On the Structure of Higher Order Voronoi Cells

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Abstract

The classic Voronoi cells can be generalized to a higher order version by considering the cells of points for which a given k-element subset of the set of sites consists of the k closest sites. We study the structure of the k-order Voronoi cells and illustrate our theoretical findings with a case study of two-dimensional higher order Voronoi cells for four points.

Keywords Higher order Voronoi cells · Structure of Voronoi cells

1 Introduction

The classic Voronoi cells partition the Euclidean space into polyhedral regions that consist of points nearest to one of the sites from a given finite set. We consider higher order (or multipoint) Voronoi cells that correspond to the subsets of points nearest to k several sites (see an illustration in Fig. 1).

To our best knowledge, the earliest mention of k-point Voronoi cells appears in [1], where a tessellation of the plane by such cells was called the *Voronoi diagram of order k*; that paper also provides bounds on the number of nonempty cells in a plane and complexity estimates for the construction of such diagrams; in [2], the complexity of constructing the higher order diagrams for line segments was studied.

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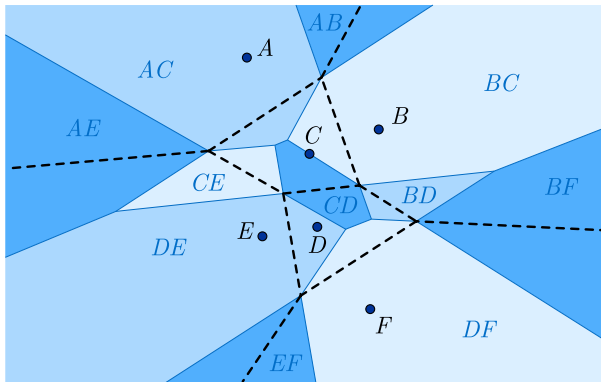[4] Institute of Mathematics and Informatics, BAS, Sofia, Bulgaria

Fig. 1 Classic and order two Voronoi diagrams on six sites (shown in dashed lines and shaded regions, respectively). Notice that some two-point combinations generate empty cells

The multipoint or k-order Voronoi diagrams discussed in this paper are one possible way to generalize the classic construction. Some notable generalizations are the cells of more general sets [2,3] , the use of non-Euclidean metrics [4–6] and the abstract cells that are defined via manifold partitions of the space rather than distance relations [7].

Much of the recent work mentioned above is focussed on the algorithmic complexity of constructing planar Voronoi diagrams of various types. In this paper, we rather focus on the *structure* of multipoint Voronoi cells and in particular obtain constructive characterizations of cells with nonempty interior, also of bounded and empty cells. We use these results for a case study of multipoint cells defined on at most four sites. We prove that—perhaps counterintuitively—some convex polygons, including triangles and cyclic quadrilaterals, cannot be such cells, and provide explicit algorithms for the construction of sites for a given cell in other cases.

Finally, we would like to mention a wealth of emerging application of higher order Voronoi cells, predominantly driven by the recent advancements in big data and mobile sensor technology. For instance, in [8] such cells are utilized in a numerical technique for smoothing point clouds from experimental data; in [9], k-order cells are used for detecting and rectifying coverage problems in wireless sensor networks; in [10], the k-order diagrams are used to analyze coalitions in the US supreme court voting decisions. A well-known application of the higher order Voronoi cells is in a k-nearest neighbor problem in spatial networks [11]; however, the practical implementations are limited due to the complexity of higher order diagrams and the lack of readily available software.

Our work is organized as follows. In Sect. 2, we study the structure of higher order Voronoi cells and in particular prove the conditions for the cell to be bounded and nonempty. In Sect. 3, we study the special case of higher order cells on no more than four sites. We will refer to the higher order cells as multipoint cells, to highlight the discrete nature of our construction.

2 High-Order Voronoi Cells in $\mathbb{R}^n$

Let $T \subset \mathbb{R}^n$ be finite and nonempty. For a nonempty proper subset $S \subset T$, we define the multipoint Voronoi cell as the set of points that are not farther from each point of S than from each point of $T \backslash S$,

$$V_T(S) := \left\{ x \in \mathbb{R}^n \; : \; \max_{s \in S} \operatorname{dist}(s, x) \leq \min_{t \in T \backslash S} \operatorname{dist}(t, x) \right\},$$

where $\operatorname{dist}(x, y)$ is the Euclidean distance function. When S is a singleton, i.e., $S = \{s\}$, the set $V_T(S)$ is a classic Voronoi cell. We abuse the notation slightly and write $V_T(s) := V_T(\{s\})$. It is evident that

$$V_T(S) = \bigcap_{s \in S} V_{\{s\} \cup (T \backslash S)}(s). \tag{1}$$

It is not difficult to observe that each multipoint Voronoi cell is a convex polyhedron, i.e., the intersection of finitely many closed half-spaces, since each cell is defined by finitely many linear inequalities. Explicitly, we have the following representation.

Proposition 2.1 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. Then, $V_T(S)$ is the intersection of $|S|(|T| - |S|)$ closed half-spaces:*

$$V_T(S) = \bigcap_{\substack{s \in S \\ t \in T \backslash S}} \left\{ x \in \mathbb{R}^n \; : \; \langle t - s, x \rangle \leq \frac{1}{2} \left(\|t\|^2 - \|s\|^2 \right) \right\}. \tag{2}$$

Proof Observe that, from the definition,

$$
\begin{aligned}
V_T(S) &= \left\{ x \in \mathbb{R}^n \; : \; \max_{s \in S} \operatorname{dist}(x, s) \leq \min_{t \in T \backslash S} \operatorname{dist}(x, t) \right\} \\
&= \bigcap_{s \in S} \left\{ x \in \mathbb{R}^n \; : \; \operatorname{dist}(x, s) \leq \min_{t \in T \backslash S} \operatorname{dist}(x, t) \right\} \\
&= \bigcap_{s \in S} \bigcap_{t \in T \backslash S} \left\{ x \in \mathbb{R}^n \; : \; \operatorname{dist}(x, s) \leq \operatorname{dist}(x, t) \right\}.
\end{aligned}
$$

Explicitly for the Euclidean distance function, we have

$$\operatorname{dist}(x, s) \leq \operatorname{dist}(x, t) \quad \Leftrightarrow \quad \|x\|^2 - 2\langle s, x \rangle + \|s\|^2 \leq \|x\|^2 - 2\langle t, x \rangle + \|t\|^2, \tag{3}$$

from where the desired representation follows.   □

As a consequence of a well-known necessary and sufficient condition for the inconsistency of an arbitrary system of linear inequalities [12, Theorem 4.4(i)], from (2) we obtain the following characterization of empty Voronoi cells.

Theorem 2.1 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. Then,*

$$V_T(S) = \emptyset$$

iff

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \mathrm{cone}\left\{ \begin{pmatrix} t - s \\ \|t\|^2 - \|s\|^2 \end{pmatrix}, \ s \in S, \ t \in T \backslash S \right\}. \tag{4}$$

We use the characterization in Theorem 2.1 to obtain two well-known statements about the classic Voronoi cells.

Corollary 2.1 *Let T be a finite subset of $\mathbb{R}^n$ with $|T| \geq 2$. Then,*

$$V_T(T \backslash \{t\}) \neq \emptyset$$

iff t is an extreme point (vertex) of $\mathrm{conv}\, T$.

Proof Suppose that there exists $s \in T$ such that $V_T(s) = \emptyset$. Then, (4) holds for $S = \{s\}$, and therefore there exist $\lambda_t \geq 0$ for $t \in T \backslash \{s\}$ such that

$$\sum_{t \in T \backslash \{s\}} \lambda_t (t - s) = 0_n \quad \text{and} \quad \sum_{t \in T \backslash \{s\}} \lambda_t \left(\|t\|^2 - \|s\|^2 \right) = -1,$$

whereby

$$\lambda_0 := \sum_{t \in T \backslash \{s\}} \lambda_t > 0 \quad \text{and} \quad s = \sum_{t \in T \backslash \{s\}} \frac{\lambda_t}{\lambda_0} t.$$

Since the square of the Euclidean norm $\|.\|^2$ is a strictly convex function, we have

$$\sum_{t \in T \backslash \{s\}} \lambda_t \|t\|^2 + 1 = \sum_{r \in T \backslash \{s\}} \lambda_r \|s\|^2 = \lambda_0 \|s\|^2 < \lambda_0 \sum_{t \in T \backslash \{s\}} \frac{\lambda_t}{\lambda_0} \|t\|^2$$

$$= \sum_{t \in T \backslash \{s\}} \lambda_t \|t\|^2,$$

which is a contradiction. Now, suppose that $V_T(T \backslash \{t\}) = \emptyset$. Denote $S := T \backslash \{t\}$. From Theorem 2.1, there exist $\lambda_s \geq 0$ for $s \in S$ such that

$$\sum_{s \in S} \lambda_s (t - s) = 0_n \quad \text{and} \quad \sum_{s \in S} \lambda_s \left(\|t\|^2 - \|s\|^2 \right) = -1,$$

whereby

$$\sum_{s \in S} \lambda_s > 0 \quad \text{and} \quad t = \sum_{s \in S} \frac{\lambda_s}{\sum_{r \in S} \lambda_r} s.$$

Hence, t is not an extreme point. If t is not an extreme point, then there exist $\lambda_s \geq 0$, $s \in S$ with $\sum_{s \in S} \lambda_s = 1$ and $t = \sum_{s \in S} \lambda_s s$. Since $\|.\|^2$ is a strictly convex function,

$$\|t\|^2 < \sum_{s \in S} \lambda_s \|s\|^2;$$

therefore, setting $\mu_s := \frac{\lambda_s}{\sum_{r \in S} \lambda_r \|r\|^2 - \|t\|^2}$ for $s \in S$, we have $\mu_s \geq 0$,

$$\sum_{s \in S} \mu_s (t - s) = 0_n \text{ and } \sum_{s \in S} \mu_s \left(\|t\|^2 - \|s\|^2 \right) = -1,$$

showing that (4) holds, which, in view of Theorem 2.1, proves the corollary.   □

The following result generalizes the "if" statement in the last part of Corollary 2.1.

