Variational Inequality Type Formulations of General Market Equilibrium Problems with Local Information

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Abstract

We suggest a new approach to creation of general market equilibrium models involving economic agents with local and partial knowledge about the system and under different restrictions. The market equilibrium problem is then formulated as a quasi-variational inequality that enables us to establish existence results for the model in different settings. We also describe dynamic processes, which fall into information exchange schemes of the proposed market model. In particular, we propose an iterative solution method for quasi-variational inequalities, which is based on evaluations of the proper market information only in a neighborhood of the current market state without knowledge of the whole feasible set and prove its convergence.

1 Introduction

Variational inequalities give a suitable common format for many applied problems arising in Economics, Mathematical Physics, Transportation, Communication Systems, Engineering and are closely related with other general problems in Nonlinear Analysis, such as fixed point, game equilibrium, and optimization problems. For this reason, their theory and solution methods are developed rather well; see, e.g., [1–5] and references therein. Nevertheless, many models describing rather complex behav-

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ior of systems with interdependent elements require a somewhat more general format, where the fixed feasible set is replaced with a set-valued mapping. This modification leads to the so-called quasi-variational inequalities; see, e.g., [1,3,6–9] and references therein. However, these problems are much more difficult for solution than the usual variational inequalities and the existing general solution methods are very expensive and applicable mostly for small dimensional problems; see, e.g., [7,10,11].

For this reason, we are interested in revealing new general classes of applications, which can be formulated as (quasi-)variational inequalities in order to derive existence properties in a unified manner and enable us to develop efficient methods for finding their solution under rather natural assumptions that may be also treated as dynamic processes converging to equilibrium states and justifying these equilibrium concepts. In this work, we intend to present implementation of this approach for general market equilibrium models. Traditionally, these models play one of the main roles in Economics. They in particular indicate ways of equilibrating different interests and opportunities of active elements (agents, participants) involved in economic systems and may serve as a basis for description of behavior of these very complex systems.

We recall that investigation of the general equilibrium models dates back to the book by Walras [12]. Since then, a number of different kinds of these models were proposed; see [3,13–17]. More detailed surveys and expositions of the basic contributions in this field can be found in [18,19]. These models usually describe markets of a great number of economic agents (customers and producers) so that actions of any separate agent cannot impact the state of the whole system; hence, any agent utilizes some integral system parameters (say, prices), rather than the information about the behavior of other separate agents. The traditional approach is based on the assumption that the agents are able to determine precisely their desirable collections of commodities for any vector of common market prices. Then, an equilibrium market state is defined from the complementarity conditions between the price and excess supply of each commodity since the usual material balance condition needs additional restrictive assumptions. At the same time, it holds for each commodity with positive equilibrium price; see, e.g., [18,19].

In this paper, we suggest some other approach to creation of general equilibrium models. The current market state is supposed to be defined by the vector of transaction quantities of the agents. These quantities satisfy the balance equation. We suppose that the agents have only partial and local knowledge about the current and future behavior of this so complicated system and evaluate their feasible volume transaction and price sets in a neighborhood of the current market state. Due to these limitations, the general market equilibrium problem is formulated as a quasi-variational inequality and corresponds to the local maximal market profit state concept. This approach enables us to utilize well developed tools from the theory of quasi-variational inequalities for establishing existence results. However, finding market equilibrium points by an iterative solution method seems rather difficult in the general case. We suggest iterative processes, which have rather natural treatment and are convergent for some classes of market equilibrium problems. In particular, we propose an iterative solution method for quasi-variational inequalities, which is based on evaluations of the market information in a neighborhood of the current market state rather than whole feasible set, which may be unknown, and prove its convergence under rather weak assumptions.

We outline now briefly the further organization of the paper. In Sect. 2, we recall some auxiliary properties and facts from the theory of quasi-variational inequalities. In Sect. 3, we describe a single commodity market equilibrium model and its variational inequality re-formulation. In Sect. 4, we describe a general multi-commodity market equilibrium model in the form of a quasi-variational inequality and give its basic properties. Implementation issues of this model are discussed in Sect. 5. Section 6 involves description and substantiation of some dynamic market processes converging to equilibrium points. In Sect. 7, we give a comparison of the presented model with the existing basic general and partial equilibrium models. Section 8 contains some conclusions.

2 Basic Preliminaries

We first recall several continuity properties of set-valued mappings.

Definition 2.1 Let W and V be convex sets in the l-dimensional Euclidean space $\mathbb{R}^l$, $W \subseteq V$, and let $T : V \to \Pi(\mathbb{R}^l)$ be a set-valued mapping. The mapping T is said to be

(a) *upper semicontinuous (u.s.c.)* on W, if for each point $v \in W$ and for each open set U such that $U \supseteq T(v)$, there is an open neighborhood $\tilde{V}$ of v such that $T(w) \subset U$ whenever $w \in \tilde{V} \bigcap W$;
(b) *lower semicontinuous (l.s.c.)* on W, if for each point $v \in W$ and for each open set U such that $U \bigcap T(v) \neq \varnothing$, there is an open neighborhood $\tilde{V}$ of v such that $U \bigcap T(w) \neq \varnothing$ whenever $w \in \tilde{V} \bigcap W$;
(c) *continuous* on W, if it is both u.s.c. and l.s.c. on W.

Here and below, $\Pi(X)$ denotes the power set of X, i.e., the family of all nonempty subsets of X. Let $\tilde{H}$ be a set in $\mathbb{R}^l$ and let $H : \tilde{H} \to \Pi(\tilde{H})$ and $G : \tilde{H} \to \Pi(\mathbb{R}^l)$ be set-valued mappings. Then, we can define the *quasi-variational inequality problem* (QVI for short): Find $x^* \in H(x^*)$ such that

$$\exists g^* \in G(x^*), \ \langle g^*, y - x^* \rangle \geq 0 \ \ \forall y \in H(x^*). \tag{1}$$

If the feasible mapping H is constant, problem (1) reduces to the usual *variational inequality problem* (VI for short).

It is well known that QVIs can be converted into the fixed point format by using the projection mapping; see [7, Theorem 5.1]. Let $\pi_X(x)$ denote the projection of a point x onto a set X.

Proposition 2.1 *Suppose that the sets $H(x)$ and $G(x)$ are nonempty, convex and compact for each $x \in \tilde{H}$. Then, QVI (1) is equivalent to the fixed point problem*

$$x^* \in \pi_{H(x^*)}[x^* - \theta G(x^*)],$$

for some $\theta > 0$.

We now give existence results for general QVIs on compact sets; see, e.g., [3, Theorem 9.14].

Proposition 2.2 *Suppose that the set $\tilde{H}$ is convex and compact, the mapping $H : \tilde{H} \to \Pi(\tilde{H})$ is continuous on $\tilde{H}$, the mapping $G : \tilde{H} \to \Pi(\mathbb{R}^l)$ is upper semi-continuous on $\tilde{H}$, the sets $H(x)$ and $G(x)$ are nonempty, convex and compact for each $x \in \tilde{H}$. Then, QVI (1) has a solution.*

Moreover, it was noticed in [8, Theorem 3] that the above conditions may be relaxed, if the feasible mapping value includes the current point.

Proposition 2.3 *Suppose that the set $\tilde{H}$ is nonempty, convex and compact, the mapping $G : \tilde{H} \to \Pi(\mathbb{R}^l)$ is upper semi-continuous on $\tilde{H}$, the set $G(x)$ is nonempty, convex and compact for each $x \in \tilde{H}$, $x \in H(x)$ for all $x \in \tilde{H}$. Then, QVI (1) has a solution.*

3 A Single Commodity Market Equilibrium Model

For the simplicity of exposition, we begin our considerations from a simple equilibrium market model of a homogeneous commodity, which was suggested in [20–22] for description of a simple market based on a suitable auction implementation mechanism. Its further development and applications are described in [23–26].

