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A Linear Scalarization Proximal Point Method for Quasiconvex Multiobjective Minimization

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Abstract

In this paper, we propose a linear scalarization proximal point algorithm for solving lower semicontinuous quasiconvex multiobjective minimization problems. Under some natural assumptions and, using the condition that the proximal parameters are bounded, we prove the convergence of the sequence generated by the algorithm and, when the objective functions are continuous, we prove the convergence to a generalized critical point of the problem. Furthermore, for the continuously differentiable case we introduce an inexact algorithm, which converges to a Pareto critical point.

Keywords Multiobjective minimization · Lower semicontinuous quasiconvex functions · Proximal point methods · Fejér convergence · Pareto–Clarke critical point

Mathematics Subject Classification 49M37 · 65K05 · 65K10 · 90C26 · 90C29

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1 Introduction

In this paper, we consider the class of problems known as multiobjective minimization problems, which involves a minimization of a set of quasiconvex objective functions. A motivation to study this problem is the consumer demand theory in microeconomy, where the quasiconvexity of each objective function is a natural condition associated with diversification of the consumption; see Sect. 3.1 for more details.

Another motivation is the multiobjective quasiconvex minimization model in location theory, where we need to find a location for an installation so that this location minimizes some functions involving some distances between the new location and each cluster set of demand points; see Sect. 4 of Apolinário et al. [1].

For other motivations, we recommend the excellent book [2], which contains a heterogeneous collection of contributions on generalized convexity and generalized monotonicity. In particular, we recommend the lecture of Chapters 2, 3, 5 and 6.

Recently, Apolinário et al. [1] have introduced an exact linear scalarization proximal point algorithm to solve the above class of problems, when each objective function is locally Lipschitz and quasiconvex. The authors proved, under some natural assumptions, that the sequence generated by the proposed algorithm is well defined and converges globally to a Pareto–Clarke critical point.

Unfortunately, the proposed algorithm cannot be applied to solve a general class of proper lower semicontinuous quasiconvex functions, in particular to solve constrained multiobjective problems or minimization problems with continuous quasiconvex functions, which are not locally Lipschitz. Moreover, for a future implementation and application, it is necessary to construct inexact versions of the proposed algorithm.

Thus, we had two motivations to develop the present paper: The first motivation was to extend the convergence properties of the linear scalarization proximal point method introduced in [1] to solve more general, probably constrained, quasiconvex multiobjective problems and the second one was to introduce an inexact algorithm, when each objective function is continuously differentiable.

Some works related to this paper are as follows:

- Bento et al. [3] introduced a proximal point algorithm for multiobjective optimization using a nonlinear scalarization function. Assuming that the objective functions are quasiconvex and continuously differentiable, the authors proved that the sequence generated by the algorithm converges to a Pareto critical point. The difference between our work and the paper of Bento et al. [3] is that in the present paper we consider a linear scalarization function instead of a nonlinear ones and another difference is that our assumptions are slightly weaker than the paper [3], because we obtain convergence results for nondifferentiable quasiconvex functions.
- Makela et al. [4] developed a multiobjective proximal bundle method for nonsmooth optimization where the objective functions are locally Lipschitz (not necessarily smooth nor convex). The authors proved that any accumulation point of the sequence is a weak Pareto efficient solution and under some assumptions, they obtained that any accumulation point is a substationary point.

– Chuong et al. [5] developed three algorithms of the so-called hybrid approximate proximal type to find Pareto optimal points for a general class of convex constrained problems of vector optimization in finite- and infinite-dimensional spaces, and proved the convergence of the sequence generated by their algorithms.

This paper proposes a linear scalarization proximal point algorithm for solving multiobjective minimization problems. Under the assumption that each objective function is a proper lower semicontinuous quasiconvex function, we prove the global convergence of the sequence generated by the algorithm to some generalized critical point of the problem and convergence to a weak Pareto efficient solution when the objective functions are continuous and the regularized proximal parameters converge to zero. Additionally, when the objective functions are differentiable, we introduce an inexact proximal algorithm and prove the convergence of sequence generated by the algorithm to a Pareto critical point of the multiobjective minimization problem.

The paper is organized as follows: In Sect. 2, we recall some concepts and basic results on multiobjective optimization, descent direction, scalar representation, quasiconvex and convex functions, Fréchet and limiting subdifferential, ϵ-subdifferential and Fejér convergence. In Sect. 3, we present the problem and we give an example of a quasiconvex model in demand theory. In Sect. 4, we introduce an exact algorithm and analyze its convergence. In Sect. 5, we present an inexact algorithm for the differentiable case and analyze its convergence. In Sect. 6, we give a numerical example of the algorithm, in Sect. 7, we give some perspectives and open problems, and in Sect. 8, we give our conclusions.

2 Preliminaries

In this section, we present some basic concepts and results that are important for the development of our work. These facts can be found, for example, in Hadjisavvas [2], Mordukhovich [6] and Rockafellar and Wets [7].

2.1 Definitions, Notations and Some Basic Results

Along this paper $\mathbb{R}^n$ denotes an Euclidean space, that is, a real vectorial space with the canonical inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ and the norm given by $||x|| = \sqrt{\langle x, x\rangle}$. Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, the *effective domain* of f we denote by $\text{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$, . If $\text{dom}(f) \neq \emptyset$, f is called proper. And f is called coercive, if $\lim_{\|x\|\to+\infty} f(x) = +\infty$. We denote by $\arg\min\{f(x) : x \in \mathbb{R}^n\}$ the set of minimizers of f and $\bar{f}$, the optimal value of problem: $\min\{f(x) : x \in \mathbb{R}^n\}$, if it exists. The function f is *lower semicontinuous* at $\bar{x}$ if for all sequence $\left\{x^l\right\}_{l\in\mathbb{N}}$ such that $\lim_{l\to+\infty} x^l = \bar{x}$ we obtain that $f(\bar{x}) \leq \liminf_{l\to+\infty} f(x^l)$.
We say that f is differentiable at $\bar{x}$, if there exists $v \in \mathbb{R}^n$ such that

$$f(x) = f(\bar{x}) + \langle v, x - \bar{x}\rangle + o(\|x - \bar{x}\|),$$

where

$$\lim_{x \to \bar{x}} \frac{o(\|x-\bar{x}\|)}{\|x-\bar{x}\|} = 0.$$

The next result ensures that the set of minimizers of a function, under some assumptions, is nonempty.

Proposition 2.1 (Rockafellar and Wets [7], Theorem 1.9) *Suppose that* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is proper, lower semicontinuous and coercive, then the optimal value* $\bar{f}$ *is finite and the set* $\arg\min\{f(x) : x \in \mathbb{R}^n\}$ *is nonempty and compact.*

Definition 2.1 Let $D \subset \mathbb{R}^n$ be a convex set and $\bar{x} \in D$. The normal cone to D at $\bar{x} \in D$ is given by

$$\mathcal{N}_D(\bar{x}) := \left\{ v \in \mathbb{R}^n : \langle v, x - \bar{x} \rangle \leq 0, \forall\, x \in D \right\}.$$

It follows an important result that involves sequences of nonnegative numbers, which will be useful in Sect. 5.

Lemma 2.1 (Polyak [8], Lemma 2.2.2.) *Let* $\{w_k\}$*,* $\{p_k\}$ *and* $\{q_k\}$ *be sequences of nonnegative real numbers. If*

$$w_{k+1} \leq (1+p_k)\, w_k + q_k, \quad \sum_{k=1}^{\infty} p_k < +\infty \ \text{ and } \ \sum_{k=1}^{\infty} q_k < +\infty,$$

then the sequence $\{w_k\}$ *is convergent.*

2.2 Multiobjective Optimization

In this subsection, we present some properties and notations on multiobjective optimization; see, for example, the books of Miettinen [9] and Luc [10] for more details. Considering the cone $\mathbb{R}^m_+ = \{y \in \mathbb{R}^m : y_i \geq 0, \forall\, i = 1, \ldots, m\}$, we define in $\mathbb{R}^m$ the following partial order $\preceq$ induced by $\mathbb{R}^m_+$: given $y, y' \in \mathbb{R}^m$, then $y \preceq y'$ if, and only if, $y' - y \in \mathbb{R}^m_+$, which is equivalent to $y_i \leq y'_i$, for all $i = 1, 2, \ldots, m$.
Given $\mathbb{R}^m_{++} = \{y \in \mathbb{R}^m : y_i > 0, \forall\, i = 1, \ldots, m\}$, we may define another relation $\prec$ induced by $\mathbb{R}^m_{++}$: $y \prec y'$, if, and only if, $y' - y \in \mathbb{R}^m_{++}$, which is equivalent to $y_i < y'_i$ for all $i = 1, 2, \ldots, m$.
Let us consider the multiobjective optimization problem (MOP) :

$$\min \left\{ G(x) : x \in \mathbb{R}^n \right\}, \tag{1}$$

where $G = (G_1, G_2, \ldots, G_m)$ with $G(x) = +\infty_{\mathbb{R}^m_+}$ (in the sense of the paper of Bolintinéanu [11]) if $x \notin \operatorname{dom}(G) := \bigcap_{i=1}^m \operatorname{dom}(G_i) \neq \emptyset$ and each $G_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, m$.