Corollary 2.2 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. If $(\text{conv } S) \cap (T \backslash S) \neq \emptyset$, then $V_T(S) = \emptyset$.*

Proof Taking $t \in (\text{conv } S) \cap (T \backslash S)$, since t is not an extreme point of conv S, by Corollary 2.1 we have $V_T(S) \subseteq V_{S \cup \{t\}}(S) = \emptyset$.   □

In fact, we can prove a more general geometric statement which yields the preceding corollary.

Theorem 2.2 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. Then,*

$$V_T(S) \neq \emptyset$$

iff there exists a closed Euclidean ball $B \supset S$ such that

$$\text{int } B \cap (T \backslash S) = \emptyset.$$

Proof First, assume that $V_T(S) \neq \emptyset$. Then, there exists $c \in V_T(S)$. Let

$$R := \max_{s \in S} \text{dist}(c, s),$$

and let B be the closed Euclidean ball of radius R centered at c, clearly $S \subset B$. Evidently, int $B \cap (T \backslash S) = \emptyset$, otherwise we would have $\text{dist}(c, t) < \text{dist}(c, s)$ for some $t \in T \backslash S$ and $s \in S$, hence, $c \notin V_T(S)$, which contradicts our choice of c. Now assume that there exists some closed Euclidean ball B such that $S \subset B$ and int $B \cap (T \backslash S) = \emptyset$. The center of the ball B is contained in $V_T(S)$, hence, $V_T(S) \neq \emptyset$.   □

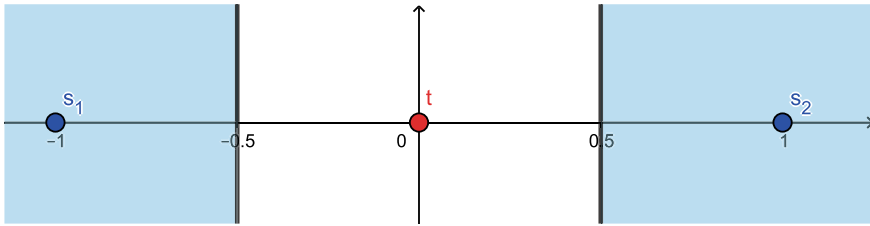We give an explicit example of an empty cell with $|S| = 2$ and $|T| = 3$.

Fig. 2 Minimal configuration for an empty cell

Example 2.1 Let $s_1 = (-1, 0)$, $s_2 = (1, 0)$, $t = (0, 0)$. It is not difficult to observe that (2) becomes

$$V_T(S) = \left\{ (x_1, x_2) : x_1 \leq -\frac{1}{2}, x_1 \geq \frac{1}{2} \right\} = \emptyset.$$

This configuration is shown in Fig. 2.

Notice that we can likewise construct an empty cell with $|T| > |S| \geq 2$ by making sure that $(T \backslash S) \cap \text{conv } S \neq \emptyset$ (using Corollary 2.2).

Theorem 2.3 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. Then, $V_T(S)$ is bounded iff*

$$\text{cone } \{t - s, s \in S, t \in T \backslash S\} = \mathbb{R}^n.$$

Proof It suffices to observe that from the linear representation (2) we obtain that the first moment cone of $V_T(S)$ is $\text{cone } \{t - s, \ s \in S, t \in T \backslash S\}$. □

Remark 2.1 It follows from Theorem 2.3 that if $n \geq 2$ and $|T| \leq 3$ all nonempty cells are unbounded.

We can strengthen the result in the preceding remark as follows.

Theorem 2.4 *Let T be a finite subset of $\mathbb{R}^n$. If*

$$|T| < 2\sqrt{n + 1}, \tag{5}$$

then for any $S \subset T$ the cell $V_T(S)$ is either empty or unbounded.

Proof Assume that $V_T(S) = \emptyset$. The proof is based on the observation that a nonempty bounded polyhedron in $\mathbb{R}^n$ must be defined by at least $n + 1$ inequalities. Let $p := |T|$, $k := |S|$. Then, the number of inequalities that feature in the representation (2) is $\phi(k) = k(p - k)$. Observe that ϕ attains its maximum at $\frac{p}{2}$ for even p and at $\frac{p-1}{2}$ for odd p. Hence, for even p

$$k(p - k) \leq \frac{p}{2} \left(p - \frac{p}{2} \right) = \frac{p^2}{4} = \left\lfloor \frac{p^2}{4} \right\rfloor,$$

and for odd p

$$k(p-k) \leq \frac{p-1}{2} \left(p - \frac{p-1}{2} \right) = \frac{p^2-1}{4} = \left\lfloor \frac{p^2}{4} \right\rfloor,$$

hence, ensuring (5) yields at most n inequalities that define each cell, and so all nonempty cells are unbounded. □

Proposition 2.2 *Let $S \subset T \subset \mathbb{R}^n$, with $|S| = 3$ and $|T| = 4$. Then, $V_T(S)$ is either empty or unbounded.*

Proof Put $T \backslash S = \{t\}$. In case $t \in \mathrm{conv}\, S$, then Corollary 2.1 gives that $V_T(S) = \emptyset$. If $t \notin \mathrm{conv}\, S$, then t can be separated from S, and by Theorem 2.3 the cell has to be unbounded. □

The following statement will be useful later for a discussion on planar quadrilateral cells.

Proposition 2.3 *Let $S \subset T \subset \mathbb{R}^2$, with $S = \{s_1, s_2\}$ and $T \backslash S = \{t_1, t_2\}$. Then*

$$V_T(S) \text{ is bounded iff } (s_1, s_2) \cap (t_1, t_2) \text{ is a singleton.}$$

Proof The configuration of the points of T in Fig. 3 means that if we take the line through s_1 and s_2, then t_1 and t_2 belong to the two opposite open half-spaces defined by this line. The same holds true if we interchange s_1 and s_2 with t_1 and t_2. From the linear representation (2), we obtain that the first moment cone M of $V_T(S)$ is equal to cone $\{t_i - s_j, i, j = 1, 2\}$. Let us consider that the configuration of the points of T is like in Fig. 3. Then, we are going to prove that $M = \mathbb{R}^2$, which, by Theorem 2.3, implies that $V_T\{S\}$ is bounded. What we are actually going to prove is the equivalent assertion that the polar cone $M°$ reduces to $\{0_2\}$. To this aim, let $p \in M°$ and assume, w.l.o.g., that $(s_1, s_2) \cap (t_1, t_2) = \{0_2\}$. Then, there exist $\lambda, \mu > 0$ such that $s_2 = -\lambda s_1$ and $t_2 = -\mu t_1$. Since $\langle p, t_1 - s_1 \rangle \leq 0$ and $\langle p, t_1 + \lambda s_1 \rangle = \langle p, t_1 - s_2 \rangle \leq 0$, we have $\langle p, t_1 \rangle \leq 0$. This inequality combined with $\langle p, -\mu t_1 - s_1 \rangle = \langle p, t_2 - s_1 \rangle \leq 0$ yields $\langle p, s_1 \rangle \geq 0$; hence, in view of $\langle p, -\mu t_1 + \lambda s_1 \rangle = \langle p, t_2 - s_2 \rangle \leq 0$, it turns out that $\langle p, t_1 \rangle = 0 = \langle p, s_1 \rangle$. Since s_1 and t_1 are linearly independent because of the assumption $(s_1, s_2) \cap (t_1, t_2) = \{0_2\}$, we conclude that $p = 0$, as was to be proved. Second, let the Voronoi cell $V_T\{s_1, s_2\}$ be bounded. Then, by Theorem 2.3, the first moment cone cone $\{t_i - s_j, i, j = 1, 2\}$ is the whole of $\mathbb{R}^2$. This implies that t_1 and t_2 are not on a common closed halfplane out of the two determined by the straight line through s_1 and s_2, as otherwise that cone would be contained in the translate of that half-space with the boundary line passing through the origin, and the same assertion holds true when we interchange t_1 and t_2 with s_1 and s_2. This rules out the possibility that conv $\{s_1, s_2, t_1, t_2\}$ be a segment, a triangle, or a quadrilateral having s_1 and s_2 as adjacent vertices. Therefore, s_1 and s_2 are opposite vertices of the quadrilateral conv $\{s_1, s_2, t_1, t_2\}$, which clearly implies that $(s_1, s_2) \cap (t_1, t_2)$ is a singleton. The proof is completed. □
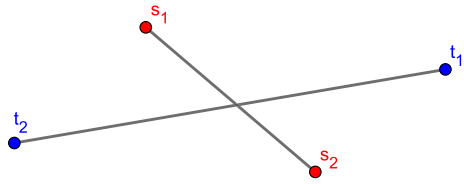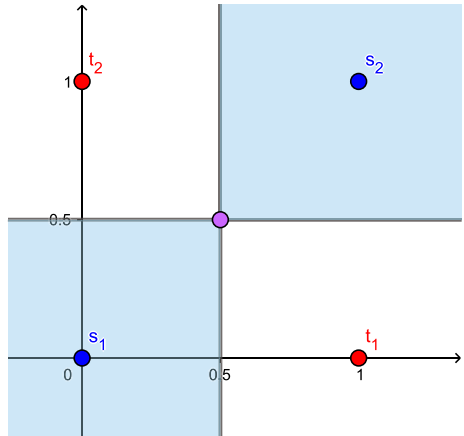
Fig. 3 Configuration for a
bounded cell