The model involves a finite number of traders and buyers of this commodity, their index sets will be denoted by I and J, respectively. Hence, the initial roles of agents are fixed. For each $i \in I$, the ith trader chooses some offer value x_i in his/her capacity segment $[\alpha'_i, \beta'_i]$ and has a price function g_i. Similarly, for each $j \in J$, the jth buyer chooses some bid value y_j in his/her capacity segment $[\alpha''_j, \beta''_j]$ and has a price function h_j. The signs of all the lower and upper bounds are arbitrary in general, but the standard choice is to set $\alpha'_i = 0$ and $\alpha''_j = 0$; hence, the upper bounds are chosen to be positive. Then, we can define the feasible set of offer/bid volumes

$$
D = \left\{ (x, y) : \sum_{i \in I} x_i = \sum_{j \in J} y_j; \right.
$$
$$
\left. x_i \in [\alpha'_i, \beta'_i], i \in I, y_j \in [\alpha''_j, \beta''_j], j \in J \right\}, \tag{2}
$$

where $x = (x_i)_{i \in I}$, $y = (y_j)_{j \in J}$. We suppose that the prices may in principle depend on offer/bid volumes of all the participants, i.e., $g_i = g_i(x, y)$ and $h_j = h_j(x, y)$. We say that a pair $(\bar{x}, \bar{y}) \in D$ constitutes a *market equilibrium point* if there exists a number $\bar{\lambda}$ such that

$$
g_i(\bar{x}, \bar{y}) \begin{cases} \geq \bar{\lambda}, & \text{if } \bar{x}_i = \alpha'_i, \\ = \bar{\lambda}, & \text{if } \bar{x}_i \in (\alpha'_i, \beta'_i), \\ \leq \bar{\lambda}, & \text{if } \bar{x}_i = \beta'_i, \quad \text{for } i \in I, \end{cases} \tag{3}
$$

and

$$h_j(\bar{x}, \bar{y}) \begin{cases} \leq \bar{\lambda}, & \text{if } \bar{y}_j = \alpha_j'', \\ = \bar{\lambda}, & \text{if } \bar{y}_j \in (\alpha_j'', \beta_j''), \\ \geq \bar{\lambda}, & \text{if } \bar{y}_j = \beta_j'', \quad \text{for } j \in J. \end{cases} \tag{4}$$

Observe that the number $\bar{\lambda}$ can be treated as a market clearing price, which equilibrates the market and yields also the offer/bid volumes for all the participants. In fact, the minimal offer (bid) volumes correspond to traders (buyers) whose prices are greater (less) than $\bar{\lambda}$, and the maximal offer (bid) volumes correspond to traders (buyers) whose prices are less (greater) than $\bar{\lambda}$. The prices of other participants are equal to $\bar{\lambda}$, and their volumes may be arbitrary within their capacity bounds, but should be subordinated to the balance equation. It follows that agents' prices at equilibrium may in general differ from the market clearing price.

In [20] (see also [21,22]), the following basic relation between the equilibrium problem (2)–(4) and a VI was established.

Proposition 3.1 (a) *If $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (3)–(4) and $(\bar{x}, \bar{y}) \in D$, then $(\bar{x}, \bar{y})$ solves VI:*
Find $(\bar{x}, \bar{y}) \in D$ such that

$$\sum_{i \in I} g_i(\bar{x}, \bar{y})(x_i - \bar{x}_i) - \sum_{j \in J} h_j(\bar{x}, \bar{y})(y_j - \bar{y}_j) \geq 0 \quad \forall (x, y) \in D. \tag{5}$$

(b) *If a pair $(\bar{x}, \bar{y}) \in D$ solves VI (5), then there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (3)–(4).*

Therefore, we can apply various results from the theory of VIs (see, e.g., [21]) for investigation and solution of the above equilibrium problem. For instance, if the set D in (1) is nonempty and bounded, the functions g_i and h_j are continuous for all $i \in I$ and $j \in J$, and then VI (5) has a solution.

4 A General Multi-commodity Market Equilibrium Model

We now present a multi-commodity extension of the model described in Sect. 3, thus also extending those in [22,24].

The model is an n-commodity market involving m economic agents, and they are producers and consumers with respect to these commodities. We suppose that the agents can strike preliminary (or virtual) bargains until an equilibrium state will be attained. Then, all these deals become fixed. Denote by $N = \{1, \ldots, n\}$ and $I = \{1, \ldots, m\}$, the index sets of commodities and agents at this market, so that $i \in I$ indicates the ith agent and $j \in N$ indicates the jth commodity. Observe that roles of the agents with respect to the commodities at the market may be changed. Next, let the vector $x^i = (x_{i1}, \ldots, x_{in})^\top$ define the current transaction quantities of commodities of the ith agent, so that $x_{ij} > 0$ means that his/her current sold volume of the jth

commodity equals x_{ij}, whereas $x_{ij} < 0$ means that his/her current purchased volume of the jth commodity equals $-x_{ij}$. The current market state is thus supposed to be completely defined by the virtual volume vector $x = (x^i)_{i \in I}$.

Behavior of economic agents may in general depends on their current various goals and restrictions. In our model, the agents determine first their current particular feasible volume transaction sets $Y_i(x) \subseteq \tilde{Y}_i$ for $i \in I$, which are supposed to be nonempty sets in $\mathbb{R}^n$ and attributed to each market state x. Similarly, the agents determine their feasible price sets $P_i(x) \subseteq \tilde{P}_i$ for $i \in I$, which are also supposed to be nonempty sets in $\mathbb{R}^n_+$ and attributed to each market state x. The sets $Y_i(x)$ and $P_i(x)$ are different for each agent i and depend on actions of all the agents in our model. The sets $\tilde{Y}_i$ and $\tilde{P}_i$, $i \in I$ give the total market transaction bounds (respectively, the total market price bounds) that may be unknown to the agents in general. These sets define the set-valued market transaction mapping $x \mapsto Y(x)$ on the set $\tilde{Y}$, where

$$Y(x) = \prod_{i \in I} Y_i(x), \quad \tilde{Y} = \prod_{i \in I} \tilde{Y}_i,$$

and the set-valued feasible market price mapping $x \mapsto P(x)$ on the set $\tilde{Y}$ where

$$P(x) = \prod_{i \in I} P_i(x), \quad P(x) \subseteq \tilde{P} = \prod_{i \in I} \tilde{P}_i \quad \forall x \in \tilde{Y}.$$

Both volume and price mappings describe the behavior of the economic agents with respect to market states. Using these sets, the agents can make some transactions and change their current transaction volumes. Next, any market transaction must also satisfy the balance equation. Hence, the particular feasible volume transaction sets together with the balance equation determine the set-valued feasible market transaction mapping $x \mapsto D(x)$ with the values

$$D(x) = \left\{ y \in \mathbb{R}^{nm} : \sum_{i \in I} y^i = \mathbf{0}, \ y^i \in Y_i(x), \ i \in I \right\}. \tag{6}$$

We say that a vector $\bar{x} \in D(\bar{x})$ is a *market equilibrium point* if

$$\exists \bar{p} \in P(\bar{x}), \ \langle \bar{p}, y - \bar{x} \rangle \geq 0 \quad \forall y \in D(\bar{x}). \tag{7}$$

This problem can be treated as an extension of VI (5) to the multi-commodity case. Therefore, the general multi-commodity market equilibrium problem is formulated as a QVI. We observe that the value

$$-\langle \bar{p}, \bar{x} \rangle = -\sum_{i \in I} \langle \bar{p}^i, \bar{x}^i \rangle$$

gives precisely the difference between the total bought and sold amount at the current market state $\bar{x}$ and current price vector $\bar{p}$ of the agents, i.e., it is the current possible

profit of the market. Therefore, the market equilibrium point $\bar{x} \in D(\bar{x})$ can be treated as the state that provides the maximal possible market profit at the current prices of the agents subject to the constraints determining the balance and current feasible transactions.