Along the paper, we use the notation $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ when $\operatorname{dom}(G_i) = \mathbb{R}^n$, for each $i = 1, \ldots, m$, and we say that G is continuous (differentiable, continuously differentiable) if each G_i is continuous (differentiable, continuously differentiable) for each $i = 1, \ldots, m$.

Definition 2.2 (*Miettinen* [9], *Definition 2.2.1*) A point $x^* \in \mathbb{R}^n$ is a Pareto optimal point or Pareto efficient solution of the problem (1), if there does not exist $x \in \mathbb{R}^n$ such that $G_i(x) \le G_i(x^*)$, for all $i \in \{1, \ldots, m\}$, and $G_j(x) < G_j(x^*)$, for at least one index $j \in \{1, \ldots, m\}$.

Definition 2.3 (*Miettinen* [9], *Definition 2.5.1*) A point $x^* \in \mathbb{R}^n$ is a weak Pareto efficient solution of the problem (1), if there does not exist $x \in \mathbb{R}^n$ such that $G_i(x) < G_i(x^*)$, for all $i \in \{1, \ldots, m\}$.

We denote by $\arg\min \{G(x) : x \in \mathbb{R}^n\}$ and by $\arg\min_w \{G(x) : x \in \mathbb{R}^n\}$ the set of Pareto efficient solutions and weak Pareto efficient solutions to the problem (1), respectively. It is easy to check that

$$\arg\min \left\{G(x) : x \in \mathbb{R}^n\right\} \subset \arg\min_w \left\{G(x) : x \in \mathbb{R}^n\right\}.$$

2.3 Pareto Critical Point and Descent Direction

Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function. Given $x \in \mathbb{R}^n$, the Jacobian of G at x, denoted by $JG(x)$, is a matrix of order $m \times n$ whose entries are defined by $(JG(x))_{i,j} = \frac{\partial G_i}{\partial x_j}(x)$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We may represent it by

$$JG(x) := [\nabla G_1(x) \nabla G_2(x) \ldots \nabla G_m(x)]^{\mathrm{T}}, x \in \mathbb{R}^n.$$

The image of the Jacobian of G at x is denoted by

$$\mathrm{Im}\,(JG(x)) := \{JG(x)\,v = (\langle \nabla G_1(x), v\rangle, \langle \nabla G_2(x), v\rangle, \ldots, \langle \nabla G_m(x), v\rangle) : v \in \mathbb{R}^n\}.$$

A necessary, but not sufficient, first-order optimality condition for the problem (1) at $x \in \mathbb{R}^n$, is

$$\mathrm{Im}\,(JG(x)) \cap \left(-\mathbb{R}^m_{++}\right) = \emptyset. \tag{2}$$

Equivalently, $\forall\, v \in \mathbb{R}^n$, there exists $i_0 = i_0(v) \in \{1, \ldots, m\}$ such that

$$\langle \nabla G_{i_0}(x), v\rangle \ge 0.$$

Definition 2.4 Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function. A point $x^* \in \mathbb{R}^n$ satisfying (2) is called a Pareto critical point.

Following from the previous definition, if a point x is not a Pareto critical point, then there exists a direction $v \in \mathbb{R}^n$ satisfying

$$JG(x)\,v \in \left(-\mathbb{R}^m_{++}\right),$$

i.e., $\langle \nabla G_i(x), v\rangle < 0,\ \forall\, i \in \{1,\ldots,m\}$. As G is continuously differentiable, then

$$\lim_{t\to 0} \frac{G_i(x+tv) - G_i(x)}{t} = \langle \nabla G_i(x), v\rangle < 0,\ \forall\, i \in \{1,\ldots,m\}.$$

This implies that v is a *descent direction* for the function G_i, i.e., there exists $\varepsilon > 0$, such that

$$G_i(x+tv) < G_i(x), \forall\, t \in (0,\varepsilon], \forall\, i \in \{1,\ldots,m\}.$$

Therefore, v is a *descent direction* for G at x, i.e., there exists $\varepsilon > 0$ such that

$$G(x+tv) \prec G(x),\ \forall\, t \in (0,\varepsilon].$$

2.4 Scalar Representation

In this subsection, we present an useful technique in multiobjective optimization, where the original vector optimization problem is replaced by a family of scalar problems.

Let $F = (F_1, F_2, \ldots, F_m)$ be a map with $F(x) = +\infty_{\mathbb{R}^m_+}$ (in the sense of the paper of Bolintinéanu [11]) if $x \notin \operatorname{dom}(F) := \bigcap_{i=1}^m \operatorname{dom}(F_i) \neq \emptyset$ where each $F_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$.

Definition 2.5 (*Luc* [10], *Definition 2.1, page 86*) A function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be a strict scalar representation of a map F when given $x, \bar{x} \in \mathbb{R}^n$:

$$F(x) \preceq F(\bar{x}) \Longrightarrow f(x) \le f(\bar{x}) \quad \text{and} \quad F(x) \prec F(\bar{x}) \Longrightarrow f(x) < f(\bar{x}).$$

Furthermore, we say that f is a weak scalar representation of F if

$$F(x) \prec F(\bar{x}) \Longrightarrow f(x) < f(\bar{x}).$$

Proposition 2.2 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Then, f is a strict scalar representation of F if, and only if, there exists a strictly increasing function $g : F(\mathbb{R}^n) \longrightarrow \mathbb{R}$ such that $f = g \circ F$.*

Proof See Luc [10] Proposition 2.3. □

Proposition 2.3 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a weak scalar representation of a vector function $F = (F_1, F_2, \ldots, F_m)$, and* $\arg\min\{f(x) : x \in \mathbb{R}^n\}$ *the set of minimizer points of f. Then, we have*

$$\arg\min\left\{f(x) : x \in \mathbb{R}^n\right\} \subseteq \arg\min_w\{F(x) : x \in \mathbb{R}^n\}.$$

Proof Let $\bar{x} \in \operatorname{argmin}\{f(x) : x \in \mathbb{R}^n\}$ and suppose that $\bar{x}$ is not a weak Pareto efficient solution of the problem $\min\{F(x) : x \in \mathbb{R}^n\}$, then there exists $x \in \mathbb{R}^n$ such that $F_i(x) < F_i(\bar{x})$, for all $i = 1, 2, \ldots, m$. From Definition 2.5 we have $f(x) < f(\bar{x})$, which is a contradiction. Therefore, $\bar{x}$ is a weak Pareto efficient solution. □

2.5 Quasiconvex and Convex Functions

In this subsection, we present some definitions of quasiconvex functions and quasiconvex multiobjective functions. These definitions and some properties can be also found in Bazaraa et al. [12], Luc [10], Mangasarian [13] and references therein.

Definition 2.6 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Then, f is called quasiconvex, if for all $x, y \in \mathbb{R}^n$, and for all $t \in [0, 1]$, it holds that

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

Definition 2.7 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Then, f is called convex, if for all $x, y \in \mathbb{R}^n$, and for all $t \in [0, 1]$, it holds that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Observe that if f is a quasiconvex function, then $\mathrm{dom}(f)$ is a convex set. On the other hand, while a convex function can be characterized by the convexity of its epigraph, a quasiconvex function can be characterized by the convexity of the lower level sets:

Definition 2.8 Let $F_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, $i = 1, 2, \ldots, m$, and $F = (F_1, \ldots, F_m)$, then F is $\mathbb{R}^m_+$-quasiconvex (convex), if each component F_i is quasiconvex (convex).

2.6 Fréchet and Limiting Subdifferentials

Definition 2.9 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function.

(a) For each $x \in \mathrm{dom}(f)$, the set of regular subgradients (also called Fréchet subdifferential) of f at x, denoted by $\hat{\partial} f(x)$, is the set of vectors $v \in \mathbb{R}^n$ such that

$$f(y) \geq f(x) + \langle v, y - x\rangle + o(\|y - x\|), \text{ where } \lim_{y \to x} \frac{o(\|y - x\|)}{\|y - x\|} = 0.$$

Or equivalently, $\hat{\partial} f(x) := \left\{ v \in \mathbb{R}^n : \liminf_{y \neq x,\ y \to x} \frac{f(y) - f(x) - \langle v, y - x\rangle}{\|y - x\|} \geq 0 \right\}$.
If $x \notin \mathrm{dom}(f)$, then $\hat{\partial} f(x) = \emptyset$.

