the same proof as in (27), (31) can be obtained.

Owing to $\hat{u}_2 = v^*\alpha$ and $\alpha^H (v^*)^2 B\alpha \leq 1$, we obtain $\hat{u}_2^H B\hat{u}_2 \leq 1$. It follows from Lemma 2.5 and $z_0^H Bz_0 = 0$ that for $u \in \mathbb{C}^n$, $Re(z_0^H Bu) = Re(u^H Bz_0) \leq (z_0^H Bz_0)^{\frac{1}{2}} (u^H Bu)^{\frac{1}{2}} = 0$. Taking $u = Bz_0$ in the above equality, one has $Re[(Bz_0)^H Bz_0] = Re\langle Bz_0, Bz_0\rangle \leq 0$ and so, $Bz_0 = 0$. Since B is positive semidefinite Hermitian matrix, then $z_0^H B = (Bz_0)^H = 0$. It therefore shows that $Re(z_0^H Bz_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2) = 0$.

(2) If $z_0^H Az_0 = 0$ and $z_0^H Bz_0 > 0$, then the desired results can be deduced by the similar argument as in the proof of case (1), and so, it is omitted.

(3) If $z_0^H Az_0 = 0$ and $z_0^H Bz_0 = 0$, then by the similar argument as in the proof of [16, Theorem 3, pp. 235-237], one can obtain the desired results. Altogether, (29)–(32) hold. □

Remark 4.3 Generally, the multipliers $\mu \in S^*$ corresponding to the constrained conditions are not necessary to be nonzero. In Theorem 4.6, we do not require the multiplier corresponding to constrained function h to be a nonzero $\widehat{\mu} \in S^*$. So, Theorem 4.6 is distinct from Theorem 3 in [16, p. 235] when f, g are perturbed by the same uncertain parameter, h is uncertain-free and $U = V$, and Theorem 6 in [17, p. 1207] when h is uncertain-free with respect to the uncertain parameter $\omega \in W$ and $Re[g(\zeta, \gamma) - (z^H Bz)^{\frac{1}{2}}] \equiv 1$.

In particular, if we set $k = 1$ in Theorem 4.6, then the following robust necessary optimality conditions for (UCFP) holds.

Corollary 4.1 *Let $\zeta_0 = (z_0, \overline{z_0}) \in F$ be a robust optimal solution of* (UCFP) *with optimal value v^*. Assume that* (UCFP) *satisfies* RCQ *at ζ_0 and $Z(\zeta_0) = \emptyset$. Then there exist $\widehat{\mu} \in S^*$, $\hat{u}_1, \hat{u}_2 \in \mathbb{C}^p$, $\widehat{\eta} \in U$, $\widehat{\gamma} \in V$ and $\widehat{\omega} \in W$ such that*

$$\begin{aligned}
&\left[\overline{\nabla_z f(\zeta_0, \widehat{\eta})} + \nabla_{\overline{z}} f(\zeta_0, \widehat{\eta})\right] - v^*\left[\overline{\nabla_z g(\zeta_0, \widehat{\gamma})} + \nabla_{\overline{z}} g(\zeta_0, \widehat{\gamma})\right] \\
&+ A\hat{u}_1 + v^* B\hat{u}_2 + \widehat{\mu}^\top \overline{\nabla_z h(\zeta_0, \widehat{\omega})} + \widehat{\mu}^H \nabla_{\overline{z}} h(\zeta_0, \widehat{\omega}) = \mathbf{0}, \\
&Re\langle h(\zeta_0, \widehat{\omega}), \widehat{\mu}\rangle = 0, \;\; \hat{u}_1^H A\hat{u}_1 \le 1, \;\; \hat{u}_2^H B\hat{u}_2 \le 1, \\
&Re(z_0^H Az_0)^{\frac{1}{2}} = Re(z_0^H A\hat{u}_1), \;\; Re(z_0^H Bz_0)^{\frac{1}{2}} = Re(z_0^H B\hat{u}_2).
\end{aligned}$$

5 Conclusions

A complex fractional programming with uncertain data is studied by the robust optimization method. The robust counterpart of (UCFP) is introduced, and then, a minimax nonfractional parametric programming reformation for the robust counterpart is presented. The equivalence between the optimal solutions of the robust counterpart and one of the minimax nonfractional parametric programming is also derived. Fritz John-type robust necessary optimality conditions for the robust optimal solution ζ_0 of (UCFP) are established under the assumption that S^* has a compact base or, $\psi_{\zeta_0}^{-1}(0) \neq W \times \{\mathbf{0}\}$. By the RCQ, Karush–Kuhn–Tucker-type robust necessary optimality conditions for the robust optimal solution ζ_0 of (UCFP) are also obtained under some suitable assumptions. The presented necessary optimality conditions improve the corresponding results in [16,17,21].

For the future research, it is interesting to investigate the robust sufficient optimality conditions, duality, stability, robust radius and algorithm of (UCFP) by the necessary optimality conditions presented in this paper. It is well known that a cone has a base if and only if its strict dual cone or the quasi-interior of its dual cone is nonempty. However, there are no sufficient conditions for ensuring a cone having a compact base. So, it is also deserved to study the sufficient conditions for a cone having a compact base.

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