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Existence Results for Noncoercive Mixed Variational Inequalities in Finite Dimensional Spaces

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Abstract

We use asymptotic analysis and generalized asymptotic functions for studying nonlinear and noncoercive mixed variational inequalities in finite dimensional spaces in the nonconvex case, that is, when the operator is nonlinear and noncoercive and the function is nonconvex and noncoercive. We provide general necessary and sufficient optimality conditions for the set of solutions to be nonempty and compact. As a consequence, a characterization of the nonemptiness and compactness of the solution set, when the operator is affine and the function is convex, is given. Finally, a comparison with existence results for equilibrium problems is presented.

Keywords Asymptotic analysis · Asymptotic functions · Noncoercive optimization · Variational inequalities · Equilibrium problems

Mathematics Subject Classification 90C25 · 90C26 · 90C30

1 Introduction

The theory of mixed variational inequalities in finite dimensional spaces has become an interesting and well-established area of research, due to its applications in several fields like economics, engineering sciences, unilateral mechanics and electronics, among others (see [1–9]).

A useful variational inequality is the *mixed variational inequality*, also called *variational inequality of the second kind*. The problem data for this type of variational inequalities consists of an operator and a function. Therefore, one can impose stronger

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assumptions on the operator in order to weaken the assumptions on the function, and vice versa. For the case in which the operator is linear (affine) and/or semicoercive and the function is proper, lsc and convex, many existence results can be found in the literature (see [1,3,4,10]). The usual techniques for proving existence results are the well-known recession (asymptotic) tools, which have been shown to be useful in the convex case (see [11–16]).

In recent years, finer estimates of asymptotic functions for the case in which the original function is nonconvex have been introduced (see [17–20]). Some of those generalized asymptotic functions were defined for the case in which the original function is quasiconvex, but they proved to be useful for more general classes of nonconvex functions.

In this paper, we use asymptotic analysis and generalized asymptotic functions for proving new necessary and sufficient optimality conditions for the mixed variational inequality. We start by studying the general case, i.e., problem (12) with a nonlinear and noncoercive operator T and a nonconvex and noncoercive function h. Afterward, we study the particular case when T is a linear (affine) operator. As a consequence, a characterization result for the linear mixed variational inequality when the function h is proper, lsc and convex will be given. This characterization generalizes many well-known result for the convex case (see [4]). Finally, since a variational inequality can be seen as an equilibrium problem (see [21] for instance), we provide a comparison with known existence results for equilibrium problems. The comparison shows that our results do not follow from them.

The paper is organized as follows. In Sect. 2, we set up notation and preliminaries. Also, we review some standard facts on asymptotic analysis and generalized convexity. In Sect. 3, we provide new optimality conditions for the mixed variational inequality in the general case. As a consequence, a characterization of the nonemptiness and compactness of the solution set for the linear mixed variational inequality in the convex case is given. Finally, a comparison with existence results for equilibrium problems is also provided.

2 Preliminaries and Basic Definitions

By $\langle \cdot, \cdot \rangle$, we denote the inner product of $\mathbb{R}^n$ and by $\|\cdot\|$ we denote a norm. For a nonempty set $K \subset \mathbb{R}^n$, its closure is denoted by $\operatorname{cl} K$, its boundary by $\operatorname{bd} K$, its topological interior by $\operatorname{int} K$, its convex hull by $\operatorname{conv} K$, its relative interior by $\operatorname{ri} K$, its polar (positive) cone by K^* and the orthogonal complement of its affine hull by $K^{\perp}$. By $B(x, \delta)$, we mean the closed ball with center at x and radius $\delta > 0$. Given $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we set $\|x\|_1 := \Sigma_{i=1}^n |x_i|$. We set $\mathbb{R}_+ := [0, +\infty[$ and $\mathbb{R}^n_+ := (\mathbb{R}_+)^n$.

We define $\operatorname{cone} K := \bigcup_{t \geq 0} tK$ as the smallest cone containing K. Given a cone P from $\mathbb{R}^n$, a compact base of the cone P is a convex and compact set B_0, such that $0 \notin B_0$, for which $P = \operatorname{cone} B$. Given $\varepsilon > 0$, we expand the cone P to P_ε by taking $P_\epsilon = \operatorname{cone}(B_0 + B(0, \varepsilon))$. Note that P_ε is a cone such that $P \subseteq P_\varepsilon$. A cone P having a compact base is called well-based. In this case, there exists $\varepsilon > 0$ such that $P_\varepsilon \neq \mathbb{R}^n$.

Given any function $h : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of h is defined by $\operatorname{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. We say that h is a proper function if $h(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\operatorname{dom} h$ is nonempty (clearly, $h(x) = +\infty$ for every $x \notin \operatorname{dom} h$). We denote by $\operatorname{epi} h := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$ the epigraph of h and by $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ the sublevel set of h at the height $\lambda \in \mathbb{R}$. By $\operatorname{argmin}_{\mathbb{R}^n} h$ we mean the set of all minimizers of h.