(b) The set of general subgradients (also called limiting subdifferential) f at $x \in \mathbb{R}^n$, denoted by $\partial f(x)$, is defined as follows:

$$\partial f(x) := \left\{ v \in \mathbb{R}^n : \exists\, x^l \to x, \quad f(x^l) \to f(x), \quad v^l \in \hat{\partial} f(x^l) \text{ and } v^l \to v \right\}.$$

Proposition 2.4 (Fermat's rule generalized) *If a proper function* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *has a local minimum at* $\bar{x} \in$ *dom*(f)*, then* $0 \in \hat{\partial} f(\bar{x})$.

Proof See Rockafellar and Wets [7] Theorem 10.1. □

Proposition 2.5 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper function. Then, the following properties are true*

(i) $\hat{\partial} f(x) \subset \partial f(x)$, *for all* $x \in \mathbb{R}^n$.
(ii) *If* f *is differentiable at* $\bar{x}$, *then* $\hat{\partial} f(\bar{x}) = \{\nabla f(\bar{x})\}$, *so* $\nabla f(\bar{x}) \in \partial f(\bar{x})$.
(iii) *If* f *is continuously differentiable in a neighborhood of* x, *then* $\hat{\partial} f(x) = \partial f(x) = \{\nabla f(x)\}$.
(iv) *If* $g = f + h$ *with* f *finite at* $\bar{x}$ *and* h *is continuously differentiable in a neighborhood of* $\bar{x}$, *then* $\hat{\partial} g(\bar{x}) = \hat{\partial} f(\bar{x}) + \nabla h(\bar{x})$ *and* $\partial g(\bar{x}) = \partial f(\bar{x}) + \nabla h(\bar{x})$.

Proof See Rockafellar and Wets [7] Exercise 8.8, page 304. □

2.7 ε-Subdifferential

We present some important concepts and results on ε-subdifferential. The theory of these facts can be found, for example, in Jofre et al. [14] and Rockafellar and Wets [7].

Definition 2.10 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function, and let ε be an arbitrary nonnegative real number. The Fréchet ε-subdifferential of f at $x \in \text{dom}(f)$ is defined by

$$\hat{\partial}_\varepsilon f(x) := \left\{x^* \in \mathbb{R}^n : \liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon\right\}. \tag{3}$$

Remark 2.1 When $\varepsilon = 0$, (3) reduces to the well-known Fréchet subdifferential, which is denoted by $\hat{\partial} f(x)$, according to Definition 2.9. More precisely,

$$x^* \in \hat{\partial} f(x), \text{ if and only if, for each } \eta > 0 \text{ there exists } \delta > 0 \text{ such that}$$
$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \eta\|y - x\|, \text{ for all } y \in x + \delta \mathrm{B},$$

where B is the closed unit ball in $\mathbb{R}^n$ centered at zero. Therefore,

$$\hat{\partial} f(x) = \hat{\partial}_0 f(x) \subset \hat{\partial}_\varepsilon f(x).$$

From Definition 5.1 of Treiman, [15],

$$x^* \in \hat{\partial}_\epsilon f(x) \Leftrightarrow x^* \in \hat{\partial}(f(.) + \epsilon\|. - x\|)(x).$$

Equivalently, $x^* \in \hat{\partial}_\epsilon f(x)$, if and only if, for each $\eta > 0$, there exists $\delta > 0$ such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + (\epsilon + \eta)\|y - x\|, \text{ for all } y \in x + \delta \mathrm{B}.$$

We now define a new kind of approximate subdifferential.

Definition 2.11 The limiting Fréchet ε-subdifferential of f at $x \in \text{dom}(f)$ is defined by

$$\partial_\varepsilon f(x) := \limsup_{y \xrightarrow{f} x} \hat{\partial}_\varepsilon f(y) \tag{4}$$

where

$$\limsup_{y \xrightarrow{f} x} \hat{\partial}_\varepsilon f(y) := \{x^* \in \mathbb{R}^n : \exists\, x_l \to x,\ f(x_l) \to f(x),\ x_l^* \to x^*$$

with $x_l^* \in \hat{\partial}_\varepsilon f(x_l)\}$.

In the case where f is continuously differentiable, the limiting Fréchet ε-subdifferential takes a very simple form, according to the following proposition

Proposition 2.6 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function at x with derivative $\nabla f(x)$. Then,*

$$\partial_\varepsilon f(x) = \nabla f(x) + \varepsilon B.$$

Proof See Jofré et al. [14] Proposition 2.8. □

2.8 Fejér Convergence

Definition 2.12 A sequence $\{y_k\} \subset \mathbb{R}^n$ is said to be Fejér convergent to a set $U \subseteq \mathbb{R}^n$ if, $\|y_{k+1} - u\| \leq \|y_k - u\|$, $\forall\, k \in \mathbb{N}$, $\forall\, u \in U$.

The following result on Fejér convergence is well known.

Lemma 2.2 *If $\{y_k\} \subset \mathbb{R}^n$ is Fejér convergent to some set $U \neq \emptyset$, then:*

(i) *The sequence $\{y_k\}$ is bounded.*
(ii) *If an accumulation point y of $\{y_k\}$ belongs to U, then $\lim_{k\to+\infty} y_k = y$.*

Proof See Schott [16] Theorem 2.7. □

3 The Problem

We are interested in solving the multiobjective optimization problem (MOP):

$$\min\{F(x) : x \in \mathbb{R}^n\}, \tag{5}$$

where $F = (F_1, F_2, \ldots, F_m)$ and each $F_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is an extended function satisfying the following assumptions:

($\mathbf{C_{1.1}}$) Each $F_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, m$, is a proper lower semicontinuous function and

$$\mathrm{dom}(F) := \bigcap_{i=1}^{m} \mathrm{dom}(F_i) \neq \emptyset.$$

($\mathbf{C_{1.2}}$) $0 \preceq F(x)$, $\forall x \in \mathbb{R}^n$, that is, $F_i(x) \geq 0$, for each $i = 1, 2, \ldots, m$.
($\mathbf{C_2}$) F is $\mathbb{R}^m_+$-quasiconvex, that is, each F_i is quasiconvex.

3.1 A Quasiconvex Model in Demand Theory

Let n be a finite number of consumer goods. A consumer is an agent who must choose how much to consume each good. An ordered set of numbers representing the amounts consumed of each good set is called vector of consumption, and denoted by $x = (x_1, x_2, \ldots, x_n)$ where x_i with $i = 1, 2, \ldots, n$, is the quantity consumed of good i. Denote by X the feasible set of these vectors, which will be called the set of consumption, usually in economic applications we have $X \subset \mathbb{R}^n_+$.

In the classical approach of demand theory, the analysis of consumer behavior starts specifying a preference relation over the set X, denoted by $\succeq$. The notation "$x \succeq y$" means that "x is at least as good as y" or "y is not preferred to x." This preference relation $\succeq$ is assumed rational, i.e., is complete because the consumer is able to order all possible combinations of goods, and transitive because consumer preferences are consistent, which means if the consumer prefers $\bar{x}$ to $\bar{y}$ and $\bar{y}$ to $\bar{z}$, then he prefers $\bar{x}$ to $\bar{z}$ (see Definition 3.B.1 of Mas-Colell et al. [17]). We say that $\succeq$ is a convex preference relation on a convex set X if for all $x, y, z \in X$ and $\lambda \in [0, 1]$ satisfying $z \succeq x$ and $z \succeq y$ we obtain $z \succeq \lambda x + (1 - \lambda) y$.

The quasiconcave model for a convex preference relation $\succeq$ is

$$\max\{\mu(x) : x \in X\},$$

where μ is the utility function representing the preference; see Papa Quiroz et al. [18] for more details. Now consider a multiple criteria, that is, consider m convex preference relations denoted by $\succeq_i$, $i = 1, 2, \ldots, m$. Suppose that for each preference $\succeq_i$, there exists an utility function, μ_i, respectively, then the problem of maximizing the consumer preference on X is equivalent to solve the quasiconcave multiobjective optimization problem

$$\max\{(\mu_1(x), \mu_2(x), \ldots, \mu_m(x)) \in \mathbb{R}^m : x \in X\}.$$

Since there is not a single point, which maximize all the functions simultaneously, the concept of optimality is established in terms of Pareto optimality or efficiency. Taking $F = (-\mu_1(x), -\mu_2(x), \ldots, -\mu_m(x))$, we obtain a minimization problem with quasiconvex multiobjective function, since each component function is quasiconvex one.

4 Exact Algorithm

In this section, to solve the problem (5), we propose a linear scalarization proximal point algorithm with quadratic regularization using the Fréchet subdifferential, denoted by **SPP** algorithm.
We consider two sequences: the proximal parameters $\{\alpha_k\}$, with $\alpha_k > 0$ for each k, and a sequence $\{z_k\} = \{(z_k^1, z_k^2, \ldots, z_k^m)\} \subset \mathbb{R}^m_+ \backslash \{0\}$ with $\|z_k\| = 1$.