Fig. 4 A singleton cell (the
intersection of the two shaded
regions)

Theorem 2.5 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. Then,*

$$\text{int } V_T(S) \neq \emptyset$$

iff

$$0_{n+1} \notin \text{conv}\left\{\begin{pmatrix} t - s \\ \|t\|^2 - \|s\|^2 \end{pmatrix}, \quad s \in S, \ t \in T \backslash S\right\}.$$

Proof The proof comes from the well-known characterization of the Slater condition for a linear system of inequalities [13, Theorem 3.1]. □

Example 2.2 Consider a system T of four points in the plane,

$$T = \{(0, 0), (1, 1), (1, 0), (0, 1)\}, \quad S = \{(0, 0), (1, 1)\}.$$

This is illustrated in Fig. 4. Using Theorems 2.1, 2.3 and 2.5, it is easy to check that $V_T(S)$ is nonempty and bounded, but int $V_T(S) = \emptyset$. Indeed, $V_T(S) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

The next statement is a specific characterization for a three-point system, which we will use in what follows.

Proposition 2.4 *Let $T, T' \subset \mathbb{R}^n$ be such that $|T| = |T'| = 3$ and T and T' differ by exactly one point (i.e., $|T \cap T'| = 2$). Let $s \in T \backslash T'$ and $s' \in T' \backslash T$. If*

$$V_T(s) \subseteq V_{T'}(s'),$$

then all points in the set $T' \cup T$ belong to the same straight line.

Proof For notational convenience, we will prove the result for $T = \{t_1, t_2, 0\}$ and $T' = \{t_1, t_2, s\}$, where the points t_1, t_2, s are all nonzero and pairwise distinct. Let

$$F := V_T(0) = \left\{ x \in \mathbb{R}^n \ : \ \langle t_j, x \rangle \leq \frac{1}{2} \left\| t_j \right\|^2, \ j = 1, 2 \right\},$$

$$F' := V_{T'}(s) = \left\{ x \in \mathbb{R}^n \ : \ \langle t_j - s, x \rangle \leq \frac{1}{2} \left(\left\| t_j \right\|^2 - \| s \|^2 \right), \ j = 1, 2 \right\}.$$

The two inequalities defining F' are consequence relations of the inequalities defining F. Therefore, there exist $\lambda_{ij} \geq 0$, $i, j = 1, 2$ such that

$$t_i - s = \sum_{j=1}^{2} \lambda_{ij} t_j, i = 1, 2 \text{ and}$$

$$\| t_i \|^2 - \| s \|^2 \geq \sum_{j=1}^{2} \lambda_{ij} \left\| t_j \right\|^2, i = 1, 2. \tag{6}$$

Hence,

$$s = (1 - \lambda_{ii}) t_i - \lambda_{i(3-i)} t_{3-i}, i = 1, 2 \text{ and} \tag{7}$$
$$\| s \|^2 \leq (1 - \lambda_{ii}) \| t_i \|^2 - \lambda_{i(3-i)} \| t_{3-i} \|^2, i = 1, 2. \tag{8}$$

We can subtract the two representations (7) of s to obtain

$$(1 - \lambda_{11} + \lambda_{21}) t_1 + (-1 - \lambda_{12} + \lambda_{22}) t_2 = 0_n.$$

If t_j, $j = 1, 2$ are linearly independent, we have

$$1 = \lambda_{11} - \lambda_{21} \text{ and } 1 = \lambda_{22} - \lambda_{12}.$$

Together with (8), this yields

$$0 \leq \| s \|^2 \leq (1 - \lambda_{11}) \| t_1 \|^2 - \lambda_{12} \| t_2 \|^2 = -\lambda_{21} \| t_1 \|^2 - \lambda_{12} \| t_2 \|^2 \leq 0,$$

which contradicts the condition $s \neq 0$. Therefore, t_j, $j = 1, 2$ are linearly dependent. Together with (6), this finishes the proof. □

Remark 2.2 This proposition means that it is impossible to enlarge a Voronoi cell of a single point in a three-point affinely independent system by moving this point.

Proposition 2.5 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty proper subset of T. If $|S| \geq 2$, then*

$$V_T(S) = \bigcup_{s \in S} \left[V_{T \setminus \{s\}}(S \setminus \{s\}) \cap V_S(s) \right]. \tag{9}$$

Proof Denote

$$A_s := V_{T \setminus \{s\}}(S \setminus \{s\}) \cap V_S(s).$$

Observe that for any $\bar{s} \in S$ we have

$$A_{\bar{s}} = \{ x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, \bar{s}) \leq \mathrm{dist}(x, s) \leq \mathrm{dist}(x, t) \quad \forall s \in S, \ t \in T \setminus S \}.$$

Evidently, $A_s \subseteq V_T(S)$ for every $s \in S$, hence, $\bigcup_{s \in S} A_s \subseteq V_T(S)$. To prove the reverse inclusion, assume that $x \in V_T(S)$. Let $\bar{s}$ be a closest point to x in S. It is evident that $x \in V_S(\bar{s})$. At the same time, it is not difficult to observe that $V_T(S) \subseteq V_{T \setminus \{\bar{s}\}}(S \setminus \{\bar{s}\})$. Hence

$$x \in V_{T \setminus \{\bar{s}\}}(S \setminus \{\bar{s}\}) \cap V_S(\bar{s}) = A_{\bar{s}},$$

and therefore $V_T(S) \subseteq \bigcup_{s \in S} A_s$. $\square$

Proposition 2.6 *Let T be a finite subset of $\mathbb{R}^n$, and let S be a nonempty and proper subset of T. If there exist $s_1, s_2 \in S$ and $t_1, t_2 \in T \setminus S$ such that the inequalities*

$$\|s_1 - x\| \leq \|t_1 - x\| \quad and \quad \|s_2 - x\| \leq \|t_2 - x\| \tag{10}$$

define the same half-space, then these inequalities are nonessential for $V_T(S)$, i.e., they can be dropped from the system (2).

Proof It is evident from equivalence (3) that these inequalities can be written as

$$\langle t_1 - s_1, x \rangle \leq \frac{1}{2} \left(\|t_1\|^2 - \|s_1\|^2 \right), \quad \langle t_2 - s_2, x \rangle \leq \frac{1}{2} \left(\|t_2\|^2 - \|s_2\|^2 \right). \tag{11}$$

If both inequalities in (10) define the same half-space, then it follows from (11) that there exists $\alpha > 0$ such that

$$t_2 - s_2 = \alpha(t_1 - s_1) \quad and \quad \frac{1}{2} \left(\|t_2\|^2 - \|s_2\|^2 \right) = \alpha \frac{1}{2} \left(\|t_1\|^2 - \|s_1\|^2 \right). \tag{12}$$

Every point $x \in V_T(S)$ also satisfies the system

$$\langle t_1 - s_2, x \rangle \le \frac{1}{2} \left(\|t_1\|^2 - \|s_2\|^2 \right), \quad \langle t_2 - s_1, x \rangle \le \frac{1}{2} \left(\|t_2\|^2 - \|s_1\|^2 \right). \quad (13)$$

Adding these inequalities together, we obtain

$$\langle t_1 - s_2 + t_2 - s_1, x \rangle \le \frac{1}{2} \left(\|t_1\|^2 - \|s_2\|^2 + \|t_2\|^2 - \|s_1\|^2 \right),$$

and using (12), we have a consequence of (13),

$$(1 + \alpha)\langle t_1 - s_1, x \rangle \le (1 + \alpha)\frac{1}{2} \left(\|t_1\|^2 - \|s_1\|^2 \right),$$

which defines the same half-space as (11).   □

Theorem 2.6 *Let $T \subset \mathbb{R}^n$ be a finite set, let $S := \{s_1, s_2\} \subset T$ be a two-point set, and let*

$$H := \{x \in \mathbb{R}^n \; : \; \|x - s_1\| = \|x - s_2\|\}$$
$$= \{x \in \mathbb{R}^n \; : \; \langle s_1 - s_2, x \rangle = \frac{1}{2}(\|s_1\|^2 - \|s_2\|^2)\}.$$