A function h with convex domain is said to be:

(a) convex if, given any $x, y \in \operatorname{dom} h$, it holds that

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y), \quad \forall\, \lambda \in]0, 1[,$$

(b) quasiconvex if, given any $x, y \in \operatorname{dom} h$, it holds that

$$h(\lambda x + (1-\lambda)y) \leq \max\{h(x), h(y)\}, \quad \forall\, \lambda \in [0, 1],$$

We mention that every convex function is quasiconvex. The continuous function $h : \mathbb{R} \to \mathbb{R}$, with $h(x) := \min\{|x|, 1\}$, is quasiconvex, without being convex. Furthermore, recall that

$$h \text{ is convex} \iff \operatorname{epi} h \text{ is a convex set.}$$
$$h \text{ is quasiconvex} \iff S_\lambda(h) \text{ is a convex set, for all } \lambda \in \mathbb{R}.$$

Remark 2.1 Let K be a nonempty and convex set from $\mathbb{R}^n$, and let $h : K \to \mathbb{R}$ be a function. Given $a \in K$ and $u \in \mathbb{R}^n$, we define $I_{a,u} := \{t \in \mathbb{R} : a + tu \in K\}$ and the function $h_{a,u} : \mathbb{R} \to \mathbb{R}$ given by $h_{a,u}(t) := h(a + tu)$. Then h is convex (resp. quasiconvex) iff $h_{a,u}$ is convex (resp. quasiconvex) on $I_{a,u}$ for all $a \in K$ and all $u \in \mathbb{R}^n$ (see [22, Page 90]).

We recall next the following class of vector-valued functions.

Definition 2.1 [23, Definition 2.2] Given a closed and convex cone P from $\mathbb{R}^m$, a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be $*$-quasiconvex for P, if $x \mapsto \langle q^*, F(x)\rangle$ is quasiconvex for all $q^* \in P^*$.

Given a matrix $A \in \mathbb{R}^{n\times n}$ (not necessarily symmetric), we say that

(a) A is positive definite, if $\langle Ax, x\rangle > 0$ for all $x \neq 0$ (when A is symmetric, this is equivalent to state that all its eingenvalues are positive).
(b) A is positive semidefinite, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$ (i.e., all its eingenvalues are nonnegative, when A is symmetric).
(c) A is K-copositive, if $\langle Ax, x\rangle \geq 0$ for all $x \in K$.
(d) A is K-copositive+, if $\langle Ax, x\rangle \geq 0$ for all $x \in K$, and $\langle Ax, x\rangle = 0$ implies that $Ax = 0$ for $x \in K$.

Clearly, every positive definite matrix is positive semidefinite, and every positive semidefinite matrix is K-copositive on any nonempty set $K \subset \mathbb{R}^n$. The converse statements are not true in general.

As was explained in [11], the notions of asymptotic cone and the associated asymptotic function have been employed in optimization theory and variational analysis in order to handle unbounded and/or nonsmooth situations, in particular when standard compactness hypotheses are absent.

Given a nonempty set K from $\mathbb{R}^n$, we define the asymptotic cone of K (see [14, Definition 2.8] or [15, Definition 2.1] for infinite dimensional spaces), as

$$K^\infty := \left\{ u \in \mathbb{R}^n : \exists\, t_n \to +\infty,\ \exists\, x_n \in K : \frac{x_n}{t_n} \to u \right\}.$$

In the case when K is closed and convex, it is well known that (see [11,12])

$$K^\infty = \{u \in \mathbb{R}^n : x_0 + \lambda u \in K,\ \forall\, \lambda \geq 0\}, \tag{1}$$

where (1) does not depend on the choice of $x_0 \in K$.

We list some basic results on asymptotic cones in the following proposition. Their proofs can be found in [11, Chapter 2].

Proposition 2.1 *Let K be a nonempty set from $\mathbb{R}^n$. Then*

(a) If $K_0 \subseteq K$ then $(K_0)^\infty \subseteq K^\infty$.
(b) $K^\infty = \{0\}$ iff K is bounded.
(c) Let $\{K_i\}_{i\in I}$ be a family of subsets of $\mathbb{R}^n$. Then $\bigcup_{i\in I}(K_i)^\infty \subseteq (\bigcup_{i\in I} K_i)^\infty$, and equality holds when $|I| < +\infty$.
(d) Let $\{K_i\}_{i\in I}$ be a family of sets contained in $\mathbb{R}^n$ satisfying $\bigcap_{i\in I} K_i \neq \emptyset$. Then

$$\left(\bigcap_{i\in I} K_i\right)^\infty \subseteq \bigcap_{i\in I} (K_i)^\infty, \tag{2}$$

and equality holds when every K_i is closed and convex.

The asymptotic function $h^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$ of a proper function h, is defined by

$$\text{epi}\, h^\infty := (\text{epi}\, h)^\infty.$$

It follows that (see [14, Definition 2.2, Remark 2.17] or [15, Definition 2.1] for infinite dimensional spaces)

$$h^\infty(u) = \inf\left\{ \liminf_{k\to+\infty} \frac{h(t_k u_k)}{t_k} : t_k \to +\infty,\ u_k \to u \right\}. \tag{3}$$

Moreover, when h is proper, lsc and convex, [11, Proposition 2.5.2] (see also [14–16]) implies that

$$h^\infty(u) = \sup_{t>0} \frac{h(x_0 + tu) - h(x_0)}{t} = \lim_{t\to+\infty} \frac{h(x_0 + tu) - h(x_0)}{t}, \tag{4}$$

and all expressions in (4) do not depend on the choice of the point $x_0 \in \operatorname{dom} h$.

A proper function h is said to be coercive if $h(x) \to +\infty$ as $\|x\| \to +\infty$, or equivalently, if $S_\lambda(h)$ is bounded for all $\lambda \in \mathbb{R}$. If $h^\infty(u) > 0$ for all $u \neq 0$, then h is coercive. In addition, if h is proper, lsc and convex, then

$$h \text{ is coercive} \iff h^\infty(u) > 0,\ \forall\, u \neq 0 \iff \operatorname*{argmin}_{\mathbb{R}^n} h \neq \emptyset \text{ and compact.} \quad (5)$$

The problem of finding an adequate definition of asymptotic function in the nonconvex case has been studied in the last years, since the standard asymptotic function is not well suited for the description of the behavior of a nonconvex function at infinity. Several attempts to deal with nonconvex functions have been made in [17–19,24,25].