SPP Algorithm

Initialization: Choose an arbitrary starting point

$$x^0 \in \mathbb{R}^n \tag{6}$$

Main Steps: Given x^k, find $x^{k+1} \in \Omega_k = \left\{x \in \mathbb{R}^n : F(x) \preceq F(x^k)\right\}$ such that

$$0 \in \hat{\partial}\left(\langle F(.), z_k\rangle + \frac{\alpha_k}{2}\| \,.\, - x^k\|^2 + \delta_{\Omega_k}(.)\right)(x^{k+1}), \tag{7}$$

where δ_{Ω_k} is the indicator function of Ω_k.

Stop criterion: If $x^{k+1} = x^k$, then stop. Otherwise, do $k \leftarrow k+1$ and return to Main Steps.

4.1 Existence of the Iterates

Theorem 4.1 *Assume that assumptions* $(\mathbf{C_{1.1}})$ *and* $(\mathbf{C_{1.2}})$ *are satisfied, then the sequence* $\left\{x^k\right\}$*, generated by the* **SPP** *algorithm, is well defined.*

Proof Let $x^0 \in \mathbb{R}^n$ be an arbitrary point given in the initialization step. Given x^k, define $\varphi_k(x) = \langle F(x), z_k\rangle + \frac{\alpha_k}{2}\left\|x - x^k\right\|^2 + \delta_{\Omega_k}(x)$, where $\delta_{\Omega_k}(.)$ is the indicator function of Ω_k. Then, we have that $\min\{\varphi_k(x) : x \in \mathbb{R}^n\}$ is equivalent to $\min\{\langle F(x), z_k\rangle + \frac{\alpha_k}{2}\left\|x - x^k\right\|^2 : x \in \Omega_k\}$. As φ_k is lower semicontinuous and coercive, then using Proposition 2.1, we obtain that there exists $x^{k+1} \in \mathbb{R}^n$, which is a global minimizer of φ_k. From Proposition 2.4, x^{k+1} satisfies:

$$0 \in \hat{\partial}\left(\langle F(.), z_k\rangle + \frac{\alpha_k}{2}\| \,.\, - x^k\|^2 + \delta_{\Omega_k}(.)\right)(x^{k+1}).$$

□

Remark 4.1 We are interested in the asymptotic convergence of the **SPP** algorithm, so we assume that $x^k \neq x^{k+1}$ for all k. If $x^k = x^{k+1}$ for some k, then from the **SPP** algorithm we have that

$$0 \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1}),$$

that is, x^{k+1} is a critical point of the optimization problem $\min\{\langle F(.), z_k\rangle : x \in \Omega_k\}$. Observe that the function $\langle F(.), z_k\rangle$ is a strict scalar representation of the map F, so if, furthermore, x^{k+1} is a minimizer of that optimization problem, then from Proposition 2.3 , x^{k+1} is a weak Pareto optimal point of the original problem (5).

4.2 Fejér Convergence Property

To obtain some desirable properties it is necessary to assume the following assumption on the function F and the initial point x^0:

($\mathbf{C_3}$) The set $\left(F(x^0) - \mathbb{R}^m_+\right) \cap F(\mathbb{R}^n)$ is $\mathbb{R}^m_+$-complete, meaning that for each sequence $\{a_k\} \subset \mathbb{R}^n$, with $a_0 = x^0$, such that $F(a_{k+1}) \preceq F(a_k)$, there exists $a \in \mathbb{R}^n$ such that $F(a) \preceq F(a_k), \ \forall\, k \in \mathbb{N}$.

Remark 4.2 The assumption ($\mathbf{C_3}$) is cited in several works involving the proximal point method for convex functions, Bonnel et al. [19], Ceng and Yao [20], and Villacorta and Oliveira [21].

Proposition 4.1 *Assume that assumptions* ($\mathbf{C_{1.1}}$) *and* ($\mathbf{C_2}$) *are satisfied. If* $x \in dom(F) \cap \Omega$ *and* $g \in \hat{\partial}\left(\langle F(.), z\rangle + \delta_\Omega(.)\right)(x)$*, with* $z \in \mathbb{R}^m_+ \setminus \{0\}$*, and* $F(y) \preceq F(x)$*, with* $y \in \Omega$*, and* $\Omega \subset \mathbb{R}^n$ *a closed and convex set, then* $\langle g, y - x\rangle \leq 0$.

Proof Let $t \in]0, 1]$, then from the $\mathbb{R}^m_+$-quasiconvexity of F and the assumption that $F(y) \preceq F(x)$, we have: $F_i(ty + (1-t)x) \leq \max\{F_i(x), F_i(y)\} = F_i(x), \ \forall\, i \in \{1, \ldots m\}$. It follows that for each $z \in \mathbb{R}^m_+ \setminus \{0\}$, we have

$$\langle F(ty + (1-t)x), z\rangle \leq \langle F(x), z\rangle . \tag{8}$$

As $g \in \hat{\partial}\left(\langle F(.), z\rangle + \delta_\Omega(.)\right)(x)$, we obtain

$$\begin{aligned}\langle F(ty + (1-t)x), z\rangle + \delta_\Omega(ty + (1-t)x) \geq \langle F(x), z\rangle + \delta_\Omega(x) + t\,\langle g, y - x\rangle \\ +o(t\,\|y - x\|).\end{aligned} \tag{9}$$

From (8) and (9), we conclude

$$t\,\langle g, y - x\rangle + o(t\,\|y - x\|) \leq 0. \tag{10}$$

On the other hand, we have $\lim\limits_{t\to 0} \frac{o(t\|y-x\|)}{t\|y-x\|} = 0$. Thus,

$$\lim_{t\to 0} \frac{o(t\,\|y - x\|)}{t} = \lim_{t\to 0} \frac{o(t\,\|y - x\|)}{t\,\|y - x\|}\,\|y - x\| = 0.$$

Thus, dividing (10) by t and taking $t \to 0$, we obtain the desired result. □

Under assumptions ($\mathbf{C_{1,1}}$), ($\mathbf{C_{1,2}}$) and ($\mathbf{C_3}$), we obtain that the set

$$E = \left\{x \in \mathbb{R}^n : F(x) \preceq F\left(x^k\right), \ \ \forall\, k \in \mathbb{N}\right\}$$

is nonempty. Furthermore, if the assumption ($\mathbf{C_2}$) is satisfied, then E is a nonempty and convex set.

Proposition 4.2 *Under assumptions* ($\mathbf{C_{1.1}}$), ($\mathbf{C_{1.2}}$), ($\mathbf{C_2}$) *and* ($\mathbf{C_3}$)*, the sequence* $\left\{x^k\right\}$*, generated by the* **SPP** *algorithm,* (6) *and* (7)*, is Fejér convergent to* E.

Proof Observe that $\forall\, x \in \mathbb{R}^n$:

$$\left\|x^k - x\right\|^2 = \left\|x^k - x^{k+1}\right\|^2 + \left\|x^{k+1} - x\right\|^2 + 2\left\langle x^k - x^{k+1}, x^{k+1} - x\right\rangle. \quad (11)$$

From Theorem 4.1, (7) and from Proposition 2.5, (iv), we have that there exists $g_k \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1})$ such that:

$$x^k - x^{k+1} = \frac{1}{\alpha_k} g_k. \quad (12)$$

Now take $x^* \in E$, then $x^* \in \Omega_k$ for all $k \in \mathbb{N}$. Combining (11) with $x = x^*$ and (12), we obtain:

$$\begin{aligned}\left\|x^k - x^*\right\|^2 &= \left\|x^k - x^{k+1}\right\|^2 + \left\|x^{k+1} - x^*\right\|^2 + \frac{2}{\alpha_k}\left\langle g_k,\ x^{k+1} - x^*\right\rangle \\ &\geq \left\|x^k - x^{k+1}\right\|^2 + \left\|x^{k+1} - x^*\right\|^2, \end{aligned} \quad (13)$$

where the last inequality follows from Proposition 4.1. Relation (13) implies that

$$0 \leq \left\|x^{k+1} - x^k\right\|^2 \leq \left\|x^k - x^*\right\|^2 - \left\|x^{k+1} - x^*\right\|^2. \quad (14)$$

Thus,

$$\left\|x^{k+1} - x^*\right\| \leq \left\|x^k - x^*\right\|. \quad (15)$$

□

Proposition 4.3 *Under assumptions* $(\mathbf{C_{1.1}})$, $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$ *and* $(\mathbf{C_3})$, *the sequence* $\left\{x^k\right\}$ *generated by the* **SPP** *algorithm,* (6) *and* (7)*, satisfies*

$$\lim_{k\to+\infty} \left\|x^{k+1} - x^k\right\| = 0.$$

Proof It follows from (15) that $\forall\, x^* \in E$, $\left\{\left\|x^k - x^*\right\|\right\}$ is a nonnegative and nonincreasing sequence and hence is convergent. Thus, the right-hand side of (14) converges to 0, when $k \to +\infty$, and the result is obtained. □

4.3 Convergence of the Iterates

In this subsection, we prove the convergence of the proposed algorithm, when F is a non differentiable vector function.