If int $V_T(S) \ne \emptyset$, *then* $H \cap \text{ri } F = \emptyset$ *for every facet F of $V_T(S)$.*

Proof Let F be the facet of $V_T(S)$ defined by the linear equation corresponding to some point $t_0 \in T$ and, say, s_1, that is, $\langle t_0 - s_1, x \rangle = \frac{1}{2} \left(\|t_0\|^2 - \|s_1\|^2 \right)$, and assume, toward a contradiction, that $H \cap \text{ri } F \ne \emptyset$. Take $\overline{x} \in H \cap \text{ri } F$. We then have

$$\langle t_0 - s_1, \overline{x} \rangle = \frac{1}{2}(\|t_0\|^2 - \|s_1\|^2).$$

Adding the equality

$$\langle s_1 - s_2, \overline{x} \rangle = \frac{1}{2}(\|s_1\|^2 - \|s_2\|^2),$$

which follows from the fact that $\overline{x} \in H$, to the preceding one, we get

$$\langle t_0 - s_2, \overline{x} \rangle = \frac{1}{2}(\|t_0\|^2 - \|s_2\|^2).$$

Since int $V_T(S) \ne \emptyset$ and $\overline{x} \in \text{ri } F$, there is exactly one half-space among those defined by the inequalities (2) such that $\overline{x}$ belongs to its boundary hyperplane (and hence to the interior of the remaining half-spaces). Hence, the linear equalities

$$\langle t_0 - s_1, x \rangle = \frac{1}{2}(\|t_0\|^2 - \|s_1\|^2) \quad (14)$$

and

$$\langle t_0 - s_2, x \rangle = \frac{1}{2}(\|t_0\|^2 - \|s_2\|^2) \tag{15}$$

define the same hyperplane, which implies the existence of a real number λ such that $t_0 - s_2 = \lambda\,(t_0 - s_1)$, so that the points t_0, s_1, s_2 are colinear. Substituting $x := \frac{t_0 + s_1}{2}$, which is a solution of (14), into (15) we have, after elementary algebraic manipulation,

$$\langle t_0 - s_2, s_1 - s_2 \rangle = 0,$$

meaning that $s_1 - s_2$ must be orthogonal to $t_0 - s_2$. This, together with the colinearity of t_0, s_1 and s_2 and the fact that these three points are distinct, yields a contradiction. □

3 Case Study

In this section, we study the special case of higher order cells on no more than four sites. With the exception of Sects. 3.5–3.7, our study will be developed for sets in $\mathbb{R}^2$. For every set $F \subset \mathbb{R}^2$ defined by four linear inequalities, we will determine whether or not there exist sets $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$. In the cases when the answer will be affirmative, we will construct the (possibly non-necessarily unique) sets S and T explicitly.

3.1 Singletons

A singleton (zero-dimensional) cell $\{c\}$ can be obtained by placing the pairs of points (s_1, s_2) and (t_1, t_2) in the opposite corners of a square centered at c. This was already discussed in Example 2.2. Note that this is a minimal representation, since we need at least three inequalities to obtain a bounded cell (cf. Theorem 2.4), and hence $|T| \geq 4$. In fact, the square can be replaced by a rectangle or a general *cyclic* quadrilateral. (We recall that a quadrilateral is said to be cyclic if all of its vertices are on a single circle, which is equivalent to the fact that the sum of opposite angles equals π.)

Proposition 3.1 *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. The following statements are equivalent:*

(a) *$V_T(S)$ is nonempty and at most one-dimensional.*
(b) *The points of T are the vertices of a cyclic quadrilateral, with the two sites of S located opposite to each other (across a diagonal).*
(c) *$V_T(S)$ is a singleton.*

Proof Throughout the proof, we use the explicit notation $S := \{s_1, s_2\}$ and $T := \{s_1, s_2, t_1, t_2\}$. a) $\Rightarrow$ b). Without loss of generality, assume that $0 \in V_T(S)$ while $\text{int } V_T(S) = \emptyset$. By Theorem 2.5, we have

$$0 = \sum_{i,j \in \{1,2\}} \lambda_{ij}(t_j - s_i), \qquad 0 = \sum_{i,j \in \{1,2\}} \lambda_{ij}(\|t_j\|^2 - \|s_i\|^2), \qquad (16)$$

where λ_{ij} are convex combination coefficients. Since $0 \in V_T(S)$, we have from (2) that

$$\|t_j\| \geq \|s_i\| \quad \forall i, j \in \{1, 2\}.$$

Without loss of generality, assume that

$$\|s_2\| \leq \|s_1\| \leq \|t_1\| \leq \|t_2\|.$$

If $\|s_1\| < \|t_1\|$, then $\|s_i\| < \|t_j\|$ for all $i, j \in \{1, 2\}$, which implies that the second equality in (16) is impossible. Hence, $\|s_1\| = \|t_1\|$. If $\|s_1\| > \|s_2\|$, then $\|t_i\| > \|s_2\|$ for all $i \in \{1, 2\}$; therefore, from (16), we get $\lambda_{21} = \lambda_{22} = 0$ and

$$s_1 = \lambda_{11}t_1 + \lambda_{12}t_2,$$

so $s_1 \in [t_1, t_2]$ and $\|s_1\|^2 = \lambda_{11}\|t_1\|^2 + \lambda_{12}\|t_2\|^2$, which holds only when $t_1 = t_2 = s_1$ by the strict convexity of the squared norm. This is impossible. Likewise, when $\|t_1\| < \|t_2\|$ we have $\lambda_{12} = \lambda_{22} = 0$, then $t_1 \in [s_1, s_2]$, which by Corollary 2.2 yields $V_T(S) = \emptyset$, again a contradiction. We have proved that $\|s_2\| = \|s_1\| = \|t_1\| = \|t_2\|$, and hence our sites are the vertices of a cyclic quadrilateral. It is now easy to observe that s_1, s_2 are located opposite to each other because, by the first equality in (16), we have $[s_1, s_2] \cap [t_1, t_2] \neq \emptyset$. b) $\Rightarrow$ c) Since the points of T lie on some circle, without loss of generality we may assume that the center of this circle is the origin, and then $\|t_1\| = \|t_2\| = \|s_1\| = \|s_2\|$. In this case, the right-hand side of system (2) is zero, and we have, for every point $x \in V_T(S)$,

$$\langle t_j - s_i, x \rangle \leq 0 \quad \forall i, j \in \{1, 2\}. \qquad (17)$$

It is evident that $x = 0$ is a solution of this system; hence, $V_T(S) \neq \emptyset$. On the other hand, from (17) it follows that $V_T(S)$ is a cone. From these facts, using Proposition 2.3 we immediately deduce that $V_T(S) = \{0\}$. The implication c) $\Rightarrow$ a) is obvious. □

Note that for the case $|S| = 2$ and $|T| = 3$, it is impossible to have a nonempty bounded cell due to Remark 2.1. This means that we do not need to consider this configuration when discussing the subsequent cases of bounded polygons.

Furthermore, in the case $|S| = 3$ and $|T| = 4$, it is impossible to have a bounded cell, as is shown in Proposition 2.2.

Since we have determined that we cannot have a nonempty bounded cell for $|T| = |S| + 1$, the only possibility to have a singleton cell is for $|S| = 2$ and $|T| = 4$. Furthermore, we can focus on the latter case when studying other bounded cells.

3.2 One-Dimensional Cells

It follows from the preceding discussion that it is impossible to obtain line segments as multipoint Voronoi cells in our setting.

Corollary 3.1 *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then, $V_T(S)$ is not one-dimensional.*

Proof It follows directly from Proposition 3.1. □

It follows from Corollary 3.1 that it is impossible to have a one-dimensional cell for $|T| = 4$, $|S| = 2$, so both rays and lines are impossible in this configuration.

Now consider the case $|T| = |S| + 1$. If $V_T(S) \neq \emptyset$, by Corollary 2.2 we must have for $\{t\} = T \backslash S$ that $t \notin \operatorname{conv} S$. By the separation theorem, this yields the existence of some d, $\|d\| = 1$ such that

$$\langle t - s, d \rangle < 0 \quad \forall s \in S,$$

which yields the existence of a sufficiently small ball $B_\varepsilon(d)$ centered at d such that

$$\langle t - s, y \rangle < 0 \quad \forall s \in S, \ \forall y \in B_\varepsilon(d).$$

Then, for any $x_0 \in V_T(S)$ and any $y \in B_\varepsilon(d)$ we have

$$\langle t - s, x_0 + y \rangle = \langle t - s, x_0 \rangle + \langle t - s, y \rangle < \langle t - s, x_0 \rangle.$$

It is hence clear from the representation in Proposition 2.1 that $x_0 + B_\varepsilon(y) \subset V_T(S)$, which gives that $V_T(S)$ is one-dimensional.