We recall that the q-asymptotic function of a proper function h is defined as

$$h_q^\infty(u) := \sup_{x \in \operatorname{dom} h} \sup_{t>0} \frac{h(x+tu) - h(x)}{t}. \quad (6)$$

Clearly, $h_q^\infty \geq h^\infty$, and equality holds when h is proper, lsc and convex. The fact that $h_q^\infty(u) > 0$ for all $u \neq 0$ does not imply the coercivity of h (see [17, Example 5.6]). Furthermore, for any proper function the q-asymptotic function is convex and positively homogeneous of degree one (see [19, Proposition 3.8]).

Given a proper, lsc and quasiconvex function h, [17, Theorem 4.7] implies that

$$h_q^\infty(u) > 0,\ \forall\, u \neq 0 \iff \operatorname*{argmin}_{\mathbb{R}^n} h \neq \emptyset \text{ and compact.} \quad (7)$$

A bifunction $f : K \times K \to \mathbb{R}$ is said to be:

(a) monotone on K, if for every $x, y \in K$;

$$f(x, y) + f(y, x) \leq 0, \quad (8)$$

(b) pseudomonotone on K, if for every $x, y \in K$;

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad (9)$$

(c) quasimonotone on K, if for every $x, y \in K$;

$$f(x, y) > 0 \implies f(y, x) \leq 0. \quad (10)$$

Every monotone bifunction is pseudomonotone, and every pseudomonotone bifunction is quasimonotone. The converse statements are not true in general.

Finally, an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on K if

$$\langle T(x) - T(y), x - y \rangle \geq 0, \quad \forall\, x, y \in K. \quad (11)$$

If the inequality in (11) is strict whenever $x \neq y$, then T is called strictly monotone.

For a further study on generalized convexity, generalized monotonicity and asymptotic analysis we refer to [17,18,26–29].

3 Mixed Variational Inequalities

In this section, we develop existence results and necessary and sufficient optimality conditions for the nonemptiness and compactness of the solution set of the mixed variational inequality in the nonconvex case.

Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set, $h : K \to \mathbb{R}$ a (finite-valued) function and $T : K \to \mathbb{R}^n$ an operator. Then the *mixed variational inequality* is defined as:

$$\text{find } \overline{x} \in K : \langle T(\overline{x}), y - \overline{x} \rangle + h(y) - h(\overline{x}) \geq 0, \quad \forall\, y \in K. \tag{12}$$

Its solution set will be denoted by $S(T; h; K)$.

An interesting particular case occurs when the operator is linear (affine). Take $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}^n$. Set $T(x) := Ax + a$ in problem (12). Then the corresponding mixed variational inequality is given by

$$\text{find } \overline{x} \in K : \langle A\overline{x} + a, y - \overline{x} \rangle + h(y) - h(\overline{x}) \geq 0, \quad \forall\, y \in K. \tag{13}$$

Its solution set will be denoted by $S(A; h; K)$.

We mention that for classical variational inequalities (i.e., with $h = 0$), continuity of T, together with compactness and convexity of K, ensure existence of solutions (see, e.g., [5, Theorem 3.1]). It is well known that these assumptions, and additionally continuity of h, are not enough in the case of mixed variational inequalities. Assume for instance that T is strictly monotone with $T(0) = 0$, $K = -K$, and h is such that $h(x) = h(-x)$ for all $x \in K$ and there exists $y \in K$ with $h(y) < h(0)$. In such a case, the mixed variational inequality lacks solutions, as we show next. Using $y = -\overline{x}$ in (12), and noting that $-\overline{x}$ belongs to K, we get

$$-2\langle T(\overline{x}), \overline{x} \rangle = -2\langle T(\overline{x}), \overline{x} \rangle + h(-\overline{x}) - h(\overline{x}) \geq 0,$$

implying that $\langle T(\overline{x}), \overline{x} \rangle \leq 0$. Since $T(0) = 0$ and T is strictly monotone, we have that $\langle T(x), x \rangle > 0$ for all $x \in K$, $x \neq 0$. Hence, $\overline{x} = 0$. Putting now $\overline{x} = 0$ in (12) leads to $h(y) - h(0) \geq 0$ for all $y \in K$, contradicting the assumption on h. We observe that these conditions on T, h and K, leading to lack of solutions, hold, e.g., for $T = I$, $h(x) = -\|x\|^2$ and $K = B(0, 1)$. Therefore, some further assumption on h, beyond continuity, is needed for existence of solutions, even for convex and compact K.

In this direction, we mention that if h is proper, lsc and convex, T is an upper semicontinuous point-to-set operator with compact and convex values, K is a compact and convex set, then problem (12) has a solution by Wang [8, Proposition 3.2].

For classes of nonconvex functions, we may apply existence results from the equilibrium problems theory. For instance, by taking

$$\varphi(x, y) := \langle T(\overline{x}), y - \overline{x} \rangle + h(y) - h(\overline{x}),$$

problem (12) is equivalent to

$$\text{find } \overline{x} \in K : \ \varphi(x, y) \geq 0, \quad \forall \, y \in K.$$

If φ is usc in its first argument, and quasiconvex in its second argument, then problem (12) has a solution on a compact and convex set K by Oettli [30, Theorem 2].