We now give analogues of equilibrium conditions (3)–(4) for the above problem. We will take the following basic assumptions.

(A1) *The sets $\tilde{Y}_i \subset \mathbb{R}^n$ are convex and compact for all $i \in I$. At each state $x \in \tilde{Y}$, the sets $Y_i(x)$ and $P_i(x)$ are nonempty, convex and compact for all $i \in I$.*

Let us define the problem of finding a feasible point $\bar{x} \in D(\bar{x})$ and a vector $\bar{\lambda} \in \mathbb{R}^n$ such that

$$\exists \bar{p} \in P(\bar{x}), \ \sum_{i \in I} \langle \bar{p}^i - \bar{\lambda}, y^i - \bar{x}^i \rangle \geq 0 \quad \forall y \in Y(\bar{x}). \tag{8}$$

Proposition 4.1 *Suppose that the assumptions in* **(A1)** *are fulfilled. Then:*

(a) *If a point $\bar{x} \in D(\bar{x})$ solves QVI (6)–(7), then there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{\lambda})$ satisfies (8).*
(b) *If a pair $(\bar{x}, \bar{\lambda}) \in D(\bar{x}) \times \mathbb{R}^n$ satisfies (8), then $\bar{x}$ solves QVI (6)–(7).*

Proof If $\bar{x} \in D(\bar{x})$ solves QVI (6)–(7), then it solves the optimization problem

$$\min \ \rightarrow \ \{ \langle \bar{p}, y \rangle \ : \ y \in D(\bar{x}) \}. \tag{9}$$

By using the suitable Karush–Kuhn–Tucker theorem for this problem (see, e.g., [27, Corollary 28.2.2]), we obtain that (8) gives its necessary and sufficient optimality condition. Conversely, let a pair $(\bar{x}, \bar{\lambda}) \in D(\bar{x}) \times \mathbb{R}^n$ satisfy (8). Using the same Karush–Kuhn–Tucker theorem, we obtain that $\bar{x}$ solves (9), hence it is a solution to QVI (6)–(7).   □

Since $Y(x)$ and $P(x)$ are Cartesian product sets, problem (8) can be replaced by the following equivalent system of partial QVIs:

$$\exists \bar{p}^i \in P_i(\bar{x}), \ \langle \bar{p}^i - \bar{\lambda}, y^i - \bar{x}^i \rangle \geq 0 \quad \forall y^i \in Y_i(\bar{x}), \quad \forall i \in I. \tag{10}$$

It is clear that (8) (or (10)) represents an analogue of the equilibrium conditions in (3)–(4) for the above multi-commodity market equilibrium model, hence the point $\bar{\lambda}$ in (8), which is nothing but the Lagrange multiplier vector corresponding to the balance constraint, should be treated as the genuine market equilibrium price vector for this model. We observe that the equilibrium prices of the agents do not in general coincide with $\bar{\lambda}$ due to the presence of the constraints determining the feasible volume transaction sets $Y_i(\bar{x})$ for all $i \in I$. This is the case even for the simplest single commodity equilibrium model due to (3)–(4). It follows that the agents cannot maintain in general the unique market prices at a given state of the market as in the usual Walrasian models.

We give the existence result for the market equilibrium problem as proper specialization of similar properties for general QVIs from Sect. 2.

(A2) *The mapping* $Y : \tilde{Y} \rightarrow \Pi(\mathbb{R}^{nm})$ *is continuous on* $\tilde{Y}$, and *the mapping* $P : \tilde{Y} \rightarrow \Pi(\mathbb{R}^{nm})$ *is upper semi-continuous on* $\tilde{Y}$.

Theorem 4.1 *Suppose that assumptions* **(A1)** *and* **(A2)** *are fulfilled. Then, QVI* (6)–(7) *has a solution.*

The assertion follows from Proposition 2.2. Moreover, it seems now natural to suppose that the feasible volume transaction set of each ith agent $Y_i(x)$ will contain the current volume vector x^i. This assumption allows us to take the existence result for the usual VIs as in Proposition 2.3. Set

$$\tilde{D} = \left\{ y \in \tilde{Y} : \sum_{i \in I} y^i = \mathbf{0} \right\};$$ (11)

cf. (2).

(A2′) *The mapping* $P : \tilde{Y} \rightarrow \Pi(\mathbb{R}^{nm})$ *is upper semi-continuous on* $\tilde{Y}$, $x \in Y(x)$ *for all* $x \in \tilde{D}$, $\tilde{D} \neq \varnothing$.

Theorem 4.2 *Suppose that assumptions* **(A1)** *and* **(A2′)** *are fulfilled. Then, QVI* (6)–(7) *has a solution.*

Proof Let us take the following VI: Find a point $\bar{x} \in \tilde{D}$ such that

$$\exists \bar{p} \in P(\bar{x}), \ \langle \bar{p}, y - \bar{x} \rangle \geq 0 \quad \forall y \in \tilde{D}.$$

Under the above assumptions, it must have a solution; see, e.g., [3, Theorem 9.9]. Due to **(A2′)**, it also solves QVI (6)–(7). □

5 Implementation Issues

The multi-commodity equilibrium model presented in Sect. 4 seems incomplete without a more detailed description of feasible transaction and price sets attributed to each state x of the market. We now intend to discuss some approaches to determination of these sets as building blocks of the whole market model. We suppose that the agents have only partial and local knowledge about the market system. Hence, they are able to only evaluate their market, industry and social goals and restrictions in a neighborhood of the current market state. That is, each agent may evaluate his/her temporal capacity bounds by using the information about the current supply or demand of some related commodities as well as their bounds. For this reason, the feasible transaction set $Y_i(x)$ of each ith economic agent is determined by his/her market capacity constraints. These constraints can reflect both personal and common market or technology relationships among several commodities that can be actively utilized by different agents. Also, the initial distribution of commodity endowments must have clear impact on the feasible transaction sets. For instance, the sets $Y_i(x)$ can be determined as intersections of personal and common obligatory feasible sets. Let $\tilde{Y}_i \subset \mathbb{R}^n$ denote the set of obligatory personal capacity constraints of the ith agent, which is independent of market states.

Also, let $W_i \subset \mathbb{R}^{nm}$ denote the set of common market volume constraints that are obligatory for the ith agent and may reflect common environmental or transportation capacity restrictions. Then, the feasible transaction set $Y_i(x)$ at the state x is defined as follows:

$$Y_i(x) = \tilde{Y}_i \bigcap Y_i'(x), \text{ where } Y_i'(x) = \left\{ y^i \in \mathbb{R}^n \ : \ (x^{-i}, y^i) \in W_i \right\}.$$

In the general case, the sets W_i may also depend on the current state x, i.e., their precise values may be unknown. Similarly, each agent may insert also the set of desirable market volume distributions in the above definition. But in the simplest case $Y_i(x)$ is a box-constrained set of the form

$$Y_i(x) = [c^i(x), d^i(x)], \tag{12}$$

where $c^i(x) \in \mathbb{R}^n$ and $d^i(x) \in \mathbb{R}^n$ give the current estimates of market capacity bounds by the ith economic agent.