Thus, we have proved the following statement.

Proposition 3.2 *Let $S \subset T \subset \mathbb{R}^n$, with $|T| = |S| + 1$. Then, $V_T(S)$ is not one-dimensional.*

3.3 Triangles

A somewhat surprising result is that a second-order Voronoi cell cannot be a triangle. As discussed previously, we only need to prove this for the case $|S| = 2$ and $|T| = 4$.

Proposition 3.3 *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then $V_T(S)$ is not a triangle.*

Proof Suppose that $V_T(S)$ is a triangle. Denote $T = \{s_1, s_2, t_1, t_2\}$, $S = \{s_1, s_2\}$. The cell $V_T(S)$ is the solution set of the linear system of inequalities

$$\langle c_{ij}, x \rangle \leq \alpha_{ij} \quad (i, j = 1, 2), \tag{18}$$

with $c_{ij} := t_i - s_j$ and $\alpha_{ij} := \frac{1}{2}\left(\|t_i\|^2 - \|s_j\|^2\right)$. Notice that

$$\begin{pmatrix} c_{22} \\ \alpha_{22} \end{pmatrix} = -\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix} + \begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix} + \begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix}. \tag{19}$$

Without loss of generality, we will assume that $0 \in \text{int } V_T(S)$, that is, $\alpha_{ij} > 0$ $(i, j = 1, 2)$. Since (18) defines a triangle, one of its inequalities, say the one corresponding to $i = 2 = j$, is redundant, so that the triangle is actually the solution set of the system consisting of the other three. We claim that the vectors $\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix}, \begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix}$ and $\begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix}$ are linearly independent. Indeed, assume the existence of $(\beta_{11}, \beta_{12}, \beta_{21}) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ such that

$$\beta_{11}\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix} + \beta_{12}\begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix} + \beta_{21}\begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix} = \begin{pmatrix} 0_2 \\ 0 \end{pmatrix}, \tag{20}$$

and let $\overline{x}$ be the vertex of $V_T(S)$ defined by

$$\langle c_{12}, \overline{x}\rangle = \alpha_{12} \text{ and } \langle c_{21}, \overline{x}\rangle = \alpha_{21}. \tag{21}$$

Then, from (20) and (21) we easily deduce that $\beta_{11}(\langle c_{11}, \overline{x}\rangle - \alpha_{11}) = 0$. Since $\langle c_{11}, \overline{x}\rangle < \alpha_{11}$, as the three sides of $V_T(S)$ do not have a common point, it follows that $\beta_{11} = 0$. By using the same argument with the other two vertices of $V_T(S)$, we deduce that $\beta_{12} = 0 = \beta_{21}$, thus proving our claim that $\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix}, \begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix}$ and $\begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix}$ are linearly independent. Since the inequality corresponding to $i = 2 = j$ is redundant, by Farkas' lemma there exist $\lambda_{ij} \geq 0$ $(i, j = 1, 2)$ such that

$$c_{22} = \lambda_{11}c_{11} + \lambda_{12}c_{12} + \lambda_{21}c_{21} \quad \text{and} \quad \alpha_{22} \geq \lambda_{11}\alpha_{11} + \lambda_{12}\alpha_{12} + \lambda_{21}\alpha_{21}. \tag{22}$$

Considering again the vertex $\overline{x}$, from (19) we obtain that

$$\alpha_{22} = -\alpha_{11} + \alpha_{12} + \alpha_{21} \leq -\langle c_{11}, \overline{x}\rangle + \langle c_{12}, \overline{x}\rangle + \langle c_{21}, \overline{x}\rangle = \langle -c_{11} + c_{12} + c_{21}, \overline{x}\rangle$$
$$= \langle c_{22}, \overline{x}\rangle \leq \alpha_{22};$$

hence, by (22), we deduce that

$$\langle c_{22}, \overline{x}\rangle = \alpha_{22} \geq \lambda_{11}\alpha_{11} + \lambda_{12}\alpha_{12} + \lambda_{21}\alpha_{21}$$
$$\geq \lambda_{11}\langle c_{11}, \overline{x}\rangle + \lambda_{12}\langle c_{12}, \overline{x}\rangle + \lambda_{21}\langle c_{21}, \overline{x}\rangle = \langle \lambda_{11}c_{11} + \lambda_{12}c_{12} + \lambda_{21}c_{21}, \overline{x}\rangle$$
$$= \langle c_{22}, \overline{x}\rangle.$$

Therefore, $\lambda_{11}\alpha_{11} + \lambda_{12}\alpha_{12} + \lambda_{21}\alpha_{21} = \alpha_{22}$, that is,

$$\begin{pmatrix} c_{22} \\ \alpha_{22} \end{pmatrix} = \lambda_{11}\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix} + \lambda_{12}\begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix} + \lambda_{12}\begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix}.$$

Comparing this equality with (19) and taking into account that the vectors $\begin{pmatrix} c_{11} \\ \alpha_{11} \end{pmatrix}$, $\begin{pmatrix} c_{12} \\ \alpha_{12} \end{pmatrix}$ and $\begin{pmatrix} c_{21} \\ \alpha_{21} \end{pmatrix}$ are linearly independent, we deduce that $\lambda_{11} = -1$, a contradiction. □

3.4 Bounded Quadrilaterals

We next show that any non-cyclic bounded convex quadrilateral is a second-order Voronoi cell with $|T| = 4$. We also prove that it is impossible for a cyclic quadrilateral to be a Voronoi cell of two points (when $|T| = 4$).

Proposition 3.4 *Let $F \subset \mathbb{R}^2$ be a non-cyclic bounded convex quadrilateral. Then, there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

Proof First, looking at Fig. 5, where F is depicted as the quadrilateral $ACBD$, we would like to mention that $\sphericalangle CAD = \alpha > 0$ and $\sphericalangle CBD = \beta > 0$ and, w.l.o.g., $\pi > \alpha + \beta$. We draw the lines through A (B) that make an angle equal to α (resp., β) with the segment AB. In this way, we get the points t_1 and t_2 as the two other vertices of the quadrilateral having those lines as their sides. We take s_1 and s_2 as the symmetric points of t_1 with respect to the lines CA and CB, respectively, and we set $S := \{s_1, s_2\}$ and $T := \{s_1, s_2, t_1, t_2\}$. The only thing that we have to prove is that $t_2 s_1 \perp AD$ and $t_2 s_2 \perp BD$. The point B is the center of the circle passing through the points t_1, t_2 and s_2. Therefore, $\sphericalangle t_1 s_2 t_2 = \pi - \beta$, which, looking at the quadrilateral $C_2 s_2 D_2 B$, implies that $t_2 s_2 \perp BD$. The point A is the center of the circle passing through the points t_1, t_2 and s_1. Therefore, $\sphericalangle t_1 s_1 t_2 = \pi - \alpha$, which, looking at the quadrilateral $C_1 s_1 D_1 A$, implies $t_2 s_1 \perp AD$. □

Observe that the algorithm does not work if $\pi \leq \alpha + \beta$. However, it is not difficult to observe that, for a non-cyclic convex quadrilateral, it is always possible to choose the corners to ensure $\alpha + \beta < \pi$. We have the following negative result.

Proposition 3.5 *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then, $V_T(S)$ is not a cyclic quadrilateral.*

Proof Assume that the Voronoi cell $V_T(S)$ of some set $S := \{s_1, s_2\}$, with $T := \{s_1, s_2, t_1, t_2\}$, is a cyclic quadrilateral. Then, each side of this quadrilateral is defined by the bisector between s_i and t_j for $i, j \in \{1, 2\}$. First, we will show that any two sides defined by the bisectors of disjoint pairs, say, $\{s_1, t_1\}$ and $\{s_2, t_2\}$, cannot be adjoint. Assume the contrary: Then, without loss of generality, the intersection u of the two bisectors is a vertex of $V_T(S)$. We have, by using the representation (2),

$$\langle t_1 - s_2, u \rangle \leq \frac{1}{2}(\|t_1\|^2 - \|s_2\|^2), \qquad \langle t_2 - s_1, u \rangle \leq \frac{1}{2}(\|t_2\|^2 - \|s_1\|^2), \quad (23)$$

and since u is the intersection of the two bisectors,

$$\langle t_1 - s_1, u \rangle = \frac{1}{2}(\|t_1\|^2 - \|s_1\|^2), \qquad \langle t_2 - s_2, u \rangle = \frac{1}{2}(\|t_2\|^2 - \|s_2\|^2). \quad (24)$$
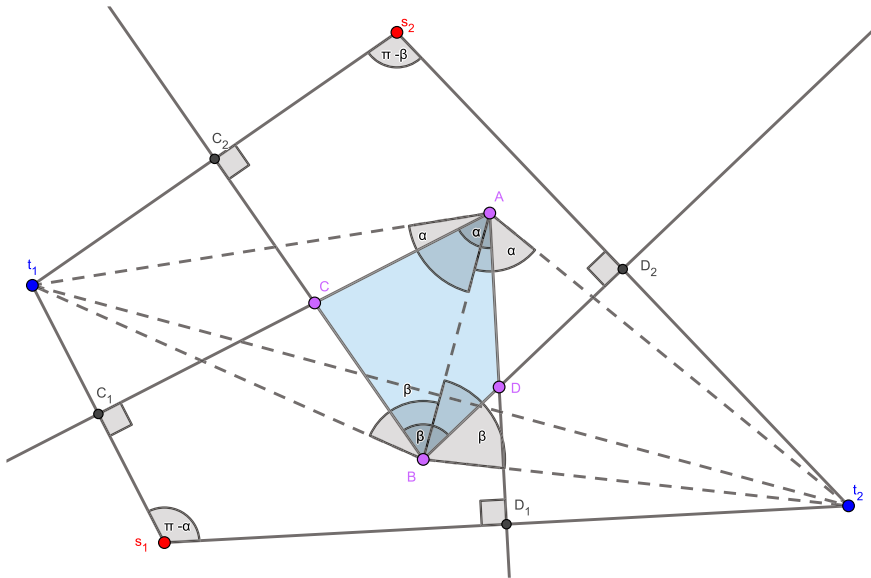
Fig. 5 Bounded quadrilateral