This situation has led to the introduction of the following property, in order to encompass all pairs of functions and operators for which problem (12) has a solution on a compact and convex set K.

Definition 3.1 [8, Definition 3.1] A pair (T, h) is said to have the mixed variational inequality property on K (MVIP on K from now on), if for every nonempty, compact and convex subset D of K, $S(T; h; D)$ is nonempty.

We present next a class of nonconvex functions for which the MVIP holds on a nonempty, closed and convex set K.

Example 3.1 Let P be a nonempty, closed, convex and pointed cone from $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ a nonempty, closed and convex set, $h : K \to \mathbb{R}$ a proper and lsc function, and $T : K \to P^*$ a continuous operator. Hence, if the function $y \mapsto (I, h)(y) := (y, h(y))$ is $*$-quasiconvex for $P \times \mathbb{R}_+$, then the pair (T, h) has the MVIP on K. Indeed, we define the function

$$\varphi_h^T(x, y) := \langle T(x), y - x\rangle + h(y) - h(x). \tag{14}$$

Here φ_h^T is usc (resp. lsc) in its first (resp. second) argument. Since $T(x) \in P^*$, $(T(x), 1) \in P^* \times \mathbb{R}_+ \subseteq (P \times \mathbb{R}_+)^*$, we have

$$y \mapsto \langle (T(x), 1), (y, h(y))\rangle = \langle T(x), y\rangle + h(y),$$

is quasiconvex for all $x \in K$. So φ_h^T is quasiconvex in its second argument and the set $\{z \in K : \varphi_h^T(x, z) < 0\}$ is convex.

Consider a nonempty, compact and convex set $D \subseteq K$. Then the equilibrium problem

$$\text{find } \overline{x} \in D : \ \varphi_h^T(\overline{x}, y) \geq 0, \quad \forall \, y \in D, \tag{15}$$

has a solution by Oettli [30, Theorem 2], i.e., the pair (T, h) has the MVIP on K.

3.1 The General Case

In order to provide an existence results for the mixed variational inequalities defined above, we consider the following assumptions:

$(T0)$ T is K-copositive operator, i.e., $\langle T(x), x\rangle \geq 0$ for all $x \in K$.
$(T1)$ T is continuous and positively homogeneous of degree one.

(Th) The pair (T, h) has the MVIP on K.

A general existence result is given below.

Theorem 3.1 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a proper and lsc function, K a nonempty, closed and convex set from $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ an operator. Suppose that assumptions $(T0)$, $(T1)$ and (Th) hold. If there exists $x_0 \in K$ such that*

$$h^{\infty}(u) - \langle T(u), x_0 \rangle > 0, \quad \forall\, u \in K^{\infty} \backslash \{0\}, \tag{16}$$

then $S(T; h; K)$ is nonempty and compact.

Proof Assume by contradiction that $S(T; h; K) = \emptyset$. Then, for every $n \in \mathbb{N}$, we define the problem VI_n as: find $\overline{x} \in K_n := K \cap B(0, n)$ such that

$$\langle T(\overline{x}), y - \overline{x} \rangle + h(y) - h(\overline{x}) \geq 0, \quad \forall\, y \in K_n. \tag{17}$$

By assumption (Th), problem VI_n has a solution, say x_n.

We claim that the sequence x_n is bounded. Indeed, assume by contradiction that $\sup_{n \in \mathbb{N}} \|x_n\| = +\infty$. Then there exists a vector $u \in K^{\infty}$ and a subsequence (still denoted by x_n) such that $\frac{x_n}{\|x_n\|} \to u \neq 0$. Thus, given any $y \in K$, there exists $n_0 \in \mathbb{N}$ with $n_0 > \|y\|$ such that

$$\begin{aligned} &\langle T(x_n), y - x_n \rangle + h(y) - h(x_n) \geq 0, \quad \forall\, n \geq n_0 \\ &\Longleftrightarrow h(x_n) \leq \langle T(x_n), y - x_n \rangle + h(y), \quad \forall\, n \geq n_0 \qquad (18) \\ &\Longrightarrow h(x_n) \leq \langle T(x_n), y \rangle + h(y), \quad \forall\, n \geq n_0. \qquad (19) \end{aligned}$$

We divide Eq. (19) by $\|x_n\|$, obtaining

$$\frac{h(x_n)}{\|x_n\|} \leq \left\langle T\left(\frac{x_n}{\|x_n\|}\right), y \right\rangle + \frac{h(y)}{\|x_n\|}, \ \forall\, n \geq n_0.$$

Since $(A0)$ holds and T is continuous, by taking the $\liminf_{n \to +\infty}$, we get

$$\liminf_{n \to +\infty} \frac{h(x_n)}{\|x_n\|} \leq \langle T(u), y \rangle.$$

Set $u_n := \frac{x_n}{\|x_n\|}$ and $t_n := \|x_n\|$, so that $x_n = t_n u_n$. Thus,

$$h^{\infty}(u) - \langle T(u), y \rangle \leq \liminf_{n \to +\infty} \frac{h(t_n u_n)}{t_n} + \langle -T(u), y \rangle \leq 0, \quad \forall\, y \in K,$$

a contradiction. Hence, the sequence x_n is bounded.