The evaluation of the current feasible price set $P_i(x)$ by the ith economic agent can be based on possible industrial capacity and social constraints and the agent's profit as the goal function. In other words, each ith agent solves the optimization problem:

$$\max_{p^i \in V_i(x)} \rightarrow \langle p^i, x^i \rangle, \tag{13}$$

where $V_i(x) \subset \mathbb{R}_+^n$ and takes its solution set as $P_i(x)$, i.e.,

$$P_i(x) = \operatorname{Arg} \max \{ \langle p^i, x^i \rangle \ : \ p^i \in V_i(x) \}.$$

The set $P_i(x)$ is non-empty if we impose certain additional assumptions on $V_i(x)$. For instance, this is the case if $V_i(x)$ is non-empty and compact. Then, $\mu_i(x) = \langle p^i, x^i \rangle$ for some $p^i \in V_i(x)$ is the optimal value function for (13). We note that $\langle p^i, x^i \rangle$ gives precisely the profit of the ith agent within his/her own prices. The feasible set $V_i(x)$ in (13) may depend on the current state x and reflect also utilization of common waste treatment plants or environmental restrictions for some industry technologies of different agents, partial budget type restrictions, common transportation capacity restrictions, application of nonlinear production technologies, utilization of special financial tools, etc.

Let us take the simplest case with independent linear production technologies. For each x^i define the index sets N_i' and N_i'', which determine the agent's supply and demand commodities. It follows that $x_{ij} \geq 0$ if $j \in N_i'$ and $x_{ij} \leq 0$ if $j \in N_i''$. Then, $V_i(x) \equiv V_i$ becomes a polyhedral set, i.e.,

$$V_i = \left\{ v^i \in \mathbb{S}_+^n \ : \ \sum_{s \in N_i''} a_{sj}^i v_{is} \geq v_{ij}, \ j \in N_i' \right\}, \tag{14}$$

where a_{sj}^i is the amount of the sth commodity used for production of one unit of the jth commodity by the ith economic agent. The inequalities in (14) mean that the minimal price p_{ij} of the jth commodity cannot be greater than the expenses for all the factors (resources) used for production of one unit of this commodity. Also,

$$\mathbb{S}_+^n = \left\{ z \in \mathbb{R}_+^n \; : \; \sum_{j=1}^n z_j = 1 \right\}$$

is the standard simplex in $\mathbb{R}^n$. Hence, the relative price scale is taken here. Then, (13)–(14) reduces to the linear programming problem:

$$\max_{p^i \in V_i} \; \to \; \langle p^i, x^i \rangle, \tag{15}$$

besides, we have

$$P_i(x) = \partial \mu_i(x^i), \tag{16}$$

i.e., the solution set of problem (14)–(15) is the sub-differential of its optimal value function $\mu_i(x^i) = \mu_i(x)$, which is clearly convex. Due to (16), QVI (6)–(7) is rewritten as follows:

$$\exists \bar{p}^i \in \partial \mu_i(\bar{x}^i), \; \forall i \in I, \; \sum_{i \in I} \langle \bar{p}^i, y^i - \bar{x}^i \rangle \geq 0 \quad \forall y = (y^i)_{i \in I} \in D(\bar{x}). \tag{17}$$

If we set

$$\mu(x) = \sum_{i \in I} \mu_i(x^i), \tag{18}$$

and take the sets $Y_i(x)$, $i \in I$, in accordance with (12), then the above QVI becomes equivalent to the quasi-optimization problem: Find a point $\bar{x} \in D(\bar{x})$ such that

$$\mu(\bar{x}) \geq \mu(y) \quad \forall y \in D(\bar{x}), \tag{19}$$

since the function μ and set $D(x)$ are convex. This means that the set of market equilibrium states coincides with the set of the points with minimal pure expenses (or maximal profit) of the whole market. If the sets V_i, $i \in I$, are nonempty, the set $D(x)$ given in (6) is nonempty and $D(x) \subseteq \tilde{Y}$ for each state $x \in \tilde{Y}$ and some convex and compact set $\tilde{Y}$, and the functions c^i and d^i are continuous on $\tilde{Y}$, then Proposition 2.2 implies that QVI (17) has a solution and this assertion is true for the quasi-optimization problem (18)–(19).

Example 5.1 Let us give an example of the feasible price mapping calculation in case (14)–(15). We take a separate agent and remove the index i for simplicity. Let $n = 3$, $N' = \{1\}$, $N'' = \{2, 3\}$, $a_{21} = 2$, $a_{31} = 3$. Then, problem (14)–(15) takes the form:

$$\max \to \{x_1 p_1 + x_2 p_2 + x_3 p_3 \ : \ 2p_2 + 3p_3 \geq p_1, \ p_1 + p_2 + p_3 = 1,$$
$$p_1 \geq 0, \ p_2 \geq 0, \ p_3 \geq 0\} \,.$$

For $x' = (1, -1, -1)^\top$, we have $P_i(x') = \{p'\}$, where $p' = (3/4, 0, 1/4)^\top$, the optimal value $\mu^* = 1/2$. For $x'' = (1, -2, -3)^\top$, we have $P_i(x'') = [p', p'']$, where $p'' = (2/3, 1/3, 0)^\top$, $\mu^* = 0$. Also, for $x''' = (1, -2, -4)^\top$ we have $P_i(x') = \{p''\}$, $\mu^* = 0$. In the general case, the feasible price value may be one of vertices or faces of the polyhedral set V_i depending of the volume vector x^i.

We observe that the non-industrial consumption commodities can be taken into account similarly. The agent should only indicate that consumption expenses per unit of the sth commodity will be covered with selling all the commodities such that $a^i_{sj} > 0$. The coefficient values a^i_{sj} will give the corresponding relative scale, i.e., the relative weights of commodities for any agent. In such a way, this model can in principle describe a pure exchange market. Similarly, common waste treatment or environment restrictions can be inserted in the model. In the general case, all these parameters and the index sets N'_i and N''_i may also depend on the current state x, i.e., the agents can change their preferences for commodities with respect to the current distribution of volumes.

We intend now to illustrate the process of creation of the whole market model by using some of the proposed partial blocks.

***Example 5.2* (A market equilibrium model with treatment of industrial wastes)**
Consider a market system with m economic agents and n commodities. Some of these economic agents are industrial firms whose outputs may contain polluted substances, which require certain treatment by using the corresponding technologies. As above, we denote by $I = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ the index sets of agents and commodities. For simplicity, we suppose that there exists a unique firm denoted as agent m having plants for treatment of these polluted substances and set $J = I \setminus \{m\}$. For each agent $i \in J$, the index sets N'_i, N''_i, and L_i of supply and demand commodities and polluted substances, respectively, are supposed to be fixed. Although polluted substances belong to outputs, we consider them as demand commodities, since the ith agent buys their treatment service from the mth agent. It follows that $L_i \subset N''_i$. Also, let

$$L = \bigcup_{i \in J} L_i$$

denote the index set of all the polluted substances and let l denote the number of these substances. Next, we suppose that firm m is a municipal (or state) property and that its prices are under regulation, i.e., its price mapping $p_m(x)$ is single-valued and given explicitly so that the unit treatment price (charge) $p_{ms}(x) = p_{ms}(x^m)$ for the sth substance may in principle depend on the volumes of all the polluted substances. The feasible volume transaction set $\tilde{Y}_m \subset \mathbb{R}^l$ is supposed to be fixed and nonempty, convex and compact. For simplicity, we suppose that all the other agents have linear production technologies and fixed capacity bounds. As above, let a^i_{sj} denote the amount of the sth commodity used for production of one unit of the jth commodity by the ith economic

agent. If $s \in L_i$, this means that the sth substance appears as a by-product of the jth commodity and needs the treatment service. Then, the feasible volume transaction set $Y_i \subset \mathbb{R}^n$ of agent $i \in J$ is defined as follows:

$$
Y_i = \left\{ y^i \in \mathbb{R}^n \ : \ y_{ij} \in [\alpha_{ij}, \beta_{ij}], \ j \in N, \right.
$$
$$
\sum_{j \in N_i'} a_{sj}^i y_{ij} + y_{is} \le 0, \ s \in N_i'' \setminus L_i,
$$
$$
\left. \sum_{j \in N_i'} a_{sj}^i y_{ij} + y_{is} = 0, \ s \in L_i \right\},
$$

and supposed to be nonempty and bounded. The feasible price set $P_i(x)$ for $i \in J$ is then represented as the sum $P_i(x) = P_i'(x) + P_i''(x)$, due to the presence of two different sources of expenses. In order to define the first term $P_i'(x)$, we take the same optimization problem (15), but modify the set V_i to be

$$
V_i = \left\{ v^i \in \mathbb{S}_+^n \ : \ \sum_{s \in N_i'' \setminus L_i} a_{sj}^i v_{is} \ge v_{ij}, \ j \in N_i' \right\},
$$

it is only supposed to be nonempty. The treatment expenses are not included in the first term, and they determine the second term components

$$
P_{ij}''(x) = \sum_{s \in L_i} a_{sj}^i p_{ms}(x^m).
$$

The feasible volume transaction set $Y_m(x)$ is defined as follows:

$$
Y_m(x) = \left\{ y^m \in \tilde{Y}_m \ : \ \sum_{i \in J} \delta_{is} x_{is} + y_{ms} = 0, \ s \in N \right\},
$$

where the set $\tilde{Y}_m \subset \mathbb{R}^n$ reflects the treatment technologies constraints,

$$
\delta_{is} = \begin{cases} 1, & \text{if } s \in L_i, \\ 0, & \text{if } s \notin L_i. \end{cases}
$$

Therefore, QVI (6)–(7) will give equilibrium states for this market system. It can be also transformed into a VI. It suffices to define the common constraint set

$$
W = \left\{ x \in \mathbb{R}^{nm} \ : \ \sum_{i \in J} \delta_{is} x_{is} + x_{ms} = 0, \ s \in L \right\},
$$

and take $Y_m(x) \equiv Y_m = \tilde{Y}_m$. Let us define the following VI: Find a point $\bar{x} \in B$ such that

$$\exists \bar{p} \in P(\bar{x}), \ \langle \bar{p}, y - \bar{x} \rangle \geq 0 \quad \forall y \in B, \tag{20}$$

where

$$B = W \cap Y, \ Y = \prod_{i \in I} Y_i, \ P(x) = \prod_{i \in I} P_i(x).$$

Under the above assumptions, VI (20) is clearly equivalent to QVI (6)–(7). Moreover, if the set B is nonempty and the treatment charge functions p_{ms} are continuous, then VI (20) has a solution; see, e.g., [3, Theorem 9.9]. Next, VI (20) admits some dimensionality reduction after the sequential substitution of the variables x_{ms} and x_{is}, $s \in L$ with the corresponding variables x_{ij}, $j \in N'_i$, $i \in J$ due to the equality constraints in W and Y_i.

It should be observed that utilization of common services with congestion effects described above is typical for modern telecommunication systems as well as for industrial and energetic systems with common transportation lines; see, e.g., [23,25,26,28–30]. Next, the price mapping $x \mapsto P(x)$ need not be potential, i.e., VI (20) cannot be in general replaced by an optimization problem, as the simple example below shows.

Example 5.3 Let us take $m = 3$, $n = 3$, $N'_1 = N'_2 = \{1\}$, $N''_1 = N''_2 = \{2, 3\}$, $L = L_1 = L_2 = \{2, 3\}$. This means that agents 1 and 2 produce commodity 1 and have two by-products (polluted substances) whose treatments are marked as commodities 2 and 3 produced by agent 3. Let $a^1_{21} = 2$, $a^1_{31} = 3$, $a^2_{21} = 1$, $a^2_{31} = 4$, $p_{32}(x) = 0.5x_{32} + 0.1x_{33}$, $p_{33}(x) = 0.2x_{33}$. It follows that

$$x_{32} = -(x_{12} + x_{22}) = 2x_{11} + x_{21}, \ x_{33} = -(x_{13} + x_{23}) = 3x_{11} + 4x_{21}.$$

Therefore,

$$p^1(x) = 2p_{32}(x) + 3p_{33}(x) = 4.4x_{11} + 4.2x_{21},$$
$$p^2(x) = p_{32}(x) + 4p_{33}(x) = 3.7x_{11} + 4.1x_{21}.$$

Since the Jacobian of the mapping $x \mapsto P(x) = (p^1(x), p^2(x))^\top$ is not symmetric, P is not the gradientmap of a function.

6 Convergence of Dynamic Processes

The existence of natural dynamic market processes converging to an equilibrium point is very essential for justification of any suggested equilibrium concept. But it is well known that substantiation of such dynamic processes is much more difficult in comparison with that of static existence results. In fact, the multi-commodity equilibrium

model from Sect. 4 is formulated as QVI (6)–(7). However, even finding its solution by an iterative solution method seems very difficult in the general case; see, e.g., [7,10,11]. In this case, we have additional restrictions due to partial and local knowledge of the agents about the system. In fact, we intend to describe a process of changing the market states that continues until attaining a solution of problem (6)–(7), thus creating a *decentralized (self-regulation) transaction mechanism*. The implementation of this mechanism is clearly attributed to a suitable information exchange scheme of this model and depends on the properties of the defined sets and mappings. Therefore, any chosen iterative solution method must admit a natural treatment of this method as a dynamic process for finding market equilibrium points in the above model together with suitable convergence conditions. For this reason, it is more suitable to select rather broad classes of equilibrium problems of form (6)–(7) and suitable natural converging dynamic processes. First of all, we observe that converging dynamic processes based on bilateral transactions were proposed for the single commodity market equilibrium model from Sect. 3; see [31–33]. We now give examples of iterative processes, which have rather natural treatment and are convergent for some classes of multi-commodity market equilibrium problems.

Let us take assumptions **(A1)** and **(A2$'$)** and suppose in addition that the set $P_i(x)$ is determined as the solution set of optimization problem (14)–(15) for all $i \in I$ and $x \in \tilde{Y}$. Then, as indicated in Sect. 5, QVI (6)–(7) can be replaced with the quasi-optimization problem (18)–(19) where the function μ and set $D(x)$ are convex. In addition, we observe that the appearance of the moving feasible set $D(x)$ reflects both relationships and restricted knowledge of the agents. Hence, problem (18)–(19) can be also treated as a relative optimization problem; see [34].

Bearing in mind Proposition 2.1, we can apply the simplest projection method

$$ x^{(k+1)} = \pi_{D(x^{(k)})}[x^{(k)} - \theta_k p^{(k)}], \ p^{(k)} \in P(x^{(k)}), \ \theta_k > 0, \ k = 0, 1, \ldots, \quad (21) $$

for finding a solution of problem (18)–(19). However, its convergence needs additional assumptions since problems (14)–(15) may have many solutions, i.e., the function μ is in general non-differentiable. This gives a quasi-optimization problem with convex non-differentiable function and creates certain difficulties for convergence properties of method (21). At the same time, extensions of more sophisticated non-differentiable optimization methods (see, e.g., [5,35,36]) do not admit a natural treatment within the above market equilibrium model and its information exchange scheme. We can take the stationary case where the agents utilize only fixed feasible transaction sets. More precisely, we take the following assumptions.