Adding the two equalities in (24) and rearranging, we obtain

$$\langle t_1 - s_2, u \rangle + \langle t_2 - s_1, u \rangle = \frac{1}{2}(\|t_1\|^2 - \|s_2\|^2) + \frac{1}{2}(\|t_2\|^2 - \|s_1\|^2).$$

Together with (23), this yields equalities in (23), and hence the four lines that define the sides of the quadrilateral must intersect at u. This is impossible; hence, the assumption is wrong. Now, let us consider the quadrilateral with vertices s_1, s_2, t_1 and t_2. Looking at Fig. 6, it is easy to see that the angles at t_1 and C are equal, and so they are the angles at t_2 and A. This means that this quadrilateral is cyclic too, that is, s_1, s_2, t_1 and t_2 lie on a circle. It follows from Proposition 3.1 that the Voronoi cell $V_T(S)$ is a singleton, which contradicts our assumption.   □

3.5 Half-spaces

A half-space cell can be obtained by putting the two points of S on a line perpendicular to the boundary line of the half-space making sure that S is in the interior of the half-space. An additional point t is placed on the same line on the opposite side of the hyperplane at the same distance from the hyperplane as the distance to the hyperplane from the farthest point in S (see Fig. 7). We hence conclude that a half-space can be constructed using $|S| = 2$ and $|T| = 3$. Observe that it is also possible to do the same construction for $|S| \in \{2, 3\}$ and $|T| = 4$. We prove this explicitly in the next statement.
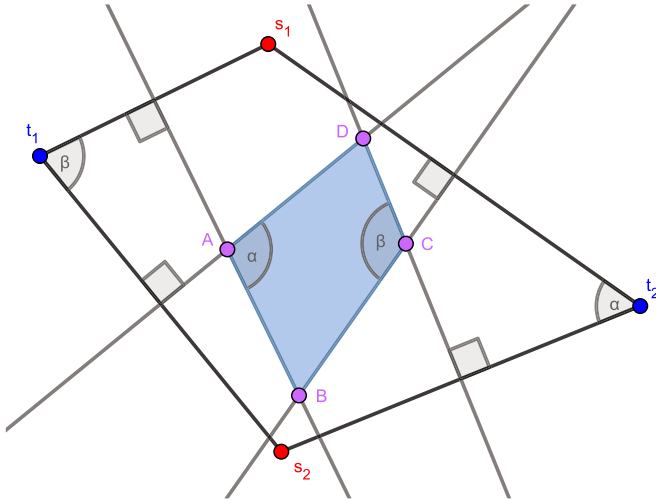
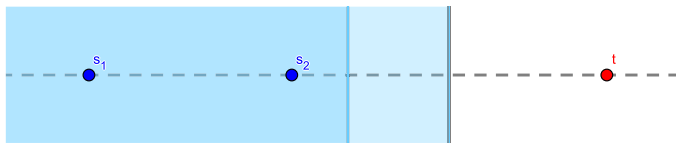**Fig. 6** An illustration to the proof of Proposition 3.5

Fig. 7 The intersection of the two half-spaces is $V_T(s_1, s_2)$

Proposition 3.6 *Let $F \subset \mathbb{R}^n$ be a half-space. Then, for any two integers $\tau > \sigma \geq 1$ there exist $S \subset T \subset \mathbb{R}^n$, with $|S| = \sigma$ and $|T| = \tau$, such that $V_T(S) = F$.*

Proof Note that any half-space F can be represented as $F = \{x \in \mathbb{R}^n \mid \langle d, x \rangle \leq \gamma\}$ for some d, $\|d\| = 1$, and $\gamma \in \mathbb{R}$. Choose any x_0 such that $\langle d, x_0 \rangle = \gamma$, and let $S := \{s_0, \ldots, s_m\}$ and $T := \{t_0, \ldots, t_p\} \cup S$, where $m = \sigma - 1$, $p = \tau - \sigma - 1$,

$$s_0 := x_0 - d, \, t_0 := x_0 + d,$$

$$s_i = x_0 - \alpha_i d, \; 0 < \alpha_i < 1 \; \forall i \in \{1, \ldots, m\}, \; t_j$$
$$:= x_0 + \beta_j d, \; \beta_j > 1 \; \forall j \in \{1, \ldots, p\}, \tag{25}$$

and the constants α_i and β_j are all different (to ensure that the sites do not coincide). We will next show that

$$V_T(S) = \{x \in \mathbb{R}^2 \mid \langle d, x \rangle \leq \gamma\}.$$

We have, from the representation in Proposition 2.1,

$$V_T(S) = \bigcap_{\substack{i \in \{0, 1, \ldots, m\} \\ j \in \{0, 1, \ldots, p\}}} \left\{ x : \langle t_j - s_i, x \rangle \leq \frac{1}{2} \left(\|t_j\|^2 - \|s_i\|^2 \right) \right\}.$$

By (25), the inequalities in the latter expression can be rewritten as

$$(\beta_j + \alpha_i)\langle d, x\rangle \leq \frac{1}{2}\left((\beta_j - \alpha_i)(\beta_j + \alpha_i) + (\beta_j + \alpha_i)\langle d, x_0\rangle\right), \quad \begin{array}{l} \forall i \in \{0, \ldots, m\}, \\ \forall j \in \{0, \ldots, p\}, \end{array}$$

where $\alpha_0 = \beta_0 = 1$. Dividing by the factor $\beta_j + \alpha_i > 0$, we have

$$\langle d, x\rangle - \langle d, x_0\rangle \leq \frac{\beta_j - \alpha_i}{2};$$

hence,

$$V_T(S) = \left\{x \; : \; \langle d, x\rangle - \gamma \leq \frac{1}{2} \cdot \min_{i,j}(\beta_j - \alpha_i)\right\} = \{x \; : \; \langle d, x\rangle \leq \gamma\}.$$

The latter set is precisely the half-space that we were aiming for.   □

3.6 Intersections of Parallel Half-spaces

We consider a set $F \subset \mathbb{R}^n$ represented by the inequalities $\alpha \leq \langle d, x\rangle \leq \beta$, with $\|d\| = 1$ and $\alpha < \beta$. One can easily check that $F = V_T(S)$ for $T := \{s_1, s_2, t_1, t_2\}$ and $S := \{s_1, s_2\}$ with, for instance, $s_1 := \frac{3\alpha+\beta}{4}d$, $s_2 := \frac{\alpha+\beta}{2}d$, $t_1 := \frac{3\alpha-\beta}{2}d$ and $t_2 := \frac{7\beta-3\alpha}{4}d$. We have shown the following result.

Proposition 3.7 *Let $F \subset \mathbb{R}^n$ be an intersection of two parallel half-spaces with opposite normals, such that F has a nonempty interior. Then, there exist $S \subset T \subset \mathbb{R}^n$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

Note that it is impossible to produce a strip with $|T| - |S| = 1$: Indeed, it is clear from the fact that all constraints are defined by parallel lines that all vectors $s - t, s \in S$, $t \in T$ should be colinear, lying on some line orthogonal to the inequalities. Now, if for the unique $t \in T \backslash S$ we have $t \in \text{conv } S$, then the cell is empty by Corollary 2.2. However, if $t \notin \text{conv } S$, then the cell has to be a half-space. Hence, the only possibility is $|T| = 4$, $|S| = 2$.

3.7 Wedges

Now, let us have a wedge $F := \{x \in \mathbb{R}^n : \langle c_i, x\rangle \leq \alpha_i, i = 1, 2\}$, with c_1 and c_2 being linearly independent. Let us take an arbitrary point from $t \in \text{int } F^- := \{x \in \mathbb{R}^n : \langle c_i, x\rangle \geq \alpha_i, i = 1, 2\}$ and construct the two symmetric points s_1 and s_2 with respect to both hyperplanes defining F. If $T := \{s_1, s_2, t\}$ and $S := \{s_1, s_2\}$, then $V_T(S) = F_T(t) = F$.