Proposition 4.4 *Under assumptions* $(\mathbf{C_{1.1}})$, $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$ *and* $(\mathbf{C_3})$, *the sequence* $\left\{x^k\right\}$ *generated by the* **SPP** *algorithm converges to some point of* E.

Proof From Proposition 4.2 and Lemma 2.2, (i), $\{x^k\}$ is bounded, and then, there exists a subsequence $\{x^{k_j}\}$ such that $\lim_{j\to+\infty} x^{k_j} = \widehat{x}$. Since $\langle F(.), z\rangle$ is lower semicontinuous function for all $z \in \mathbb{R}^m_+\backslash\{0\}$, then $\langle F(\widehat{x}), z\rangle \le \liminf_{j\to+\infty}\langle F(x^{k_j}), z\rangle$. On the other hand, $x^{k+1} \in \Omega_k$ so $\langle F(x^{k+1}), z\rangle \le \langle F(x^k), z\rangle$. Furthermore, from assumption $(\mathbf{C_{1.2}})$ the function $\langle F(.), z\rangle$ is bounded below for each $z \in \mathbb{R}^m_+\backslash\{0\}$, and then, the sequence $\{\langle F(x^k), z\rangle\}$ is nonincreasing and bounded below, hence convergent. Therefore,

$$\langle F(\widehat{x}), z\rangle \le \liminf_{j\to+\infty}\left\langle F(x^{k_j}), z\right\rangle = \lim_{j\to+\infty}\left\langle F(x^{k_j}), z\right\rangle$$
$$= inf_{k\in\mathbb{N}}\left\{\left\langle F(x^k), z\right\rangle\right\} \le \left\langle F(x^k), z\right\rangle.$$

It follows that $\langle F(x^k) - F(\widehat{x}), z\rangle \ge 0, \forall\, k \in \mathbb{N}, \forall\, z \in \mathbb{R}^m_+\backslash\{0\}$. We conclude that $F(x^k) - F(\widehat{x}) \in \mathbb{R}^m_+$, i.e., $F(\widehat{x}) \preceq F(x^k), \forall\, k \in \mathbb{N}$. Therefore, $\widehat{x} \in E$ and by Lemma 2.2, (ii), we get the result. □

4.4 Convergence to a Weak Pareto Efficient Solution

In this subsection, assuming that F is also continuous and the sequence $\{\alpha_k\}$ converges to zero, then we obtain the convergence of the sequence $\{x^k\}$ to a weak Pareto efficient solution.

Theorem 4.2 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous vector function satisfying assumptions $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$ and $(\mathbf{C_3})$. If $\lim\limits_{k\to+\infty} \alpha_k = 0$ and the iterations are given in the form*

$$x^{k+1} \in arg\ min\left\{\langle F(x), z_k\rangle + \frac{\alpha_k}{2}\left\|x - x^k\right\|^2 : x \in \Omega_k\right\}, \tag{16}$$

then the sequence $\{x^k\}$ converges to a weak Pareto efficient solution of the problem (5).

Proof Let $x^{k+1} \in \arg\min\left\{\langle F(x), z_k\rangle + \frac{\alpha_k}{2}\left\|x - x^k\right\|^2 : x \in \Omega_k\right\}$, this implies that

$$\left\langle F(x^{k+1}), z_k\right\rangle + \frac{\alpha_k}{2}\left\|x^{k+1} - x^k\right\|^2 \le \langle F(x), z_k\rangle + \frac{\alpha_k}{2}\left\|x - x^k\right\|^2, \tag{17}$$

$\forall\, x \in \Omega_k$. Since the sequence $\{x^k\}$ converges to some point of E, then there exists $x^* \in E$ such that $\lim_{k\to+\infty} x^k = x^*$. Since that $\{z_k\}$ is bounded because $\|z_k\| = 1$, there exists a subsequence $\{z_{k_l}\}_{l\in\mathbb{N}}$ such that $\lim\limits_{l\to+\infty} z_{k_l} = \bar{z}$, with $\bar{z} \in \mathbb{R}^m_+\backslash\{0\}$ (as $||z_{k_l}|| = 1$ and from the continuity of the norm $||.||$ we have that $||\bar{z}|| = 1$ and so $\bar{z} \ne 0$). Taking $k = k_l$ in (17), we have

$$\left\langle F(x^{k_l+1}), z_{k_l}\right\rangle + \frac{\alpha_{k_l}}{2}\left\|x^{k_l+1} - x^{k_l}\right\|^2 \le \left\langle F(x), z_{k_l}\right\rangle + \frac{\alpha_{k_l}}{2}\left\|x - x^{k_l}\right\|^2. \tag{18}$$

$\forall\, x \in E$. As

$$\frac{\alpha_{k_l}}{2}\left\|x^{k_{l+1}} - x^{k_l}\right\|^2 \rightarrow 0 \text{ and } \frac{\alpha_{k_l}}{2}\left\|x - x^{k_l}\right\|^2 \rightarrow 0 \text{ when } l \rightarrow +\infty$$

and from the continuity of F, taking $l \rightarrow +\infty$ in (18), we obtain

$$\left\langle F(x^*), \bar{z}\right\rangle \leq \langle F(x), \bar{z}\rangle\,, \forall\, x \in E. \tag{19}$$

Thus, $x^* \in \arg\min\{\langle F(x), \bar{z}\rangle : x \in E\}$. Now, $\langle F(.), \bar{z}\rangle$, with $\bar{z} \in \mathbb{R}^m_+ \backslash \{0\}$ is a strict scalar representation of F, so a weak scalar representation, then by Proposition 2.3 we have that $x^* \in \arg\min_w \{F(x) : x \in E\}$.

We shall prove that $x^* \in \arg\min_w \{F(x) : x \in \mathbb{R}^n\}$. Suppose by contradiction that $x^* \notin \arg\min_w \{F(x) : x \in \mathbb{R}^n\}$, then there exists $\widetilde{x} \in \mathbb{R}^n$ such that

$$F(\widetilde{x}) \prec F(x^*). \tag{20}$$

So for $\bar{z} \in \mathbb{R}^m_+ \backslash \{0\}$, it follows that

$$\langle F(\widetilde{x}), \bar{z}\rangle < \left\langle F(x^*), \bar{z}\right\rangle. \tag{21}$$

Since $x^* \in E$, from (20) we conclude that $\widetilde{x} \in E$. Therefore, from (19) and (21) we obtain a contradiction. □

4.5 Convergence to a Generalized Critical Point

In this subsection, we prove the convergence of the sequence $\{g^k\}$ to 0, where

$$g^k \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1}).$$

We call the above result as convergence to a generalized critical point.

Theorem 4.3 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous vector function satisfying assumptions* $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$ *and* $(\mathbf{C_3})$. *If $0 < \alpha_k < \tilde{\alpha}$, then the sequence $\{x^k\}$ generated by the* **SPP** *algorithm,* (6) *and* (7)*, satisfies*

$$\lim_{k \rightarrow +\infty} g^k = 0,$$

where $g^k \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1})$.

Proof From Theorem 4.1, (7) and from Proposition 2.5, (iv), there exists a vector $g_k \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1})$ such that $g^k = \alpha_k(x^k - x^{k+1})$. Since $0 < \alpha_k < \tilde{\alpha}$, then

$$0 \leq \|g^k\| \leq \tilde{\alpha}\left\|x^k - x^{k+1}\right\|. \tag{22}$$

From Proposition 4.3, $\lim_{k\to+\infty} \left\|x^{k+1} - x^k\right\| = 0$, and from (22) we have $\lim_{k\to+\infty} g^k = 0$.
□

Remark 4.3 If F is continuously differentiable, the **SSP** algorithm coincides with the algorithm proposed in [1] and so we obtain the global convergence of the sequence.

5 An Inexact Proximal Algorithm

In this section, we present an inexact version of the **SPP** algorithm, which we denote by **ISPP** algorithm, when F satisfies the following assumption:

($\mathbf{C_4}$) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a continuously differentiable vector function on $\mathbb{R}^n$.

5.1 ISPP Algorithm

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector function satisfying the assumptions ($\mathbf{C_2}$) and ($\mathbf{C_4}$), and consider two sequences: the proximal parameters $\{\alpha_k\}$, with $\alpha_k > 0$ for each k, and a sequence $\{z_k\} = \{(z_k^1, z_k^2, \ldots, z_k^m)\} \subset \mathbb{R}^m_+ \backslash \{0\}$ with $\|z_k\| = 1$.