It follows that there exists a subsequence x_n^1 such that $x_n^1 \to \overline{x}$. Note that for all $y \in K$, there exists $N_0 \in \mathbb{N}$ such that

$$\langle T(x_n^1), y - x_n^1 \rangle + h(y) - h(x_n^1) \geq 0, \quad \forall\, n \geq N_0.$$

By the lower semicontinuity of h and the continuity of T, we have

$$\begin{aligned} h(\overline{x}) \leq \liminf_{n\to+\infty} h(x_n^1) &\leq \liminf_{n\to+\infty} \Big(\langle T(x_n^1), y - x_n^1\rangle + h(y)\Big) \\ &= \langle T(\overline{x}), y - \overline{x}\rangle + h(y), \quad \forall\, y \in K. \end{aligned}$$

Hence, $\overline{x} \in S(T; h; K)$.

For the boundedness of $S(T; h; K)$, assume by contradiction that there exists a sequence $x_n \in S(T; h; K)$ with $\|x_n\| \to +\infty$, thus $\frac{x_n}{\|x_n\|} \to u \in K^{\infty}$, $u \neq 0$. Then we proceed as above to get the contradiction.

Finally, set $x_n \in S(T; h; K)$ such that $x_n \to \overline{x}$. Since h is lsc, the function $x \mapsto \langle T(x), y - x\rangle + h(y) - h(x)$ is usc. So

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \left(\langle T(x_n), y - x_n\rangle + h(y) - h(x_n)\right) \\ &\leq \langle T(\overline{x}), y - \overline{x}\rangle + h(y) - h(\overline{x}), \quad \forall\, y \in K. \end{aligned}$$

Therefore, $S(T; h; K)$ is compact and the proof is complete. □

If $T(x) = Ax + a$, then we have the following corollary:

Corollary 3.1 *Let A be a K-copositive matrix, $a \in \mathbb{R}^n$ and $T(x) := Ax + a$. If there exists $x_0 \in K$ such that*

$$h^{\infty}(u) + \langle a - A^{\top} x_0, u\rangle > 0, \quad \forall\, u \in K^{\infty}\backslash\{0\}, \tag{20}$$

then $S(A; h; K)$ is nonempty and compact.

Proof Since assumption (Th) holds immediately for $T(x) = Ax + a$, we repeat the proof of Theorem 3.1 until Eq. (18). So, for every $y \in K$, there exists $n_0 \in \mathbb{N}$ such that

$$h(x_n) \leq \langle Ax_n + a, y - x_n\rangle + h(y), \quad \forall\, n \geq n_0 \tag{21}$$

$$\leq \langle Ax_n, y\rangle + \langle a, y - x_n\rangle + h(y), \quad \forall\, n \geq n_0, \tag{22}$$

where (22) follows from the K-copositiveness of A. We claim that the sequence x_n is bounded. Indeed, assume by contradiction that $\sup_{n\in\mathbb{N}} \|x_n\| = \infty$. Then, we divide Eq. (22) by $\|x_n\|$, obtaining

$$\frac{h(x_n)}{\|x_n\|} \leq \left\langle A\frac{x_n}{\|x_n\|}, y\right\rangle + \left\langle a, \frac{y}{\|x_n\|} - \frac{x_n}{\|x_n\|}\right\rangle + \frac{h(y)}{\|x_n\|}, \quad \forall\, n \geq n_0.$$

So,

$$h^{\infty}(u) \leq \langle Au, y\rangle + \langle a, -u\rangle,\ \forall\, y \in K \iff h^{\infty}(u) + \langle a - A^{\top} y, u\rangle \leq 0, \quad \forall\, y \in K,$$

sequence is bounded and $S(A; h; K)$ is nonempty.
The compactness of the solution set follows as in the proof of Theorem 3.1. □

As far as we know, Theorem 3.1 applies to a class of problems for which no other existence result does. We illustrate this remark with the following example.

Example 3.2 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(x) := (T_1(x), \dots, T_n(x))$ with $T_i(x) = \max\{0, x_i\}$ for all $i \in \{1, \dots, n\}$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be given by,

$$h(x) = \begin{cases} \|x\|_1^2, & \text{if } x_i < 0 \text{ for all } i, \\ \min\{1, \|x\|_1^2\}, & \text{elsewhere.} \end{cases}$$

Let K be a nonempty, closed, convex and pointed cone in $\mathbb{R}^n$ for which $e_i \notin K$ for all $i \in \{1, 2, \dots, n\}$ (no canonical vectors belongs to K) and such that is not contained in the positive orthant $\mathbb{R}^n_+$ and its intersection with the negative orthant is the null vector $K \cap \mathbb{R}^n_- = \{0\}$.

Note that the operator T is neither affine nor semicoercive (see Remark 3.1(i) below), and that h is a lsc function which is neither convex nor coercive. So, the existence results in [1–4,6–10] and references therein can not be applied.

On the other hand, T is continuous, positively homogeneous of degree one and $\mathbb{R}^n$-copositive. Furthermore, $T(u) \in \mathbb{R}^n_-$ iff $u_i \le 0$ for all $i \in \{1, 2, \dots, n\}$. Given $u \in \mathbb{R}^n$, an easy calculation shows that

$$h^{\infty}(u) = \begin{cases} +\infty, & \text{if } u_i < 0 \text{ for all } i, \\ 0, & \text{elsewhere.} \end{cases}$$

Observe that, with this construction, $\varphi : K \times K \to \mathbb{R}$ given by

$$\begin{aligned} \varphi(x, y) &:= \langle T(x), y - x\rangle + h(y) - h(x), \\ &= \langle T(x), y\rangle + h(y) - (h(x) + \langle T(x), x\rangle), \end{aligned} \tag{23}$$

is quasiconvex on K in its second argument. Indeed, given $y_0 \in K$ and $u \in \mathbb{R}^n$, the functions $t \mapsto h_{y_0,u}(t) = h(y_0 + tu)$ and $t \mapsto \psi_{y_0,u}(t) = \langle T(x), y_0 + tu\rangle$ are nondecreasing on $I_{y_0,u} := \{t \in \mathbb{R} : y_0 + tu \in K\}$, so its sum is nondecreasing too, i.e., quasiconvex on $I_{y_0,u}$ by Hadjisavvas [31, Proposition 4.9]. Hence, φ is quasiconvex on K by Remark 2.1. Therefore, it follows from [30, Theorem 2] that (T, h) has the MVIP on K.