It is possible to add any number of more sites s to S in such a way that the half-spaces $\|s - x\| \leq \|t - x\|$ include the original wedge.
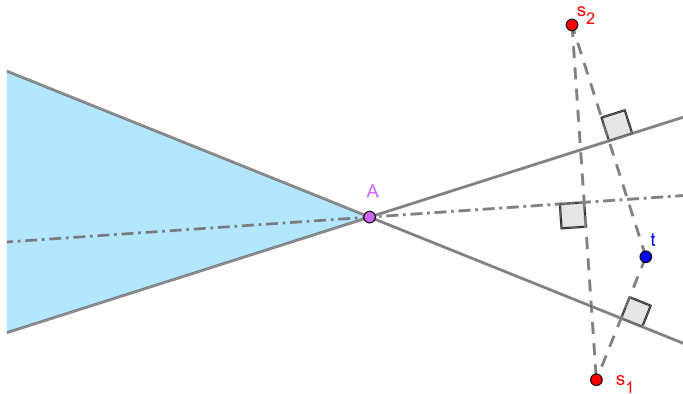
Fig. 8 Construction of the set of sites for a wedge

Proposition 3.8 *Let*

$$F := \left\{ x \in \mathbb{R}^n \mid \langle v_i, x \rangle \leq b_i, \quad i = 1, 2 \right\},$$

with $v_1, v_2 \in \mathbb{R}^n$ linearly independent unit vectors and $b_1, b_2 \in \mathbb{R}$. Then, there exist $S \subset T \subset \mathbb{R}^n$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.

Proof Without loss of generality, we assume that $b_i = 0$ $(i = 1, 2)$. Set $\beta := \langle v_1, v_2 \rangle$, $i' := 3 - i$ for $i = 1, 2$, and define

$$s_i := -v_i - 2v_{i'} \quad (i = 1, 2).$$

Let t_i be the point symmetric to s_i with respect to the hyperplane defined by $\langle x, v_{i'} \rangle = 0$, that is,

$$t_i := -v_i + 2 (1 + \beta) v_{i'} \quad (i = 1, 2),$$

and define $S := \{s_1, s_2\}$ and $T := \left\{ s_1, s_2, t_1, t_2 \right\}$. We have $|T| = 4$, since $\beta > -1$. Given that

$$\|t_i\|^2 = 5 + 4\beta = \|s_i\|^2 \quad (i = 1, 2),$$

one has

$$V_T(S) = \left\{ x \in \mathbb{R}^n \mid \langle t_i - s_i, x \rangle \leq 0, \ \langle t_i - s_{i'}, x \rangle \leq 0, \quad i = 1, 2 \right\}. \quad (26)$$

Hence, to prove that $V_T(S) = F$, it will suffice to show that, for $i = 1, 2$, the inequality $\langle x, t_i - s_i \rangle \leq 0$ is equivalent to $\langle x, v_{i'} \rangle \leq 0$ and that the remaining two inequalities in (26) are redundant. The first assertion is obvious, since $t_i - s_i = 2 (2 + \beta) v_{i'}$ and $\beta > -1$. Let us now prove that the inequalities $\langle t_i - s_{i'}, x \rangle \leq 0$ are redundant, that
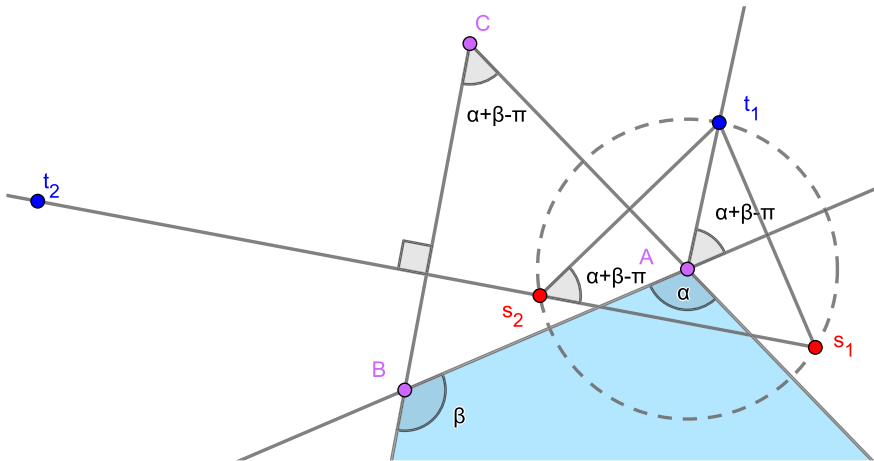
Fig. 9 Unbounded polygon with three sides

is, that they are consequences of the system $\langle v_i, x \rangle \leq 0 \ (i = 1, 2)$. But this is also immediate, since $t_i - s_{i'} = v_i + (3 + 2\beta) v_{i'}$ and $\beta > -1$. $\qquad\square$

3.8 Unbounded Polygons with Three Sides

Our next construction is for an unbounded polygon like the shadowed one in Fig. 9, with three sides, which is unbounded and has non-parallel sides.

Proposition 3.9 *Let F be an unbounded convex polygon with non-parallel sides and two vertices. Then, there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

Proof In the notation of Fig. 9, where F is depicted as the shaded region, we would like to mention that $0 < \alpha < \pi$, $0 < \beta < \pi$, $\alpha + \beta > \pi$ and $\beta \geq \alpha$. If we consider the triangle ABC, the angle at C is $\alpha + \beta - \pi$. Now, we draw a line through the point A, which makes an angle $\alpha + \beta - \pi$ with the line BA. On this line, we consider an arbitrary point t_1, sufficiently close to A. Let us take the symmetric points s_1 and s_2 of t_1 with respect to AB and AC, respectively. We shall prove that the line through the points s_1, s_2 is perpendicular to BC. For this purpose, we would like to mention that the point A is the center of the circle passing through the points s_1, s_2 and t_1. This means that $\sphericalangle s_1 s_2 t_1 = \alpha + \beta - \pi$. If we consider the quadrilateral formed by the lines BC, AC, $s_1 s_2$ and $t_1 s_2$, we get the desired fact. At the end we construct the site t_2 as the symmetric point of s_1 with respect to the line BC and set $S := \{s_1, s_2\}$ and $T := \{s_1, s_2, t_1, t_2\}$. $\qquad\square$

Proposition 3.10 *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then, $V_T(S)$ is not an unbounded polygon with parallel sides and just two vertices.*

Proof Similarly to the proof of Proposition 3.3, suppose that $V_T(S)$ is an unbounded convex polygon with parallel sides and just two vertices, denote $S = \{s_1, s_2\}$ and

$T = \{s_1, s_2, t_1, t_2\}$, and suppose further, w.l.o.g., that the redundant inequality in the linear system

$$\langle t_i - s_j, x \rangle \leq \frac{1}{2} \left(\|t_i\|^2 - \|s_j\|^2 \right) \quad (i, j = 1, 2) \tag{27}$$

(which has $V_T(S)$ as its solution set) is the one corresponding to $i = 2 = j$. Then, by Farkas' Lemma, there exist $\gamma_{11}, \gamma_{12}, \gamma_{21} \geq 0$ such that

$$t_2 - s_2 = \gamma_{11}(t_1 - s_1) + \gamma_{12}(t_1 - s_2) + \gamma_{21}(t_2 - s_1) \tag{28}$$

and

$$\alpha_{22} \geq \gamma_{11}\alpha_{11} + \gamma_{12}\alpha_{12} + \gamma_{21}\alpha_{21}, \tag{29}$$

where $\alpha_{ij} := \frac{1}{2} \left(\|t_i\|^2 - \|s_j\|^2 \right)$. From (28) we obtain that $t_2 - s_2$ belongs to

$$K_1 := \text{cone}\{t_1 - s_1, t_1 - s_2, t_2 - s_1\}.$$

On the other hand, two of the three vectors $t_1 - s_1$, $t_1 - s_2$ and $t_2 - s_1$ are exterior normals to the parallel sides, and hence they make an angle of π. Those two vectors cannot be $t_1 - s_1$ and $t_1 - s_2$, because with such a configuration $V_T(S)$ would be empty. We will consider the two remaining cases. We start with the case when the two vectors are $t_1 - s_1$ and $t_2 - s_1$, say $t_1 - s_1 = -\mu(t_2 - s_1)$, with $\mu > 0$. Since in this case s_2 does not belong to the straight line determined by s_1, t_1 and t_2 (because the side of $V_T(S)$ contained in the line $\langle t_1 - s_2, x \rangle = \frac{1}{2} \left(\|t_1\|^2 - \|s_2\|^2 \right)$ is not parallel to the other two sides), there is a circle containing s_2, t_1 and t_2, and we may assume, w.l.o.g., that the center of this circle is 0_2. Then, $\alpha_{11} = \alpha_{21} > 0$ and $\alpha_{12} = \alpha_{22} = 0$, thus, in view of (29), we have $\gamma_{11} = 0 = \gamma_{12}$ and then, by (28),

$$\gamma_{12}(t_1 - s_2) = t_2 - s_2 = -(t_1 - s_1) + t_1 - s_2 + t_2 - s_1$$
$$= t_1 - s_2 + (1 + \mu)(t_2 - s_1),$$

which yields

$$t_2 - s_1 = \frac{\gamma_{12} - 1}{1 + \mu}(t_1 - s_2).$$

This is impossible because the vectors $t_1 - s_1$, $t_1 - s_2$ and $t_2 - s_1$ are not all colinear. It only remains to consider the case when $t_1 - s_2$ and $t_2 - s_1$ are the vectors that make an angle of π. Since $t_1 - s_1$ is not colinear with these two vectors, we have that the cones K_1 and