Initialization: Choose an arbitrary starting point

$$x^0 \in \mathbb{R}^n. \tag{23}$$

Main Steps: Given x^k, define the function $\Psi_k : \mathbb{R}^n \to \mathbb{R}$ such that $\Psi_k(x) = \langle F(x), z_k \rangle$ and consider $\Omega_k = \left\{x \in \mathbb{R}^n : F(x) \preceq F(x^k)\right\}$. Find $x^{k+1} \in \Omega_k$ satisfying

$$0 \in \hat{\partial}_{\epsilon_k} \Psi_k(x^{k+1}) + \alpha_k \left(x^{k+1} - x^k\right) + \nu_k, \tag{24}$$

$$\sum_{k=1}^{\infty} \delta_k < +\infty, \tag{25}$$

where $\nu_k \in \mathcal{N}_{\Omega_k}(x^{k+1})$, $\delta_k = \max\left\{\frac{\varepsilon_k}{\alpha_k}, \frac{\|\nu_k\|}{\alpha_k}\right\}$, $\varepsilon_k \geq 0$, and $\hat{\partial}_{\varepsilon_k}$ is the Fréchet ε_k-subdifferential.
Stop criterion: If $x^{k+1} = x^k$, then stop. Otherwise, do $k \leftarrow k+1$ and return to Main Steps.

5.2 Existence of the Iterates

Proposition 5.1 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector function satisfying the assumptions* ($\mathbf{C_{1.2}}$), ($\mathbf{C_2}$) *and* ($\mathbf{C_4}$)*. Then, the sequence $\left\{x^k\right\}$ generated by the* **ISPP** *algorithm is well defined.*

Proof Let $x^0 \in \mathbb{R}^n$ given by (23) and considering x^k fixed, analogous to the proof of Theorem 4.1, there exists x^{k+1} satisfying

$$0 \in \hat{\partial}\left(\Psi_k(.) + \frac{\alpha_k}{2}\| . - x^k\|^2 + \delta_{\Omega_k}(.)\right)(x^{k+1}).$$

From Proposition 2.5, (iii) and (iv), we obtain

$$0 \in \hat{\partial}\Psi_k(x^{k+1}) + \alpha_k\left(x^{k+1} - x^k\right) + \mathcal{N}_{\Omega_k}(x^{k+1}).$$

From Remark 2.1, x^{k+1} satisfies (24) with $\varepsilon_k = 0$. □

Remark 5.1 As in the exact algorithm, we are interested in the asymptotic convergence of the **ISPP** algorithm, so we assume that $x^k \neq x^{k+1}$ for all k. If $x^k = x^{k+1}$ for some k, then from the algorithm we have that

$$0 \in \nabla\Psi_k(x^{k+1}) + \mathcal{N}_{\Omega_k}(x^{k+1}) + \epsilon_k B(0, 1).$$

From Proposition 2.5, (iv), it is equivalent to

$$0 \in \hat{\partial}\left(\langle F(.), z_k\rangle + \delta_{\Omega_k}(.)\right)(x^{k+1}) + \epsilon_k B(0, 1),$$

that is, x^{k+1} is a approximate critical point of the optimization problem $\min\{\langle F(.), z_k\rangle : x \in \Omega_k\}$. Observe that the function $\langle F(.), z_k\rangle$ is a strict scalar representation of the map F, so if, furthermore, x^{k+1} is a minimizer of that optimization problem, then from Proposition 2.3, x^{k+1} is a weak Pareto optimal point of the original problem (5).

The next proposition gives a necessary condition for quasiconvex differentiable vector functions.

Proposition 5.2 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable quasiconvex vector function and $x, z \in \mathbb{R}^n$. If $F(x) \preceq F(z)$, then $\langle \nabla F_i(z), x - z\rangle \leq 0$, $\forall\, i \in \{1, \ldots, m\}$.*

Proof Since F is $\mathbb{R}^m_+$-quasiconvex, each F_i, $i = 1, \ldots, m$, is quasiconvex. Then, the result follows from the classical characterization of the scalar differentiable quasiconvex functions; see Mangasarian [13], p.134. □

Proposition 5.3 *Let $\left\{x^k\right\}$ be a sequence generated by the* **ISPP** *algorithm. If the assumptions* $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, $(\mathbf{C_4})$ *and* (25) *are satisfied, then for each $\hat{x} \in E$, $\{\left\|\hat{x} - x^k\right\|^2\}$ converges and $\{x^k\}$ is bounded.*

Proof From (24), there exist $g_k \in \hat{\partial}_{\varepsilon_k}\Psi_k(x^{k+1})$ and $\nu_k \in \mathcal{N}_{\Omega_k}(x^{k+1})$ such that

$$0 = g_k + \alpha_k\left(x^{k+1} - x^k\right) + \nu_k.$$

It follows that for any $x \in \mathbb{R}^n$, we obtain

$$\langle -g_k, x - x^{k+1}\rangle + \alpha_k\langle x^k - x^{k+1}, x - x^{k+1}\rangle = \langle \nu_k, x - x^{k+1}\rangle \leq \|\nu_k\|\|x - x^{k+1}\|.$$

Therefore,

$$\langle x^k - x^{k+1}, x - x^{k+1}\rangle \leq \frac{1}{\alpha_k}\left(\langle g_k, x - x^{k+1}\rangle + \|v_k\|\|x - x^{k+1}\|\right). \tag{26}$$

Note that $\forall\ x \in \mathbb{R}^n$, see equation (17) of [1]:

$$\left\|x - x^{k+1}\right\|^2 - \left\|x - x^k\right\|^2 \leq\ 2\left\langle x^k - x^{k+1}, x - x^{k+1}\right\rangle. \tag{27}$$

From (26) and (27), we obtain

$$\left\|x - x^{k+1}\right\|^2 - \left\|x - x^k\right\|^2 \leq \frac{2}{\alpha_k}\left(\langle g_k, x - x^{k+1}\rangle + \|v_k\|\|x - x^{k+1}\|\right). \tag{28}$$

On the other hand, let $\Psi_k(x) = \langle F(x), z_k\rangle$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable vector function, then $\Psi_k : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with gradient denoted by $\nabla\Psi_k$. From Proposition 2.6, we have

$$\partial_{\varepsilon_k}\Psi_k(x) = \nabla\Psi_k(x) + \varepsilon_k B, \tag{29}$$

where B is the closed unit ball in $\mathbb{R}^n$ centered at zero. Furthermore, $\hat{\partial}_{\varepsilon_k}\Psi_k(x) \subset \partial_{\varepsilon_k}\Psi_k(x)$, (see (2.12) in Jofré et al. [14]). As $g_k \in \hat{\partial}_{\epsilon_k}\Psi_k(x^{k+1})$, we have that $g_k \in \partial_{\epsilon_k}\Psi_k(x^{k+1})$, then

$$g_k = \nabla\Psi_k(x^{k+1}) + \varepsilon_k h_k,$$

with $\|h_k\| \leq 1$. Now take $\hat{x} \in$ E, then

$$\begin{aligned}\langle g_k, \hat{x} - x^{k+1}\rangle &= \left\langle \nabla\Psi_k(x^{k+1}) + \varepsilon_k h_k\ ,\ \hat{x} - x^{k+1}\right\rangle \\ &= \sum_{i=1}^{m}\left\langle \nabla F_i(x^{k+1})\ ,\ \hat{x} - x^{k+1}\right\rangle (z_k)_i + \varepsilon_k\left\langle h_k\ ,\ \hat{x} - x^{k+1}\right\rangle. \end{aligned}\tag{30}$$

From Proposition 5.2, we conclude that (30) becomes

$$\langle g_k, \hat{x} - x^{k+1}\rangle \leq \varepsilon_k\left\langle h_k\ ,\ \hat{x} - x^{k+1}\right\rangle \leq \varepsilon_k\|\hat{x} - x^{k+1}\|. \tag{31}$$

Using in the inequality $\|\hat{x} - x^{k+1}\|^2 + \frac{1}{4} \geq \|\hat{x} - x^{k+1}\|$, it follows

$$\|\hat{x} - x^{k+1}\| \leq \left(\|\hat{x} - x^{k+1}\|^2 + \frac{1}{4}\right). \tag{32}$$

Consider $x = \hat{x}$ in (28), using (31), (32) and the condition (25) we obtain

$$\left\|\hat{x} - x^{k+1}\right\|^2 - \left\|\hat{x} - x^k\right\|^2 \leq \frac{2}{\alpha_k}\left(\varepsilon_k + \|\nu_k\|\right)\|\hat{x} - x^{k+1}\| \\ \leq 4\delta_k \left\|\hat{x} - x^{k+1}\right\|^2 + \delta_k.$$

Thus,

$$\left\|\hat{x} - x^{k+1}\right\|^2 \leq \left(\frac{1}{1-4\delta_k}\right)\left\|\hat{x} - x^k\right\|^2 + \frac{\delta_k}{1-4\delta_k}. \tag{33}$$