(B1) *At each state $x \in \tilde{Y}$, it holds that $Y_i(x) = \tilde{Y}_i$, where the set $\tilde{Y}_i \subset \mathbb{R}^n$ is convex and compact for each $i \in I$. The set $\tilde{D}$ in (11) is non-empty.*

(B2) *At each state $x \in \tilde{Y}$ the set $P_i(x)$, is determined as the solution set of optimization problem (14)–(15), where the set V_i is non-empty for all $i \in I$.*

Then, QVI (6)–(7) reduces to the set-valued VI: Find a vector $\bar{x} \in D$ such that

$$ \exists \bar{p} \in P(\bar{x}), \ \langle \bar{p}, y - \bar{x} \rangle \geq 0 \quad \forall y \in \tilde{D}. \quad (22) $$

Moreover, due to (16) this problem is equivalent to the convex optimization problem:

$$\min_{x \in \tilde{D}} \to \mu(x);$$

cf. (19). It is clear that VI (22), (11) has a solution under the assumptions in **(B1)**–**(B2)**. This result follows, e.g., from Theorem 4.1. Next, we can find a solution of VI (22), (11) by using the fixed point iterate (21) that now reduces to the sub-gradient projection method:

$$x^{(k+1)} = \pi_{\tilde{D}}[x^{(k)} - \theta_k p^{(k)}], \ p^{(k)} \in \partial\mu(x^{(k)}), \ \theta_k > 0, \ k = 0, 1, \ldots, \quad (23)$$

with the standard step-size rules:

$$\sum_{k=0}^{\infty} \theta_k = \infty, \ \sum_{k=0}^{\infty} \theta_k^2 < \infty. \quad (24)$$

Its convergence property is well known and can be deduced, e.g., from Lemma 2.1 in [37, Chapter V].

Proposition 6.1 *Suppose that assumptions* **(B1)**–**(B2)** *are fulfilled. If a sequence* $\{x^{(k)}\}$ *is subordinated to rules* (23)–(24)*, it converges to a solution of problem* (22)*,* (11)*.*

Nevertheless, implementation of each iterate may be not so easy within the information exchange scheme of the multi-commodity market model. Bearing in mind (12), we specialize the market capacity bounds as follows.

(B1′) *At each state* $x \in \tilde{Y}$*, it holds that* $Y_i(x) = \tilde{Y}_i = [c^i, d^i]$ *for all* $i \in I$*. The set* $\tilde{D}$ *in* (11) *is non-empty.*

Clearly, **(B1′)**–**(B2)** imply **(B1)**–**(B2)**. According to the sub-gradient projection method, the agents first determine their price vector $p^{(k)}$ for the current market state $x^{(k)}$ from the optimization problems of form (14)–(15). Afterward, they find the next state $x^{(k+1)}$ from (23). This problem now decomposes into n independent single commodity market equilibrium problems of form (5) with affine price functions. In fact, the jth commodity market problem is written as VI: Find $x_{(j)}^{(k+1)} \in \tilde{D}_j$ such that

$$\sum_{i \in I} (p_{ij}^{(k)} + \theta_k^{-1}(x_{ij}^{(k+1)} - x_{ij}^{(k)}))(y_{ij} - x_{ij}^{(k+1)}) \geq 0 \quad \forall y_{(j)} \in \tilde{D}_j, \quad (25)$$

where $y_{(j)} = (y_{1j}, \ldots, x_{mj})^{\top}$ and

$$\tilde{D}_j = \left\{ y_{(j)} \in \mathbb{R}^m \ : \ \sum_{i \in I} y_{ij} = 0, \ y_{ij} \in [c_{ij}, d_{ij}], \ i \in I \right\}. \quad (26)$$

This problem can be solved by simple solution algorithms including the bilateral exchanges; see [31,32]. Therefore, process (23) can be naturally implemented within usual market mechanisms.

We still intend to present the simple method for finding a solution of QVI (6)–(7) within the assumptions from **(A1)** and **(A2′)** with moving feasible transaction sets.

(C1) *The sets $\tilde{Y}_i \subset \mathbb{R}^n$ are convex and compact for all $i \in I$. At each state $x \in \tilde{Y}$, the set $Y_i(x) \subseteq \tilde{Y}_i$ is convex and compact for each $i \in I$.*

In general, the set-valuedness of the price mappings P_i may create certain difficulties for the agents since the clear choice of unique prices is more suitable. For this reason, we suppose that each ith economic agent bearing in mind some most suitable reference price vector $\bar{v}^i$ chooses his/her feasible price value $P_i(x)$ at state x, i.e., problem (14)–(15) is replaced by the optimization problem

$$\max_{p^i \in V_i} \to \sum_{j=1}^{n} \left\{ p_{ij}x_{ij} - 0.5\beta_i(p_{ij} - \bar{v}_{ij})^2 \right\}, \tag{27}$$

where $\beta_i > 0$ is a suitable weight parameter chosen by the ith agent. Since it has the unique solution $p^i(x^i) = p^i(x)$, the mapping $x \mapsto P_i(x)$ is now single-valued for each $i \in I$ and so is $x \mapsto P(x)$. If we denote by $\eta_i(x^i)$ the optimal value of problem (27), (14), then the function η_i will be convex and differentiable and $\eta_i'(x^i) = p^i(x^i)$. In turn, $p(x) = (p^i(x))_{i \in I}$ is the gradient of the convex function

$$\eta(x) = \sum_{i \in I} \eta_i(x^i). \tag{28}$$

In other words, we take the following assumptions.

(C2) *At each state $x \in \tilde{Y}$, the point $p^i(x) = p^i(x^i)$ is determined as a unique solution of optimization problem (27), (14), where the set V_i is non-empty for all $i \in I$.*

(C3) *At each state $x \in \tilde{D}$, it holds that $x \in Y(x)$, the set $\tilde{D}$ is non-empty, the mapping $D : \tilde{Y} \to \Pi(\tilde{D})$ is lower semi-continuous on $\tilde{Y}$.*

Then, QVI (6)–(7) reduces to the single-valued problem: Find a vector $\bar{x} \in D(\bar{x})$ such that

$$\langle p(\bar{x}), y - \bar{x} \rangle \geq 0 \quad \forall y \in D(\bar{x}). \tag{29}$$

Its solution can be, however, deduced from the usual VI: Find a vector $\bar{x} \in \tilde{D}$ such that

$$\langle p(\bar{x}), y - \bar{x} \rangle \geq 0 \quad \forall y \in \tilde{D}, \tag{30}$$

where the set $\tilde{D}$ is defined in (11). From **(C1)** and **(C3)**, it follows that $\tilde{D}$ is nonempty, convex and compact. From **(C2)**, it follows that the mapping p is continuous on $\tilde{Y}$. Hence, VI (30) has a solution $\bar{x} \in \tilde{D}$. Due to **(C3)**, we have $\bar{x} \in Y(\bar{x})$, therefore, $\bar{x} \in D(\bar{x})$. This means that $\bar{x}$ is a solution to QVI (29). We observe that the reverse implication (29) $\Longrightarrow$ (30) does not hold in general.

Proposition 6.2 *Suppose that assumptions **(C1)**–**(C3)** are fulfilled. Then, QVI (29) has a solution.*

Moreover, VI (30) is equivalent to the convex smooth optimization problem:

$$\min_{x \in \tilde{D}} \to \eta(x),$$

where $\eta(x)$ is defined in (28). It follows that we can in principle apply one of various well known smooth optimization methods (see, e.g., [38]) for finding a solution of QVI (29). However, most of them are too sophisticated and do not fall into the above market information exchange framework. For instance, even simple projection and conditional gradient methods with line-search or utilization of a priori information cannot be taken due to the indicated information restrictions. For this reason, we will present a dynamic process based on the parametric conditional gradient method without line-search, which was proposed in [39] for custom optimization problems. Given a point $x \in \tilde{D}$, we define the auxiliary problem

$$\min_{y \in D(x)} \to \langle p(x), y \rangle. \tag{31}$$

We denote by $Z(x)$ the solution set of problem (31), thus defining the set-valued mapping $x \mapsto Z(x)$. Observe that $Z(x)$ is determined only by the local information from $D(x)$ rather than $\tilde{D}$ and that the set $Z(x)$ is always non-empty, convex, and compact.