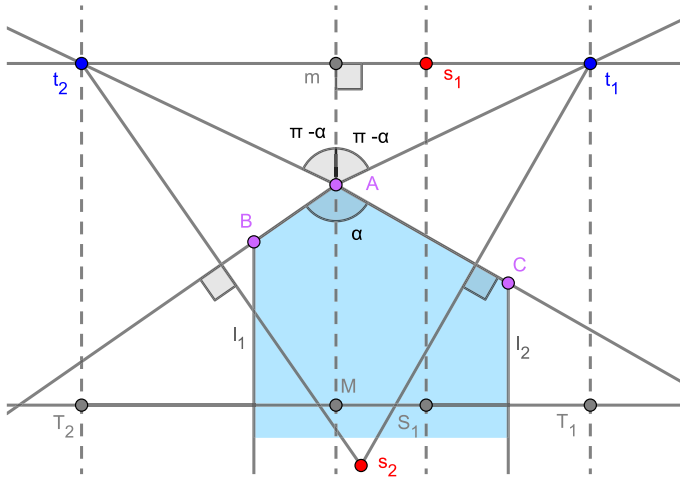$$K_2 := \text{cone}\{-(t_1 - s_1), t_1 - s_2, t_2 - s_1\}$$

Fig. 10 Unbounded polygon with parallel sides

are opposite halfplanes defined by the line containing the vectors $t_1 - s_2$ and $t_2 - s_1$. The fact that $t_2 - s_2 \in K_1 \cap \text{int } K_2$ provides a contradiction. $\qquad\square$

We note here that it is possible to have an unbounded polygon with three sides for the case $|S| = 3$ and $|T| = 4$ if and only if the unbounded sides are non-parallel. Indeed, in this case we can first build a wedge that defines the two unbounded sides (see Sect. 3.7) and then add an extra site to define the extra inequality. For the case of unbounded parallel sides, it is clear that the point t should at the same time lie outside of each of these parallel sides, which is impossible.

3.9 Unbounded Polygons with Four Sides

In the next proposition, we shall consider the case of an unbounded quadrilateral in $\mathbb{R}^2$ with two parallel sides, shown in Fig. 10.

Proposition 3.11 *Let $F \subset \mathbb{R}^2$ be an unbounded convex quadrilateral with two parallel sides. Then, there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

Proof In the notation of Fig. 10, where F is depicted as the unbounded convex polygon with vertices A, B and C, we have $l_1 \parallel l_2$ and $0 < \alpha < \pi$, and the point A belongs to the interior of the parallel band. First, we move, parallel to the line l_1, the point A to an arbitrary point M. Through M, we draw a line perpendicular to l_1. Take a point S_1 on the same line, with the same distance to l_2 as M is to l_1. Now, we consider the symmetric point T_1 and T_2 of S_1 with respect to l_2 and l_1, respectively. In this way, M is the midpoint of the segment $T_1 T_2$. Next, we translate the line through the points T_1 and T_2 in such a way that, defining t_1, t_2, m and s_1 as the orthogonal projections of T_1, T_2, M and S_1 onto the resulting translated line, one has $\sphericalangle t_1 A t_2 = 2\pi - 2\alpha$ and $\sphericalangle m A t_2 = \pi - \alpha$. For the last site, we take s_2 as the symmetric point of t_2 with respect to the line through the points A and B. We shall prove that the line through the points t_1,
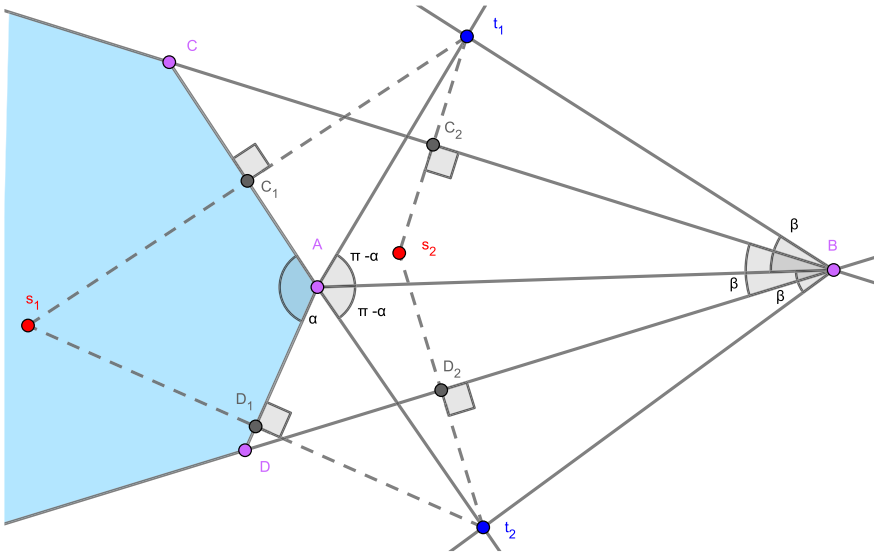
Fig. 11 Configuration for a general unbounded quadrilateral

s_2 is perpendicular to AC. For this purpose, we observe that the point A is the center of the circle passing through the points t_1, t_2 and s_2. This means that $\sphericalangle t_2 s_2 t_1 = \pi - \alpha$. If we consider the quadrilateral formed by the lines BA, AC, $t_1 s_2$ and $t_2 s_2$, we get the desired fact. At the end, after having found the desired sets $S := \{s_1, s_2\}$ and $T := \{s_1, s_2, t_1, t_2\}$, we would like to mention that this construction is impossible if $\alpha = \pi$. $\qquad\square$

We next consider the case of an arbitrary unbounded convex quadrilateral with non-parallel sides, presented in Fig. 11 .

Proposition 3.12 *Let F be an unbounded convex quadrilateral with no sides parallel. Then, there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

Proof First, looking at Fig. 11, where F is depicted as the shaded region, we obtain the point B as the intersection of the unbounded sides of the convex quadrilateral with vertices CAD. We denote $\sphericalangle CAD = \alpha$ and $\sphericalangle CBD = \beta$ and observe that $\pi > \alpha > \beta > 0$. We next draw the lines that make angles $\pi - \alpha$ with AB at the point A and those that make angles β with the same segment at B. We then get the points t_1 and t_2 as the two other vertices of the quadrilateral determined by these four lines. We take s_1 and s_2 as the symmetric points of t_1 with respect to CA and CB, respectively. To see that $V_T(S) = F$ for $S := \{s_1, s_2\}$ and $T := \{s_1, s_2, t_1, t_2\}$, the only thing that we have to prove is that $t_2 s_1 \perp AD$ and $t_2 s_2 \perp BD$. The pòint B is the center of the circle passing through the points t_1, t_2 and s_2. Therefore, $\sphericalangle t_1 s_2 t_2 = \pi - \beta$, which, looking at the quadrilateral $C_2 s_2 D_2 B$, implies that $t_2 s_2 \perp BD$. The point A is the center of the circle passing through the points t_1, t_2 and s_1. Therefore, $\sphericalangle t_1 s_1 t_2 = \pi - \alpha$, which, looking at the quadrilateral $C_1 s_1 D_1 A$, implies $t_2 s_1 \perp AD$. $\qquad\square$

4 Conclusions

We have obtained constructive characterizations of properties pertaining to higher order Voronoi cells and applied these characterizations to a case study of cells of order at least two in the system of four sites.

In particular, the results obtained in the preceding subsections show that for $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, the resulting cell $V_T(S)$ can be any of the following sets: the empty set, a singleton, a non-cyclic bounded convex quadrilateral, a halfplane, an intersection of parallel halfplanes with opposite normals and a nonempty interior, an angle, an unbounded polygon with non-parallel sides and just two vertices, or an unbounded convex quadrilateral. All the remaining possibilities for sets in $\mathbb{R}^2$ defined by four linear inequalities, namely, a singleton, a one-dimensional set, a triangle, a cyclic quadrilateral and an unbounded quadrilateral with parallel sides and just two vertices are unfeasible.

The restrictions on the possible shapes of higher order cells discovered in our case study are consistent with the challenges encountered in the design of numerical methods for constructing higher order cells (cf. [14]), where the key assumption of general position ensures the absence of cyclic quadrilateral configurations). Thus, a natural direction for the future study is to obtain general structural results on the shapes of higher order cells, which will in turn inform the design of algorithms. For instance, we would like to know whether the convex hull of affinely independent points can be represented as a higher order Voronoi cell (generalizing our result on the impossibility of a triangular cell), what higher-dimensional configurations produce cells with empty interiors, and what are the possible dimensions of higher order Voronoi cells in $\mathbb{R}^n$.

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