The condition (25) guarantees that

$$\delta_k < \frac{1}{4}, \quad \forall k > k_0,$$

where k_0 is a natural number sufficiently large, and so,

$$1 \leq \frac{1}{1-4\delta_k} \leq 1 + 2\delta_k < 2, \quad \text{for } k \geq k_0,$$

combining with (33), results in

$$\left\|\hat{x} - x^{k+1}\right\|^2 \leq (1 + 2\delta_k)\left\|\hat{x} - x^k\right\|^2 + 2\delta_k. \tag{34}$$

Since $\sum_{i=1}^{\infty} \delta_k < \infty$, applying Lemma 2.1 in the inequality (34), we obtain the convergence of $\{\|\hat{x} - x^k\|^2\}$, for each $\hat{x} \in E$, which implies that there exists $M \in \mathbb{R}_+$, such that $\left\|\hat{x} - x^k\right\| \leq M, \ \ \forall\, k \in \mathbb{N}$. Now, since that $\|x^k\| \leq \|x^k - \hat{x}\| + \|\hat{x}\|$, we conclude that $\{x^k\}$ is bounded, and so, we guarantee that the set of accumulation points of this sequence is nonempty. □

5.3 Convergence of the ISPP Algorithm

Proposition 5.4 (Convergence to some Point of E) *If the assumptions* $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$ *and* $(\mathbf{C_4})$ *are satisfied, then the sequence* $\left\{x^k\right\}$ *generated by the* **ISPP** *algorithm converges to some point of the set* E.

Proof As $\left\{x^k\right\}$ is bounded, then there exists a subsequence $\left\{x^{k_j}\right\}$ such that $\lim_{j\to+\infty} x^{k_j} = \hat{x}$. Since F is continuous in $\mathbb{R}^n$, then the function $\langle F(.), z\rangle$ is also continuous in $\mathbb{R}^n$ for all $z \in \mathbb{R}^m$, in particular, for all $z \in \mathbb{R}^m_+\backslash\{0\}$, and $\langle F(\widehat{x}), z\rangle = \lim_{j\to+\infty}\left\langle F(x^{k_j}), z\right\rangle$. On the other hand, we have that $F(x^{k+1}) \preceq F(x^k)$, and so, $\left\langle F(x^{k+1}), z\right\rangle \leq \left\langle F(x^k), z\right\rangle$ for all $z \in \mathbb{R}^m_+\backslash\{0\}$. Furthermore, the function $\langle F(.), z\rangle$ is bounded below, for each $z \in \mathbb{R}^m_+\backslash\{0\}$, then the sequence $\left\{\left\langle F(x^k), z\right\rangle\right\}$ is nonincreasing and bounded below, thus convergent. So,

$$\langle F(\widehat{x}), z\rangle = \lim_{j\to+\infty}\left\langle F(x^{k_j}), z\right\rangle = \lim_{k\to+\infty}\left\langle F(x^k), z\right\rangle = inf_{k\in\mathbb{N}}\left\{\left\langle F(x^k), z\right\rangle\right\}$$
$$\leq \left\langle F(x^k), z\right\rangle, \forall k\in\mathbb{N}.$$

It follows that $F(x^k) - F(\widehat{x}) \in \mathbb{R}^m_+$, i.e., $F(\widehat{x}) \preceq F(x^k)$, $\forall\, k\in\mathbb{N}$. Therefore, $\widehat{x}\in E$. Now, from Proposition 5.3, we have that the sequence $\{\|\widehat{x} - x^k\|\}$ is convergent, and since $\lim_{k\to+\infty}\|x^{k_j} - \hat{x}\| = 0$ (because $\{x^{k_j}\}$ converges to $\hat{x}$), we conclude that $\lim_{k\to+\infty}\|x^k - \widehat{x}\| = 0$, i.e., $\lim_{k\to+\infty} x^k = \widehat{x}$. □

Theorem 5.1 *Suppose that the assumptions* $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$ *and* $(\mathbf{C_4})$ *are satisfied. If* $0 < \alpha_k < \widetilde{\alpha}$, *then the sequence* $\{x^k\}$ *generated by the* **ISPP** *algorithm*, (23), (24) *and* (25), *converges to a Pareto critical point of the problem* (5).

Proof From Proposition 5.4 there exists $\widehat{x}\in E$ such that $\lim_{j\to+\infty} x^k = \widehat{x}$. Furthermore, as the sequence $\{z^k\}$ is bounded, there exists $\{z^{k_j}\}_{j\in\mathbb{N}}$ such that $\lim_{j\to+\infty} z^{k_j} = \bar{z}$, with $\bar{z}\in\mathbb{R}^m_+\setminus\{0\}$. From (24) there exists $g_{k_j}\in\hat{\partial}_{\varepsilon_{k_j}}\Psi_{k_j}(x^{k_j+1})$, with $g_{k_j} = \nabla\Psi_{k_j}(x^{k_j+1}) + \varepsilon_{k_j}h_{k_j}$ with $\|h_{k_j}\|\leq 1$, and $\nu_{k_j}\in\mathcal{N}_{\Omega_{k_j}}(x^{k_j+1})$, such that

$$0 = \sum_{i=1}^m \nabla F_i(x^{k_j+1})(z_{k_j})_i + \varepsilon_{k_j}h_{k_j} + \alpha_{k_j}\left(x^{k_j+1} - x^{k_j}\right) + \nu_{k_j}. \tag{35}$$

Since $\nu_{k_j}\in\mathcal{N}_{\Omega_{k_j}}(x^{k_j+1})$, then

$$\left\langle \nu_{k_j}\,,\; x - x^{k_j+1}\right\rangle \leq\; 0,\;\; \forall\, x\in\Omega_{k_j}. \tag{36}$$

Take $\bar{x}\in E$. By definition of E, $\bar{x}\in\Omega_k$, for all $k\in\mathbb{N}$, so $\bar{x}\in\Omega_{k_j}$. Combining (36) with $x=\bar{x}$ and (35), we have

$$\begin{aligned}0 &\leq \left\langle \sum_{i=1}^m \nabla F_i(x^{k_j+1})(z_{k_j})_i\;,\;\bar{x} - x^{k_j+1}\right\rangle + \varepsilon_{k_j}\left\langle h_{k_j}\;,\;\bar{x} - x^{k_j+1}\right\rangle\\ &\quad + \alpha_{k_j}\left\langle x^{k_j+1} - x^{k_j}\;,\;\bar{x} - x^{k_j+1}\right\rangle\\ &\leq \left\langle \sum_{i=1}^m \nabla F_i(x^{k_j+1})(z_{k_j})_i\;,\;\bar{x} - x^{k_j+1}\right\rangle + \varepsilon_{k_j}M\\ &\quad + \tilde{\alpha}|\langle x^{k_j+1} - x^{k_j}\;,\;\bar{x} - x^{k_j+1}\rangle|,\end{aligned} \tag{37}$$

where M is a constant such that $\left\langle h_{k_j}\;,\;\bar{x} - x^{k_j+1}\right\rangle \leq M$.

Observe that, $\forall\, x \in \mathbb{R}^n$:

$$\left\|x^{k+1} - x^k\right\|^2 = \left\|x - x^k\right\|^2 - \left\|x - x^{k+1}\right\|^2 + 2\left\langle x^k - x^{k+1}, x - x^{k+1}\right\rangle. \quad (38)$$

Now, from (26) with $x = \bar{x} \in E$, and (31), we obtain

$$\left\langle x^k - x^{k+1}, \bar{x} - x^{k+1}\right\rangle \leq \left\|\bar{x} - x^{k+1}\right\| \left(\frac{\varepsilon_k}{\alpha_k} + \frac{\|\nu_k\|}{\alpha_k}\right) \leq 2M\delta_k.$$

Thus, from (38), with $x = \bar{x}$, we have

$$0 \leq \left\|x^{k+1} - x^k\right\|^2 \leq \left\|\bar{x} - x^k\right\|^2 - \left\|\bar{x} - x^{k+1}\right\|^2 + 4M\delta_k. \quad (39)$$

Since that the sequence $\left\{\left\|\bar{x} - x^k\right\|\right\}$ is convergent and $\sum_{i=1}^{\infty} \delta_k < \infty$, from (39) we conclude that $\lim_{k\to+\infty} \left\|x^{k+1} - x^k\right\| = 0$. Furthermore, as

$$0 \leq \left\|x^{k_j+1} - \bar{x}\right\| \leq \|x^{k_j+1} - x^{k_j}\| + \|x^{k_j} - \bar{x}\|, \quad (40)$$

we obtain that the sequence $\{\left\|\bar{x} - x^{k_j+1}\right\|\}$ is bounded.
Thus, returning to (37), since $\lim_{k\to+\infty} \varepsilon_k = 0$, $\lim_{j\to+\infty} x^k = \widehat{x}$ and $\lim_{j\to+\infty} z^{k_j} = \bar{z}$, taking $j \to +\infty$, we obtain

$$\sum_{i=1}^{m} \bar{z}_i \left\langle \nabla F_i(\widehat{x})\ ,\ \bar{x} - \widehat{x}\right\rangle \geq 0. \quad (41)$$