Finally, there exists $x_0 \in K$ (actually, for all $x_0 \in K\backslash\{0\}$) such that

$$h^{\infty}(u) + \langle T(u), x_0\rangle > 0, \quad \forall\, u \in K^{\infty}\backslash\{0\}. \tag{24}$$

Hence, the solution set $S(T; h; K)$ is nonempty and compact by Theorem 3.1.

Remark 3.1 (i) In [2,7,10], the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be semicoercive, meaning that there exists $c_0 > 0$ such that

$$\langle T(x), x\rangle \ge c_0\|x\|^2, \quad \forall\, x \in \mathbb{R}^n. \tag{25}$$

Clearly, every semicoercive operator T is K-copositive.
(ii) If h is proper, lsc and convex, then Corollary 3.1 coincide with [4, Theorem 8].

3.2 Affine Operators

In this subsection, we study problem (13). As a first result, we have the following necessary condition.

Proposition 3.1 *If $S(A; h; K) \neq \emptyset$, then $\sup_{y \in K} \langle Ay + a, u\rangle + h_q^{\infty}(u) \geq 0$ for all $u \in K^{\infty}$. In addition, if A is symmetric, then*

$$\begin{aligned} S(A; h; K) \neq \emptyset \text{ and compact } &\Longrightarrow h_q^{\infty}(u) + \langle a, u\rangle > 0, \\ &\forall\, u \in (K^{\infty} \cap \text{ Ker } A)\backslash\{0\}. \end{aligned} \tag{26}$$

Proof Assume by contradiction that there exists $u_0 \in K^{\infty}, u_0 \neq 0$, such that $\sup_{y \in K} \langle Ay + a, u_0\rangle + h_q^{\infty}(u_0) < 0$, that is,

$$\begin{aligned} &\langle Ax + a, u_0\rangle + \frac{h(x + tu_0) - h(x)}{t} < 0, \quad \forall\, x \in K,\ \forall\, t > 0 \\ &\quad \Longleftrightarrow \langle Ax + a, (x + tu_0) - x\rangle + h(x + tu_0) - h(x) < 0, \quad \forall\, x \in K,\ \forall\, t > 0. \end{aligned}$$

The contradiction follows by taking $x = \overline{x} \in S(A; h; K)$.

For (26): Assume by contradiction that there exists $u_0 \in K^{\infty} \cap \text{Ker } A$, with $u_0 \neq 0$, such that

$$\begin{aligned} &\langle a, u_0\rangle + h_q^{\infty}(u_0) \leq 0 \Longleftrightarrow t\langle a, u_0\rangle + h(x + tu_0) - h(x) \leq 0, \quad \forall x \in K, \forall t > 0 \\ &\quad \Longleftrightarrow -t\langle A(x + tu_0) + a, u_0\rangle + h(x) - h(x + tu_0) \geq 0, \quad \forall x \in K, \forall t > 0, \end{aligned} \tag{27}$$

because $\langle A(x + tu_0), u_0\rangle = \langle (x + tu_0), Au_0\rangle = 0$.

By adding the term $\pm\langle Ax + a, y - x\rangle \pm h(y)$ to Eq. (27), it follows that

$$\begin{aligned} 0 \leq\ &(\langle A(x + tu_0) + a, y - (x + tu_0)\rangle + h(y) - h(x + tu_0)) \\ &-(\langle Ax + a, y - x\rangle + h(y) - h(x)), \quad \forall\, x, y \in K,\ \forall\, t > 0. \end{aligned} \tag{28}$$

Take $x = \overline{x} \in S(A; h; K)$ in (28). Then

$$0 \leq \langle A(\overline{x} + tu_0) + a, y - (\overline{x} + tu_0)\rangle + h(y) - h(\overline{x} + tu_0), \quad \forall\, y \in K,\ \forall\, t > 0,$$

i.e., $\overline{x} + tu_0 \in S(A; h; K)$ for all $t > 0$, contradicting the fact that $S(A; h; K)$ is compact. □

Remark 3.2 Assumption (16), which is a kind of weak coercivity condition, is connected with the necessary condition NC for existence of solutions given in [32, Proposition 2.6] in an infinite dimensional setting.

The following result may be found in [4, Proposition 3.2].

Proposition 3.2 *If $x_1, x_2 \in S(A; h; K)$, then $\langle A(x_1 - x_2), x_1 - x_2\rangle \leq 0$. In addition, if A is symmetric and positive semidefinite, then $x_1 - x_2 \in$ Ker A.*

The positive semidefinite assumption on the matrix A cannot be replaced by a K-copositive or K-copositive+ assumption, since $x_1, x_2 \in K$ does not imply (necessarily) that $x_1 - x_2 \in K$.