Method (PCGM)

Initialization: Choose a point $w^{(0)} \in \tilde{D}$, a number $\beta \in (0, 1)$, and a positive sequence $\{\delta_s\} \to 0$. Set $s = 1$.

Step 0 For the given number s, choose a positive sequence $\{\tau_{l,s}\}$ such that $\tau_{l,s} \in (0, 1)$ and $\{\tau_{l,s}\} \to 0$ as $l \to \infty$. Set $k = 0$, $l = 0$, $x^{(0)} = w^{(s-1)}$, and choose a number $\theta_0 \in (0, \tau_{0,s}]$.

Step 1 Find a point $y^{(k)} \in Z(x^{(k)})$. If

$$\langle p(x^{(k)}), x^{(k)} - y^{(k)} \rangle \geq \delta_s, \tag{32}$$

go to Step 2. Otherwise, set $w^{(s)} = x^{(k)}$, $s = s + 1$ and go to Step 0. *(Restart)*

Step 2 Set $d^{(k)} = y^{(k)} - x^{(k)}$, $x^{(k+1)} = x^{(k)} + \theta_k d^{(k)}$. If

$$\langle p(x^{(k+1)}), d^{(k)} \rangle \leq \beta \langle p(x^{(k)}), d^{(k)} \rangle, \tag{33}$$

take $\theta_{k+1} \in [\theta_k, \tau_{l,s}]$. Otherwise, take $\theta_{k+1} \in (0, \min\{\theta_k, \tau_{l+1,s}\}]$ and set $l = l + 1$. Afterward, set $k = k + 1$ and go to Step 1.

Note that each outer iteration (stage) in s contains some number of inner iterations in k with the fixed tolerance δ_s. Completing each stage, that is marked as restart, leads to decrease in its value. Note that the choice of the parameters $\{\tau_{l,s}\}$ can be in principle independent for each stage s. Also, by (32), we have $\langle p(x^{(k)}), d^{(k)} \rangle \leq -\delta_s < 0$ in (33). It follows that

$$\eta(x^{(k+1)}) - \eta(x^{(k)}) \leq \theta_k \langle p(x^{(k+1)}), d^{(k)} \rangle \leq \beta \theta_k \langle p(x^{(k)}), d^{(k)} \rangle$$
$$\leq -\beta \theta_k \delta_s, \tag{34}$$

besides, $x^{(k)} \in \tilde{D}$ for each k.

We show that each stage is well defined.

Proposition 6.3 *Suppose that assumptions* **(C1)–(C3)** *are fulfilled. Then, the number of iterations at each stage s is finite.*

Proof Fix any s and suppose that the sequence $\{x^{(k)}\}$ is infinite. Then, the number of changes of index l is also infinite. In fact, otherwise we have $\theta_k \geq \bar{\theta} > 0$ for k large enough, hence (34) gives $\eta(x^{(k+t)}) \leq \eta(x^{(k)}) - t\beta\bar{\theta}\delta_s \to -\infty$ as $t \to \infty$, for k large enough, which is a contradiction. Therefore, there exists an infinite subsequence of indices $\{k_l\}$ such that

$$\langle p(x^{(k_l+1)}), d^{(k_l)} \rangle > \beta \langle p(x^{(k_l)}), d^{(k_l)} \rangle, \tag{35}$$

where $d^{(k_l)} = y^{(k_l)} - x^{(k_l)}$. Besides, it holds that $\theta_{k_l} \in (0, \tau_{l,s}]$, $\theta_{k_l+1} \in (0, \tau_{l+1,s}]$, where

$$\lim_{l \to \infty} \tau_{l,s} = 0.$$

Both the sequences $\{x^{(k)}\}$ and $\{y^{(k)}\}$ belong to the bounded set $\tilde{D}$ and hence have limit points. Without loss of generality, we can suppose that the subsequence $\{x^{(k_l)}\}$ converges to a point $\bar{x}$ and the corresponding subsequence $\{y^{(k_l)}\}$ converges to a point $\bar{y}$. Due to (32), we have

$$\langle p(\bar{x}), \bar{y} - \bar{x} \rangle = \lim_{l \to \infty} \langle p(x^{(k_l)}), y^{(k_l)} - x^{(k_l)} \rangle \leq -\delta_s. \tag{36}$$

At the same time, taking the limit $l \to \infty$ in (35), we obtain

$$\langle p(\bar{x}), \bar{y} - \bar{x} \rangle \geq \beta \langle p(\bar{x}), \bar{y} - \bar{x} \rangle,$$

i.e., $(1 - \beta)\langle p(\bar{x}), \bar{y} - \bar{x} \rangle \geq 0$, which is a contradiction with (36). □

We are ready to prove convergence of the whole method.

Theorem 6.1 *Suppose that assumptions* **(C1)–(C3)** *are fulfilled. Then:*

(i) *The number of changes of index k at each stage s is finite.*
(ii) *The sequence $\{w^{(s)}\}$ generated by method (PCGM) has limit points, all these limit points are solutions of QVI* (29).

Proof Assertion (i) has been obtained in Proposition 6.3. By construction, the sequence $\{w^{(s)}\}$ is bounded, hence it has limit points. By definition, for each s it holds that

$$\langle p(w^{(s)}), y - w^{(s)} \rangle \geq -\delta_s \quad \forall y \in D(w^{(s)}). \tag{37}$$

Take an arbitrary limit point $\bar{w}$ of $\{w^{(s)}\}$, then

$$\bar{w} = \lim_{t \to \infty} w^{(s_t)},$$

for some subsequence $\{w^{(s_t)}\}$, hence $\bar{w} \in \tilde{D}$. Then, $D(\bar{w})$ is nonempty and $\bar{w} \in D(\bar{w})$. Take any $\bar{y} \in D(\bar{w})$, then there exists a sequence of points $\{y^{(s_t)}\} \to \bar{y}$, $y^{(s_t)} \in D(w^{(s_t)})$, since the mapping D is lower semi-continuous on $\tilde{Y}$. Setting $s = s_t$ and $y = y^{(s_t)}$ in (37) and taking the limit $t \to \infty$, we obtain

$$\langle p(\bar{w}), \bar{y} - \bar{w} \rangle \geq 0.$$

This means that $\bar{w}$ is a solution of QVI (29). Assertion (ii) is true.   □

Again, implementation of each iterate may be not so easy in general. Bearing in mind (12), we specialize the feasible transaction sets and market capacity bounds as follows.

(C1′) *At each state* $x \in \tilde{Y} = \prod\limits_{i \in I} \tilde{Y}_i$, *it holds that* $Y_i(x) = \tilde{Y}_i \bigcap [c^i(x), d^i(x)]$, *where* $\tilde{Y}_i = [g^i, h^i]$ *for all* $i \in I$, *the set* $\tilde{Y}$ *is bounded.*

Clearly, **(C1′)**, **(C2)–(C3)** imply **(C1)–(C3)**. Following (PCGM), the agents determine their price vector $p^{(k)}$ for the current market state $x^{(k)}$ in accordance with **(C2)**. However, changing the stage does not require calculation of the price since the real state remains the same. This means that the agents only adjust their current tolerances for determination of the sufficient decrease in the pure expenses of the market; see (34). The main part of the iteration consists in solution of problem (31) at $x^{(k)}$. This problem clearly decomposes into n independent single commodity market equilibrium problems of form (5) with fixed prices. In fact, the jth commodity market problem is now written as follows:

$$\min_{y_{(j)} \in D_j} \to \sum_{i \in I} p_{ij}^{(k)} y_{ij},$$

where

$$D_j = \left\{ y_{(j)} \in \mathbb{R}^m : \sum_{i \in I} y_{ij} = 0, \ y_{ij} \in [c_{ij}^{(k)}, d_{ij}^{(k)}], \ i \in I \right\},$$

$c_{ij}^{(k)}$ and $d_{ij}^{(k)}$ are the corresponding capacity bounds; cf. (25)–(26). This problem can be solved by simple arrangement algorithms. Therefore, (PCGM) can be naturally implemented within usual market mechanisms.