From the quasiconvexity of each component function F_i, for each $i \in \{1, \ldots, m\}$, we have that
$\left\langle \nabla F_i(\widehat{x})\ ,\ \bar{x} - \widehat{x}\right\rangle \leq 0$ and because $\bar{z} \in \mathbb{R}^m_+ \backslash \{0\}$, from (41), we obtain

$$\sum_{i=1}^{m} \bar{z}_i \left\langle \nabla F_i(\widehat{x})\ ,\ \bar{x} - \widehat{x}\right\rangle = 0. \quad (42)$$

Without loss of generality, consider the set $J = \{i \in I : \bar{z}_i > 0\}$, where $I = \{1, \ldots, m\}$. Thus, from (42), for all $\bar{x} \in E$ we have

$$\left\langle \nabla F_i(\widehat{x})\ ,\ \bar{x} - \widehat{x}\right\rangle = 0,\ \forall\ i \in J. \quad (43)$$

Now we will show that $\widehat{x}$ is a Pareto critical point.
Suppose by contradiction that $\widehat{x}$ is not a Pareto critical point, then there exists a direction $v \in \mathbb{R}^n$ such that $JF(\widehat{x})v \in -\mathbb{R}^m_{++}$, i.e.,

$$\left\langle \nabla F_i(\widehat{x}), v\right\rangle < 0, \forall\, i \in \{1, \ldots, m\}. \quad (44)$$

Therefore, v is a descent direction for the multiobjective function F in $\widehat{x}$, so, $\exists\, \varepsilon > 0$ such that

$$F(\widehat{x} + \lambda v) \prec F(\widehat{x}),\ \forall\, \lambda \in (0, \varepsilon]. \tag{45}$$

Since $\widehat{x} \in E$, then from (45) we conclude that $\widehat{x} + \lambda v \in E$. Thus, from (43) with $\bar{x} = \widehat{x} + \lambda v$, we obtain:

$$\langle \nabla F_i(\widehat{x})\ ,\ \widehat{x} + \lambda v - \widehat{x} \rangle = \langle \nabla F_i(\widehat{x})\ ,\ \lambda v \rangle = \lambda \langle \nabla F_i(\widehat{x})\ ,\ v \rangle = 0.$$

It follows that $\langle \nabla F_i(\widehat{x})\ ,\ v \rangle = 0$ for all $i \in J$, contradicting (44). Therefore, $\widehat{x}$ is Pareto critical point of the problem (5). □

6 A Numerical Result

In this subsection, we give a simple numerical example of the **SPP** algorithm showing the functionality of the proposed method. For that, we use a Intel Core i5 computer 2.30 GHz, 3GB of RAM, Windows 7 as operational system with SP1 64 bits and we implement our code using MATLAB software 7.10 (R2010a).

Example 6.1 Consider the following multiobjective minimization problem

$$\min \left\{ (F_1(x_1, x_2), F_2(x_1, x_2)) : (x_1, x_2) \in \mathbb{R}^2 \right\}$$

where $F_1(x_1, x_2) = -e^{-x_1^2 - x_2^2} + 1$ and $F_2(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2$. This problem satisfies the assumptions $(\mathbf{C_{1.2}})$, $(\mathbf{C_2})$ and $(\mathbf{C_4})$. We can easily verify that the points $\bar{x} = (0, 0)$ and $\hat{x} = (1, 2)$ are Pareto efficient solutions of the problem.

We take $x^0 = (-1, 3)$ as an initial point and given $x^k \in \mathbb{R}^2$, the main step of the **SPP** algorithm is to find a critical point (local minimum, local maximum or a saddle point) of the following problem

$$\begin{aligned} \min g(x_1, x_2) = (-e^{-x_1^2 - x_2^2} + 1) z_1^k &+ \left((x_1 - 1)^2 + (x_2 - 2)^2 \right) z_2^k \\ &+ \frac{\alpha_k}{2} \left((x_1 - x_1^k)^2 + (x_2 - x_2^k)^2 \right) \end{aligned}$$

s.t :

$$\begin{aligned} &x_1^2 + x_2^2 \leq (x_1^k)^2 + (x_2^k)^2 \\ &(x_1 - 1)^2 + (x_2 - 2)^2 \leq (x_1^k - 1)^2 + (x_2^k - 2)^2. \end{aligned}$$

In this example, we consider $z_k = \left(z_1^k, z_2^k \right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ and $\alpha_k = 1$, for each k. We take $z^0 = (2, 3)$ as the initial point to solve all the subproblems using the MATLAB function fmincon (with interior point algorithm), and we consider the stop criterion

k	$N[x^k]$	$x^k = (x_1^k, x_2^k)$	$\|\|x^k - x^{k-1}\|\|$	$\sum F_i(x^k) z_i^k$	$F_1(x_1^k, x_2^k)$	$F_2(x_1^k, x_2^k)$
1	10	(0.17128, 2.41010)	1.31144	1.30959	0.99709	0.85496
2	10	(0.65440, 2.16217)	0.54302	0.80586	0.99392	0.14574
3	9	(0.85337, 2.05877)	0.22423	0.71983	0.99303	0.02496
4	7	(0.93534, 2.01588)	0.09251	0.70518	0.99284	0.00443
5	7	(0.96912, 1.99814)	0.03816	0.70268	0.99279	0.00096
6	7	(0.98305, 1.99080)	0.01574	0.70226	0.99277	0.00037
7	7	(0.98879, 1.98776)	0.00649	0.70219	0.99277	0.00028
8	7	(0.99115,1.98651)	0.00268	0.70217	0.99276	0.00026
9	7	(0.99213, 1.98599)	0.00110	0.70217	0.99276	0.00026
10	7	(0.99253, 1.98578)	0.00046	0.70217	0.99276	0.00026
11	7	(0.99270, 1.98569)	0.00019	0.70217	0.99276	0.00026
12	7	(0.99277,1.98565)	0.00008	0.70217	0.99276	0.00026

$||x^{k+1} - x^k|| < 0.0001$ to finish the algorithm. The numerical results are given in the following table:
The above table shows that we need $k = 12$ iterations to solve the problem; $N[x^k]$ denotes the inner iterations of each subproblem to obtain the point x^k, for example to obtain the point $x^3 = (0.85337, 2.05877)$ we need $N[x^3] = 9$ inner iterations. Observe also that in each iteration, we obtain $F(x^k) \succeq F(x^{k+1})$ and the function $\langle F(x^k), z^k \rangle$ is nonincreasing.

7 Perspectives and Open Problems

To reduce considerably the computational cost in each iteration of the **SPP** algorithm, it is necessary to consider the unconstrained iteration

$$0 \in \hat{\partial} \left(\langle F(.), z_k \rangle + \frac{\alpha_k}{2} \| \, . \, - x^k \|^2 \right) (x^{k+1}), \tag{46}$$

which is more practical than (7). One natural condition to obtain (46) is that $x^{k+1} \in \text{int}\Omega_k$ (interior of Ω_k), because in this case (7) becomes (46). So, we believe that a variant of the **SPP** algorithm may be an interior proximal point method. Thus, a future work may be the introduction of an interior variable metric proximal point method to solve the problem (5).

On the other hand, the **SPP** algorithm may be applied to solve a class of linear multiobjective problems. In fact, if the objective functions F_i, $i = 1, 2, \ldots, m$, are defined as $F_i(x) = {c_i}^T x$, if x satisfies $Ax = b$, $x \in \mathbb{R}^n_+$ and $F_i(x) = +\infty$ otherwise, where $c_i \in \mathbb{R}^n_+$, $b \in \mathbb{R}^m$ are given vectors and $A \in \mathbb{R}^{m \times n}$ is a $m \times n$ matrix, then Theorem 4.2 assures that the sequence $\{x^k\}$ converges to a weak Pareto efficient solution of the problem

$$\min\{F(x) = (c_1^T x, c_2^T x, \ldots, c_m^T x) : Ax = b, x \in \mathbb{R}^n_+\}.$$

The extension of the above theorem when $\{\lambda_k\}$ does not converge to zero is an open question.

Other future works may be the extension of the proposed algorithm to solve more general constrained vector minimization problems using the class of proximal distances and also to obtain conditions for the finite convergence of the **SPP** algorithm for the quasiconvex case.

8 Conclusions

This paper introduces an exact linear scalarization proximal point algorithm, denoted by **SPP** algorithm, to solve arbitrary extended multiobjective quasiconvex minimization problems and in the differentiable case, it is presented an inexact version of the proposed algorithm.

Our paper may be considered as a first attempt to develop a proximal point method to solve constrained multiobjective problems with quasiconvex objective functions. Future works should improve the result obtained in Theorem 4.3 targeting real applications, for example, in demand theory.

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