Since K is a closed and convex set, K^∞ is a closed and convex cone. Given $\varepsilon > 0$, we expand K^∞ to K^∞_ε as defined in Sect. 2. Consider the following assumptions:

($A0$) K^∞ is well-based, and A is symmetric and K^∞_ϵ-copositive+.
($h0$) h is proper, lsc and $0 \leq \liminf_{\|x\| \to +\infty} \frac{h(x)}{\|x\|^2}$.

Another existence result is given below. Note that in comparison with Corollary 3.1, we impose stronger conditions on h in order to weaken the assumptions on $T(x) = Ax + a$.

Theorem 3.2 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function, and K a nonempty, closed and convex set from $\mathbb{R}^n$. Suppose that assumptions ($A0$) and ($h0$) hold. Then*

$$\begin{aligned} & h^\infty(u) + \langle a, u\rangle > 0, \ \forall\, u \in (K^\infty \cap \text{ Ker } A)\backslash\{0\} \\ & \Longrightarrow \ S(A; h; K) \neq \emptyset \ \text{ and compact.} \end{aligned} \tag{29}$$

Proof We repeat the proof given in Corollary 3.1 until Eq. (21). So, for every $y \in K$, there exists $n_0 \in \mathbb{N}$ such that

$$h(x_n) \leq \langle Ax_n, y - x_n\rangle + \langle a, y - x_n\rangle + h(y), \quad \forall\, n \geq n_0. \tag{30}$$

We claim that the sequence x_n is bounded. Indeed, assume by contradiction that $\sup_{n\in\mathbb{N}} \|x_n\| = +\infty$, then $\frac{x_n}{\|x_n\|} \to u \in K^\infty$ with $u \neq 0$.

Dividing by $\|x_n\|^2$, we have

$$\frac{h(x_n)}{\|x_n\|^2} \leq \left\langle A\frac{x_n}{\|x_n\|} + \frac{a}{\|x_n\|}, \frac{y}{\|x_n\|} - \frac{x_n}{\|x_n\|}\right\rangle + \frac{h(y)}{\|x_n\|^2}, \quad \forall\, n \geq n_0.$$

Since ($h0$) and ($A0$) holds, we have

$$0 \leq \liminf_{n\to+\infty} \frac{h(x_n)}{\|x_n\|^2} \leq \langle Au, -u\rangle \implies \langle Au, u\rangle \leq 0 \implies u \in \text{ Ker } A.$$

Now, we divide Eq. (30) by $\|x_n\|$, then

$$\begin{aligned} \frac{h(x_n)}{\|x_n\|} &\leq \left\langle A\frac{x_n}{\|x_n\|}, y - x_n\right\rangle + \left\langle a, \frac{y - x_n}{\|x_n\|}\right\rangle + \frac{h(y)}{\|x_n\|}, \quad \forall\, n \geq n_0 \\ &= \left\langle A\frac{x_n}{\|x_n\|}, y\right\rangle - \|x_n\| \left\langle A\frac{x_n}{\|x_n\|}, \frac{x_n}{\|x_n\|}\right\rangle + \left\langle a, \frac{y - x_n}{\|x_n\|}\right\rangle + \frac{h(y)}{\|x_n\|}, \quad \forall\, n \geq n_0. \end{aligned}$$

Since ($A0$) holds, there exists $n_1 \in \mathbb{N}$ (with $n_1 \geq n_0$) such that for all $n \geq n_1$

$$\left\langle A\frac{x_n}{\|x_n\|}, \frac{x_n}{\|x_n\|}\right\rangle \geq 0 \iff -\|x_n\| \left\langle A\frac{x_n}{\|x_n\|}, \frac{x_n}{\|x_n\|}\right\rangle \leq 0. \tag{31}$$

Hence,

$$\frac{h(x_n)}{\|x_n\|} \leq \left\langle A\frac{x_n}{\|x_n\|}, y\right\rangle + \left\langle a, \frac{y-x_n}{\|x_n\|}\right\rangle + \frac{h(y)}{\|x_n\|}, \quad \forall\, n \geq n_1.$$

Taking $\liminf_{n\to+\infty}$ and using the fact that $u \in \operatorname{Ker} A$, we obtain

$$\liminf_{n\to+\infty} \frac{h(x_n)}{\|x_n\|} \leq \langle a, -u\rangle.$$

Thus, $h^\infty(u) + \langle a, u\rangle \leq 0$, a contradiction. Hence, the sequence x_n is bounded and $S(A; h; K)$ is nonempty. The compactness of $S(A; h; K)$ follows as always. □

The breadth of our previous existence result is analyzed next.

Remark 3.3 (i) The fact that A is K_ε^∞-copositive+ (or K^∞-copositive+) does not imply that A is K_ε-copositive+ (or K-copositive+) in the general case. Indeed, consider the closed and convex set $K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq \frac{1}{2}(x_2)^2\}$, and the symmetric matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $K^\infty = \mathbb{R}_+ \times \{0\}$, for $u = (1, 0) \in K^\infty$ we have $\langle Au, u\rangle = 1$, i.e., A is K_ε^∞-copositive+ for some small enough $\varepsilon > 0$, but A is not K-copositive+ (take $(x, y) = (\frac{1}{2}, -1) \in K$).
(ii) Every proper, lsc and convex function h satisfies $(h0)$. Indeed, since h is convex, there exists an affine function $l(x) := \langle b, x\rangle + \beta$, with $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, such that $h(x) \geq \langle b, x\rangle + \beta$ for all $x \in \mathbb{R}^n$. Then

$$\liminf_{\|x\|\to+\infty} \frac{h(x)}{\|x\|^2} \geq \liminf_{\|x\|\to+\infty} \frac{\langle b, x\rangle + \beta}{\|x\|^2} = 0.$$

We refer to [4, Chapter 4] for the convex case.
(iii) Every proper, lsc and coercive function h satisfies assumption $(h0)$. Indeed, suppose to the contrary that $\liminf_{\|x\|\to+\infty} \frac{h(x)}{\|x\|^2} < 0$. Then, we have $\liminf_{\|x\|\to+\infty} h(x) < 0$, a contradiction.
The uniqueness of the solution follows easily when A is positive definite.