It should be also noticed that (PCGM) has rather low computation expenses per iteration and provides opportunities for computations in a distributed manner, where computational units possess only limit data sets and information transmission flows are also limited. These multi-agent computational procedures are developed very intensively; see, e.g., [40,41].

We illustrate the proposed market model by a simple example.

Example 6.1 **(A market equilibrium model with production delay).** Consider a market system with m economic agents and n commodities. We suppose that the agents can utilize the purchased commodities only in the next time period, so that they utilize

in the current production only the endowments b^i, $i \in I$ from the previous period. This delay is typical for inter-temporal market models; see, e.g., [18,19]. Bearing in mind the previous endowments, each agent i is able to determine some current reference endowment vector $\bar{b}^i$. Similarly, taking into account the previous active prices, each agent i is able to determine some current reference price vector $\bar{v}^i$. Next, each agent i has the total market transaction set $\tilde{Y}_i = [g^i, h^i]$ such that $\bar{b}^i \in \tilde{Y}_i$ and the tolerance mapping $x^i \mapsto u^i(x^i)$ such that $u^i(x^i) \to \mathbf{0}$ as $x^i \to g^i$ or $x^i \to h^i$. Then, agent i determines the current market transaction set $Y_i(x) = \tilde{Y}_i \bigcap ([\bar{b}^i, x^i] + u^i(x^i))$ at state x. The current market price vector $p^i(x) = p^i(x^i)$ at state x is then naturally determined as a unique solution of optimization problem (27), (14). This means that the current output commodity prices are evaluated together with the next input commodity prices. QVI (29) will give equilibrium states for this market system. If the set $\tilde{Y}$ is bounded, the set V_i is non-empty, and the tolerance mapping u^i is continuous for each $i \in I$, then conditions (**C1′**), (**C2**), and (**C3**) hold. Hence, by Theorem 6.1, QVI (29) has a solution and each limit point of the sequence $\{w^{(s)}\}$ generated by method (PCGM) is a market equilibrium state.

7 Relationships with Some Other Basic Equilibrium Models

In this section, we intend to make a comparison of the presented model with the existing basic general and partial equilibrium models.

We recall that the classical Walrasian type general equilibrium models describe a market of a great number of economic agents, so that actions of any separate agent cannot impact the state of the whole system and traditionally belong to the perfect competition market models. These models are based on the assumption that any agent does not utilize particular information about the behavior of the others, but accepts the common market price values and then determines precisely his/her personal demand and/or supply values. These values create the general market excess supply values that have certain impact on the prices. Namely, a negative (positive) market excess supply of a commodity forces its price to increase (decrease), thus defining the so-called tâtonnement process; see [12,19]. Hence, the model essentially exploits (in fact, postulates) the assumption that there exists a common (market) price value for each commodity at any moment that can be recognized by any separate agent. Then, the budget constraint of any consumer involves just these common prices, which is usually active for his/her optimal choice due to the basic non-satiability property of the utility function. Similarly, any producer maximizes his/her profit, which is again determined by the same common prices. Next, the equilibrium conditions in the classical models are determined as complementarity relationships between the prices and market excess supply values; see, e.g., [18,19].

Our model is based on the assertion that the agents of a general market do not possess in fact a sufficient information about the current and future behavior of this so complicated system. That is, information for their decision making can be only partial and local. This means that each agent can evaluate his/her opportunities only in a neighborhood of the current market state by using his/her technology peculiarities and capacities, some integral parameters of the market (say, the current total

supply or demand for certain commodities), and some information on other agents due to the transaction process. For instance, this lack of information about the market participants is typical for contemporary telecommunication-based systems; see, e.g., [29,42]. Besides, the behavior of each economic agent depends on his/her current market, industry and social goals and restrictions. For this reason, the assumption that there exists an "ideal broker" who is able to determine precisely the common market price for each commodity at any moment and report simultaneously all these prices to all the agents seems rather restrictive. Similarly, it seems too difficult requirement for an agent to precisely fulfil the budget constraint with common prices also because of the data uncertainty. It should be noted that common constraints may appear due to the complexity of the system, these constraints can reflect common market, transportation, and/or technology relationships that are actively utilized by different agents. For instance, markets with joint constraints arise often in telecommunication and energy sectors; see, e.g., [28,30,42].

For this reason, we suppose that the agents will utilize their own prices that may differ from the current market prices even at an equilibrium state. These prices are values of their feasible price mappings attributed to the current market state that is determined by the current distribution of commodities. Instead of the exact budget constraint, any agent should simply indicate current sources that will cover his/her consumption expenses for each purchased commodity, as described in Sect. 5. Similarly, the agents determine feasible transaction sets dependent of the current market state. This approach leads to a quasi-variational inequality formulation of the market equilibrium state, which corresponds to the local maximal market profit in a neighborhood of the current state. It was proved in Sect. 4 that the proposed market equilibrium model also involves common market prices as the Lagrange multipliers corresponding to the common balance constraint, but they are in general different from the equilibrium prices of the agents; see Proposition 4.1. It also follows that the agents may have positive endowments at equilibrium due to the market capacity bounds as in the single commodity case.

We observe that most imperfectly competitive (or partial) equilibrium models (see, e.g., [43]) and some general market equilibrium models (see, e.g., [9,16]) are formulated as non-cooperative game-theoretic problems. Due to the common balance constraint in any market, the agents are mutually dependent, which contradicts to the custom non-cooperative game setting with equal and independent players. But in the imperfect competition case, actions of each agent can change the state of the whole market so that agents' utility functions depend on these actions (strategies). In order to avoid the mentioned drawback, the imperfect competition models are based on uneven roles of market sides. Namely, one side of the market (say, consumers) is presented by a common price (inverse demand) function, whereas each agent from the other side (producer) can change the state of the whole system by choosing his/her supply, so that the common balance is always satisfied implicitly. Therefore, the model becomes a game of producers and its solution is usually determined as its Nash equilibrium point. However, this is not the case for the general market equilibrium models with equal agents, which are then formulated as extended non-cooperative game-theoretic problems; see, e.g., [9,16]. The presence of the joint binding constraints creates certain difficulties for players and needs in a special non-trivial mechanism for attaining

such an equilibrium point; see, e.g., [44]. Therefore, our quasi-variational inequality formulation of the market equilibrium problem seems rather natural and suitable for different applications.

8 Conclusions

We described a new class of general market equilibrium models involving economic agents with restricted knowledge about the system. Due to these limitations, the market equilibrium problem was formulated as a quasi-variational inequality. We then deduced existence results for the model from the theory of quasi-variational inequalities. Besides, we indicated some ways of implementation of the proposed model in different settings and made comparisons with the other basic general equilibrium models. We also presented decentralized dynamic processes converging to equilibrium points within information exchange schemes of the market model. In particular, we proposed an iterative solution method for quasi-variational inequalities, which is based on evaluations of the market information in a neighborhood of the current market state rather than whole feasible set, which may be unknown, and proved its convergence.

Investigations of these models and methods can be continued in several directions. For instance, it seems worthwhile to adjust them to different applications, which involve explicit technology, environment and transportation capacity restrictions and develop new suitable dynamic processes.

The datasets generated during and/or analyzed during the current study are available from the author on reasonable request.

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