Corollary 3.2 *Let A be a symmetric matrix. Suppose that assumption $(h0)$ holds. Then:*

(i) *If A is positive semidefinite, then problem (13) has a solution. Moreover, if $x_1, x_2 \in S(A; h; K)$, then $x_1 - x_2 \in \operatorname{Ker} A$ and $\langle a, x_1 - x_2\rangle = h(x_2) - h(x_1)$.*
(ii) *If A is positive definite, then problem (13) has a unique solution.*

Proof Since $T(x) = Ax + a$ is continuous, (i) follows in a way similar to [4, Corollary 8(c)] and for (ii), we simply apply Proposition 3.2. □

By using Remark 3.3(ii), we obtain another corollary:

Corollary 3.3 *Suppose that assumption* (*A*0) *holds. If h is proper, lsc and convex, then*

$$\begin{aligned} & h^{\infty}(u) + \langle a, u \rangle > 0, \ \forall\, u \in (K^{\infty} \cap \ \mathrm{Ker}\, A)\backslash\{0\} \\ & \Longleftrightarrow \ S(A; h; K) \neq \emptyset \ \text{ and compact}. \end{aligned} \tag{32}$$

Proof (⇒) Immediately from Theorem 3.2.

(⇐) We just apply (26), noting that $h_q^{\infty} = h^{\infty}$ whenever h is proper, lsc and convex. □

The previous corollary provides more information about the solution set than well-known results (see [4, Chapter 4] for the convex case). Note that the usual assumption $0 \in K$ was not needed (see [1,2,7]).

3.3 A Comparison with Equilibrium Problems

As was noted in [21], equilibrium problems encompass several problems found in fixed point theory, optimization and nonlinear analysis, e.g., minimization problems, linear complementary problems, minimax problems and variational inequalities among others. Hence, a comparison between our previous existence results for mixed variational inequalities with existence results for equilibrium problem should be made.

Let K be a nonempty, closed and convex set from $\mathbb{R}^n$, and $f : K \times K \to \mathbb{R}$ be a function. The equilibrium problem (see [21]) is defined as

$$\text{find } \overline{x} \in K : \ f(\overline{x}, y) \geq 0, \quad \forall\, y \in K. \tag{33}$$

Define $f_h^A : K \times K \to \mathbb{R}$ as

$$f_h^A(x, y) := \langle Ax + a, y - x \rangle + h(y) - h(x). \tag{34}$$

In order to apply the existence results for equilibrium problems given in [20,33–35] and references therein, the function f_h^A should be pseudomonotone or quasimonotone (see Sect. 2). That is the case when A is positive semidefinite.

Proposition 3.3 *If A is positive semidefinite, then f_h^A is monotone.*

Proof Indeed, observe that for all $x, y \in K$, we have

$$\begin{aligned} f_h^A(x, y) + f_h^A(y, x) &= \langle Ax + a, y - x \rangle + \langle Ay + a, x - y \rangle \\ &= \langle Ax - Ay, y - x \rangle = -\langle A(x - y), x - y \rangle. \end{aligned}$$

Since A is positive semidefinite, $f_h^A(x, y) + f_h^A(y, x) \leq 0$ for all $x, y \in K$, i.e., f_h^A is monotone. □

If A is not positive semidefinite, then f_h^A is not necessarily pseudomonotone, as the following example shows.

Example 3.3 We consider $a = 0$ and the symmetric matrix A given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Take the closed and convex set $K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq \frac{1}{2}(x_2)^2\}$. By Remark 3.3(i), we known that A is K_ε^∞-copositive+ for $\varepsilon > 0$ small enough.

Consider the quasiconvex function $h(x_1, x_2) := -(x_1)^3$. Take $x, y \in K$ given by $x = (1, 0)$ and $y = (\frac{1}{2}, 1)$. Then

$$f_h^A(x, y) = \langle Ax, y - x\rangle + h(y) - h(x) = \left\langle (1, 0), \left(-\frac{1}{2}, 1\right)\right\rangle - \frac{1}{8} + 1 = \frac{3}{8} > 0,$$
$$f_h^A(y, x) = \langle Ay, x - y\rangle + h(x) - h(y) = \left\langle \left(\frac{1}{2}, -1\right), \left(\frac{1}{2}, -1\right)\right\rangle - 1 + \frac{1}{8} = \frac{3}{8} > 0.$$

Therefore, f_h^A is not pseudomonotone.

As a consequence, Theorem 3.2 applies for classes of variational inequality problems for which the existence results presented in [20,33–35] cannot be applied.

4 Conclusions

We established new existence results for noncoercive mixed variational inequalities in the nonconvex case. By using the structure of the problem, finer optimality conditions under weaker assumptions on the operator and the function are provided. If the function is convex, a full characterization of the nonemptiness and compactness of the solution set is given, when the operator is affine. In a subsequent work, we hope to deal with the case when the function is quasiconvex. However, since the sum of quasiconvex functions is not necessarily quasiconvex even when one of them is a linear (affine) function, this problem is expected to be harder.

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